Published by Institute of Physics Publishing for SISSA/ISAS
Received: *February 7, 2003*
Accepted: *February 24, 2003*

Effective actions near singularities*

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Abstract: We study the heterotic string compactified on $K3 \times T^2$ near the line $T = U$, where the effective action becomes singular due to an SU(2) gauge symmetry enhancement. By 'integrating in' the light $W^{\pm}$ vector multiplets we derive a quantum corrected effective action which is manifestly SU(2) invariant and non-singular. This effective action is found to be consistent with a residual $\mathrm{SL}(2, \mathbb{Z})$ quantum symmetry on the line $T = U$. In an appropriate decompactification limit, we recover the known SU(2) invariant action in five dimensions.

Keywords: Superstrings and Heterotic Strings, Gauge Symmetry, Supergravity Models, Supersymmetric Effective Theories.

*Work supported by: DFG — The German Science Foundation, GIF — the German-Israeli Foundation for Scientific Research, European RTN Program HPRN-CT-2000-00148 and the DAAD — the German Academic Exchange Service.

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Contents

1. Introduction

The low energy effective action of string theories is of particular importance, since it captures the interactions of the light string excitations. Its derivation has a long history and has been continuously refined. The main idea is to integrate out the heavy string modes and to derive an effective action of only the light modes below the mass scale of the heavy excitations.

Generically, this low energy effective action features a set of moduli scalar fields whose vacuum expectation values are not determined since they correspond to flat directions of the effective potential. In compactifications of the ten-dimensional string theories on compact Ricci-flat manifolds, Y, some of the moduli have a geometrical meaning in that they correspond to deformations of the metric on Y that preserve the Ricci-flatness. These deformations can be viewed as coordinates of the moduli space, $\mathcal{M}$, of Y. Unfortunately, all couplings in the low energy effective theory depend on these undetermined vacuum expectation values and therefore phenomenological predictions are difficult to extract. One expects that non-perturbative effects generate a potential for the moduli fields and thus dynamically lift this vacuum degeneracy.

It has become clear for some time that interesting physics is 'hidden' at special points in the moduli space where some couplings in the effective action become singular. From a mathematical point of view, these singularities often arise at points (or subspaces) of the moduli space where the compactification manifold Y develops a singularity. From a physical point of view, the singularities generically are due to heavy fields that become massless at the locus of the singularity. Integrating these fields out of the effective theory is thus not legitimate in this region of the moduli space, and this inconsistency manifests itself as a singularity in some of the effective couplings.

In general, a consistent, i.e. non-singular, effective action cannot be derived over the entire moduli space, since, as described above, some of the fields are only light at particular points (or subspaces) of the moduli space. Their mass, M, is a nontrivial function of the moduli, and thus M varies over the moduli space. However, locally near a given singularity where some of the fields are light and M approaches zero, one can choose to not integrate out these light fields and derive a non-singular effective action in the vicinity of $M = 0$, i.e. near the region of the former singularity. Differently put — and that is the way we will proceed in this paper — one can start from the singular action and locally 'integrate the light modes back in'.[1]

The purpose of this paper is to perform this 'integrating in' procedure in some detail in a model where the singular effective action is known exactly. More specifically, we consider the heterotic string compactified on $K3 \times T^2$, which leads to an effective theory with $N = 2$ supersymmetry in four space-time dimensions ($d = 4$). This class of string backgrounds is believed to be dual to type IIA string theory compactified on $K3$ fibered Calabi-Yau threefolds [2]–[7]. As a consequence, some of the couplings of the low energy effective theory are known exactly.

The low energy $N = 2$ supergravity theory contains, apart from the gravitational multiplet, a set of n_V vector multiplets and n_H hypermultiplets. Both multiplets contain scalar fields which can be viewed as the coordinates of the moduli space $\mathcal{M}$. As a consequence of $N = 2$ supersymmetry, this moduli space factorizes: $\mathcal{M} = \mathcal{M}_V \times \mathcal{M}_H$, where $\mathcal{M}_V$ is spanned by the scalars in the vector multiplets, and $\mathcal{M}_H$ is spanned by the scalars in the hypermultiplets. Due to this factorization, one can discuss each component separately, and in this paper we will only focus on $\mathcal{M}_V$. $\mathcal{M}_V$ is constrained to be a special Kähler manifold [8, 9], that is, $\mathcal{M}_V$ is endowed with a Kähler metric which can be expressed in terms of a holomorphic prepotential $\mathcal{F}$ (see appendix A).

For a certain class of compactifications, $\mathcal{F}$ is known exactly, and here we focus on a very specific model known as the STU-model. It corresponds to a compactification of the heterotic string on a $K3 \times T^2$ manifold with instanton numbers $(14, 10)$. This model is non-perturbatively dual to the IIA string compactified on the Calabi-Yau threefold $Y_{1,1,2,8,12}(24)$, which is an elliptic fibration over the second Hirzebruch surface $\mathbb{F}_2$ [10]–[12]. The generic low energy effective theory contains $n_V = 3$ vector multiplets (whose complex scalar fields we denote by S, T, U) and $n_H = 244$ neutral hypermultiplets. The hypermultiplets play no role in the following and will be consistently ignored. The gauge

[1]The term 'integrating in' was coined in [1] in a slightly different context.

group at a generic point in the moduli space is $G = \mathrm{U}(1)^4$, where the additional gauge boson is the $N = 2$ graviphoton. At the subspace $T = U$, G is enlarged to $G = \mathrm{U}(1)^3 \times \mathrm{SU}(2)$, at $T = U = 1$ it is further enhanced to $G = \mathrm{U}(1)^2 \times \mathrm{SU}(2)^2$, and at $T = U = \rho \equiv e^{i\pi/6}$ one finally has $G = \mathrm{U}(1)^2 \times \mathrm{SU}(3)$. In other words, at these special points additional gauge bosons (or rather $N = 2$ vector multiplets) become massless and enhance the everywhere existing abelian to a non-abelian gauge symmetry. From the point of view of the heterotic string, this is the usual perturbative gauge symmetry enhancement due to additional massless Kaluza-Klein and winding modes for particular values of the moduli of the two-torus. This symmetry enhancement does not survive non-perturbative quantum corrections [13, 4], and we will therefore restrict our considerations to the perturbative heterotic string only.[2]

The one-loop prepotential $\mathcal{F}^{(1)}$ is singular at $T = U$, which signals the existence of the additional light states. As we show in this paper, it is possible to derive an effective action valid near $T = U$ which is non-singular and which contains the $W^{\pm}$ gauge bosons of the SU(2). Following the approach of [15], we will not do this via a microscopic string theory calculation, but rather by using symmetry arguments to reconstruct the non-singular theory from the well-known [16]–[18] singular effective action, in which the $W^{\pm}$ bosons are integrated out.

Our motivation for this work is to study compactifications of string theory and M-theory in situations where the effective action becomes singular due to the presence of additional light modes. Such extra states might be either of perturbative or non-perturbative origin. In this paper, we consider the heterotic perturbative mechanism of SU(2) enhancement, as explained above. The natural next step will be to consider conifold singularities in compactifications of type II string theory on Calabi-Yau threefolds [19]. In this case the additional states are non-perturbative and descend from D-branes wrapped on a vanishing cycle. Since heterotic string compactifications on $K3 \times T^2$ are dual to type II compactifications on Calabi-Yau threefolds, we expect that the results of this paper will be useful for the study of conifolds.

There are further, even more interesting cases one might wish to consider. One line of developement will be the study of extremal transitions, where one has, in contrast to conifold transitions, an unbroken non-abelian gauge symmetry at the transition point [20, 21]. Another interesting extension is to add background flux, which can resolve the singularity [22] and create a hierarchically small scale [23, 24]. Besides Calabi-Yau compactifications one should also try to study $N = 1$ supersymmetric G_2-compactifications of M-theory along similar lines. Here the understanding of singular manifolds is mandatory, as smooth G_2-compactification do not lead to non-abelian gauge groups or chiral fermions [25]. We hope to return to these issues in later publications.

[2]From the point of view of the type IIA string, the SU(2) gauge symmetry enhancement is, at the classical level, due to a shrunken two-cycle of the Calabi Yau three-fold, with the wrapped D2 brane giving rise to the $W^{\pm}$ bosons. Unlike its M-theory analogue [14, 11, 15], this geometrical singularity (and hence the symmetry enhancement) does not survive quantum corrections. These quantum corrections can be calculated in the dual IIB picture using mirror symmetry. The mirror threefold develops a conifold singularity and one gets massless hypermultiplets with magnetic or dyonic charge, which come from D3 branes wrapping the vanishing three-cycle [19].

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This paper is organized as follows. In section 2, we briefly recall the results for the perturbative prepotential in the STU-model. In section 3, we then derive the tree level action for the effective theory near $T = U$ with the $W^{\pm}$ gauge bosons (and their superpartners) included. We explicitly compute the potential and the masses of the $N = 2$ $W^{\pm}$ supermultiplets. In section 4 we determine the one-loop corrections to this effective action near $T = U$. Based on general arguments, we first show (section 4.1) that the minimum of the quantum corrected potential does not change and the masses only receive multiplicative corrections which are entirely due to corrections of the Kähler potential. In analogy with $N = 1$ supergravity, it is possible to define a 'holomorphic mass' which remains uncorrected. In section 4.2, we determine the loop-corrected prepotential by using the known result of the STU-model. We 'undo' the integrating out procedure of the $W^{\pm}$ gauge bosons by subtracting their threshold corrections to the SU(2) gauge couplings. This way we derive a non-singular quantum corrected prepotential for the SU(2) gauge theory. In section 5 we check that this prepotential transforms appropriately under the expected residual quantum duality symmetry $\mathrm{SL}(2,\mathbb{Z})$ and that it does have the proper singularities at points in the moduli space where further gauge enhancement occurs. This is an independent check on our procedure. Finally, in section 6, we decompactify the theory to five space-time dimensions and establish the consistency with the results of [15]. Some technical details are relegated to three appendices. In appendix A, we supply the necessary facts of $N = 2$ supergravity. In appendix B, we assemble some useful formulae about the polylog series, while in appendix C we review properties of modular forms.

2. Preliminaries: review of the STU-model

Let us first recall a few facts about the STU-model [16]–[18] and [4, 7, 26, 6]. At the string tree level it is characterized by the prepotential[3]

$$\mathcal{F}^{(0)} = -STU = -S(T_+^2 - T_-^2)\,, \tag{2.1}$$

where $T_\pm \equiv \frac{1}{2}(T \pm U)$.

The quantum correction of the vector multiplet couplings can be parameterized by corrections to this prepotential. They only appear at 1-loop (generating a correction $\mathcal{F}^{(1)}$) and non-perturbatively (generating a contribution $\mathcal{F}^{(NP)}$). Therefore, the quantum corrected $N = 2$ prepotential obeys the expansion $\mathcal{F} = \mathcal{F}^{(0)} + \mathcal{F}^{(1)} + \mathcal{F}^{(NP)}$. $\mathcal{F}^{(1)}$ is known from a heterotic computation [16]–[18], while $\mathcal{F}^{(NP)}$ is known from the duality to IIA on the Calabi-Yau threefold $Y_{1,1,2,8,12}(24)$ [4, 18]. As we restrict ourselves to the perturbative heterotic string, only the one-loop correction $\mathcal{F}^{(1)}$ is of interest. For the rest of this paper we neglect the non-perturbative correction $\mathcal{F}^{(NP)}$ and, by abuse of notation, simply write

$$\mathcal{F} = \mathcal{F}^{(0)} + \mathcal{F}^{(1)}\,. \tag{2.2}$$

We already displayed $\mathcal{F}^{(0)}$ in (2.1). For Re $T >$ Re U, $\mathcal{F}^{(1)}$ is given by [18]

$$\mathcal{F}^{(1)} = -\frac{1}{12\pi}U^3 - \frac{1}{(2\pi)^4}Li_3\left(e^{-2\pi(T-U)}\right) - \frac{1}{(2\pi)^4}\sum_{k,l=0}^{\infty} c_1(kl)Li_3\left(e^{-2\pi(kT+lU)}\right), \tag{2.3}$$

[3] For a review of $N = 2$ supergravity see appendix A.

where the third polylog Li_3 is defined in appendix B, and the coefficients $c_1(kl)$ can be found in [18]. As was pointed out in [27, 16, 18], $\mathcal{F}^{(1)}$ is only determined up to a quadratic polynomial in the variables $1, iT, iU, TU$ with purely imaginary coefficients. Adding any such polynomial simply amounts to a shift in theta angles; we will come back to this ambiguity in section 4.2.

$\mathcal{F}^{(1)}$ is largely determined by its quantum symmetries. The STU-model has the perturbative quantum symmetry $\mathrm{SO}(2,2;\mathbb{Z})$, which includes the exchange $\sigma : T \leftrightarrow U$ as well as the duality group $\mathrm{SL}(2,\mathbb{Z})_T \times \mathrm{SL}(2,\mathbb{Z})_U$. Here, $\mathrm{SL}(2,\mathbb{Z})_T$ acts on T, U as

$$T \rightarrow \frac{aT - ib}{icT + d}, \qquad U \rightarrow U\,, \qquad ad - bc = 1\,, \quad a, b, c, d, \in \mathbb{Z}\,, \tag{2.4}$$

whereas the action of $\mathrm{SL}(2,\mathbb{Z})_U$ is obtained by exchanging T with U. As a consequence of this symmetry, the third derivative $\partial_T^3 \mathcal{F}^{(1)}$ is a modular form of weight $(+4, -2)$, while $\partial_U^3 \mathcal{F}^{(1)}$ is a modular form of weight $(-2, +4)$ under $\mathrm{SL}(2,\mathbb{Z})_T \times \mathrm{SL}(2,\mathbb{Z})_U$. They are given by [16, 17]

$$\begin{aligned}
\partial_T^3 \mathcal{F}^{(1)} &= \frac{+1}{2\pi} \frac{E_4(iT)\, E_4(iU) E_6(iU) \eta^{-24}(iU)}{j(iT) \,-\, j(iU)}\,, \\
\partial_U^3 \mathcal{F}^{(1)} &= \frac{-1}{2\pi} \frac{E_4(iU) E_4(iT) E_6(iT) \eta^{-24}(iT)}{j(iT) \,-\, j(iU)}\,,
\end{aligned} \tag{2.5}$$

where the modular forms E_4, E_6, η, j are defined in appendix C.

As we discuss more explicitly in sections 4 and 5, $\mathcal{F}^{(1)}$ is singular at $T = U$ due to gauge symmetry enhancement:

$$\begin{aligned}
T = U: &\quad \mathrm{U}(1) \times \mathrm{U}(1) \rightarrow \mathrm{U}(1) \times \mathrm{SU}(2)\,, \\
T = U = 1: &\quad \mathrm{U}(1) \times \mathrm{U}(1) \rightarrow \mathrm{SU}(2) \times \mathrm{SU}(2)\,, \\
T = U = \rho: &\quad \mathrm{U}(1) \times \mathrm{U}(1) \rightarrow \mathrm{SU}(3)\,.
\end{aligned} \tag{2.6}$$

At these points additional massless vector multiplets appear which should not have been integrated out of the effective action and which are the origin of the singular couplings (2.5).

3. The tree level effective action

Our goal in this paper is to derive a non-singular effective action that gives an accurate description of the theory near the surface $T = U$, where the SU(2) gauge symmetry enhancement occurs. In this section, we restrict ourselves to the tree level approximation of this effective theory. This sets our notation and prepares the discussion of the one-loop corrections.

Let us begin with our notation regarding the spectrum. Near the surface $T = U$, the set of light fields in the low energy effective action has to be enlarged to also include the $W^{\pm}$ bosons (along with their superpartners). The effective action is thus an $N = 2$ supergravity theory with $3 + 2 = 5$ vector multiplets in which SU(2) is realized as a Yang-Mills-type gauge symmetry. Three of these five vector multiplets have to transform in the adjoint representation of SU(2), and we use $C^a, a = 1, 2, 3$, to denote the complex scalar fields of this triplet. We choose to identify C^1 and C^2 with the scalar superpartners of the $W^\pm$ bosons:

$$W^\pm = C^1 \pm iC^2 \,. \tag{3.1}$$

The scalar field C^3 then has to be identified with $T_- = \frac{1}{2}(T-U)$, whose vacuum expectation value triggers the symmetry breaking $\mathrm{SU}(2) \longrightarrow U(1)$ via a supersymmetric Higgs effect. In addition to the triplet C^a, there are two SU(2) singlet vector multiplets, and at tree level the scalars of these singlet multiplets can be chosen to coincide with the moduli S and T_+.[4] The scalar fields (C^a, S, T_+) are 'special' coordinates of a symplectic section (X^I, F_J) $(I, J = 0, 1, \ldots, 5)$, which in our conventions (see appendix A for details) means that

$$\frac{X^j}{X^0} = t^j = (iC^a, iS, iT_+)\,, \qquad j = 1, \ldots, 5\,. \tag{3.2}$$

In the following, we use the subscript 'in' to label all actions, prepotentials, etc., where the two $W^\pm$ bosons have been 'integrated in'. More explicitly, $S_{\rm in}$ denotes the full perturbative effective action near $T = U$ with the $W^\pm$ bosons included, while $\mathcal{F}_{\rm in}$ refers to the underlying prepotential. The corresponding tree level quantities are denoted by $S^{(0)}_{\rm in}$ and $\mathcal{F}^{(0)}_{\rm in}$, respectively.

The prepotential $\mathcal{F}^{(0)}_{\rm in}$ is a holomorphic function of the variables (C^a, S, T_+). Its defining property is that the corresponding action, $S^{(0)}_{\rm in}[C^a, S, T_+]$, should reproduce the action $S^{(0)}[S, T_+, T_-]$ encoded in the prepotential $\mathcal{F}^{(0)} = -S(T_+^2 - T_-^2)$ of eq. (2.1), when $C^{1,2}$ (and their superpartners) are integrated out. At tree level, no threshold effects can occur, and integrating out these two multiplets simply means to set them equal to zero in the action $S^{(0)}_{\rm in}[C^a, S, T_+]$ or, equivalently, in the prepotential $\mathcal{F}^{(0)}_{\rm in}(C^a, S, T_+)$.[5] As the 'in-theory' is to be SU(2) invariant, the triplet C^a can only appear via its SU(2) invariant combination C^aC^a, and integrating out $C^{1,2}$ at tree level is thus tantamount to making the replacement

$$(C^aC^a) \longrightarrow (T_-)^2 \tag{3.3}$$

everywhere in $\mathcal{F}^{(0)}_{\rm in}(C^a, S, T_+)$. Conversely, $\mathcal{F}^{(0)}_{\rm in}(C^a, S, T_+)$ can simply be obtained from $\mathcal{F}^{(0)}(S, T_+, T_-) = -S(T_+^2 - T_-^2)$ by the inverse substitution

$$(T_-)^2 \longrightarrow (C^aC^a)\,. \tag{3.4}$$

We therefore arrive at

$$\mathcal{F}^{(0)}_{\rm in} = -S[T_+^2 - C^aC^a] \tag{3.5}$$

as the tree level prepotential with the two $W^\pm$ bosons included.

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[4] In general, the two singlets have to be invariant under the Weyl twist $C^3 \to -C^3$, which is the only remnant of the SU(2) symmetry after C^1 and C^2 have been integrated out. According to the identification $C^3 = T_-$, the Weyl twist is equivalent the exchange symmetry $\sigma : T \leftrightarrow U$. At string tree level, both S and T_+ are σ-invariant and can therefore be identified with the two SU(2) singlets, as we did above.

[5] The consistency of this truncation is guaranteed because $C^{1,2}$ form a doublet of an obvious $\mathrm{SO}(2) \subset \mathrm{SU}(2)$.

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As a consequence of $N=2$ supersymmetry, $S^{(0)}_{\rm in}[C^a, S, T_+]$ is completely determined by (3.5) and can in principle be worked out in all detail using the relations reviewed in appendix A. For the rest of this section, let us content ourselves with a short discussion of some of the quantities that play a role in the integrating out process.

Consider first the Kähler potential, K, of the scalar manifold, $\mathcal{M}_V$. Using (A.2), one obtains for K

$$K = -\log(S+\bar{S}) - \log Y\,, \qquad Y \equiv (T_+ + \bar{T}_+)^2 - (C^a + \bar{C}^a)(C^a + \bar{C}^a)\,. \tag{3.6}$$

This is the Kähler potential of the symmetric space $\mathcal{M}_V = \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(2,4)}{\mathrm{SO}(2)\times\mathrm{SO}(4)}$ with isometry group $\mathrm{ISO}(\mathcal{M}_V) = \mathrm{SU}(1,1)\times\mathrm{SO}(2,4)$. The form (3.6) corresponds to a parametrization in which only the subgroup $\mathrm{SO}(1,1)\times\mathrm{SO}(1,3)$ is a manifest symmtry of the Kähler potential.[6] The Yang-Mills-type gauge group of the theory is to be identified with the SU(2) subgroup of the SO(1,3) factor. On the homogeneous coordinates, X^I, this SU(2) acts as

$$\begin{aligned} \delta X^a &= \Lambda^b \epsilon_{bca} X^c \\ \delta X^0 &= \delta X^4 = \delta X^5 = 0\,, \end{aligned} \tag{3.7}$$

where we have identified the structure constants $f_{ab}^{c} = \epsilon_{abc}$. On the scalar manifold, $\mathcal{M}_V$, the corresponding SU(2) isometries are generated by Killing vectors (k_b^a, k_b^+, k_b^S):

$$\begin{aligned} \delta C^a &= \Lambda^b k_b^a \\ \delta T_+ &= \Lambda^b k_b^+ \\ \delta S &= \Lambda^b k_b^S\,. \end{aligned} \tag{3.8}$$

From the relation (3.2), one reads off[7]

$$k_a^b = \epsilon_{abc} C^c, \qquad k_a^+ = k_a^S = 0\,. \tag{3.9}$$

The Killing vectors enter the covariant derivatives of the scalar fields (see (A.5)), as well as the scalar potential,

$$V = 2\, e^K (X^I k_I^{\bar{\imath}}) g_{\bar{\imath}j}\, (\bar{X}^J k_J^j)\,. \tag{3.10}$$

Obviously, V is positive semi-definite, and zero if and only if

$$(\bar{X}^J k_J^j) = 0\,. \tag{3.11}$$

In view of (3.9), this means that the vacua of the theory correspond to field configurations with

$$[C, C^\dagger] = 0\,, \qquad \text{where } C \equiv C^a \sigma^a\,. \tag{3.12}$$

Thus, any vacuum can be brought to the form $\langle C^1\rangle = \langle C^2\rangle = 0$ by means of an SU(2) transformation. As the gaugino variations are proportional to the quantity $(\bar{X}^J k_J^j)$ [8, 9], all these vacua also preserve the $N=2$ supersymmetry (and also exhaust all $N=2$ supersymmetric Minkowski vacua).

[6]The SO(1,1) factor acts as $S \to \lambda^2 S$, $(C^a, T_+) \to (\lambda^{-1} C^a, \lambda^{-1} T_+)$.

[7]Using eqs. (A.7) and (A.8), one arrives at the same result.

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Remembering $C^3 = T_-$, we have therefore, at tree level, simply rediscovered that, modulo SU(2) transformations, (T_-, S, T_+) indeed parametrize the flat directions of the scalar potential and that all of the corresponding vacua are $N = 2$ supersymmetric.

Let us close this section with a short discussion of the tree-level masses of the $W^\pm$ bosons and their scalar superpartners in these vacua. The mass of any scalar field arises from non-vanishing second derivatives of the scalar potential. Combining (3.6), (3.9) and (3.10), the tree-level scalar potential is

$$\begin{aligned} V &= e^K \frac{4}{Y}[(\bar{C}^a C^a)^2 - (\bar{C}^a \bar{C}^a)(C^b C^b)] \\ &= \frac{4}{(S+\bar{S})Y^2}[(\bar{C}^a C^a)^2 - (\bar{C}^a \bar{C}^a)(C^b C^b)] \\ &= \frac{1}{2(S+\bar{S})Y^2}\mathrm{tr}([C, C^\dagger]^2)\,. \end{aligned} \tag{3.13}$$

At $\langle C^1 \rangle = \langle C^2 \rangle = 0$, the only non-vanishing second derivatives of this potential are

$$\begin{aligned} \partial_1 \partial_1 V|_{\langle C^1 \rangle = \langle C^2 \rangle = 0} &= -e^K \frac{8}{Y}(\bar{C}^3)^2 \\ \partial_1 \partial_{\bar{1}} V|_{\langle C^1 \rangle = \langle C^2 \rangle = 0} &= e^K \frac{8}{Y}|C^3|^2 \\ \partial_{\bar{1}} \partial_{\bar{1}} V|_{\langle C^1 \rangle = \langle C^2 \rangle = 0} &= -e^K \frac{8}{Y}(C^3)^2 \end{aligned} \tag{3.14}$$

and similarly for the derivatives with respect to C^2 and $\bar{C}^2$.

In order to diagonalize these mass matrices, one decomposes C^1 and C^2 into the real fields parallel and perpendicular to $\langle C^3 \rangle$:

$$\begin{aligned} C^1(x) &= a^1(x)\langle C^3 \rangle + ib^1(x)\langle C^3 \rangle \\ C^2(x) &= a^2(x)\langle C^3 \rangle + ib^2(x)\langle C^3 \rangle\,. \end{aligned} \tag{3.15}$$

One then finds

$$\frac{\partial^2 V}{\partial b^1 \partial b^1}\Big|_{\langle C^1 \rangle = \langle C^2 \rangle = 0} = \frac{\partial^2 V}{\partial b^2 \partial b^2}\Big|_{\langle C^1 \rangle = \langle C^2 \rangle = 0} = 32\frac{e^K}{Y}|C^3|^4 \tag{3.16}$$

with all other derivatives vanishing. Taking into account

$$\begin{aligned} g_{1\bar{j}}|_{\langle C^1 \rangle = \langle C^2 \rangle = 0} &= \delta_{1j}\,\frac{2}{Y}\,, \\ g_{2\bar{j}}|_{\langle C^1 \rangle = \langle C^2 \rangle = 0} &= \delta_{2j}\,\frac{2}{Y}\,, \end{aligned} \tag{3.17}$$

the kinetic terms of C^1 and C^2 simplify to

$$\begin{aligned} \mathcal{L}_{\rm kin} &= -\frac{2}{Y}\,\partial_\mu C^1 \partial^\mu \bar{C}^1 - \frac{2}{Y}\,\partial_\mu C^2 \partial^\mu \bar{C}^2 \\ &= -\frac{2}{Y}\,|C^3|^2(\partial_\mu a^1 \partial^\mu a^1 + \partial_\mu b^1 \partial^\mu b^1 + \partial_\mu a^2 \partial^\mu a^2 + \partial_\mu b^2 \partial^\mu b^2)\,, \end{aligned} \tag{3.18}$$

and we see that only $b^{1,2}$ obtain a mass, but not $a^{1,2}$ or any other scalar field. The mass of the corresponding canonically normalized scalar fields is given by

$$M^2 = 8e^K|C^3|^2 = 8e^K|T_-|^2\,. \tag{3.19}$$

This mass formula agrees with the one obtained from string theory, which reads [28]:

$$M^2 = \frac{4}{\alpha'}\frac{|T-U|^2}{(T+\overline{T})(U+\overline{U})} = \frac{16}{\alpha'}\frac{|T_-|^2}{Y}\,. \tag{3.20}$$

To see the agreement, one has to reinstall the gravitational coupling κ, which we have set to unity throughout, to convert string units into gravitational units using [29] $g^2\alpha'\kappa = 4$ and to express the heterotic string coupling g through the vev of the dilaton, $1/g^2 = \langle S+\overline{S}\rangle/2$.

In analogy with $N=1$ theories, we define a holomorphic mass, m, through the relation $M^2 = 8e^K|m(T)|^2$, which for the case at hand implies[8]

$$m = T_-\,. \tag{3.21}$$

As the vacua with $\langle C^1\rangle = \langle C^2\rangle = 0$ preserve the full $N=2$ supersymmetry, one should observe a supersymmetric Higgs effect in which the vector fields $A_\mu^{1,2}$ (i.e., the $W^\pm$ bosons) absorb the massless components $a^{1,2}$ and acquire the same mass as their scalar superpartners $b^{1,2}$ (3.19). And indeed, the mass term for the vector fields arises from the square of the covariant derivative

$$D_\mu C^a = \partial_\mu C^a - k_b^a A_\mu^b \tag{3.22}$$

which leads to the mass matrix

$$M_{ab}^2 = 2g^2 g_{c\bar{d}} k_a^c \bar{k}_b^{\bar{d}}\,, \tag{3.23}$$

where $g^2 = -2\langle \mathrm{Im}\mathcal{N}_{11}\rangle^{-1} = -2\langle \mathrm{Im}\mathcal{N}_{22}\rangle^{-1}$ ensures the correct canonical normalization. For $\langle C^1\rangle = \langle C^2\rangle = 0$, one has

$$g^{-2} = -\frac{\langle \mathrm{Im}\mathcal{N}_{11}\rangle}{2} = -\frac{\langle \mathrm{Im}\mathcal{N}_{22}\rangle}{2} = \frac{\langle S+\bar{S}\rangle}{2} \tag{3.24}$$

so that one indeed obtains the same mass matrices as above:

$$M_{ab}^2 = 8e^K\ \mathrm{diag}(|C^3|^2, |C^3|^2, 0) = 8e^K\ |T_-|^2\ \mathrm{diag}(1,1,0) \tag{3.25}$$
$$m_{ab} = T_-\ \mathrm{diag}(1,1,0)\,. \tag{3.26}$$

As m depends holomorphically on the moduli, it should not receive loop corrections. We will verify this in section 4.1.

[8]The mass parameters of an $N=1$ superpotential are necessarily holomorphic and a similar feature holds for masses generated via a supersymmetric Higgs effect [30]. Such holomorphic mass parameters are of importance due to their non-renormalization properties. In $N=2$ theories one can analogously define a holomorphic mass, and, as we will see in section 4.1, this mass is not renormalized.

4. The one-loop corrections

In this section we go beyond the tree level approximation and determine the one-loop corrections to the effective action $S_{\rm in}^{(0)}$ described in the previous section. Our result will be an action $S_{\rm in}$ that describes the full low energy dynamics of the perturbative heterotic string near $T = U$. This action again involves the coupling of five vector multiplets to $N = 2$ supergravity and exhibits an SU(2) gauge symmetry. Several properties of this action follow already from its gauge invariance and can be inferred without a detailed knowledge of all the couplings. We list these general properties in section 4.1 before we construct the complete theory with all the detailed couplings in section 4.2.

4.1 Some general properties of $S_{\rm in}$

Just as in the tree level case, three of the five vector fields transform in the adjoint representation of SU(2), while the remaining two have to be SU(2) inert. The scalars of the triplet are again denoted by C^a $(a = 1, 2, 3)$, with $C^{1,2}$ corresponding to the $W^\pm$ bosons and $C^3 = T_-$. In the tree level approximation, we could choose S and T_+ as the scalar fields of the two singlet multiplets, because both are classically invariant under the Weyl twist $\sigma : T \leftrightarrow U$, which is the only remnant of the SU(2) gauge symmetry once the $C^{1,2}$ are integrated out. At one loop, this is still true for T_+, however, S now becomes a multivalued function on the moduli space and transforms non-trivially under the perturbative duality group $SO(2, 2; \mathbb{Z})$ [16]. More precisely, using [16, eq. (4.27)], one finds

$$\sigma : S \longrightarrow S - \frac{1}{2\pi} T_- \, . \tag{4.1}$$

Thus, S can no longer serve as one of the SU(2) singlets. Fortunately, it is easy to construct a σ-invariant linear combination out of $\{S, T_+, T_-\}$:[9]

$$\hat{S} := S - \frac{1}{4\pi} T_- \, . \tag{4.2}$$

The two singlet scalars are therefore chosen to be $(\hat{S}, T_+)$.

We assume that $(C^a, \hat{S}, T_+)$ are special coordinates of a symplectic section for which a holomorphic prepotential — denoted by $\mathcal{F}_{\rm in}(C^a, \hat{S}, T_+)$ — exists.[10] The relation to the notation of appendix A is given by

$$\frac{X^j}{X^0} = t^j = (iC^a, i\hat{S}, iT_+) \, , \qquad j = 1, \ldots, 5 \, . \tag{4.3}$$

[9] The most general σ-invariant linear combination would be $\hat{S} + aT_+$ with a arbitrary. $\hat{S}$ is singled out by the property $\hat{S}\Big|_{T_-=0} = S\Big|_{T_-=0}$, a property that simplifies some of the equations in section 5. Note that neither $\hat{S}$ nor $\hat{S} + aT_+$ is the 'invariant dilaton' $S^{\rm inv}$ described in [16]. The invariant dilaton $S^{\rm inv}$ is a highly non-linear function of the moduli that is invariant under the full duality group $SO(2, 2; \mathbb{Z})$. $\hat{S}$, by contrast, is only invariant under the Weyl twist σ. It is the part of $S^{\rm inv}$ that is *linear* in the moduli. As such, $\hat{S}$ is a proper 'special coordinate', i.e., a scalar field of an $N = 2$ vector multiplet, a property not shared by the full invariant dilaton $S^{\rm inv}$.

[10] This assumption is supported by the tree level approximation discussed in the previous section and the self-consistency of our one-loop result (sections 4 and 5). Using the tree level approximation, however, one can also show that the rank two gauge groups at $T = U = 1$ and $T = U = \rho$ (which are beyond the scope of the present paper) can*not* be manifestly realized in a symplectic basis with a prepotential.

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Thus far, the only difference to the tree level case is that the singlet t^4 is to be identified with $i\hat{S}$ instead of iS. It is therefore not surprising that many of the general conclusions we drew in section 3 go through for the one-loop corrected theory as well. In particular, the SU(2) transformation properties of the X^I remain formally the same,

$$\begin{aligned} \delta X^a &= \Lambda^b \epsilon_{bca} X^c \\ \delta X^0 &= \delta X^4 = \delta X^5 = 0 \,, \end{aligned} \tag{4.4}$$

which, together with the analogue of eqs. (3.8), implies that the Killing vectors do not get renormalized:

$$k_a^b = \epsilon_{abc} C^c \,, \qquad k_a^+ = k_a^{\hat{S}} = 0 \,. \tag{4.5}$$

Similarly, one can repeat large parts of the analysis of the scalar potential,

$$V = 2 \; e^K (X^I k_I^{\bar{\imath}}) g_{\bar{\imath} j} \, (\bar{X}^J k_J^j) \,. \tag{4.6}$$

Because of the manifestly positive semidefinite form, the vacua are again given by $\langle \bar{X}^J k_J^j \rangle = 0$, which, in the light of (4.5), again implies that any Minkowski vacuum can be brought to the form $\langle C^1 \rangle = \langle C^2 \rangle = 0$ by means of an SU(2) rotation. These vacua are also the supersymmetric ones, and we see that the one-loop corrections might change the shape of the scalar poptential, but not its ground states.

So far, everything we "derived" in this section was completely independent of the prepotential $\mathcal{F}_{\rm in}$ and solely based on the assumed SU(2) gauge invariance and the underlying $N = 2$ supersymmetry. The SU(2) symmetry, however, also restricts the possible form of the prepotential, which in turn allows us to make further statements about the theory without the detailed knowledge of the prepotential.

More precisely, the SU(2) gauge symmetry of the theory requires that $\mathcal{F}_{\rm in}(C^a, \hat{S}, T_+)$ be SU(2) invariant. Consequently, the triplet C^a can only appear via the SU(2) invariant combination $(C^a C^a)$ (or powers thereof). Hence, the prepotential has to be of the general form

$$\mathcal{F}_{\rm in}(C^a, \hat{S}, T_+) = \sum_{n=0}^{\infty} H_n(\hat{S}, T_+)(C^a C^a)^n \,, \tag{4.7}$$

where $H_n(\hat{S}, T_+)$ denotes a set of as yet undetermined functions of the singlets $\hat{S}$ and T_+. Determining these functions will be the content of sections 4.2, 5 and 6, but a number of statements already follow from the general form (4.7). As an example, let us again consider the masses of the scalar fields.

First note that in a vacuum with $\langle C^1 \rangle = \langle C^2 \rangle = 0$, the metric components $g_{1\bar{\jmath}}$ and $g_{2\bar{\jmath}}$ simplify to

$$\begin{aligned} g_{1\bar{\jmath}} &= \delta_{1j} e^K [(\mathcal{F}_{\rm in})_{11} + (\bar{\mathcal{F}}_{\rm in})_{11}] \,, \\ g_{2\bar{\jmath}} &= \delta_{2j} e^K [(\mathcal{F}_{\rm in})_{11} + (\bar{\mathcal{F}}_{\rm in})_{11}] \,, \end{aligned} \tag{4.8}$$

where $\langle(\mathcal{F}_{\rm in})_{11}\rangle = \langle(\mathcal{F}_{\rm in})_{22}\rangle$ has been used. A closer inspection of (4.6) then reveals that the only non-vanishing second derivatives of V are

$$\begin{aligned} \partial_1\partial_1 V\Big|_{\langle C^1\rangle=\langle C^2\rangle=0} &= -4e^K g_{\bar{2}2}(\bar{C}^3)^2\,, \\ \partial_1\partial_{\bar{1}} V\Big|_{\langle C^1\rangle=\langle C^2\rangle=0} &= 4e^K g_{\bar{2}2}|C^3|^2\,, \\ \partial_{\bar{1}}\partial_{\bar{1}} V\Big|_{\langle C^1\rangle=\langle C^2\rangle=0} &= -4e^K g_{\bar{2}2}(C^3)^2 \end{aligned} \tag{4.9}$$

and analogously for the derivatives with respect to C^2, $\bar{C}^2$ (remembering $\langle g_{\bar{2}2}\rangle = \langle g_{\bar{1}1}\rangle$). These mass matrices are again diagonalized by a decomposition as in (3.15). In terms of the corresponding fields $a^{1,2}$ and $b^{1,2}$, the only non-vanishing derivatives are then

$$\frac{\partial^2 V}{\partial b^1 \partial b^1}\Big|_{\langle C^1\rangle=\langle C^2\rangle=0} = \frac{\partial^2 V}{\partial b^2 \partial b^2}\Big|_{\langle C^1\rangle=\langle C^2\rangle=0} = 16e^K g_{\bar{2}2}|C^3|^4\,. \tag{4.10}$$

Taking into account the corresponding kinetic terms,

$$\begin{aligned} \mathcal{L}_{\rm kin} &= -g_{1\bar{1}}\partial_\mu C^1\partial^\mu \bar{C}^1 - g_{2\bar{2}}\partial_\mu C^2\partial^\mu \bar{C}^2 \\ &= \frac{1}{2}(2g_{\bar{1}1}|C^3|^2)(\partial_\mu a^1\partial^\mu a^1 + \partial_\mu b^1\partial^\mu b^1 + \partial_\mu a^2\partial^\mu a^2 + \partial_\mu b^2\partial^\mu b^2)\,, \end{aligned} \tag{4.11}$$

one obtains for the masses of the corresponding canonically normalized scalar fields

$$M^2 = 8e^K|C^3|^2 = 8e^K|T_-|^2\,. \tag{4.12}$$

This has the same form as in the tree level approximation, but the non-holomorphic Kähler potential K now contains quantum corrections. The *holomorphic* mass, m, however, is the same as it was at tree level,

$$m = T_-\,, \tag{4.13}$$

as anticipated in section 3.

As the vacua with $\langle C^1\rangle = \langle C^2\rangle = 0$ are $N=2$ supersymmetric, we expect the vector fields $A_\mu^{1,2}$ (i.e., the $W^\pm$ bosons) to acquire the same mass (4.12) as their scalar superpartners by absorbing the massless fields $a^{1,2}$ in a supersymmetric Higgs effect. This is again easy to verify: Just as in the tree level case, the mass term for the vector fields arises from the square of the covariant derivative

$$D_\mu C^a = \partial_\mu C^a - k^a_b A^b_\mu \tag{4.14}$$

which leads to the mass term

$$-g_{2\bar{2}}|C^3|^2(A^1_\mu)^2 - g_{1\bar{1}}|C^3|^2(A^2_\mu)^2\,. \tag{4.15}$$

The corresponding kinetic terms are (see eq. (A.1))

$$\frac{1}{4}\frac{(\bar{\mathcal{F}}_{11}+\mathcal{F}_{11})}{4}F^1_{\mu\nu}F^{\mu\nu 1} + \frac{1}{4}\frac{(\bar{\mathcal{F}}_{22}+\mathcal{F}_{22})}{4}F^2_{\mu\nu}F^{\mu\nu 2} \tag{4.16}$$

so that, remembering (4.8), one indeed obtains the same mass matrices as above

$$M^2_{ab} = 8e^K\ \mathrm{diag}(|C^3|^2, |C^3|^2, 0) = 8e^K\ |T_-|^2\ \mathrm{diag}(1,1,0) \tag{4.17}$$

$$m_{ab} = T_-\ \mathrm{diag}(1,1,0)\,. \tag{4.18}$$

4.2 Determining $\mathcal{F}_{\text{in}}$

We are now ready to determine the complete perturbative prepotential $\mathcal{F}_{\text{in}}(C^a, \hat{S}, T_+)$ that encodes the non-singular effective action $S_{\text{in}}[C^a, \hat{S}, T_+]$. The defining property of $S_{\text{in}}[C^a, \hat{S}, T_+]$ is that integrating out C^1 and C^2 and going over to the variables (S, T_+, T_-) should reproduce the singular action $S[S, T_+, T_-]$ based on the perturbative prepotential $\mathcal{F}(S, T_+, T_-) = \mathcal{F}^{(0)} + \mathcal{F}^{(1)}$ given in section 2.

As we are now going beyond tree level, integrating out C^1 and C^2 is no longer equivalent to simply setting these fields equal to zero. Instead, one now also has to take into account threshold effects that arise from Feynman diagrams in which C^1 and C^2 (or their superpartners) run in loops.

In practice this means that $\mathcal{F}(S, T_+, T_-)$ is obtained from $\mathcal{F}_{\text{in}}(C^a, \hat{S}, T_+)$ in a two-step process (see also [15]): First one sets $C^1 = C^2 = 0$ in $\mathcal{F}_{\text{in}}$. This will then yield an auxiliary prepotential $\mathcal{F}_{\text{in}}^{\text{truncated}} \equiv \mathcal{F}_{\text{in}}|_{C^1=C^2=0}$ which only depends on $(C^3, \hat{S}, T_+)$ (or, alternatively, on (S, T_+, T_-)). If there were no threshold effects, this would already be the prepotential $\mathcal{F}$ in which the $W^{\pm}$ bosons have been integrated out. If threshold effects do exist, however, $\mathcal{F}_{\text{in}}^{\text{truncated}}$ and $\mathcal{F}$ will differ by an additional term $\delta\mathcal{F}$ which subsumes all effective interactions that are generated by diagrams with C^1 and C^2 running in loops, i.e., one has

$$\mathcal{F} = \mathcal{F}_{\text{in}}^{\text{truncated}} + \delta\mathcal{F}\,. \tag{4.19}$$

In our case, the threshold corrections introduce a logarithmic dependence on the holomorphic mass of the $W^{\pm}$ gauge bosons into the (wilsonian) gauge couplings g_W [31, 30, 16]

$$\delta g_W^{-2} = -\frac{b}{16\pi^2}\log|m|^2\,, \tag{4.20}$$

where b is the one-loop coefficient of the β-function. The definition of the wilsonian gauge coupling is exactly as in $N = 1$ supergravity where g_W is determined by a holomorphic function [30]. In $N = 2$ supergravity g_W^{-2} is determined by the matrix of second derivatives of $\mathcal{F}$ [16] and for the case at hand we find[11]

$$\delta g_W^{-2} = \frac{1}{4}(\partial_-^2\,\delta\mathcal{F} + \bar{\partial}_-^2\,\delta\bar{\mathcal{F}}) = \frac{1}{4\pi^2}\log|m|^2\,, \tag{4.21}$$

where we used $b_{\text{SU}(2)} = -4$. Note that the definition of m includes the choice of a (field-independent) cut-off scale which in supergravity has to be proportional to the Planck mass. This in turn implies that the right hand side of (4.21) is defined only up an arbitrary additive constant. Using (4.13) this implies

$$\delta\mathcal{F} = \frac{1}{2\pi^2}\,T_-^2\log T_- + A_2 T_-^2 + A_1(T_+)T_- + A_0(T_+)\,, \tag{4.22}$$

where A_2 is the arbitrary constant while $A_1(T_+), A_0(T_+)$ are a priori undetermined functions of T_+. The prepotentials with and without the $W^{\pm}$ bosons are thus related by

$$\mathcal{F} = \mathcal{F}_{\text{in}}^{\text{truncated}} + \frac{1}{2\pi^2}\,T_-^2\log T_- + A_2 T_-^2 + A_1(T_+)T_- + A_0(T_+)\,. \tag{4.23}$$

[11] Note that the non-holomorphic piece in the definition (A.4) of $\mathcal{N}_{IJ}$ does not contribute to the harmonic wilsonian gauge couplings.

As we see, integrating out $W^{\pm}$ introduces a logarithmic singularity into $\mathcal{F}$ while $\mathcal{F}_{\rm in}$ has to be non-singular. Thus we can now go backwards and compute $\mathcal{F}_{\rm in}^{\rm truncated}$ by first subtracting the logarithmic divergence $\delta\mathcal{F}$ from $\mathcal{F}$, which is given by (2.3) and (2.1). $\mathcal{F}_{\rm in}$ is then obtained from $\mathcal{F}_{\rm in}^{\rm truncated}$ by replacing every T_-^2 by C^aC^a. For this to be possible, T_- should appear in $\mathcal{F}_{\rm in}^{\rm truncated}$ only in terms of *even* powers (cf. (4.7)). We will see whether this is indeed the case.

The first step is therefore to expand $\mathcal{F}$ near $T = U$ in order to isolate the logarithmic singularity. The last term in (2.3) is manifestly non-singular in this limit while the second term can be expanded using (B.13). In the region $\mathrm{Re}T_- > 0$ one finds

$$\begin{aligned}\mathcal{F} = &-S(T_+^2 - T_-^2) - \frac{1}{12\pi}(T_+ - T_-)^3 - \frac{1}{(2\pi)^4}\zeta(3) + \frac{1}{24\pi}T_- - \frac{3}{4\pi^2}T_-^2 - \frac{1}{3\pi}T_-^3 + O(T_-^4) + \\ &+ \frac{1}{2\pi^2}T_-^2\log(4\pi T_-) - \frac{1}{(2\pi)^4}\sum_{k,l=0}^{\infty} c_1(kl) Li_3\left(e^{-2\pi[(k+l)T_+ + (k-l)T_-]}\right). \end{aligned} \tag{4.24}$$

In appendix B (eqs. (B.13), (B.14)) we show that the terms denoted by $O(T_-^4)$ involve at most *even* powers $(T_-)^{2n}$ with $n \geq 2$ The last term in (4.24) is analytic near $T_- = 0$ and manifestly invariant under $T_- \to -T_-$. Viewed as a power series in T_-, it therefore also contains only *even* powers of T_-. Furthermore, the only non-analytic piece is the logarithmic term

$$\frac{1}{2\pi^2}T_-^2\log(4\pi T_-)\,. \tag{4.25}$$

Remembering (4.23), we see that in $\mathcal{F}_{\rm in}^{\rm truncated}$ this term is precisely canceled by the one-loop threshold correction $\delta\mathcal{F}$. Thus, as desired, $\mathcal{F}_{\rm in}^{\rm truncated}$ is analytic near $T_- = 0$ and reads

$$\begin{aligned}\mathcal{F}_{\rm in}^{\rm truncated} = &-S(T_+^2 - T_-^2) - \frac{1}{12\pi}(T_+ - T_-)^3 - \frac{1}{(2\pi)^4}\zeta(3) + \frac{1}{24\pi}T_- - \frac{1}{3\pi}T_-^3 + \\ &+ O(T_-^4) - \frac{1}{(2\pi)^4}\sum_{k,l=0}^{\infty} c_1(kl) Li_3\left(e^{-2\pi[(k+l)T_+ + (k-l)T_-]}\right) - \\ &- \left[A_2 + \frac{3}{4\pi^2} - \frac{1}{2\pi^2}\log(4\pi)\right]T_-^2 - A_1(T_+)T_- - A_0(T_+)\,. \end{aligned} \tag{4.26}$$

As mentioned above, the full prepotential $\mathcal{F}_{\rm in}$ is now obtained from $\mathcal{F}_{\rm in}^{\rm truncated}$ by reversing the truncation of the $W^{\pm}$ bosons. At tree level, this was done by simply promoting every T_-^2 to the SU(2) invariant combination C^aC^a. However, a closer inspection of (4.26) reveals that this is not possible here, because there are *cubic* powers of T_- which cannot cancel against any other term (there are also linear terms in T_-, but we will discuss them later).

The source of this problem is of course that we are still working with the variables (S, T_+, T_-) that were suitable for the tree level approximation. As explained in section 4.1, the loop corrected version instead requires working with the quantities $(\hat{S}, T_+, T_-)$ in terms of which the Weyl twist σ becomes diagonal. Only in terms of the variables $(\hat{S}, T_+, T_-)$ should one expect the prepotential $\mathcal{F}_{\rm in}^{\rm truncated}$ to be even in T_-. And indeed, inserting (4.2) into (4.26), one obtains

$$\begin{aligned}\mathcal{F}_{\rm in}^{\rm truncated} &= -\hat{S}(T_+^2 - T_-^2) - \frac{1}{3\pi}T_+^3 + \frac{1}{4\pi}T_+(T_+^2 - T_-^2) - \frac{1}{(2\pi)^4}\zeta(3) + \frac{1}{24\pi}T_- + O(T_-^4) - \\ &\quad - \frac{1}{(2\pi)^4}\sum_{k,l=0}^{\infty} c_1(kl) Li_3\left(e^{-2\pi[(k+l)T_+ + (k-l)T_-]}\right) - \\ &\quad - \left[A_2 + \frac{3}{4\pi^2} - \frac{1}{2\pi^2}\log(4\pi)\right] T_-^2 - A_1(T_+)T_- - A_0(T_+)\,. \end{aligned} \tag{4.27}$$

We see that the disturbing cubic terms in T_- have indeed disappeared.

Let us now turn to the terms quadratic in T_-. As we discussed above, the constant A_2 is undetermined by the subtraction procedure, and so one is free to choose $A_2 = -\frac{3}{4\pi^2} + \frac{1}{2\pi^2}\log(4\pi)$ in order to simplify eq. (4.27).

The linear term

$$\frac{1}{24\pi}T_-\,, \tag{4.28}$$

on the other hand, only leads to a constant shift in one of the numerous theta angles. Such a term can always be neglected, because it is part of the ambiguity [27, 16, 18] in the prepotential $\mathcal{F}$ we have mentioned below eq. (2.3). The same is true for the real part of a possible constant term in $A_1(T_+)$ as well as for a possible linear term in $A_1(T_+)$ with imaginary coefficient. All other terms in $A_1(T_+)$, however, have to vanish from the outset in order for T_- to appear only with even powers. Modulo irrelevant changes in theta angles, we have thus derived

$$A_1(T_+) \equiv 0\,. \tag{4.29}$$

The full prepotential $\mathcal{F}_{\rm in}$ is then obtained by simply replacing every T_-^2 in (4.27) by C^aC^a.

It remains to determine the unknown function $A_0(T_+)$. In principle, this could be done in a similar way as in our discussion following eq. (4.19) by considering the couplings F_{++}, F_{+0} and F_{00}. As the two multiplets we integrate out are not charged with respect to the corresponding vector fields A_μ^+ and A_μ^0, the gauge couplings of the latter should not feel the shift $\delta\mathcal{F}$ and remain unchanged in the integrating out process. This would suggest $A_0(T_+) \equiv 0$. In the following two sections, we will see that this expectation is supported by a completely independent line of argument. As we will show, $A_0(T_+)$ is already strongly constrained by the quantum symmetry and the proper large radius limit.

5. Quantum symmetries of $\mathcal{F}_{\rm in}$

At $T_- = 0$ the original $\mathrm{SL}(2,\mathbb{Z})_T \times \mathrm{SL}(2,\mathbb{Z})_U$ quantum symmetry reduces to the diagonal $\mathrm{SL}(2,\mathbb{Z})_+$ acting on T_+ and T_- as

$$T_+ \to \frac{aT_+ - ib}{icT_+ + d}\,, \qquad T_- \to T_-\,. \tag{5.1}$$

This symmetry should be respected by $\mathcal{F}_{\rm in}$, which we explicitly check in this section. In addition, this consistency check will confirm that A_2 is a constant and further constrain $A_0(T_+)$ in eq. (4.27).

The generic form of $\mathcal{F}_{\rm in}$ is given in (4.7), which, near $T_- = 0$, can be approximated by the first two terms

$$\mathcal{F}_{\rm in}(\hat{S}, T_+, C^a) = H_0(\hat{S}, T_+) + H_1(\hat{S}, T_+)C^aC^a + \mathcal{O}(C^4)\,. \tag{5.2}$$

Since we have already determined the tree level contribution to these functions in (3.5), we can parameterize H_0 and H_1 more conveniently by

$$H_0(\hat{S}, T_+) = -\hat{S}T_+^2 + h(T_+)\,, \qquad H_1(\hat{S}, T_+) = \hat{S} + f(T_+)\,, \tag{5.3}$$

where h can be viewed as the loop corrections to the Kähler potential of T_+ (also known as the four-dimensional Green-Schwarz term [16, 17]), while $f(T_+)$ is the one-loop correction to the SU(2) gauge coupling. As we will show, appropriate derivatives of h and f transform as modular forms which can be computed from (2.5).

Let us first focus on $h(T_+)$. For this coupling there is a closely related computation we can make use of. In ref. [32] the two-parameter ST-model was investigated. Its prepotential including one-loop corrections is given by $\mathcal{F} = -ST^2 + h(T)$ with an $\mathrm{SL}(2,\mathbb{Z})$ acting on T. Using arguments outlined in [16], it was shown in [32] that $\partial_T^5 h(T)$ has to be a modular form of weight $+6$. The exact same arguments can be used here to conclude that $\partial_+^5 h(T_+)$ has to be a modular form of weight $+6$. Furthermore, in the ST-model the second derivative $\partial_T^2 h$ has a logarithmic singularity at $T = 1$ which arises from the fact that additional massless states appear at $T = 1$ which lead to a gauge symmetry enhancement. However, in our case the second derivative $\partial_+^2 h(T_+)$ has a logarithmic singularity both at $T_+ = 1$ and $T_+ = \rho$, since charged states become massless at both points. The coefficient of the singularity is set by the β-function of the gauge group opening up. More precisely, one has

$$\partial_+^2 h(T_+) = +\frac{b_{\rm SU(2)}}{4\pi^2}\log(T_+ - 1) + \frac{b_{\rm SU(3)}}{4\pi^2}\log(T_+ - \rho) + \text{finite}\,. \tag{5.4}$$

Using $b_{\rm SU(2)} = -4$ and $b_{\rm SU(3)} = -6$ this implies

$$\partial_+^5 h(T_+) = -\frac{2}{\pi^2}\frac{1}{(T_+ - 1)^3} - \frac{3}{\pi^2}\frac{1}{(T_+ - \rho)^3} + \text{finite}\,. \tag{5.5}$$

In order to check the above singularity structure and the modular properties, we will now compute $\partial_+^5 h$ by relating it to the modular forms (2.5). Truncating out C^1, C^2 from (5.2) gives

$$\mathcal{F}_{\rm in}^{\rm truncated}(\hat{S}, T_+, T_-) = -\hat{S}(T_+^2 - T_-^2) + h(T_+) + f(T_+)\,T_-^2 + \mathcal{O}(T_-^4)\,. \tag{5.6}$$

Hence,

$$\begin{aligned} h(T_+) &= \mathcal{F}_{\rm in}^{\rm truncated}|_{T_-=\hat{S}=0} \\ &= (\mathcal{F} - \delta\mathcal{F})|_{T_-=S=0} \\ &= (\mathcal{F}^{(1)} - \delta\mathcal{F})|_{T_-=0}\,, \end{aligned} \tag{5.7}$$

where we have used (4.19) and (4.2) in the second and $\mathcal{F}^{(0)}|_{S=0} = 0$ in the third line. Using the explicit form (4.22) of $\delta\mathcal{F}$, we then obtain

$$\partial_+^5 h = \partial_+^5 \mathcal{F}^{(1)}|_{T_-=0} - \partial_+^5 A_0(T_+)\,. \tag{5.8}$$

In order to compute $\partial_+^5 \mathcal{F}^{(1)}|_{T_-=0}$ it is convenient to define

$$I_\pm = (\partial_T^3 \pm \partial_U^3)\mathcal{F}^{(1)}\,, \tag{5.9}$$

where the third derivatives of $\mathcal{F}^{(1)}$ are given in (2.5). Expanding $I_\pm$ near $T_- = 0$ one has

$$\begin{aligned} I_+ &= a_0(T_+) + a_2(T_+)\,T_-^2 + O(T_-^4) \\ I_- &= a_{-1}T_-^{-1} + a_1(T_+)\,T_- + O(T_-^3)\,, \end{aligned} \tag{5.10}$$

where[12]

$$\begin{aligned} a_{-1} &= \frac{1}{4\pi^2}\,, \\ a_0 &= -\frac{1}{4\pi}\,E_2 - \frac{1}{4\pi}\,\frac{E_4^2}{E_6}\,, \\ a_1 &= \frac{23}{216}\,E_4 + \frac{1}{8}\,{E_2}^2 + \frac{1}{4}\,\frac{E_2E_4^2}{E_6} - \frac{4}{27}\,\frac{E_6^2}{E_4^2}\,, \\ a_2 &= -\frac{19\pi}{432}\,E_2\,E_4 + \frac{23\pi}{216}\,E_6 - \frac{19\pi}{432}\,E_2^3 - \frac{19\pi}{144}\,\frac{E_2^2E_4^2}{E_6} + \frac{\pi}{144}\,\frac{E_4^3}{E_6} + \frac{4\pi}{27}\,\frac{E_6^2E_2}{E_4^2} - \frac{\pi}{24}\,\frac{E_4^4E_2}{E_6^2}\,. \end{aligned} \tag{5.11}$$

Expressing $\partial_+^5 \mathcal{F}^{(1)}|_{T_-=0}$ in terms of derivatives of $I_\pm$, we obtain after some straightforward algebra

$$\begin{aligned} \partial_+^5 \mathcal{F}^{(1)}|_{T_-=0} &= \Big(4\partial_+^2 I_+ + \frac{3}{2}\partial_-^2 I_+ - \frac{9}{2}\partial_-\partial_+ I_-\Big)\Big|_{T_-=0} \\ &= 4\partial_+^2 a_0 - \frac{9}{2}\partial_+ a_1 + 3a_2 \\ &= -2\pi\Big(\frac{E_4^6}{E_6^3} - \frac{23}{18}\frac{E_4^3}{E_6} + \frac{8}{18}\frac{E_6^3}{E_4^3} - \frac{1}{6}E_6\Big)\,, \end{aligned} \tag{5.12}$$

where the last equation used repeatedly the derivatives of modular forms given in appendix C. As expected $\partial_+^5 \mathcal{F}^{(1)}$ is indeed a modular form of weight $+6$. It also is closely related to the corresponding quantity for the ST-model computed in [32] but differs in the structure of the singularities to which we turn to now.

In (5.5) we determined the singularities of $\partial_+^5 h$ which differs from $\partial_+^5 \mathcal{F}^{(1)}|_{T_-=0}$ by the so far unknown $\partial_+^5 A_0(T_+)$ (c.f. (5.8)). However, as we are going to see shortly $\partial_+^5 \mathcal{F}^{(1)}|_{T_-=0}$ has precisely the right singularity structure and modular properties to be exactly *equal* to $\partial_+^5 h$ so that $\partial_+^5 A_0(T_+)$ has to vanish identically. First of all, it is easy to see that $\partial_+^5 \mathcal{F}^{(1)}|_{T_-=0}$ does have a triple pole at $T_+ = 1$ and $T_+ = \rho$. Using (C.5) and expanding

$$\begin{aligned} E_6(iT_+) &= iE_6'(T_+ - 1) + \cdots \\ E_4(iT_+) &= iE_4'(T_+ - \rho) + \cdots \end{aligned} \tag{5.13}$$

[12]This and some of the following calculations have been performed using Maple.

we infer that near $T_+ = 1$ the leading singularity is

$$\partial_+^5 \mathcal{F}^{(1)}|_{T_-=0} \to -2\pi i \frac{E_4^6(i)}{E_6'^3(i)(T_+-1)^3} = -\frac{2}{\pi^2}\frac{1}{(T_+-1)^3} \tag{5.14}$$

which is indeed consistent with (5.5). Similarly, at $T_+ = \rho$ the leading singularity is

$$\partial_+^5 \mathcal{F}^{(1)}|_{T_-=0} \to -\frac{8\pi i}{9}\frac{E_6^3(\rho)}{E_4'^3(\rho)(T_+-\rho)^3} = -\frac{3}{\pi^2}\frac{1}{(T_+-\rho)^3}\,, \tag{5.15}$$

again consistent with (5.5). Finally, from the dual type IIA vacua we know that for large T_+ the prepotential is at most a cubic polynomial and hence

$$\lim_{T_+\to\infty} \partial_+^5 h = 0\,. \tag{5.16}$$

Using (C.4) we indeed check

$$\lim_{T_+\to\infty} \partial_+^5 \mathcal{F}^{(1)}|_{T_-=0} = 2\pi\left(1 - \frac{23}{18} + \frac{8}{18} - \frac{1}{6}\right) = 0\,. \tag{5.17}$$

We thus conclude

$$\partial_+^5 h = \partial_+^5 \mathcal{F}^{(1)}|_{T_-=0} \Longrightarrow \partial_+^5 A_0(T_+) \equiv 0 \tag{5.18}$$

so that $A_0(T_+)$ can be at most a quartic polynomial in T_+.

In a similar fashion we can compute $f(T_+)$. Using (5.6), (4.19), (4.2) and $\mathcal{F}^{(0)}|_{S=0} = 0$, one first derives

$$\begin{aligned} f(T_+) &= \frac{1}{2}[\partial_-^2 \mathcal{F}_{\rm in}^{\rm truncated}]_{T_-=\hat{S}=0} \\ &= \frac{1}{2}[\partial_-^2(\mathcal{F} - \delta\mathcal{F})]_{T_-=S=0} \\ &= \frac{1}{2}[\partial_-^2(\mathcal{F}^{(1)} - \delta\mathcal{F})]_{T_-=0}\,, \end{aligned} \tag{5.19}$$

so that

$$\partial_+ f = \frac{1}{2}\partial_+\partial_-^2 \mathcal{F}^{(1)}|_{T_-=0}\,. \tag{5.20}$$

Furthermore, using (5.9), (5.7) and (5.20), one easily verifies

$$\begin{aligned} a_0 \equiv I_+\Big|_{T_-=0} &= \frac{1}{4}\left[\partial_+^3\mathcal{F}^{(1)} + 3\partial_+\partial_-^2\mathcal{F}^{(1)}\right]_{T_-=0} \\ &= \frac{1}{4}\left[\partial_+^3 h + \partial_+^3 A_0\right] + \frac{3}{2}\,\partial_+ f\,. \end{aligned} \tag{5.21}$$

Differentiating twice yields (remembering $\partial_+^5 A_0 = 0$)

$$\partial_+^3 f = \frac{2}{3}\partial_+^2 a_0 - \frac{1}{6}\partial_+^5 h\,. \tag{5.22}$$

From (5.11) and (5.12) and repeated use of (C.14) we compute

$$\partial_+^3 f = -\frac{\pi}{108}\,E_2^3 + \frac{\pi}{4}\,E_2E_4 - \frac{2\pi}{9}\,E_6 - \frac{\pi}{36}\,\frac{E_2^2E_4^2}{E_6} - \frac{\pi}{6}\,\frac{E_4^4E_2}{E_6^2} + \frac{\pi}{36}\,\frac{E_4^3}{E_6} + \frac{4\pi}{27}\,\frac{E_6^3}{E_4^3}\,. \tag{5.23}$$

As we see, this expression is not a modular form and singular both at $T_+ = 1$ and at $T_+ = \rho$. However, as was stressed in ref. [16] at one-loop the dilaton S transforms under modular transformations. Nevertheless, it is possible to define a modular invariant, non-singular dilaton $S^{\rm inv}$ via [16]

$$S^{\rm inv} = S - \frac{1}{2}\partial_T \partial_U \mathcal{F}^{(1)} - \frac{1}{8\pi^2}\log[j(iT) - j(iU)]\,. \tag{5.24}$$

To see the modular properties of f we need to separate f into the part which is redefined into $S^{\rm inv}$ and the left over piece $f^{\rm cov}$ defined by

$$\hat{S} + f = S^{\rm inv} + f^{\rm cov}\,. \tag{5.25}$$

Using the same strategy as before, we find

$$\partial_+^3 S^{\rm inv}\Big|_{T_-=0} = -\frac{1}{8}\partial_+^5 h + \frac{1}{4}\partial_+^3 f - \frac{1}{8\pi^2}\partial_+^3 \log[\partial_+ j]\,. \tag{5.26}$$

Taking the third derivative of (5.25) evaluated at $T_- = 0$ and inserting into (5.26) we arrive at

$$\partial_+^3 f^{\rm cov} = \frac{3}{4}\partial_+^3 f + \frac{1}{8}\partial_+^5 h + \frac{1}{8\pi^2}\partial_+^3 \log[\partial_+ j]\,. \tag{5.27}$$

Using (5.12) and (5.23) this finally gives

$$\partial_+^3 f^{\rm cov} = \frac{1}{4\pi^2}\partial_+^3 \log E_4(T_+)\,. \tag{5.28}$$

We see that for $f^{\rm cov}$ the result considerably simplified compared to (5.23) and both the modular properties and the singularity structure qualitatively changed. The right hand side of (5.28) is singular only at $T_+ = \rho$ corresponding to the enhancement $\mathrm{U}(1)\times\mathrm{SU}(2) \to \mathrm{SU}(3)$ as expected. At $T_+ = 1$ on the other hand the enhancement is $\mathrm{U}(1) \times \mathrm{SU}(2) \to \mathrm{SU}(2) \times \mathrm{SU}(2)$ and no charged states contribute to the SU(2) gauge couplings which is already present at $T = U$. Thus $f^{\rm cov}$ has to be finite at $T_+ = 1$ which is indeed satisfied by the $\log E_4$-term.

Let us close this section by checking the modular properties of $f^{\rm cov}$. From (5.2) we infer that $f(T_+)$ plays the role of the one-loop corrections to the SU(2) gauge coupling. In $N = 2$ supergravity the gauge couplings obey [16]

$$g^{-2} = \mathrm{Re}(\hat{S} + f) + \frac{b}{16\pi^2}K(S, T_+, C^a = 0)\,, \tag{5.29}$$

where $K(S, T_+, C^a = 0)$ is the tree level Kähler potential obtained from (3.6)

$$K = -\log(S + \bar{S}) - 2\log(T_+ + \bar{T}_+)\,. \tag{5.30}$$

K transforms under (5.1) according to

$$K \to K + 2\log|icT_+ + d|^2\,. \tag{5.31}$$

Since the physical gauge couplings g have to be modular invariant the combination $\hat{S}+f = S^{\rm inv} + f^{\rm cov}$ has to compensate the transformation (5.30). Since $S^{\rm inv}$ is modular invariant by construction, the transformation law of $f^{\rm cov}$ is fixed to be

$$f^{\rm cov} \to f^{\rm cov} - \frac{b}{4\pi^2}\log(icT_+ + d) = f^{\rm cov} + \frac{1}{\pi^2}\log(icT_+ + d)\,, \tag{5.32}$$

where the last equation used $b_{\rm SU(2)} = -4$. Thus we conclude

$$f^{\rm cov} = \frac{1}{4\pi^2}\log E_4(T_+) \tag{5.33}$$

which is consistent with both (5.28) and (5.30) and fixed only up to an arbitrary constant, which can be identified with the ambiguous constant A_2 in (4.22).

6. The large radius limit

In this section, we perform the large radius limit of the theory and show the consistency of our results with the results obtained in ref. [15][13] for $d = 5$. In this limit, one circle of the T^2 is decompactified, and one obtains heterotic string theory on $K3 \times S^1$ [14]. The low energy limit of this theory is five-dimensional, $N = 2$ supergravity coupled to $n_V - 1$ vector multiplets and n_H hypermultiplets, where n_V, n_H count four-dimensional supermultiplets. As before, the hypermultiplets can be consistently ignored. The couplings of five-dimensional vector multiplets to $N = 2$ supergravity are encoded in a cubic prepotential [35]. The vector multiplet moduli T^i are real, rather than complex, and the moduli space is a cubic hypersurface,

$$\mathcal{V}(T^i) = \frac{1}{6}C_{ijk}T^iT^jT^k \stackrel{!}{=} 1\,, \tag{6.1}$$

where $\mathcal{V}(T^i)$ is the five-dimensional prepotential, the C_{ijk} denote a set of constants, and $i = 1, \ldots, n_V$. The underlying structure is often referred to as 'very special geometry' [36].

Dimensional reduction of the five-dimensional supergravity theory defined by (6.1) over a circle of radius R gives four-dimensional $N = 2$ supergravity with n_V vector multiplets and a 'very special' (i.e., purely cubic) four-dimensional prepotential:

$$\mathcal{F}(t^i) = \frac{1}{6}C_{ijk}t^it^jt^k\,. \tag{6.2}$$

Here, the t^i are complex scalars whose real parts are related to the five-dimensional scalars by

$$\mathrm{Re}t^i = RT^i\,. \tag{6.3}$$

The imaginary parts of the t^i arise from the internal components of the corresponding gauge fields. This means that in a meaningful decompactification limit the imaginary parts cannot have a vev, and the t^i have to be restricted to real values. In the rest of this section, this will always be assumed, i.e., from now on t^i stands for Re t^i, and inequalities

[13] In fact, the results of [15] were the motivation for the present analysis.

such as $S > T > U$, should be read as Re $S >$ Re $T >$ Re U, etc. With t^i restricted to real values, (6.1) and (6.3) imply

$$\mathcal{F}(t^i) = \frac{1}{6} C_{ijk} t^i t^j t^k = R^3 \,. \tag{6.4}$$

Whereas a five-dimensional prepotential must be purely cubic, a four-dimensional prepotential is, in general, allowed to be an arbitrary holomorphic function of the t^i (possibly with singularities on special loci). Therefore (6.2) and (6.4) only represent the pure supergravity contribution that can be (and typically is) subject to further stringy corrections:

$$\mathcal{F}(t^i) = \frac{1}{6} C_{ijk} t^i t^j t^k + \cdots = R^3 + \cdots \,. \tag{6.5}$$

If, however, such a four-dimensional prepotential can be obtained by dimensional reduction from five dimensions, then these corrections must vanish in the decompactification limit $R \to \infty$:

$$\lim_{R\to\infty} R^{-3} \mathcal{F}(t^i) = \mathcal{V}(T^i) \,. \tag{6.6}$$

In order to make contact with [15], we need to switch to a slightly different parameterization of the prepotential. This corresponds, in the notation of [6], to going from 'heterotic' to 'type IIA' conventions:

$$S \to 4\pi S \,, \qquad \mathcal{F} \to -4\pi \mathcal{F} \,. \tag{6.7}$$

In this convention, the prepotential without the $W^{\pm}$ bosons (eqs. (2.1), (2.3)) takes the form

$$\mathcal{F}_{>} = STU + \frac{1}{3} U^3 + \frac{2}{(2\pi)^3} Li_3 \left(e^{-2\pi(T-U)} \right) + \frac{2}{(2\pi)^3} \sum_{k,l=0}^{\infty} c_1(kl) Li_3 \left(e^{-2\pi(kT+lU)} \right) . \tag{6.8}$$

We indicated by our notation that this expression is valid in the Weyl chamber[14] $S > T > U$. In order to perform the decompactification limit inside this Weyl chamber we set $S = Rs$, etc. and take $R \to \infty$, while keeping $s > t > u$ fixed. Using (B.15), we find

$$\lim_{R\to\infty} R^{-3} \mathcal{F}(S,T,U) = \mathcal{V}(s,t,u) = stu + \frac{1}{3} u^3 \,, \tag{6.9}$$

which is precisely the same prepotential as one obtains directly in five-dimensional heterotic string theory [14].

The five-dimensional scalars s, t, u are subject to the hypersurface constraint (6.1). One can express them in terms of two unconstrained scalars. The natural choice for these scalars are the five-dimensional heterotic dilaton ϕ, or, equivalently, the five-dimensional

[14] As explained in [18] the BPS states form an infinite dimensional Lie algebra and one can generalize the notion of a Weyl chamber, which is familiar from simple Lie algebras. In particular, $T > U$ and $T < U$ define the two Weyl chambers of an SU(2) subalgebra. The corresponding group is the gauge group which is un-Higgsed at $T = U$.

heterotic string coupling $g_{(5)} = \sqrt{2\pi/\phi}$ and the radius r of the remaining circle. The relation between the contrained scalars s, t, u and the uncontrained scalars ϕ, r is [14]:

$$s = \frac{\phi}{2\pi} - \frac{\sqrt{2\pi}}{3\sqrt{\phi}r^3}\,, \qquad t = \frac{\sqrt{2\pi}r}{\sqrt{\phi}}\,, \qquad u = \frac{\sqrt{2\pi}}{\sqrt{\phi}r}\,. \tag{6.10}$$

The regime $s > t > u > 0$ thus corresponds to $1 < r/\sqrt{\alpha'} < (2\phi)^{3/2}$ [14], i.e., the radius of the circle is larger than the self-dual radius $\sqrt{\alpha'}$ and the heterotic string is weakly coupled.

Let us now take a different decompactification limit, where the hierarchy between the moduli T, U is reversed: $S > U > T$. The four-dimensional prepotential in this region is found by analytical continuation of (6.8) using the connection formula (B.8) for the polylogarithm:

$$\begin{aligned}\mathcal{F}_< = STU &+ \frac{1}{3}U^3 + \frac{1}{3}(T-U)^3 - \frac{i}{2}(T-U)^2 - \frac{1}{6}(T-U) + \\ &+ \frac{2}{(2\pi)^3}Li_3\left(e^{-2\pi(U-T)}\right) + \frac{2}{(2\pi)^3}\sum_{k,l=0}^{\infty} c_1(kl)Li_3\left(e^{-2\pi(kT+lU)}\right).\end{aligned} \tag{6.11}$$

Note that (6.8) and (6.11) differ by polynomial terms that come from the analytical continuation of $Li_3\left(e^{-2\pi(T-U)}\right)$. The additional cubic term will survive in the decompactification limit although the polylogarithm itself goes to zero. Thus one of the polylogarithmic terms leaves a subtle shadow in the decompactification limit. Note that it is precisely this term which is responsible for the fact that the prepotentials in the two Weyl chambers are not just related by exchanging T and U. Just as the non-trivial monodromy around $T = U$, this is caused by the threshold corrections corresponding to the two charged vector multiplets which become massless on this line.

The last term in (6.11), which contains an infinite number of further polylogarithmic terms, is manifestly invariant under the exchange of T and U and does not contribute to the monodromy of the prepotential around $T = U$. It contains the contributions of the infinitely many other BPS states of the heterotic string. This term is non-universal, in the sense that it depends on details of the BPS spectrum. For example, it will be different for the closely related model with instanton numbers $(13, 11)$. In contrast, the first polylogarithmic term is universal in the sense that it is fully determined by the fact that we have SU(2) enhancement on the line $T = U$.

We can now take the decompactification limit $S > U > T \to \infty$ and obtain

$$\mathcal{V}_< = stu + \frac{1}{3}u^3 + \frac{1}{3}(t-u)^3 = stu + \frac{1}{3}t^3 + (tu^2 - t^2u)\,, \tag{6.12}$$

valid for $s > u > t$. Comparing (6.9) and (6.12) we see that the two prepotentials differ by the term $\frac{1}{3}(t-u)^3$. This difference vanishes at $t = u$, so the prepotential itself is a continuous function at $t = u$. The resulting couplings in the lagrangian, however, are discontinuous, because they depend on derivatives of the prepotential. These discontinuities in the couplings are the analogues of the logarithmic branch cuts present in four dimensions.

Using five-dimensional field theory, one can show that the difference $\frac{1}{3}(t-u)^3$ of the two prepotentials precisely corresponds to the threshold corrections of two charged vector

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multiplets [11, 37, 15]. This shows that at $t = u$ the U(1) gauge group corresponding to the scalar $t - u$ is enhanced to SU(2). This result was in fact first found in [14] by using perturbative heterotic string theory for the compactification on $K3 \times S^1$. In terms of five-dimensional heterotic variables, the second Weyl chamber $s > u > t$ corresponds to a small radius $r < \sqrt{\alpha'}$ of the circle, and one recognizes the the SU(2) enhancement as the usual SU(2) gauge symmetry enhancement at the self-dual radius $r = \sqrt{\alpha'}$ of the circle.

Using the field redefinition $s \to s+u-t$, the prepotential (6.12) takes the form $stu+\frac{1}{3}t^3$ given in [14]. The fact that this form actually involves a field redefinition is crucial for the study of space-time geometries where the scalars evolve dynamically from $t > u$ to $t < u$. Indeed, a naive use of $\mathcal{V}_> = stu + \frac{1}{3}u^3$ for $t > u$ and $\mathcal{V}_< = stu + \frac{1}{3}t^3$ for $t < u$, with the same s in both Weyl chambers, leads to artificial space-time singularities, which are absent when the correct continuation (6.12) is used [38, 39].

Let us now consider a third decompactification limit, where we keep $T_- = \frac{1}{2}(T - U)$ small, so that we stay in the vicinity of the enhancement locus. In this case, we should keep the charged vector multiplets and work with the prepotential $\mathcal{F}_{\text{in}}$ or, for simplicity, with the truncated version $\mathcal{F}_{\text{in}}^{\text{truncated}}$ (4.26). In the conventions used in this section, $\mathcal{F}_{\text{in}}^{\text{truncated}}$ takes the following form:[15]

$$\begin{aligned}\mathcal{F}_{\text{in}}^{\text{truncated}} &= S(T_+^2 - T_-^2) + \frac{1}{3}(T_+ - T_-)^3 + \frac{2}{(2\pi)^3}\zeta(3) - \frac{1}{6}T_- + \frac{4}{3}T_-^3 + \\ &\quad + O(T_-^4) + \frac{2}{(2\pi)^3}\sum_{k,l=0}^{\infty} c_1(kl) Li_3\left(e^{-2\pi[(k+l)T_+ + (k-l)T_-]}\right) + \\ &\quad + 4\pi\left([A_2 + \frac{3}{4\pi^2} - \frac{1}{2\pi^2}\log(4\pi)]T_-^2 + A_0(T_+)\right). \end{aligned} \tag{6.13}$$

We will now show that one gets a consistent decompactification limit if one first takes T_- to zero and then takes S, T_+ to infinity. In order to keep track of the behaviour of the prepotential away from the special locus $t_- = 0$, we use that the five-dimensional prepotential is purely cubic and perform the limit at the level of the third derivatives:

$$\lim_{R\to\infty}\lim_{T_-\to 0}\frac{\partial^3 \mathcal{F}_{\text{in}}^{\text{truncated}}}{\partial t^i \partial t^j \partial t^k} = \frac{\partial^3 \mathcal{V}_{\text{in}}^{\text{truncated}}}{\partial T^i \partial T^j \partial T^K}\,, \tag{6.14}$$

where $t^i = S, T_+, T_-$ and $T^i = s, t_+, t_-$.

We illustrate this by computing the term cubic in t_-. This term is particularly important because it encodes the five-dimensional threshold corrections. From (6.13) we find:

$$\lim_{R\to\infty}\lim_{T_-\to 0}\frac{\partial^3 \mathcal{F}}{\partial T_- \partial T_- \partial T_-} = 6\cdot\left(-\frac{1}{3} + \frac{4}{3}\right) = 6\,. \tag{6.15}$$

This implies

$$\mathcal{V}_{\text{in}}^{\text{truncated}} = t_-^3 + \cdots\,, \tag{6.16}$$

where we used (B.2) and (B.5) to show that the contribution of the polylogarithms vanishes in the limit.

[15] At this point, we have already used $A_1(T_+) \equiv 0$.

Looking at the other third derivatives and ignoring the term $A_0(T_+)$ for the moment, we find

$$\mathcal{V}_{\rm in}^{\rm truncated} = s(t_+^2 - t_-^2) + \frac{1}{3}(t_+ - t_-)^3 + \frac{4}{3}t_-^3\,. \tag{6.17}$$

To compare (6.17) with (6.9) and (6.12) we switch to the variables s, t, u:

$$\mathcal{V}_{\rm in}^{\rm truncated} = stu + \frac{1}{3}u^3 + \frac{1}{6}(t-u)^3 = \frac{1}{2}(\mathcal{V}_> + \mathcal{V}_<)\,. \tag{6.18}$$

This is precisely the truncated five-dimensional prepotential derived in [15]. Just as in the four-dimensional case (section 4.2), a manifestly gauge invariant form is obtained by introducing $\hat{s} = s - t_-$, which is the five-dimensional limit of the Weyl-invariant dilaton $\hat{S} = S - T_-$(4.2):

$$\mathcal{V}_{\rm in}^{\rm truncated} = \hat{s}(t_+^2 - t_-^2) + \frac{1}{3}t_+^3 + t_+ t_-^2\,. \tag{6.19}$$

In this basis all fields transform covariantly under the Weyl twist: $\hat{s}$ and t_+ are invariant, and t_- is mapped to $-t_-$. As t_- only enters through the invariant t_-^2, the 'untruncated' prepotential is again obtained via a substitution of the form $t_-^2 \to (c_1^2 + c_2^2 + c_3^2)$, where c_i transform in the adjoint representation of SU(2) [15].

We have thus shown that our decompactification limit of $\mathcal{F}_{\rm in}^{\rm truncated}$ is consistent with the purely five-dimensional result (6.18) obtained in [15] provided that the function $A_0(T_+)$ does not contribute to this limit, i.e., provided that $\lim_{R\to\infty} \partial_+^3 A_0(T_+) = 0$. From section 5, we already know that $A_0(T_+)$ can be at most a quartic polynomial in T_+. The five-dimensional decompactification limit now tells us that $A_0(T_+)$ can, in fact, be at most a *quadratic* polynomial: $A_0(T_+) = c_0 + c_1 T_+ + c_2 T_+^2$. As mentioned earlier, these remaining terms are expected to vanish as well, because they would give rise to changes in the gauge couplings of the spectator vector fields A_μ^+ and A_μ^0 when the $W^\pm$ multiplets are integrated out.

7. Conclusions

In this paper, we have shown, in an explicit example, how to determine a non-singular effective action near a singular subspace of the moduli space of a string compactification. The key feature of this effective action is that it includes modes that are massive at a generic point in the moduli space but become massless at the singularity. Starting from a singular effective action where such modes have been integrated out, we carefully integrated them back in and in this way derived an effective action valid in the vicinity of the singularity, or, in other words, in a region of the moduli space where these modes are still light. Using a combination of field-theoretical reasoning (the general structures of $N = 2$ Yang-Mills-supergravity actions and of threshold corrections) together with some (in fact, little) knowledge of the underlying microscopical string physics (only the type of the additional massless states was needed) and symmetry arguments (the residual T-duality at a fixed point) turned out to be sufficient to determine the effective theory up to a few irrelevant integration constants.

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It was clear from the outset that such a non-singular effective action should exist, but we find it interesting and useful to carry out this derivation explicitly and determine a complete and consistent description of the low energy physics near the singularity. Our calculations have exhibited many features which we expect to be generic. In particular, we have seen that although the prepotential is a very complicated function, which involves an infinite number of polylogarithmic functions, integrating in the charged vector multiplets basically amounts to adding a term of the form $\delta\mathcal{F} \simeq T_-^2 \log T_-$. This term is fixed by pure field theory arguments, and is complelely determined by the knowledge that two charged vector multiplets become massless at $T_- = 0$. As we have seen, the resulting theory nevertheless has the correct global properties on the moduli space, i.e., it has the correct singularities at the points $T_+ = 1, \rho$ of higher gauge symmetry enhancement and exhibits the residual modular symmetry $\mathrm{SL}(2,\mathbb{Z})_+$. Modular symmetries are related to the presence of infinitely many massive string states. What our results thus demonstrate is that the field theory reasoning employed here is able to capture such stringy properties. Also note that it is not completely obvious that (2.3) does not contain odd powers of T_- apart from the first and the third. Since the infinitely many massive modes do not interfere with the integrating in procedure we expect that the methods developed in this paper can be applied to other cases as well.

For example, it would be interesting to extend our results to conifold points and conifold transitions in type II string compactifications on Calabi-Yau threefolds. This corresponds to situations where hypermultiplets become massless, and we must distinguish between statements about the vector multiplet sector and about the hypermultiplet sector. The vector multiplet sector is still determined by its prepotential, but now the singularities and monodromies of the generic prepotential are not due to SU(2) gauge symmetry enhancement, but to massless monopoles and dyons. Integrating in these hypermultiplets must remove the non-trivial monodromies of the prepotential around the conifold locus. Given the monodromies, one should be able to integrate in the hypermultiplets in the same way as the vector multiplets considered in this paper. Note that although the SU(2) gauge symmetry is never restored, it nevertheless leaves its imprint in the monopole and dyon monodromies, as explained in [26].

It is much harder to say anything concrete about the hypermultiplet sector, due to our lack of knowledge about generic quaternionic manifolds. Whereas we can start in the vector multiplet sector from a known prepotential, the metric on the hypermultiplet moduli space is not known for the STU-model. Therefore, any extension of our knowledge on this sector of the theory is extremely valuable. One interesting question is the structure of the metric and of the scalar potential in the effective theory where the monopole hypermultiplets have been integrated in. The particular structure of the scalar potential corresponds to a non-generic gauging of the supergravity lagrangian (since one still has many flat directions) and requires the hypermultiplet manifold to have specific isometries. It should be interesting to investigate this in detail.

Another direction is the investigation of higher rank non-abelian gauge groups in the perturbative heterotic string. This will require a generalization of the present formalism, since, at least in our example, these higher gauge symmetry enhancements cannot be described in a basis for the symplectic section where a prepotential exists. A reformulation purely in terms of sections should also be useful for studying non-abelian gauge symmetry enhancement in type II Calabi-Yau compactifications [20, 21]. Persuing this line of developement, we expect to get a better understanding of the relation between gauged supergravity, the geometry of Calabi-Yau manifolds and M-theory.

A. $N = 2$ gauged supergravity in $d = 4$

In this appendix we collect some facts about gauged $N = 2$ supergravity in $d = 4$ [33, 8, 34, 9]. A generic spectrum contains the gravitational multiplet which contains the graviton $g_{\mu\nu}, \mu, \nu = 0, \ldots, 3$ and the graviphoton A^0_μ as bosonic components. In addition there can be n_V vector multiplets which feature n_V vector bosons A^i_μ and n_V complex scalars $t^i, i = 1, \ldots, n_V$ as bosonic components. Finally there are n_H hypermultiplets which contain $4n_H$ real scalars $q^u, u = 1, \ldots, 4n_H$. The bosonic part of the effective action reads [9][16]

$$S = \int \frac{1}{2}R - g_{i\bar{\jmath}} D_\mu t^i D^\mu \bar{t}^{\bar{\jmath}} - h_{uv}\partial_\mu q^u \partial^\mu q^v + \frac{1}{8}\,\mathrm{Im}\mathcal{N}_{IJ} F^I_{\mu\nu} F^{J\mu\nu} + \frac{1}{4}\,\mathrm{Re}\mathcal{N}_{IJ} F^I \wedge F^J - V\,, \quad \text{(A.1)}$$

where R is the Einstein term and h_{uv} is the metric on a quaternionic manifold, $\mathcal{M}_H$, spanned by the scalars q^u in the hypermultiplets. As this part of the action is of no importance for this paper we do not discuss them any further and instead refer the reader to the literature [9]. The metric $g_{i\bar{\jmath}}$ is the metric on a special Kähler manifold, $\mathcal{M}_V$, spanned by the scalars t^i. Being special Kähler, $g_{i\bar{\jmath}}$ can be derived from a Kähler potential via $g_{i\bar{\jmath}} = \partial_i \bar{\partial}_{\bar{\jmath}} K$, where K is not an arbitrary real function but determined in terms of a holomorphic prepotential F according to

$$K = -\log\left[i\bar{X}^I(\bar{t})F_I(X) - iX^I(t)\bar{F}_I(\bar{X})\right]. \quad \text{(A.2)}$$

The $X^I, I = 0, \ldots, n_V$ are $(n_V + 1)$ holomorphic functions of the t^i. F_I abbreviates the derivative, i.e. $F_I \equiv \partial F(X)/\partial X^I$ and $F(X)$ is a homogeneous function of X^I of degree 2, i.e. $X^I F_I = 2F$. Using this homogeneity property one can go to special coordinates defined by $X^0 = 1, X^i = t^i$. In this parameterization the Kähler potential can be written as

$$K = -\log\left[2(\mathcal{F} + \bar{\mathcal{F}}) - (t^i + \bar{t}^i)(\mathcal{F}_i + \bar{\mathcal{F}}_i)\right], \quad \text{(A.3)}$$

where $\mathcal{F} = i(X^0)^{-2}F(X)$.

$F^I_{\mu\nu}$ are the field strength of the gauge bosons. The gauge coupling functions $\mathcal{N}$ are defined in terms of the prepotential according to

$$\mathcal{N}_{IJ} = \bar{F}_{IJ} + 2i\,\frac{\mathrm{Im}F_{IK}\mathrm{Im}F_{JL}X^K X^L}{\mathrm{Im}F_{LK}X^K X^L}\,. \quad \text{(A.4)}$$

The covariant derivatives are given by

$$D_\mu t^i = \partial_\mu t^i - k^i_I A^I_\mu\,, \quad \text{(A.5)}$$

[16]Our normalizations coincide with those of [16].

where $k^i_I(t)$ are Killing vectors which generate isometries on $\mathcal{M}_V$[17]

$$\delta t^i = \Lambda^I k^i_I(t)\,. \tag{A.6}$$

As a consequence of the Killing equation and the Kähler geometry of $\mathcal{M}_V$, the $k^i_I(t)$ are constrained to be holomorphic, i.e. $\bar\partial k^i_I(t) = 0$ and furthermore can be solved in terms of Killing prepotentials P_I

$$k^i_I(t) = g^{i\bar{j}}\partial_{\bar{j}} P_I\,. \tag{A.7}$$

The P_I in turn are determined by

$$P_I = e^K(F_J f^J_{IK}\bar{X}^K + \bar{F}_J f^J_{IK} X^K)\,, \tag{A.8}$$

where the f^J_{IK} are the structure constants of the symmetry group. Finally, the potential is expressed in terms of the Killing vectors and reads

$$V = 2\, e^K X^I \bar{X}^J g_{\bar{i}j}\, k^{\bar{i}}_I k^j_J\,. \tag{A.9}$$

B. Polylogology

In this appendix we assemble facts and useful formulae for polylogarithmic functions, as they can be found, for example, in refs. [40, 18, 41].

For $0 < z < 1$, the k-th polylog is defined by the series expansion

$$Li_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}\,. \tag{B.1}$$

It can be continued to a multivalued function on the complex plane. Polylogarithmic functions with different values of k are related by the equation

$$z\frac{d}{dz} Li_k(z) = Li_{k-1}(z)\,. \tag{B.2}$$

Whereas the first polylog is related to the logarithm,

$$Li_1(z) = -\log(1-z)\,, \tag{B.3}$$

the polylogs with $k \leq 0$ are algebraic functions:

$$Li_0(z) = \frac{z}{1-z}\,, \qquad Li_k(z) = \left(z\frac{d}{dz}\right)^{-k} \frac{z}{1-z} \;\text{ for } k \leq -1\,. \tag{B.4}$$

From (B.2) one can derive integral representations for the higher polylogs, $k \geq 1$, but we will not need them. But in order to describe the behaviour of the prepotential in the decompactification limit and on the enhancement locus we need the following special values:[18]

$$Li_k(0) = 0\,, \qquad (\forall k \in \mathbb{Z}) \qquad \text{and} \qquad Li_k(1) = \zeta(k)\,, \quad \text{for} \quad k > 1\,. \tag{B.5}$$

[17] Of course it is also possible to gauge isometries on the quaternionic manifold $\mathcal{M}_H$ but since this does not occur in the present models we do not discuss this situation here.

[18] $Li_{-k}(z)$ has a k-th order pole at $z = 1$ for $k > 0$, whereas $Li_0(z)$ diverges logarithmically for $z \to 1$.

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The connection formula [40] relates the values at z and $1/z$:

$$Li_k(z) + (-1)^k Li_k(\frac{1}{z}) = -\frac{(2\pi i)^k}{k!} B_k\left(\frac{\log(z)}{2\pi i}\right), \qquad \text{for} \quad k > 0\,, \tag{B.6}$$

where $B_k(\cdot)$ are the Bernoulli polynomials.[19] For Li_3 one finds

$$Li_3(z) - Li_3(\frac{1}{z}) = -\frac{1}{6}\log^3(z) - \frac{i\pi}{2}\log^2(z) + \frac{\pi^2}{3}\log(z)\,. \tag{B.7}$$

For our purposes it is more natural to work with the variable x, where $z = e^x$. In the main part of the paper, x is a modulus or a linear combination of moduli, and $x = 0 \Leftrightarrow z = 1$ corresponds to gauge symmetry enhancement, while $x \to \infty \Leftrightarrow 1/z \to 0$ corresponds to the decompactification limit. In terms of the variable x, formula (B.7) becomes [18]

$$Li_3(e^x) = Li_3(e^{-x}) + \frac{\pi^2}{3}x - \frac{i\pi}{2}x^2 - \frac{1}{6}x^3\,. \tag{B.8}$$

The function $Li_3(e^{-x})$ has a logarithmic branch point at $x = 0$. Since this limit is relevant for the study of gauge symmetry enhancement, it is useful to have an expansion of the form

$$Li_3(e^{-x}) \simeq p(x) + q(x)\log(x) \text{ for } x \to 0\,, \tag{B.9}$$

where $p(x)$ and $q(x)$ are power series,

$$p(x) = \sum_{j=0}^{\infty} p_j x^j \qquad \text{and} \qquad q(x) = \sum_{j=0}^{\infty} q_j x^j\,. \tag{B.10}$$

This expansion can be analytically continued to an expansion for $Li_3(e^x)$, using $\log(-x) = \log(x) + i\pi$. Plugging this into the connection formula (B.8) and comparing term by term one finds:

$$q_0 = q_1 = 0\,, \qquad q_2 = -\frac{1}{2}\,, \qquad q_3 = q_4 = q_5 = \cdots = 0 \tag{B.11}$$

and

$$p_1 = -\frac{\pi^2}{6}\,, \qquad p_3 = \frac{1}{12}\,, \qquad p_5 = p_7 = p_9 = \cdots = 0\,. \tag{B.12}$$

The coefficients p_{2i}, $i = 0, 1, 2, \ldots$ can be obtained using (B.1). p_0 is fixed by $Li_3(1) = \zeta(3)$ while the other coefficients can be found by comparing derivatives of (B.1) with (B.9). In particular, the second derivative fixes $p_2 = \frac{3}{4}$. Combining all our results we have [41, 18].[20]

$$Li_3(e^{-x}) \simeq p(x) - \frac{1}{2}x^2\log(x)\,, \tag{B.13}$$

where

$$p(x) = \zeta(3) - \frac{\pi^2}{6}x + \frac{3}{4}x^2 + \frac{1}{12}x^3 + O(x^{2n})\,, \quad n = 2, 3, 4, \ldots\,. \tag{B.14}$$

[19] There is an analogous equation for $k \leq 0$, where the right hand side is zero.

[20] Our formula specifies some terms which were not displayed in [41, 18] Ref. [41] suggests the existence of terms of the form $O(x^3)\log(x)$, but as we have seen these are absent. Our formula is consistent with eq. (8.5) of [18] after the change of variables $x = -\log(1-y)$. We thank G. Cardoso for discussions on this issue.

Note that the higher terms in $p(x)$ are even powers of x. The odd powers, except the linear and the cubic term, are ruled out by the connection formula. This is important for our discussion of gauge symmetry.

To analyze the decompactification limit we need the first formula of (B.5), or, being more precise about the asymptotics,

$$Li_3(e^{-x}) \simeq e^{-x}\,, \qquad \text{for} \qquad x \to \infty\,. \tag{B.15}$$

C. Modular forms

The modular group is defined by the following transformation:[21]

$$T \to \frac{aT - ib}{icT + d}\,, \qquad ad - bc = 1\,, \qquad a, b, c, d \in \mathbb{Z}\,. \tag{C.1}$$

On the fundamental domain of this transformation there are two fixed points at $T = 1$ and $T = \rho \equiv e^{i\pi/6}$.

A modular form $E_k(iT)$ of weight k is defined to be holomorphic and to obey the transformation law

$$E_k(iT) \to (icT + d)^k E_k(iT)\,. \tag{C.2}$$

One can show that there are no modular forms of weight 0 and 2, while at weight 4 and 6 one has the Eisenstein functions

$$\begin{aligned} E_4(q) &\equiv 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n} = 1 + 240q + 2160q^2 \ldots\,, \\ E_6(q) &\equiv 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n} = 1 - 504q - 16632q^2 \ldots\,, \end{aligned} \tag{C.3}$$

where $q \equiv e^{-2\pi T}$. From their definition one immediately infers that they have been normalized such that

$$\lim_{T\to\infty} E_4 = 1 = \lim_{T\to\infty} E_6\,. \tag{C.4}$$

Furthermore, both function have no pole on the fundamental domain and E_4 has exactly one simple zero at $T = \rho$, while E_6 has one simple zero at $T = 1$

$$E_4(i) \neq 0\,, \qquad E_4(i\rho) = 0\,, \qquad E_6(i) = 0\,, \qquad E_6(i\rho) \neq 0\,. \tag{C.5}$$

One can construct modular forms of arbitrary even weight from products of these two Eisenstein functions.

A modular form which vanishes at $T = \infty$ is called a cusp form. There is no cusp form of weight $r < 12$ and for $r = 12$ there is the unique cusp form η^{24} where

$$\eta(q) \equiv q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)\,, \qquad \eta^{24} = \frac{E_4^3 - E_6^2}{1728} \tag{C.6}$$

is the Dedekind η-function. (η does not vanish at ρ or i.)

[21] It is common to choose a different convention for T where real and imaginary part are exchanged. More precisely, for $\tau = iT$ one has $\tau \to \frac{a\tau+b}{c\tau+d}$.

One can also construct modular invariant functions but such a function necessarily has a pole somewhere on the fundamental domain. The j-function defined by

$$j(q) \equiv \frac{E_4^3}{\eta^{24}} = \frac{E_6^2}{\eta^{24}} + 1728 \frac{E_4^3}{E_4^3 - E_6^2} = q^{-1} + 744 + 196884q + \cdots \tag{C.7}$$

has a simple pole at $T = \infty$ and a triple zero at $T = \rho$.

Finally, the Eisenstein series E_2 is defined by

$$E_2(iT) = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} \,. \tag{C.8}$$

$E_2(iT)$ is holomorphic, but not quite a modular form:

$$E_2 \to (icT + d)^2 E_2(iT) + \frac{6c}{\pi i}(icT + d) \,. \tag{C.9}$$

The derivative of a modular form is in general not a modular form. But using the transformation properties of E_2 one defines the modular covariant derivative

$$Df(iT) := f'(iT) - k \,\frac{\pi i}{6} E_2(iT) f(iT) \,, \tag{C.10}$$

where the prime denotes differentiation with the respect to the argument, i.e. $f'(iT) \equiv -i\partial_T f(iT)$. The covariant derivative maps modular forms of degree k to modular forms of degree $k+2$. Its action on normalized Eisenstein series is

$$DE_k(iT) = -k \,\frac{\pi i}{6} E_{k+2}(iT) \tag{C.11}$$

for $k = 4, 6, \ldots$. This can be used to express derivatives of Eisenstein series in terms of the Eisenstein series themselves. For E_2' we also have a relation:

$$E_2' - \frac{\pi i}{6} E_2 E_2 = -\frac{\pi i}{6} E_4 \,. \tag{C.12}$$

Also note that all higher Eisenstein series E_k, with $k = 8, 10, 12, \ldots$ are homogenous polynomials in E_4, E_6 (the ring of modular forms is generated by E_4, E_6). For example:

$$E_8 = E_4^2 \,, \qquad E_{10} = E_4 E_6 \,. \tag{C.13}$$

In the text we need the following derivatives:

$$\begin{aligned} E_4' &= \frac{2\pi i}{3}(E_4 E_2 - E_6) \,, \\ E_6' &= i\pi(E_6 E_2 - E_8) = i\pi(E_6 E_2 - E_4^2) \,, \\ j' &= -2\pi i \; j \,\frac{E_6}{E_4} = -2\pi i E_4^2 E_6 \eta^{-24} \,. \end{aligned} \tag{C.14}$$

The logarithmic derivative of η^{-24} is proportional to E_2:

$$(\eta^{-24})' = -2\pi i \eta^{-24} E_2 \,. \tag{C.15}$$

As we just saw the derivative of a modular form is not a modular form, since it does not satisfy eq. (C.2) in general. An exception is the derivative $\partial_T^n F_{1-n}$ which transforms according to

$$\partial_T^n F_{1-n} \to (icT + d)^{(n+1)} \partial_T^n F_{1-n} \tag{C.16}$$

and thus is a modular form of weight $n+1$.

Acknowledgments

This work is supported by DFG — The German Science Foundation — within the "Schwerpunktprogramm Stringtheorie" and by GIF — the German-Israeli Foundation for Scientific Research, the European RTN Program HPRN-CT-2000-00148 and the DAAD — the German Academic Exchange Service.

We have greatly benefited from conversations with G. Cardoso and A. Micu.

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Received: *March 25, 2003*

Addendum

After completion of this paper, we became aware of related work in refs. [42, 43], where similar phenomena in $N = 1$ supersymmetric non-linear sigma models are analyzed. We thank Jan-Willem van Holten for drawing our attention to these references.

Added references

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