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Universal Calabi-Yau Algebra: Classification and Enumeration of Fibrations

F. Anselmo[1], **J. Ellis**[1], **D. V. Nanopoulos**[2] and **G. Volkov**[1,3]

[1] *CERN, CH-1211 Geneva, Switzerland*

[2] *George P. and Cynthia W. Mitchell Institute of Fundamental Physics,*
Texas A & M University, College Station, TX 77843-4242, USA
Astroparticle Physics Group, Houston Advanced Research Center (HARC),
The Mitchell Campus, The Woodlands, TX 77381, USA
Chair of Theoretical Physics, Academy of Athens, Division of Natural Sciences,
28 Panepistimiou Avenue, Athens 10679, Greece

[3] *LAPP TH, Annecy-Le-Vieux, France, and*
St Petersburg Nuclear Physics Institute, Gatchina, 188300 St Petersburg, Russia

Abstract

We apply a universal normal Calabi-Yau algebra to the construction and classification of compact complex n-dimensional spaces with $SU(n)$ holonomy and their fibrations. This algebraic approach includes natural extensions of reflexive weight vectors to higher dimensions and a 'dual' construction based on the Diophantine decomposition of invariant monomials. The latter provides recurrence formulae for the numbers of fibrations of Calabi-Yau spaces in arbitrary dimensions, which we exhibit explicitly for some Weierstrass and K3 examples.

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1 The Algebraic Way to Unify Calabi-Yau Geometry

Geometrical ideas play ever-increasing rôles in the quest to unify all the fundamental interactions. They were introduced by Einstein in the formulation of general relativity, and extended to higher dimensions by Kaluza and Klein in order to include electromagnetism. To explain the appearance of electromagnetism, it was enough to introduce just one extra dimension with the topology of a circle, exploiting the geometrical equivalence between the $U(1)$ gauge group and the circle. In order to find a corresponding geometrical origin for the full Yang-Mills symmetries of the Standard Model, namely $SU(3) \times SU(2) \times U(1)$, one needs to consider more complicated geometrical structures.

The modern approach to gauge symmetry is based on string theory, whose underlying geometrical nature is still mysterious, but includes an enormous extension of general coordinate invariance in ten or eleven dimensions. This is large enough to include the gauge symmetries of the Standard Model in four dimensions, if one compactifies six surplus dimensions on a suitable complex three-dimensional space, called a Calabi-Yau manifold [1]. In the original compactifications of weakly-coupled ten-dimensional heterotic string theory, the resulting four-dimensional gauge group would be some subgroup of E_6.

This construction has since been extended to various non-perturbative constructions. It has been realized that additional gauge-group factors may appear in suitable singular limits of the Calabi-Yau manifold [2]. For example, one approach to non-perturbative string theory is based on twelve-dimensional F theory, which may be compactified to four dimensions on a complex four-dimensional space with $SU(4)$ holonomy. We also note that many CY_3 or CY_4 spaces can be obtained as Complete-Intersection Calabi-Yau (CICY) spaces, i.e, as projections inside higher-dimensional CY_n, motivating further studies of the latter also for $n > 4$.

These geometrical ideas exemplify the physics interest of classifying systematically spaces with holonomy groups in the series $SU(n), SO(n)$ and $Sp(n)$, as well as G_2 and Spin(7). Listings are available of special cases such as $K3$ spaces with $SU(2)$ holonomy and CY_3 spaces with $SU(3)$ holonomy [3, 4], and there are also many results for other holonomy groups [5, 6]. Ideally, one would like to classify these spaces in a systematic way, much as Cartan provided an algebraic classification of Lie groups [7]. This is, of course, a very ambitious programme, for which only partial results are available.

We have proposed an algebraic approach [8, 9] to the problem of classifying complex CY_n manifolds with $SU(n)$ holonomy, which is based on their identifications with the loci of zeroes of polynomials in suitable complex projective spaces, and their complete intersections. These complex projective spaces in different dimensions are characterized by 'reflexive' projective weight vectors $\vec{k}$. Our approach has been based on the systematic extension of lower-dimensional projective vectors to higher dimensions, and their combination via binary, ternary, quaternary, etc., algebraic operations of increasing 'arity'.

We have recently proposed [10] a supplement to this approach which is based on a 'dual' approach via the monomials $x^{\vec{\mu}_\alpha} = x_1^{\mu_{1\alpha}} x_2^{\mu_{2\alpha}} ... x_{n+1}^{\mu_{(n+1)\alpha}}$ in the quasihomogeneous coordinates $x_1, ..., x_{n+1}$ that obey a 'duality' condition $\vec{\mu}_\alpha \cdot \vec{k} = [d]$. Specifically, we showed how CY_n spaces could be obtained by the Diophantine decomposition of simple invariant monomials, a technique that gives immediate insights into the fibrations of higher-dimensional Calabi-Yau spaces involving lower-dimensional Calabi-Yau spaces.

In this Letter, we summarize briefly the essential aspects of this new Diophantine algebraic approach to the systematic classification of Calabi-Yau spaces, demonstrating its complementarity to our previous expansion technique. In particular, we present a number of explicit results for the numbers of fibrations of Calabi-Yau spaces in arbitrary numbers of complex dimensions. These results support our claim to have formulated a 'Universal Calabi-Yau Algebra' [11] capable in principle of decoding the full Calabi-Yau genome. Moreover, as we comment at the end, the techniques used here could in principle also be used to classify the series of spaces with $SO(n)$.

2 The Arity-Dimension Structure of Universal Calabi-Yau Algebra

The starting point for our algebraic classification of Calabi-Yau spaces has been the construction of 'reflexive' weight vectors $\vec{k}$, whose components specify complex quasihomogeneous projective spaces $CP^n(k_1, k_2, ..., k_{n+1})$. These have $(n+1)$ quasihomogeneous coordinates

$x_1, ..., x_{n+1}$, which are subject to the following identification:

$$(x_1, \ldots, x_{n+1}) \sim (\lambda^{k_1} \cdot x_1, \ldots, \lambda^{k_{n+1}} \cdot x_{n+1}). \tag{1}$$

A general quasihomogeneous polynomial of degree $[d]$ is a linear combination

$$\wp = \sum_{\vec{\mu}_\alpha} c_{\vec{\mu}_\alpha} x^{\vec{\mu}_\alpha} \tag{2}$$

of monomials $x^{\vec{\mu}_\alpha} = x_1^{\mu_{1\alpha}} x_2^{\mu_{2\alpha}} ... x_{n+1}^{\mu_{(n+1)\alpha}}$ with the condition:

$$\vec{\mu}_\alpha \cdot \vec{k} = [d]. \tag{3}$$

A d-dimensional Calabi-Yau space X_d can be given by the locus of zeroes of a transversal quasihomogeneous polynomial $\wp$ of degree $deg(\wp) = [d] : [d] = \sum_{j=1}^{n+1} k_j$ in such a complex projective space $CP^n(\vec{k}) \equiv CP^n(k_1, ..., k_{n+1})$ [3]:

$$X \equiv X^{(n-1)}(k) \equiv \{\vec{x} = (x_1, ..., x_{n+1}) \in CP^n(k) | \wp(\vec{x}) = 0\}. \tag{4}$$

This algebraic projective variety is irreducible if and only if its polynomial $\wp$ is irreducible. A hypersurface will be smooth for almost all choices of polynomials. To obtain Calabi-Yau d-folds, one should choose reflexive weight vectors (RWVs) related to reflexive Batyrev polyhedra [12]. Other examples of compact Calabi-Yau manifolds can be obtained as the complete intersections (CICY) of such quasihomogeneous polynomial constraints:

$$X_{CICY}^{(n-r)} = \{\vec{x} = (x_1, \ldots x_{n+1}) \in CP^n \,|\, \wp_1(\vec{x}) = \ldots = \wp_r(\vec{x}) = 0\}, \tag{5}$$

where each polynomial $\wp_i$ is determined by some extended weight vector $\vec{k}_i$, $i = 1, \ldots, r$, where r is the arity

These RWVs that specify the polynomials $\wp_i$ may be classified using the natural extensions of lower-dimensional vectors and their combinations via binary, ternary, etc., operations ω_r, as illustrated in Fig. 1. The Universal Calabi-Yau Algebra (UCYA) structure of reflexive weight vectors (RWVs) in different dimensions depends on two integer parameters: the dimension n of the RWVs, and the *arity* r of the combination operation ω_r.

A useful technique for constructing Calabi-Yau spaces in any number of dimensions is to visualize the various possible monomials $(x_1^{\mu_1} x_2^{\mu_2} ... x_n^{\mu_{n+1}})_\alpha$ as points $m_\alpha = (\mu_1, ..., \mu_{n+1})_\alpha$ in

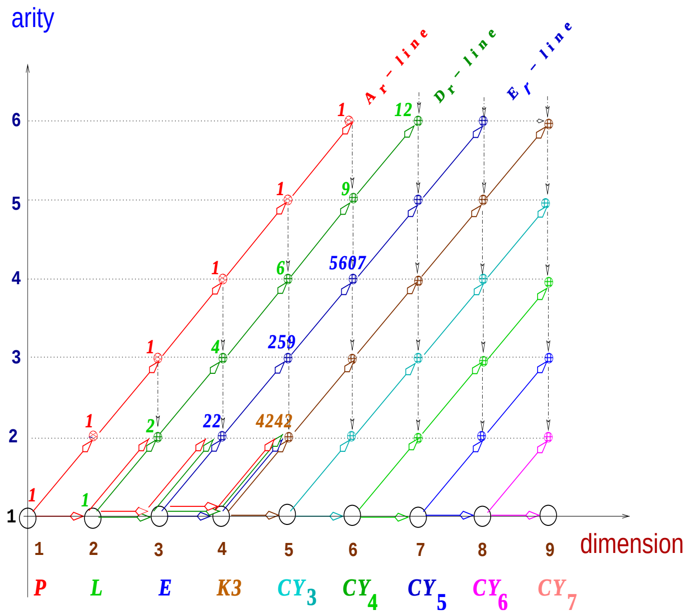

Figure 1: *The arity-dimension plane for complex manifolds with $SU(n)$ holonomy, showing the numbers of eldest vectors/chains obtained by normal extensions of RWVs, including complete results for CY_3 and lower-dimensional spaces, and partial results for CY_4 and CY_5 spaces.*

the Z_{n+1} integer lattice of an n-dimensional polyhedron [1]. This feature enabled us to introduce a complementary algebraic approach to the construction of Calabi-Yau spaces, based on the construction of suitable monomials $\vec{\mu}$ obeying the 'duality' condition: $\vec{k} \cdot \vec{\mu}_\alpha = d$. This construction supplements the previous geometrical method related to Batyrev polyhedra, and enables one to calculate the numbers of eldest vectors, and hence chains, in arbitrary dimensions. We have verified explicitly that the eldest vectors found in the two different ways agree in several instances for both CY_3 and CY_4 spaces [10], providing increased confidence in our results. The study of the Calabi-Yau equations and the associated hypersurfaces via the remarkable composite properties of invariant monomials (IMs) provides an algebraic alternative to reflexive polyhedron techniques.

Central rôles are played in this approach by composite lower-dimensional structures

[1] Using this technique, Batyrev [12] demonstrated by explicit construction how to associate a mirror polyhedron to each Calabi-Yau space. This approach also established in a very elegant way the corresponding mirror duality among Calabi-Yau spaces, in which CICY spaces play an essential role.

within CY d-folds, which can be seen by the algebraically dual ways of expansions using weight vectors $\vec{k}$ and IMs. By analogy with the Galois normal extension of fields, we term the first way of expanding weight vectors a *normal* extension, and the dual decomposition in terms of IMs we call the *Diophantine* expansion. These two expansion techniques are consistently combined in our algebraic approach, whose composition rules exhibit explicitly the internal structure of the Calabi-Yau algebra. Our method is closely connected to the well-known Cartan method for constructing Lie algebras, and reveals various structural relationships between the sets of Calabi-Yau spaces of different dimensions. We interpret our approach as revealing a 'Universal Calabi-Yau Algebra' [11] for the following reasons: 'Universal' because it may, in principle, be used to generate all Calabi-Yau manifolds of any dimension with all possible substructures, and 'Algebra' because it is based on a sequence of binary and higher n-ary operations on weight vectors and monomials.

This Universal Calabi-Yau Algebra (UCYA) acts upon the set of reflexive weight vectors in all dimensions, $A_n \equiv \{\mathrm{RWV}(n)\}$, and the corresponding set of invariant monomials, $\{IM(n)\}$, which is 'dual' to A_n in the sense of (3). The IMs are the minimal set of monomials determining the eldest vector and its chain, and thence the full list of weight vectors in the corresponding chain. According to their degrees $a = 2, 3, 4, ...$ we term them conics, cubics, etc.. The structure of each sloping line in the dimension-arity plane is determined by the corresponding set of IMs, e.g., the first A_r line is determined by the unit monomials E_n, the second by conics and linear monomials, the third line by cubics and quartics, taking into account also conics, etc.. The number of IMs is much less the full set of monomials $\vec{m}_\alpha : 1 \leq \alpha \leq \alpha_{max}$ that determine the Calabi-Yau equation. To construct them, we can start from the unit IM in some dimension n and then, via a Diophantine expansion, generate related conic IMs, cubic IMs, quartic IMs, etc.. This process may then be continued by studying in turn the Diophantine expansions of conic IMs, of cubic IMs, etc..

We note in addition that the algebraic-geometry realization [3, 13] of Coxeter-Dynkin diagrams provides a general characterization of the possible structures in singular limits of Calabi-Yau hypersurfaces, which are associated with possible gauge groups. Thus, a deeper understanding of the origins of gauge invariance provides an additional motivation for studying string vacua via our unification of the complex geometry of $d = 1$ elliptic curves, complex tori, $K3$ manifolds, CY_3, CY_4, etc. This point is illustrated in Figs. 1, where the points on the the first three sloping lines, labelled A_r (red), D_r (green) and E (blue), correspond to those d-folds that are characterized by the 'maximal' quotient A, D, E singularities, respectively [2]. This characterization of the types of singularities is directly connected to the degrees of the associated monomials - linear, conics, cubics, quartics, etc., that appear along the corresponding sloping lines.

In summary: the UCYA provides a two-parameter classification of CY_d spaces in terms of arity r and dimension n, which is based on the following ingredients:

- Universal composition rules,
- Normal expansions and Diophantine decompositions,
- Mirror symmetry,
- Singularities and their links with Cartan-Lie algebras.

We have shown that this algebraic approach leads us to a natural formalism for a unified description of complex geometry in all dimensions, including $K3$ spaces and Calabi-Yau d-folds for any d [8, 10].

As an example of the extension procedure for RWVs, as applied to $K3$ manifolds, we classified [8] the 95 different possible weight vectors $\vec{k}$ in 22 binary chains generated by pairs of extended vectors, which included 90 of the total, and 4 ternary chains generated by triplets of extended vectors, which yielded 91 weight vectors, of which 4 were not included in the binary chains. The one remaining $K3$ weight vector was found in a quaternary chain [8]. This algebraic construction provides a convenient way of generating all the $K3$ weight vectors, and arranging them in chains of related vectors whose overlaps yield further indirect relationships.

[2] To be more precise, the D line includes also A-type singularities, and the E line includes also D-type and A-type singularities.

3 The Classification and Enumeration of Fibrations

We now show how our technique for building higher-dimensional Calabi-Yau spaces systematically out of lower-dimensional ones enables us also to enumerate explicitly their fibrations. As examples, we showed previously [8, 9, 10] how our construction reveals elliptic and $K3$ fibrations of CY_3 manifold [3]. We now present some further results derived via the new description of CY_d spaces based on the structures of the set of invariant monomials (IMs). Recurrence relations for conic, cubic and quartic monomials give us the formulae for the numbers of IMs and hence fibrations in arbitrary dimensions, leading us to a complete solution for the fibrations of CY_d spaces along the D_r- and E_r-lines in Fig. 1. These results confirm that, in the framework of the UCYA, the Calabi-Yau 'genome' can in principle be solved completely. As we explain in more detail below, the IMs determine completely the fibration structures of the 22 $K3$ chains mentioned earlier, as shown in Fig. 2:

$$\begin{aligned} \{IM\}_4 \quad &\mapsto \quad \Big(1\cdot\{4\}_\triangle\Big)+\Big(\mathbf{2}\cdot\{\mathbf{10}\}_{\boldsymbol{\triangle}}\Big) \\ &+ \quad \Big(2\cdot\{5\}_\triangle + 1\cdot\{5\}_\square\Big) \\ &+ \quad \Big(\mathbf{4}\cdot\{\mathbf{9}\}_{\boldsymbol{\triangle}} + 2\cdot\{9\}_\square\Big) \\ &+ \quad \Big(\mathbf{7}\cdot\{\mathbf{7}\}_{\boldsymbol{\triangle}} + 1\cdot\{7\}_\square\Big) \\ &+ \quad \Big(1\cdot\{6\}_\square\Big)+\Big(1\cdot\{8\}_\square\Big) \\ &\mapsto \quad \{22\}, \end{aligned} \tag{6}$$

where we label planar sections through Batyrev polyhedra via the number of points they contain and their geometric shapes: $\{N\}_{\triangle,\square}$, etc.. This expansion in terms of fibration structures for $K3$ spaces may be extended to more general CY_d spaces, via recurrence relations. Each of the terms $\{10,4,...\}_{\triangle,\square,...}$ in the expansion has its own recurrence relation, of which we give below several examples, namely those indicated in bold script above: $\mathbf{2}\cdot\{\mathbf{10}\}_{\boldsymbol{\triangle}}$, etc., providing complete results in any number of dimensions for the numbers of CY_d spaces with

[3] Our approach may also be used to obtain the projective weight vector structure of a mirror manifold, starting from those of a given Calabi-Yau manifold.

these particular fibrations. A similar recurrence formula could in principle be derived for any analogous fibration.

In the five-dimensional case corresponding to CY_3 spaces, we have derived the types and numbers of IMs which determine the structures of the full 259 (161 irreducible) chains, which are similar to those for the $K3$ case above:

$$
\begin{aligned}
\{IM\}_5 \;\mapsto\;& \Big(9 \cdot \{4\}_{\triangle} + \mathbf{4} \cdot \{\mathbf{10}\}_{\boldsymbol{\triangle}}\Big) \\
+\;& \Big(16 \cdot \{5\}_{\triangle} + 5 \cdot \{5\}_{\square} + 1 \cdot \{5\}_{\square'}\Big) \\
+\;& \Big(\mathbf{11} \cdot \{\mathbf{9}\}_{\boldsymbol{\triangle}} + 5 \cdot \{9\}_{\square} + 1 \cdot \{9\}_{\square'}\Big) \\
+\;& \Big(\mathbf{28} \cdot \{\mathbf{7}\}_{\boldsymbol{\triangle}} + 7 \cdot \{7\}_{\square} + 1 \cdot \{7\}_{Quint}\Big) \\
+\;& \Big(8 \cdot \{6\}_{\square} + 1 \cdot \{6\}_{Quint}\Big) \\
+\;& \Big(6 \cdot \{8\}_{\square} + 1 \cdot \{8\}_{Quint}\Big) \\
\mapsto\;& \{161\}
\end{aligned}
\tag{7}
$$

In the six-dimensional case corresponding to CY_4 spaces, out of the 5,607 6-dimensional 4-vector chains, just 2,111 are independent. We find the following classification of their fibrations:

$$
\begin{aligned}
\{IM\}_6 \;\mapsto\;& \Big(37 \cdot \{4\}_{\triangle} + \mathbf{7} \cdot \{\mathbf{10}\}_{\boldsymbol{\triangle}}\Big) \\
+\;& \Big(66 \cdot \{5\}_{\triangle} + 27 \cdot \{5\}_{\square} + 6 \cdot \{5\}_{\square'}\Big) \\
+\;& \Big(\mathbf{24} \cdot \{\mathbf{9}\}_{\boldsymbol{\triangle}} + 11 \cdot \{9\}_{\square} + 5 \cdot \{9\}_{\square'}\Big) \\
+\;& \Big(\mathbf{84} \cdot \{\mathbf{7}\}_{\boldsymbol{\triangle}} + 28 \cdot \{7\}_{\square} + 5 \cdot \{7\}_{Quint} + 1 \cdot \{7\}_{Sixt}\Big) \\
+\;& \Big(36 \cdot \{6\}_{\square} + 5 \cdot \{6\}_{Quint}\Big) \\
+\;& \Big(21 \cdot \{8\}_{\square} + 5 \cdot \{8\}_{Quint}\Big) \\
\mapsto\;& \{2111\}.
\end{aligned}
\tag{8}
$$

Similar expressions can be derived for any desired dimension.

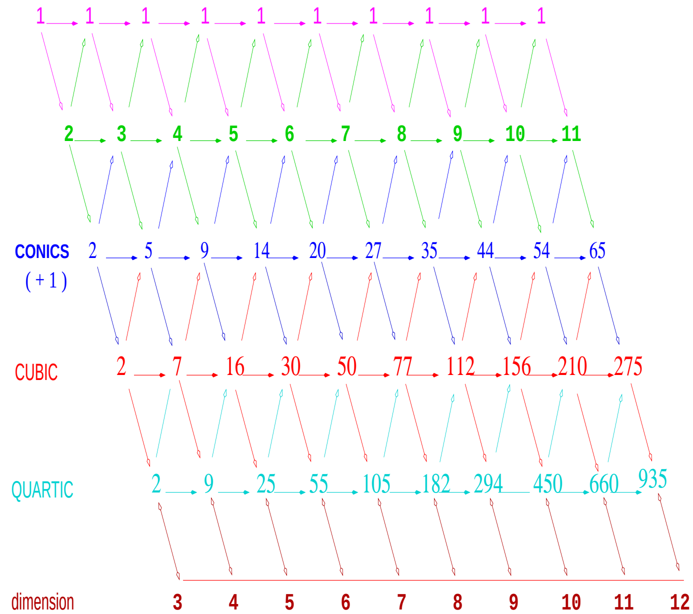

Figure 2: *Lattice illustrating recurrence relations for the numbers of conic, cubic and quartic monomials.*

Illustrating the derivation, we recall that the first A_r line in Fig. 1 is characterized by the unit monomials $E_n = (1, ...,)$. The CY_d spaces along the second D_r line with arity $r = (n-1)$ have 'almost trivial' substructures, i.e., 'circles', whose reflexive polyhedra are linear. These may be classified and enumerated by Diophantine expansions of the unit monomials $E_n = (1, ..., 1)_n$ in terms of pairs of conics $C_{i(n)}$ and $C_{j(n)}$, which should satisfy the following Diophantine property:

$$\frac{1}{2}(C_{i(n)} + C_{j(n)}) = E_n, \tag{9}$$

where the index n notes the dimension being considered. This Diophantine expansion yields the following numbers of possible different types of conic monomials in any dimension n,

$$N_{conics} = \frac{(n)(n-1)}{2}, \tag{10}$$

as may easily be shown by induction. As we discuss later, the structures of the CY_d spaces on the next third E_r line are correspondingly determined by Diophantine expansions in terms of cubic and quartic IMs, either directly: $(P_1 + P_2 + P_3)/3 = E_n$, or in two steps: $(Q_{j1} + Q_{j2})/2 = C_j$ where $\vec{k}^{ext} \cdot Q_a = d$ and $\vec{k}^{ext} C_j = d$.

In order to enumerate the IMs and the corresponding chains of Calabi-Yau spaces, which are given by suitable pairs (9) of conics, one solves the following Diophantine equations:

$$\vec{k}^{i,ext} \cdot C_{1(n)} = \vec{k}^{i,ext} \cdot E_n = d(\vec{k}^{i,ext}). \tag{11}$$

To give sense to these equations and, consequently, to evaluate the finite numbers of chains and their eldest vectors in the case of arity $r = (n-1)$, we first recall that, in the UCYA, the points on this line in the arity-dimension plane are determined by n-dimensional extensions of the two eldest vectors $\vec{k}_1 = (1)$ and $\vec{k}_2 = (1,1)$. This means that the possible values of $d(\vec{k}^{i(ex)})$ in these equations are only 1 and 2, whereas the components of the extended vectors can only be 0 or 1. Due our algebra, this second sloping line is determined also only by extensions of the weight vectors (1) and (1,1), so their components can include only one or two units. It is then simple to verify by induction the following recurrence formula for the numbers of chains along the second diagonal line in the arity-dimension plane:

$$\begin{aligned} N_{chains} &= k \cdot (k+1), \qquad if \;\; n = (2k+1) \\ N_{chains} &= k^2, \qquad if \;\; n = (2k). \end{aligned} \tag{12}$$

Thus, along the line $r = (n-1)$, the numbers of the eldest vectors and chains in dimensions $n = 2, 3, 4, ...$ are the following: 1, 2, 4, 6, 9, 12, 16, 20, 25, 30, 36, 42, 49, 56, 64, 72, 81, 90, 100, 110, 121, 132, 144, ...

Extending our previous approach to the third line in Fig. 1, the first step is to enumerate the cubic and quartic monomials, from which we can find all the IMs along this E_r line. The appearance of cubic monomials is connected with the following new Diophantine condition for the expansion of the unit monomials E_n of the A_r line:

$$E_n \mapsto \{P_1, P_2, P_3 | \frac{1}{3}(P_1 + P_2 + P_3) = E_n\}. \tag{13}$$

However, the set of appropriate cubic monomials is somewhat more restricted. Similarly, the appearance of quartic monomials is connected with the possible Diophantine expansion of the conic monomials $C_{i(n)}$ of the second D_r line:

$$C_{i(n)} \;\mapsto\; \{P_1, P_2 | \frac{1}{2}(P_1 + P_2) = C_{i(n)}\}. \tag{14}$$

Again, there are some further restrictions on the list of possible quartic monomials, which we do not discuss here.

As indicated in Fig. 2, there are recurrence formulae for the numbers of monomials in any dimension, which are obvious for the leading (red and green) lines in the arity-dimension plane. The resulting expressions for the numbers of cubic and quartic monomials are, respectively:

$$\begin{aligned} N_{cubics} &= \frac{1}{6}(n-2)(n-1)(n+3) \\ N_{quartics} &= \frac{1}{24}(n-2)(n-1)(n)(n+5) \end{aligned} \tag{15}$$

There are remarkable links between the numbers of conics, cubics and quartics. For example, to obtain the number of quartics in dimension n, one should sum over all the cubics in dimensions $3, 4, ..., n$, i.e., $N^{(n)}_{Quart} = \sum_{i=3}^{i=n} N^{(i)}_{Cub}$. Thus, as seen in Fig. 2, the number 105 of quartic monomials in the septic Calabi-Yau case can be represented as follows: $2_{dim=3} + 7_{dim=4} + 16_{dim=5} + 30_{dim=6} + 50_{dim=7}$.

Based on Fig. 2, one can convince oneself that there also exist n-dimensional recurrence formulae for the numbers of IMs along other diagonal lines in any dimension, as we have found for the first two lines on the arity-dimension plot in Fig. 1. However, the situation can become complicated, because, in the construction of the cubic and quartic IMs, one must also take into account conic and conic + cubic monomials, respectively. In the case of Calabi-Yau spaces with Weierstrass fibres, it is also important to know the list of sextic monomials, which is given by the following recurrence formula:

$$C^{n-3}_{n+2} = \frac{(n+2)!}{(n-3)!5!}, \tag{16}$$

where $n \geq 3$ is the dimension of the weight-vector space. Via these sextic monomials and three- and four-fold Diophantine decompositions of the unit vector E_n, we then obtain the following expression for the number of Weierstrass IMs, $\{3\}$ and $\{4\}$:

$$N_W(n) = N_{\{7\}_\Delta}(n) = C^{n-3}_{n+3} = \frac{(n+3)!}{(6!)(n-3)!}, \tag{17}$$

valid for all dimensions.

Similarly, one can find a recurrence relation for $\{IM\} = \{9\}_\Delta$, which is constructed from two quartic monomials, one conic and E_n. In this case the difference of the two quartic monomials should be divisible by four. Taking into account all possible quartics, after some effort, one can find the following formula for the number of these IMs:

$$N_{\{9\}_\Delta}(n) = \frac{1}{3} \cdot (n-2)(n^2 - 4n + 6). \tag{18}$$

This expression gives the following numbers: $1, 4, 11, 24, 45, 76, 119, 176, 249, ...$ for $n = 3, 4, 5, 6, 7, 8, 9, 10, 11, ..,$ respectively. We find analogously the recurrence relation

$$N_{\{10\}_\Delta}(n) = N_{\{10\}_\Delta}(n-1) + n(3), \tag{19}$$

where $n(3)$ denotes the number of the ways of decomposing n into three positive integers.

This way of getting the recurrence formula for Calabi-Yau spaces with elliptic fibres $\{10\}_\Delta$ can be extended to the cases of CY_d spaces with $K3$ fibres, such as those described by $\vec{k}_4 = (1,1,1,1)[4]$ whose algebraic equation includes 35 monomials. The IM_4 for this $K3$ space contains the four quartic monomials P_1, P_2, P_3, P_4 obeying the Diophantine equation: $(P_1+P_2+P_3+P_4)/4 = E_4$. These monomials have in addition one very important condition: $P_i - P_j$ should be divisible by 4 for each choice of $i, j = 1, 2, 3, 4, i \neq j$. The types of different n-dimensional IM_4, describing the $CY_d : n = d+2 \geq 4$ spaces with such $\{35\}_\Delta$ fibres are constructed only from the numbers 4 and 0, and possibly the unit. Similarly to the case of the third E_r sloping line, the recurrence formulae for these IMs is determined by the expansions of positive integer numbers in terms of four positive integers. Indeed, for each slope line there is a recurrence formula of the type

$$N_{...\Delta}(n) = N_{...\Delta}(n-1) + n(p), \tag{20}$$

where p is the number of the sloping line, and $N_{...\Delta}(n_{min}) = 2 : n_{min} \geq 3$. The numbers of many other desired IMs can be established in a similar way.

We give finally the example of CY_d spaces on the fourth ($K3$) line with an intersection manifold determined by $\vec{k} = (1,1,4,6)[12]$, corresponding to such a fibre in the mirror manifold. All these CY_d spaces can be constructed by Diophantine expansions of unit monomials $E_n = (1, ..., 1) \rightarrow \{P_1, P_2, P_3, P_4 | 1/4(P_1 + P_2 + P_3 + P_4) = E_n\}$, where

$n = d + 2$. They can also be obtained by Diophantine expansions of conic monomials $C_2 = (2, 2, ..., 2, 0)$ appearing along the second sloping line, together with $C_1 = (0, ..., 0, 2)$: $C_2 \to \{P_1, P_2, P_3 | 1/3(P_1 + P_2 + P_3) = C_2\}$. Finally, these CY_d spaces can also be obtained by Diophantine expansions of the cubic or quartic monomials of the third E_r line: $(Cub) \to \{P_1, P_2 | 1/2(P_1 + P_2) = (Cub)\}$. The lists of such CY_d can be determined in a similar way to the list of CY_d spaces with Weierstrass fibres, with the difference that, instead of sextic monomials, we consider IMs of twelfth degree in any fixed variable. The corresponding formula for the number of CY_d spaces with $(1, 1, 4, 6)[12]$ intersections is:

$$N_{\{39\}_\Delta}(n) = C_{n+3}^{n-4} = \frac{(n+3)!}{(7!)(n-4)!}, \tag{21}$$

where $n \geq 5$ and the index 39 corresponds to the total number of monomials in the intersection.

4 Conclusions and Prospects

In previous papers [8, 9, 10], we have described two aspects of a Universal Calabi-Yau Algebra that enables one in principle to classify and enumerate Calabi-Yau spaces and their fibrations in any number of dimensions. One aspect of this algebraic approach is the normal extension of reflective weight vectors to higher dimensions [8, 9], and the other is the Diophantine expansion of invariant monomials, which was first discussed in [10], and has been further developed in this paper. As discussed here, this invariant-monomial approach enables one to enumerate systematically the numbers of interesting fibrations of Calabi-Yau manifolds in arbitrary dimensions. The same approach could be adapted to discuss in a similar way manifolds with $SO(n)$ holonomy, a topic extending beyond the scope of this paper.

We close by noting that the basic ideas of the Universal Calabi-Yau Algebra can be accommodated within the general theory of operads [14]. We leave this and the relation of our work to generalizations of algebraic concepts for a future discussion [15].

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