Effective Actions Near Singularities: The STU-Model [1]

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Abstract: We derive the low energy effective action of the STU-model in four and five dimensions near the line $T = U$, where $SU(2)$ gauge symmetry enhancement occurs. By 'integrating in' the light $W^{\pm}$ bosons together with their superpartners, the quantum corrected effective action becomes non-singular at $T = U$ and manifestly $SU(2)$ invariant. The four-dimensional theory is found to be consistent with modular invariance and the five-dimensional decompactification limit.

1 Introduction

One of the, at first sight, somewhat irritating features of string theories (or M-theory) is that their mathematical consistency requires the existence of more than four space-time dimensions. This apparent discrepancy with observation is conventionally accomodated by demanding the additional dimensions to be compact. Assuming the compact part of space-time and the string length to be sufficiently small, the resulting spectrum allows a clear separation between very heavy modes and modes that are very light, or massless. If one is only interested in low energy processes, the heavy modes can be integrated out, and one is left with an effective, lower-dimensional field theory that describes all low-energy phenomena.

The set of light fields of such a string compactification typically includes a number of scalar fields whose vacuum expectation values (vevs) are not fixed by a perturbative scalar potential. The vevs of some of these moduli fields have a simple geometrical interpretation in that they parameterize deformations of the compactification manifold that cost no energy.

For certain values of these deformation parameters, one sometimes encounters the phenomenon that some of the generically heavy modes suddenly become light. As a classical example, consider a string compactification that involves an S^1 factor. The radius, R, of the circle is in general not fixed by perturbative string physics and thus gives rise to a modulus, ϕ, whose vev is proportional to the radius: $\langle\phi\rangle = aR$. At $\langle\frac{\phi}{a}\rangle = \sqrt{\alpha'}$, the circle is at its self-dual radius, and certain otherwise massive Kaluza-Klein and winding modes become massless (and lead to an $SU(2)$ gauge symmetry enhancement).

As another, non-perturbative, example (which can, however, sometimes be dual to the previous one (see below)), consider a compactification manifold with a non-trivial p-cycle.

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If the string (or M-)theory in question has appropriate solitonic objects (p-branes), these may wrap around the cycle and give rise to pointlike particles in the uncompactified dimensions [1]. The mass of such particles is proportional to the volume of the p-cycle. When the size of the cycle corresponds to a modulus in the low energy effective theory, a vanishing vev for this modulus field corresponds to the cycle being collapsed to zero size. The wrapped brane then gives rise to additional massless states in the lower-dimensional theory at that particular point in the moduli space.

Clearly, away from these special points, such extra states are heavy and should be integrated out of the effective action. Near the points in moduli space where they become light, however, it is inaccurate, or rather inconsistent, to neglect them in the low energy theory. This inconsistency is typically reflected in couplings becoming singular or discontinuous when the modulus reaches the special value at which additional fields become massless. In order to obtain a consistent and non-singular action, one would have to avoid integrating out the extra light modes, at least in the region of the moduli space where they are light.

This can, in principle, be achieved in two different ways:
(i) Either one uses a microscopic string calculation in order to determine all the low energy couplings of the extra modes, or
(ii) one hopes that the generic low energy effective action without the extra states is sufficiently well-known and that there are sufficiently many symmetries involved so that one can "integrate the extra modes back in" by simply exploiting all these symmetries.

The second ("bottom-up") method might not always be applicable, but if it is, it can be much simpler than the first, especially, when the microscopic formulation is not so well understood (such as, e.g., in the case of M-theory). In this talk I summarize two non-trivial examples [2, 3] in which the second method has been carried out in full detail. For applications in the context of geometric phase transitions, I refer to T. Mohaupt's talk [4].

2 The STU-model in five dimensions

The model we are going to study first is the $E_8 \times E_8$ heterotic string on $K3 \times S^1$ with instanton numbers $(14, 10)$ [5]. This model is believed to be dual [6, 7] to M-theory compactified on the Calabi-Yau threefold $Y_{1,1,2,8,12}(24)$ [8, 9, 10], which is an elliptic fibration over the second Hirzebruch surface $\mathbb{F}_2$. The generic low energy effective action of this compactification describes the coupling of two Abelian vector multiplets and 244 neutral hypermultiplets to five-dimensional (5D), $\mathcal{N} = 2$ (i.e., minimal) supergravity. The hypermultiplets play no rôle in the following and will be consistently truncated out. The bosonic part of the Lagrangian is of the form

$$\begin{aligned} e^{-1}\mathcal{L}_{\text{bosonic}} &= -\frac{1}{2}R - \frac{1}{4}\overset{\circ}{a}_{IJ}F^I_{\mu\nu}F^{\mu\nu J} - \frac{1}{2}g_{xy}(\partial_\mu\phi^x)(\partial^\mu\phi^y) \\ &\quad + \frac{e^{-1}}{6\sqrt{6}}C_{IJK}\varepsilon^{\mu\nu\rho\sigma\lambda}F^I_{\mu\nu}F^J_{\rho\sigma}A^K_\lambda, \end{aligned} \tag{1}$$

where ϕ^x $(x = 1, 2)$ are two real scalars, and the index $I = 0, 1, 2$ collectively labels the graviphoton and the vector fields from the two vector multiplets. The completely symmetric tensor C_{IJK} in the FFA term of (1) is independent of the scalar fields and completely determines the entire theory via a cubic polynomial (or "prepotential") [11]

$$\mathcal{V}(h) := C_{IJK}h^I h^J h^K \tag{2}$$

in three real variables h^I $(I = 0, 1, 2)$. $\mathcal{V}(h)$ endows the auxiliary space $\mathbb{R}^3$ spanned by the h^I with a metric,

$$a_{IJ}(h) := -\frac{1}{3}\frac{\partial}{\partial h^I}\frac{\partial}{\partial h^J}\ln\mathcal{V}(h). \tag{3}$$

The two-dimensional target space, $\mathcal{M}$, of the scalar fields ϕ^x can then be represented as the hypersurface [11]

$$\mathcal{V}(h) = C_{IJK}h^I h^J h^K = 1 \tag{4}$$

with g_{xy} being the pull-back of (3) to $\mathcal{M}$. The quantity $\overset{\circ}{a}_{IJ}(\phi)$ appearing in (1), finally, is given by the restriction of a_{IJ} to $\mathcal{M}$: $\overset{\circ}{a}_{IJ}(\phi) = a_{IJ}|_{\mathcal{V}=1}$. Due to these relations, the physical requirement that g_{xy} and $\overset{\circ}{a}_{IJ}$ be positive definite imposes constraints on the possible prepotentials $\mathcal{V}$.

For our particular string compactification, these constraints are, of course, satisfied, and the corresponding prepotential reads [5]

$$\mathcal{V}(S, T, U) = STU + \frac{1}{3}U^3, \tag{5}$$

where, as is common in the literature, the letters S, T, U are used instead of the variables h^0, h^1, h^2. This prepotential is valid in the region $T > U$. For $T < U$, it has to be replaced by [5, 2]

$$\mathcal{V}(S, T, U) = STU + \frac{1}{3}U^3 + \frac{1}{3}(T-U)^3. \tag{6}$$

Obviously, (5) and (6) combine to form a continuous function $\mathcal{V}$ at $T = U$. The couplings in the Lagrangian (1), however, depend on derivatives of $\mathcal{V}$, and these are not all continuous at $T = U$. The physical reason for these discontinuities is exactly as described in the Introduction: At $\langle T - U\rangle = 0$, the circle of the heterotic compactification manifold $K3 \times S^1$ is at its self-dual radius, and two additional vector multiplets (containing two $W^{\pm}$ bosons) become light and restore an $SU(2)$ gauge symmetry. In the dual M-theory picture, this very same situation corresponds to a collapsed 2-cycle in the Calabi-Yau threefold, and the two additional vector multiplets are supplied by the zero modes of wrapped M2 branes [12]. Thus, near $T = U$, a complete low energy effective theory should also contain these two additional vector multiplets. Let us use $\mathcal{L}_{\rm in}$ do denote the Lagrangian of this more complete theory ("in", because the two extra vector multiplets have been "integrated back in"). $\mathcal{L}_{\rm in}$ has to have the following properties:
(i) Just as $\mathcal{L}$ in eq. (1), it is based on a cubic polynomial, but now this is a polynomial in $3 + 2 = 5$ variables. We call this polynomial $\mathcal{V}_{\rm in}$.
(ii) Whereas $\mathcal{L}$ is Abelian, $\mathcal{L}_{\rm in}$ exhibits $SU(2)$ as a Yang-Mills-type gauge symmetry. This implies that three of the five variables entering $\mathcal{V}_{\rm in}$ transform in the adjoint of $SU(2)$; the other two are $SU(2)$ singlets. We use C^a $(a = 1, 2, 3)$ to denote the $SU(2)$ triplet and Z^1 and Z^2 for the two singlets. The polynomial $\mathcal{V}_{\rm in}(Z^1, Z^2, C^a)$ has to be $SU(2)$ invariant.
(iii) The metrics g_{xy} and $\overset{\circ}{a}_{IJ}$ that follow from $\mathcal{V}_{\rm in}$ have to be positive definite.
(iv) Integrating out the two extra multiplets from $\mathcal{L}_{\rm in}$ should reproduce $\mathcal{L}$ and the underlying prepotential $\mathcal{V}(S, T, U)$ (eqs. (5), (6)).

In [2], it was shown that, modulo reparameterizations, $\mathcal{V}_{\rm in}(Z^1, Z^2, C^a)$ (and with it $\mathcal{L}_{\rm in}$) is completely fixed by (i)-(iv). The approach was to view the integrating out process $\mathcal{V}_{\rm in} \to \mathcal{V}$ as a two-step procedure, in which, in the language of Feynman diagrams, the two extra multiplets represented by, say, C^1 and C^2 are first removed as external lines by setting them equal to zero in $\mathcal{V}_{\rm in}$. This yields an intermediate (unphysical) prepotential

$$\mathcal{V}_{\rm in}^{\rm truncated} := \mathcal{V}_{\rm in}|_{C^1=C^2=0}. \tag{7}$$

In a second step, one then has to take into account that the two extra multiplets can also occur as internal lines and run in loops. Integrating them out will thus also produce additional effective interactions among the remaining fields. These induced interactions are subsumed in an additional contribution $\delta\mathcal{V}$, so that

$$\mathcal{V} = \mathcal{V}_{\rm in}^{\rm truncated} + \delta\mathcal{V}. \tag{8}$$

In order to determine $\mathcal{V}_{\rm in}$, one simply has to go backwards. As explained in [12, 13], $\delta\mathcal{V}$ is simply given by a one-loop correction to the FFA term in $\mathcal{L}$, which in our case turns out to be $\delta\mathcal{V} = -(T-U)^3/6$, implying

$$\mathcal{V}_{\rm in}^{\rm truncated} = STU + \frac{1}{3}U^3 + \frac{1}{6}(T-U)^3. \tag{9}$$

This truncated prepotential now has to be "untruncated", i.e., the variables (S, T, U) have to be transformed to (Z^1, Z^2, C^3), and C^1 and C^2 have to be appropriately re-inserted. Using the list of admissible polynomials given in [14], one can show that, modulo obvious linear transformations, there is essentially only one way to do that [2]: The two $SU(2)$ singlets are given by $Z^1 = S - (T-U)/2$ and $Z^2 = (T+U)/2$, whereas $C^3 = (T-U)/2$. In terms of these variables, $\mathcal{V}_{\rm in}^{\rm truncated}$ becomes quadratic in C^3, and re-introducing C^1 and C^2 is simply done by $SU(2)$ covariantization: $(C^3)^2 \to [(C^1)^2 + (C^2)^2 + (C^3)^2]$.

3 The STU-model in four dimensions

In the previous section, we have reconstructed the low energy effective theory of a five-dimensional string (or M-theory) compactification near an $SU(2)$ enhancement line in the moduli space. The one-loop threshold effects allowed by $\mathcal{N} = 2$ supersymmetry added a certain degree of non-triviality to this exercise. Nevertheless, one might wonder how much this construction relied on the purely cubic form of the prepotential in five dimensions. Let us therefore, in this section, consider the same theory compactified to four dimensions. In other words, we are now considering the heterotic string on $K3 \times T^2$ with instanton numbers (14,10) or, equivalently, type IIA string theory on the Calabi-Yau manifold $Y_{1,1,2,8,12}(24)$ [8, 10, 15, 16, 17]. The generic low energy effective theory describes 4D, $\mathcal{N} = 2$ supergravity coupled to three Abelian vector multiplets and 244 neutral hypermultiplets. Again, the hypermultiplets can be ignored, and the vector multiplet couplings are summarized in terms of a prepotential, $\mathcal{F}$ [18]. Just as in five dimensions, this prepotential depends on three variables (S, T, U). This time, however, the fields (S, T, U) are complex, rather than real, and they do not have to satisfy a hypersurface constraint of the form (4). Furthermore, the prepotential $\mathcal{F}$ no longer has to be cubic. Instead, $\mathcal{F}$ can now be a rather arbitrary holomorphic function of the moduli (with possible logarithmic branch cuts and singularities). For $\mathrm{Re}\, S > \mathrm{Re}\, T > \mathrm{Re}\, U > 0$, $\mathcal{F}$ turns out to be [17]

$$\mathcal{F} = STU + \frac{1}{3}U^3 + \frac{2}{(2\pi)^3} Li_3\left(e^{-2\pi(T-U)}\right) + \frac{2}{(2\pi)^3}\sum_{k,l=0}^{\infty} c_1(kl) Li_3\left(e^{-2\pi(kT+lU)}\right) + \mathcal{F}^{(NP)}\ , \tag{10}$$

where Li_3 denotes the third polylog (see, e.g., [17]), and the coefficients $c_1(kl)$ can be found in [17]. The term $\mathcal{F}^{(NP)}$ summarizes non-perturbative corrections.

The prepotential (10) has a logarithmic singularity along the surface $T = U$, where two additional vector multiplets become massless and enhance the gauge group to $U(1)^3 \times SU(2)$. At $T = U = 1$ and $T = U = e^{i\pi/6}$, the gauge group is further enhanced to, respectively, $U(1)^2 \times SU(2)^2$ and $U(1)^2 \times SU(3)$. In contrast to the 5D case, these symmetry enhancements do not survive non-perturbative quantum corrections [8, 19], and we therefore have to restrict our considerations to the perturbative heterotic string, dropping from now on the term $\mathcal{F}^{(NP)}$.

We will now try to derive an effective action, $\mathcal{L}_{\rm in}$, that includes the two additional light vector multiplets near $T = U$ and describes the $SU(2)$ gauge symmetry enhancement [3]. Except at the discrete points where further gauge symmetry enhancement occurs, we expect this effective theory to be non-singular. The underlying prepotential will be called $\mathcal{F}_{\rm in}$. Just as its five-dimensional analogue, $\mathcal{F}_{\rm in}$ is a function of 3+2=5 variables: one $SU(2)$ triplet C^a $(a = 1, 2, 3)$ and two singlets, Z^1 and Z^2. Again, we expect $\mathcal{F}$ and $\mathcal{F}_{\rm in}$ to be related by a relation of the form

$$\mathcal{F} = \mathcal{F}_{\rm in}^{\rm truncated} + \delta\mathcal{F}, \tag{11}$$

where $\mathcal{F}_{\rm in}^{\rm truncated} := \mathcal{F}_{\rm in}|_{C^1=C^2=0}$, and $\delta\mathcal{F}$ subsumes the threshold effects.

Let us introduce $T_\pm := (T \pm U)/2$. Obviously, T_- represents the order parameter of the symmetry breaking $SU(2) \to U(1)$, and when we identify $(C^1 \pm iC^2)$ with the scalar superpartners of the $W^\pm$-bosons (as we implicitly did above), we have to identify $C^3 = T_-$. The mass of the $W^\pm$ bosons is then proportional to the vev of $|T_-|$. Integrating them out induces a one-loop threshold correction to the gauge coupling of the vector field A_μ^-, i.e., the superpartner of the scalar field T_-. This correction is of the form [15, 20, 21] $\delta g^{-2} \sim log|m|^2 \sim log|T_-|^2$. The quantity g^{-2} is essentially the second derivative of the prepotential, and if one inserts all prefactors correctly[3], one deduces that

$$\delta\mathcal{F} = -\frac{2}{\pi}T_-^2 \log T_- + A_2 T_-^2 + A_1(T_+)T_- + A_0(T_+). \tag{12}$$

Here, A_2 is an arbitrary constant related to the cut-off scale, and $A_1(T_+)$ and $A_0(T_+)$ are a priori undermined functions.

Using (10) and (11), one can now solve for $\mathcal{F}_{\rm in}^{\rm truncated}$. Expanding the first polylogarithmic term in (10), one then finds that the logarithmic singularity in $\mathcal{F}$ is precisely cancelled by $\delta\mathcal{F}$ as given in (12) [3]. Thus, $\mathcal{F}_{\rm in}^{\rm truncated}$ is regular at $T = U$, as it should.

It remains to "untruncate" $\mathcal{F}_{\rm in}^{\rm truncated}$ to obtain the desired function $\mathcal{F}_{\rm in}$. Just as in five dimensions, this is done by first going over from the variables (S, T, U) (or (S, T_+, T_-)) to the $SU(2)$ covariant variables (Z^1, Z^2, C^3) and then replacing everywhere $(C^3)^2$ by $[(C^1)^2 + (C^2)^2 + (C^3)^2]$. Modulo obvious linear combinations, the right change of variables turns out to be the same as in five dimensions [3]: $Z^1 = S - T_-$, $Z^2 = T_+$, $C^3 = T_-$. Modulo theta angles, one then finds that [3], in terms of the variables (Z^1, Z^2, C^3), C^3 only appears with *even* powers, provided that the as yet undetermined function $A_1(T_+)$ vanishes identically: $A_1(T_+) \equiv 0$. The substitution $(C^3)^2 \to [(C^1)^2 + (C^2)^2 + (C^3)^2]$ can then be readily performed.

It remains to determine the remaining unknown function $A_0(T_+)$ that appeared in eq. (12). As is shown in [3], a diagonal $SL(2, \mathbb{Z})$ subgroup of the perturbative quantum symmetry $SL(2, \mathbb{Z})_T \times SL(2, \mathbb{Z})_U$ remains unbroken on the line $T = U$ in the 'in' theory and implies $\partial_+^5 A_0(T_+) \equiv 0$. Hence, $A_0(T_+)$ can at most be a quartic polynomial in T_+.

Taking the 5D decompactification limit of $\mathcal{L}_{\text{in}}$, one can then even show that $A_0(T_+)$ can be at most quadratic in T_+ [3].

To sum up, up to a polynomial of the form $A_0(T_+) = a_0 + a_1 T_+ + a_2 T_+^2$, the prepotential $\mathcal{F}_{\text{in}}$ can be reconstructed using only symmetry arguments and some 4D quantum field theory reasoning. In order to fix the remaining three coefficients, a few more couplings have to be considered. In any case, it shows that the method we used successfully in 5D, can also be applied in 4D.

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