Solvable model of strings in a time-dependent plane-wave background

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Abstract

We investigate a string model defined by a special plane-wave metric $ds^2 = 2dudv - \lambda(u)x^2du^2 + dx^2$ with $\lambda = \frac{k}{u^2}$ and k=const > 0. This metric is a Penrose limit of some cosmological, Dp-brane and fundamental string backgrounds. Remarkably, in Rosen coordinates the metric has a "null cosmology" interpretation with flat spatial sections and scale factor which is a power of the light-cone time u. We show that: (i) This spacetime is a Lorentzian homogeneous space. In particular, it admits a boost isometry $u' = \ell u$, $v' = \ell^{-1}v$ similar to Minkowski space. (ii) It is an exact solution of string theory when supplemented by a u-dependent dilaton such that the corresponding effective string coupling $e^{\phi(u)}$ goes to zero at $u = \infty$ and at the singularity $u = 0$, reducing back-reaction effects. (iii) The classical string equations in this background become linear in the light-cone gauge and can be solved explicitly in terms of Bessel's functions, and thus the string model can be directly quantized. This allows one to address the issue of singularity at the string-theory level. We examine the propagation of first-quantized point-particle and string modes in this time-dependent background. Using an analytic continuation prescription we argue that the string propagation through the singularity can be smooth.

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1 Introduction

Some major problems in string cosmology are the nature of cosmological singularity, the initial conditions or the possibility of a pre-bigbang region and the definition observables (see, e.g., [1, 2, 3, 4] for a recent reviews and references). Similar issues arise in many *time-dependent* backgrounds. So to address them one may start with examples which, in contrast to "realistic" (FRW or de Sitter) cosmological backgrounds, are easier to embed to and analyze in string theory.

Much of our understanding of closed string theory is based on few examples of exactly solvable models. Solvability in this context means that it is possible to find explicitly the solutions to the classical string equations, perform a canonical quantization, determine the spectrum of the Hamiltonian operator and possibly compute some of the simplest scattering amplitudes. Many of such models are constructed by defining closed string theory on a certain background including non-vanishing p-form field strengths and non-trivial dilaton. In superstring theory, such backgrounds are usually supergravity solutions that preserve a large number of space-time supersymmetries. There are three broad classes of known exactly solvable closed string theories:

(i) Strings on flat space and its various compact and non-compact orbifolds, as well as models related to those on flat-space by formal coordinate and T-duality transformations (see, e.g., [5]);

(ii) Strings on compact and non-compact (gauged) WZW backgrounds and their orbifolds. These models have non-vanishing NS-NS two-form gauge potential and a dilaton;

(iii) Strings on plane-wave backgrounds with non-vanishing NS-NS and/or R-R form gauge potentials.

A common characteristic of all of the above models is that the string background does not receive α' corrections in an essential way (for the case (iii) see [6, 7]). In addition, the classical string equations can be solved explicitly and thus the theory can be quantized. The case (iii) also includes the recently found maximally supersymmetric (BFHP) solution [8] of IIB string theory and the Lorentzian symmetric spaces [9].

1.1 Why plane wave models ?

Searching for new examples of solvable string theories in *time-dependent* backgrounds it is natural to study in detail strings propagating in a generic plane-wave space-times with the metric

$$ds^2 = 2dudv + A_{ij}(u)x^i x^j du^2 + dx^2 \ , \tag{1.1}$$

where dx^2 denotes the standard metric in the Euclidean space $\mathbb{E}^d$ and $x \in \mathbb{E}^d$. Here the corresponding light cone gauge fixed action is quadratic and therefore the string equations are linear in the transverse coordinates x^i and admit at least a formal solution. Properties of strings propagating in such plane-wave backgrounds have been previously investigated, in particular, in [7]. [1]

[1] There is a more general class of pp-wave models recently discussed in [10, 11] for which the non-constant component of the metric is no longer quadratic in the x^i coordinates as it was in the plane-wave case. These models may also be solvable in the cases when they can be related to integrable two-dimensional field theories.

The solutions of string theory in the plane-wave background for which the matrix $A=(A_{ij})$ is *constant* negative semi-definite *and* the dilaton is constant was discussed extensively in the literature, both in the case of non-vanishing NS-NS 2-form gauge potential [12, 13, 14]) and in the case of non-vanishing R-R p-form gauge potentials [15, 16, 17, 18]. Such plane-waves are Lorentzian symmetric spaces [9] and one of them is the BFHP solution [8]. The light cone gauge fixed string action does not explicitly depend on the world-sheet time and describes a collection of free massive bosons and fermions with mass matrix A. The most obvious and simplest generalization is to make the "mass-matrix" A dependent on u and include appropriate non-vanishing form field strengths and/or a non-constant dilaton. Examples of such models have already appeared in, e.g., [7, 19, 20, 21].

Some of our motivations for investigating plane-wave backgrounds with non-constant matrix A are the following.

(1) Strings in standard cosmological backgrounds, like, for example, the metric of FRW universe with time dependent scale factor $a(t)$ and dilaton $\phi(t)$, are difficult to solve because (i) such backgrounds receive α' corrections and (ii) classical string equations are non-linear. Plane-wave backgrounds also exhibit effective time dependence, but they are much simpler to study than cosmological models; they may be viewed as curved-space generalizations of previously studied flat null orbifold and null brane models [22, 23, 24]. Moreover, plane wave backgrounds are related to cosmological backgrounds via a Penrose limit procedure in two different ways. As explained in [19], taking Penrose limit along radial direction of the FRW universe gives a plane wave metric of the type (1.1). Alternatively, with any $(d+1)$-dimensional cosmological metric $ds^2=-dt^2+g_{ij}(t)dx^i dx^j$ we can associate a $(d+2)$-dimensional plane-wave metric of the form (in Rosen coordinates) $ds^2=2dudv+g_{ij}(u)dx^i dx^j$ by performing a Penrose limit along one extra dimension. This suggests a "null cosmology" interpretation for the plane-wave metric. All this may be viewed as an indication of some relevance of the study of the plane-wave backgrounds for investigations of string cosmology. [2]

(2) Plane waves appear as Penrose limits of various p-brane and "non-conformal" generalizations of $AdS_5\times M^5$ backgrounds [8, 19].[3] One particularly simple class of examples, which will be a special case of models discussed below, is the Penrose limit [19, 26] of the near-horizon geometries of the fundamental string and Dp-brane backgrounds. Study of these examples may shed light on some aspects of string-string interactions in curved space. Indeed, in the standard quantization of strings in a given background the back-reaction effect of the various string states on the geometry of space-time is usually ignored.[4] It is obviously desirable to learn how to take the back-reaction into account.

[2]The well-known differences with standard cosmology are of course the presence of supersymmetry and the absence of particle creation and vacuum polarization for a plane-wave metric. Also, plane-wave metrics have no horizons [25] and the nature of singularity is different: all scalar curvature invariants (though not components of the curvature) vanish for a plane wave metric while they blow up at a singularity of a cosmological metric.

[3]Plane-wave backgrounds may be viewed also as Penrose-type limits of pp-wave models. In fact, they can be thought of as quadratic approximations originating from expansions near the null geodesic associated with an extremum of the non-trivial component of the pp-wave metric. For example, in the case with $g_{uu}=-m^2(x_1^2-x_2^2)^2$ considered in [5] one is to expand near the vacuum line $x_1=a,\ x_2=a$.

[4]A justification for this is that the ten-dimensional Newton constant scales as $G_{10}\sim g_s^2$ in terms of the string coupling constant g_s. Therefore, at small string coupling, the effect on the geometry of a mass

To model this, one may consider a source string with a large mass located at some point in space-time and study quantum strings propagating in the background produced by the source.

(3) Plane-wave models are also good examples to study the role of non-constant dilaton in the context of first-quantized string theory. Non-trivial dilaton appears in many simple static (e.g., p-brane) and non-static (e.g., cosmological) backgrounds. Dilaton couples to string world sheet through the 2-d curvature term [27], and its role is to ensure that the resulting two-dimensional stress tensor is traceless at the quantum level, i.e. that the two-dimensional theory is conformal. In the light cone gauge this effectively amounts to cancelling the anomalous contribution of the "transverse" string coordinates to the expectation value of the light cone Hamiltonian. We shall illustrate this below on a plane-wave example.

1.2 A homogeneous plane wave

Despite apparent simplicity of generic plane-wave backgrounds (1.1) the corresponding string theory model is hard to analyse explicitly using analytic methods. A further important simplification occurs if we consider the following special case of the isotropic ($A_{ij} = -\lambda(u)\delta_{ij}$) plane-wave metric (1.1)

$$ds^2 = 2dudv - \lambda(u)x^2du^2 + dx^2 \ , \qquad \lambda(u) = \frac{k}{u^2} \ , \quad k = \text{const} \ . \tag{1.2}$$

This case has several remarkable features. The presence of an apparent singularity at $u = 0$ makes this model an interesting laboratory for a study of the issue of initial singularity in time-dependent backgrounds. This is apparent in Rosen coordinates where the metric (1.2) for $k < \frac{1}{4}$ has a simple "null cosmology" form

$$ds^2 = 2du d\mathrm{v} + a^2(u)d\mathrm{x}^i d\mathrm{x}^i \ , \qquad a(u) = u^\mu \ , \tag{1.3}$$

where $\mu = \frac{1}{2}(1 - \sqrt{1-4k})$. At the same time, the classical string equations here can be solved explicitly in terms of Bessel's functions and thus the model is much more under analytic control than in generic plane wave case.

For different values of the constant k the plane wave metric (1.2) is a Penrose limit of the FRW metric [19], as well of the near-horizon regions of Dp-brane backgrounds ($k = \frac{(7-p)(p-3)}{16}$) and the fundamental string background ($k = \frac{3}{16}$) [19, 26]. All plane waves admit a Heisenberg group of isometries. In the isotropic case we have also invariance under orthogonal rotations in $\mathbb{E}^d$. The above plane-wave metric (1.2) is special in that it admits an additional *scaling isometry* $u \to \ell u, \quad v \to \ell^{-1}v$, which is the same $SO(1,1)$ Lorentz symmetry present in the flat-space limit of (1.1).[5] This symmetry implies *scale-invariance* of the geometry: there is no dimensional parameter like radius, so the components of the curvature R_{uiuj} get their canonical dimension 2 from $1/u^2$ dependence on coordinates only.

M which is measured by $G_{10}M$ is small. This argument holds also for certain non-perturbative states of string theory, like D-branes, for which masses scale as $M \sim g_s^{-1}$ with the string coupling.

[5]While this symmetry will be formally broken by the dilaton, it will have important consequences for the solution of string theory in which, as we shall see below, the dilaton will play rather limited role.

Another consequence of scale invariance is the independence of light cone Hamiltonian on $p^+ = p^u = p_v$ in contrast to the string model in [15] based on the BFHP solution.

It turns out that the space-time associated with the plane-wave (1.2) is a conformally flat *homogeneous* Lorentzian space. The plane-wave spacetime with metric (1.2) restricted to $u > 0$ is geodesically incomplete even though it is homogeneous. This is unlike the de-Sitter, AdS and BFHP plane-wave [8] which are Lorentzian symmetric spaces and smooth.

The presence of the singularity at $u = 0$ is in agreement with the standard argument [7] (see also [28] for a recent review) that all the metrics (1.1) with $A_{ij}(u)$ divergent at certain value of u, say $u = 0$, are singular: (i) the geodesic deviation equation is governed by the curvature components $R_{uiuj} = A_{ij}(u)$ and so the tidal forces are infinite at $u = 0$; (ii) time-like geodesics reach the point $u = 0$ in finite proper time, and so the space-time without $u = 0$ point is geodesically incomplete.

As we will see from the detailed analysis of the geometrical structure of the space associated with (1.2), all time-like geodesics of the region $u > 0$ focus at $u = 0$, $x = 0$ at finite proper time. Some of these geodesics cannot be extended to the $u < 0$ region but some other *can*. In Rosen coordinates the line $u = 0$, $x = 0$ is mapped to a hyperplane located at the origin of the light-cone "time" $u = 0$. An alternative global construction of the homogeneous plane wave space, based on group theory, allows one to identify this hyperplane as a set of fixed points of the scaling symmetry mentioned above. In the coordinates where (1.2) is conformal to (d+2)-dimensional Minkowski spacetime, conformal coordinates, the homogeneous plane wave space can be identified either with the Minkowski space with the singularity located at the origin of a light-cone coordinate or with a strip in Minkowski space bounded by two hyperplanes located at two opposite values of a light-cone coordinate with the singularity located at the origin of this coordinate.

Our suggestion is to consider the spacetime (1.2) for the whole range $(-\infty, +\infty)$ of the light-cone time u, and (in the spirit of the "null cosmology" interpretation) view it as a universe which starts as flat space at $u \to -\infty$, collapses at $u = 0$ and then expands again to a flat universe at $u \to \infty$. The global definition of string theory on this space will then depend on choosing appropriate boundary conditions at $u = 0$. As already mentioned above, the present problem thus has a similarity with string cosmology set-up, with the advantage of the absence of α'-corrections and a possibility of an exact solution of first-quantized string model.[6]

Indeed, a remarkable feature of this special example (1.2) is that the corresponding classical string equations can be solved *explicitly* so that the theory can be canonically quantized in a straightforward way (cf. [7, 29, 21]). This allows one to study string dynamics on this background in much detail. As in any time-dependent background, we are dealing with non-stationary quantum-mechanical problem and thus the main observables are not masses of string states but the time-depedent expectation values and transition amplitudes.

[6] In the light cone gauge, the equations for τ-dependence of string transverse coordinates x may be interpreted as two-dimensional scalar field equations in a cosmological background (with "scale-invariant" choice of τ-dependent mass term).

1.3 Structure of the paper

The rest of the paper is organised as follows.

In sections 2 and 3 we explore the geometrical structure of the space-time associated with the plane wave metric (1.2). In section 2 we find the corresponding Killing vectors and demonstrate that this spacetime is a homogeneous space. Then in section 3 we give its global construction and, starting with the form of the metric in Rosen coordinates, describe its conformal compactification. We find also the equation for the conformal boundary.

In section 4 we embed the metric (1.2) into string theory, i.e. describe a general class of exact string backgrounds (defining 2-d conformal theories) with metric (1.2) and non-trivial dilaton and/or 5-form fluxes. We mention generalizations of these models which have no singularity at $u = 0$ in both the spacetime metric and dilaton. We also discuss relations between plane-wave backgrounds and cosmological models.

In sections 5 and 6 we turn to quantum theory of particles and strings in the background of (1.2) supported by u-dependent dilaton. In section 5 we start with solving the Klein-Gordon equation for a scalar field, both in Rosen and Brinkmann coordinates. For fixed $p^+ \equiv p^u = p_v$ (in the Fourier representation in u) this equation may be interpreted as time-dependent ($\tau \sim u$) Schrödinger equation representing light cone gauge dynamics of a relativistic particle theory. The light cone Hamiltonian is that of a collection of oscillators with time-dependent frequency $\omega = \frac{\sqrt{k}}{\tau}$. We explain how to diagonalize the Hamiltonian in terms of proper set of creation/annihilation operators and compute its expectation values. We also demonstrate that the decrease of the effective string coupling $e^{\phi(u)}$ reduces back reaction near the singularity.

In section 6 we solve the classical (super)string equations in the plane-wave background (1.2) in the light cone gauge and then canonically quantize the theory. We show that, as in the point-particle case, the string light cone Hamiltonian operator which describes an infinite collection of oscillators with time-dependent frequencies can be put into a "diagonal" form. We discuss the role of the dilaton in maintaining the conformal invariance of the quantum theory. We then study the creation of excitation modes as string approaches the singularity at $u = 0$. The total number of created modes depends on a choice of vacuum in Fock space; choosing the usual vacuum of free massless particles at infinity leads to a divergent result near the singularity, but another choice gives no mode creation at all. We then study string propagation from $u = -\infty$ to $u = +\infty$ through the singularity and describe an analytic continuation prescription that leads to the conclusion that the transition amplitude from "in" state at $u = -\infty$ to "out" state at $u = \infty$ is trivial, i.e. the string can actually pass through the singularity, and the final string state at $u = \infty$ is the same as the original state at $u = -\infty$.

Section 7 contains some concluding remarks.

In appendix A we point out that the space (1.2) admits also a group-manifold structure with (1.2) being a left-invariant metric. In appendix B we write down the homogeneous plane wave metric in several different coordinate systems. In appendix C we present the solution for geodesics of the space (1.2) and give the expression for the geodesics that go through the singularity. In appendix D we discuss the form of the Penrose diagram for this plane wave spacetime.

2 Geometry of a class of homogeneous plane-wave space-times

In this section we shall first find the isometries of the plane-wave metric (1.2) and then show that, like the metric (1.1) with A_{ij}=const in [8], this is a Lorentzian G/H *homogeneous space*. However, unlike the BFHP solution [8], this space is singular despite being homogeneous. In contrast to the Euclidean homomogeneous spaces, the Lorentzian ones can be singular, i.e. geodesically incomplete. This plane wave space is thus a special case in the class of metrics (1.1) with singular $A_{ij}(u)$ which generically describe singular spaces [7].

2.1 Isometries : Killing vectors

To find the Killing vectors of the plane-wave background (1.2), we shall begin with more general plane-wave metric

$$ds^2 = 2dudv - \lambda(u)x^2du^2 + ds^2(\mathbb{E}^d) \tag{2.1}$$

where $x \in \mathbb{E}^d$ and λ is a function of u only. After some computation, it is easy to see that such a metric (2.1) admits the following Killing vectors:

$$\begin{aligned} T &= \partial_v \\ X_i &= a\partial_i - \partial_u a x_i \partial_v \\ R_{ij} &= x_i\partial_j - x_j\partial_i \end{aligned} \tag{2.2}$$

where $a(u)$ satisfies

$$\partial_u^2 a + \lambda a = 0 \tag{2.3}$$

We raise and lower indices i, j in the $\mathbb{E}^d$ directions with respect to the Euclidean metric. The Killing vectors R_{ij} are associated to orthogonal rotations in $\mathbb{E}^d$. Since the equation for $a(u)$ is second order, it is determined up to two arbitrary constants. Therefore, there are total of $1 + 2d + \frac{1}{2}d(d-1) = 1 + \frac{1}{2}d(d+3)$ isometries. For the special choice of

$$\lambda(u) = \frac{k}{u^2} \tag{2.4}$$

for the plane-wave metric (1.2), there are additional isometries associated with the scaling of the light-cone coordinates

$$u \to \ell u\ , \qquad v \to \ell^{-1} v\ . \tag{2.5}$$

The special choice of λ allows us also to determine the function $a(u)$ explicitly. The solutions of (2.3) are qualitatively different in the three cases: (i) $0 < k < \frac{1}{4}$, (ii) $k > \frac{1}{4}$ and (iii) $k = \frac{1}{4}$.

The Killing vectors of the metric (1.2) for $0 < k < \frac{1}{4}$ are

$$\begin{aligned} T &= \partial_v \\ X_i &= u^\nu \partial_i - \nu u^{\nu-1} x_i \partial_v \\ \tilde{X}_i &= u^{1-\nu} \partial_i - (1-\nu) u^{-\nu} x_i \partial_v \\ D &= u\partial_u - v\partial_v \\ R_{ij} &= x_i \partial_j - x_j \partial_i \ , \end{aligned} \tag{2.6}$$

where D is the generator associated with the scaling symmetry and

$$\nu \equiv \frac{1+\sqrt{1-4k}}{2} \ . \tag{2.7}$$

We shall mostly focus on the case where $0 < k < \frac{1}{4}$. It is this range that appears in the Penrose limits of the FRW metrics, Dp-branes ($3 < p < 7$) and of the fundamental string background [19, 26]. In the latter case $k = \frac{3}{16}$ and thus $\nu = \frac{3}{4}$.

It is straightforward to compute the Lie bracket algebra of the Killing vectors for $0 < k < \frac{1}{4}$ to find that the non-vanishing commutators are as follows:

$$\begin{aligned} [D, T] &= T \\ \left[X_i, \tilde{X}_j\right] &= (2\nu - 1)\delta_{ij} T \\ [D, X_i] &= \nu X_i \\ \left[D, \tilde{X}_i\right] &= (1-\nu)\tilde{X}_i \\ [R_{ij}, X_k] &= X_i \delta_{jk} - X_j \delta_{ik} \\ \left[R_{ij}, \tilde{X}_k\right] &= \tilde{X}_i \delta_{jk} - \tilde{X}_j \delta_{ik} \\ [R_{ij}, R_{kl}] &= \delta_{jk} R_{il} - \delta_{ik} R_{jl} + \delta_{il} R_{jk} - \delta_{jl} R_{ik} \ . \end{aligned} \tag{2.8}$$

The algebra of isometries of the metric in (1.2) is similar to that of the BFHP plane-wave. The metric (2.1) with $\lambda = \frac{k}{u^2}$ (i.e. (1.2)) and that of BFHP plane-wave have the same number of isometries. The two isometry algebras also contain a Heisenberg subalgebra with n position and n momentum generators. They both also have an external generator of automorphims which however acts differently on the rest of generators in the two cases. In the BFHP case, the external automorphism of the Heisenberg algebra rotates the X_i Killing vectors to $\tilde{X}_i$ ones and vice versa. It is a compact generator. In the case of (2.4) we are investigating here the external automorphism D rotates X_i generators to themselves and acts similarly on the $\tilde{X}_i$ generators. It also rotates the central generator of the Heisenberg algebra T. It is a non-compact generator.

Next, let us consider the case where $k > \frac{1}{4}$. The Killing vectors T, D and R_{ij} are the same as those in the case of the metric (1.2) with $k < \frac{1}{4}$ above. Solving (2.3) for real $a(u)$, we find that the Killing vectors X_i and $\tilde{X}_i$ can be expressed as

$$X_i = u^{\frac{1}{2}} \cos(\gamma \ln u)\partial_i - x_i u^{-\frac{1}{2}} (\frac{1}{2}\cos(\gamma \ln u) - \gamma \sin(\gamma \ln u))\partial_v$$

$$\tilde{X}_i = u^{\frac{1}{2}}\sin(\gamma\ln u)\partial_i - x_i u^{-\frac{1}{2}}(\frac{1}{2}\sin(\gamma\ln u) + \gamma\cos(\gamma\ln u))\partial_v \ , \qquad (2.9)$$

where $\gamma = \sqrt{k - \frac{1}{4}}$. The Lie bracket algebra of the Killing vectors in this case is

$$\begin{aligned} [D,T] &= T \\ \left[X_i, \tilde{X}_j\right] &= \gamma\delta_{ij}T \\ [D,X_i] &= \frac{1}{2}X_i - \gamma\tilde{X}_i \\ \left[D,\tilde{X}_i\right] &= \frac{1}{2}\tilde{X}_i + \gamma X_i \ . \end{aligned} \qquad (2.10)$$

The commutators involving the generators R_{ij} are the same as in (2.8). D is again an outer automorphism of the Heisenberg algebra.

Finally, in the $k = \frac{1}{4}$ case, the Killing vectors T, D and R_{ij} are the same as for $k < \frac{1}{4}$ and $k > \frac{1}{4}$ above. To find the analogue of the Killing vectors X_i and $\tilde{X}_i$, we observe that the two independent solutions of (2.3) for $k > \frac{1}{4}$ are $a = u^{\frac{1}{2}}$ and $a = u^{\frac{1}{2}}\ln u$. Then

$$\begin{aligned} X_i &= u^{\frac{1}{2}}\partial_i - \frac{1}{2}x_i u^{-\frac{1}{2}}\partial_v \\ \tilde{X}_i &= u^{\frac{1}{2}}\ln u\partial_i - x_i u^{-\frac{1}{2}}(\frac{1}{2}\ln u + 1)\partial_v \ . \end{aligned} \qquad (2.11)$$

The Lie bracket algebra of the Killing vectors in this case is

$$\begin{aligned} [D,T] &= T \\ \left[X_i, \tilde{X}_j\right] &= -\delta_{ij}T \\ [D,X_i] &= \frac{1}{2}X_i \\ \left[D,\tilde{X}_i\right] &= \frac{1}{2}\tilde{X}_i + X_i \ . \end{aligned} \qquad (2.12)$$

The rest of the algebra is again as in (2.8).

In all the above cases the number of Killing vectors is $2 + \frac{1}{2}d(d+3)$. Therefore, a 10-dimensional metric (1.2), $d = 8$, admits a group of isometries of dimension fourty six.

2.2 Homogeneous space structure

Let us now proceed to show that the space-time of the plane-wave (1.2) is a homogeneous Lorentzian space. It is worth starting with summarizing some of the elementary properties of homogeneous spaces. Let G/H be a homogeneous space. The Lie algebra $\mathbf{g}$ of the group G decomposes as $\mathbf{g} = \mathbf{h} + \mathbf{m}$, where $\mathbf{h}$ is the Lie algebra of the subgroup H and $\mathbf{m}$ is the rest which is identified with the tangent space of G/H at the origin, with $[\mathbf{h},\mathbf{h}] \subset \mathbf{h}$ and $[\mathbf{m},\mathbf{m}] \subset \mathbf{h} + \mathbf{m}$. If $[\mathbf{h},\mathbf{m}] \subset \mathbf{m}$, then G/H is called *reductive*; G/H space is called *symmetric* if it is reductive and $[\mathbf{m},\mathbf{m}] \subset \mathbf{h}$.

Let G/H be a reductive homogeneous space. Consider a local section s of $G \to G/H$. Then we write $s^{-1}ds = e + \omega = e^m t_m + \omega^a t_a$, where $\{t_m\}$ is a basis in $\mathbf{m}$ and $\{t_a\}$ is a basis in $\mathbf{h}$. As is well known, e is the left-invariant frame on G/H and ω is the canonical connection. The structure equations are

$$\begin{aligned}\mathcal{R}: &= d\omega + \omega \wedge \omega = -e \wedge e|_{\mathbf{h}} \\ \mathcal{T}: &= De := de + \omega \wedge e - e \wedge \omega = -e \wedge e|_{\mathbf{m}}\end{aligned} \tag{2.13}$$

where $\mathcal{R}$ is the curvature of the canonical connection and $\mathcal{T}$ is the torsion. The torsion vanishes for symmetric spaces. A metric is invariant under the left action of G on G/H provided it is associated with an H-invariant quadratic form on $\mathbf{m}$, i.e. a quadratic form B such that $B([t_a, t_m], t_n) + B(t_m, [t_a, t_n]) = 0$. The metric on G/H is constructed by using the quadratic form B and the invariant frame e.

To demonstrate that the space-time with the plane-wave metric (1.2) is homogeneous, we have to identify the Lie algebra of the subgroup H and compute the frame e and canonical connection ω. We shall begin with the case $0 < k < \frac{1}{4}$. The Lie algebra $\mathbf{g}$ of the group G is generated by the elements $X_i, \tilde{X}_i, D, T$ which are taken to satisfy the Lie commutators

$$\begin{aligned}\left[X_i, \tilde{X}_j\right] &= -(2\nu - 1)\delta_{ij} T, \qquad && [D, T] = -T \\ [D, X_i] &= -\nu X_i, && \left[D, \tilde{X}_i\right] = -(1-\nu)\tilde{X}_i \ .\end{aligned} \tag{2.14}$$

We have used the same letters to denote the generators of $\mathbf{g}$ and the Killing vectors of (1.2). Note the sign difference in the commutator of the Killing vectors and that of the generators of $\mathbf{g}$ (cf. (2.8)).

Next, we define $\mathbf{h} =< \alpha X_i + \beta \tilde{X}_i >$, where $< ... >$ denotes a linear span, i.e. $h_i = \alpha X_i + \beta \tilde{X}_i$, for some non-vanishing constants α, β which we shall specify later. Then we define $\mathbf{m} =< m_i, D, T >$, where we take $m_i = -[h_i, D]$. The non-vanishing commutators of $\mathbf{g}$ in terms of this new basis are

$$\begin{aligned}[h_i, m_j] &= \delta_{ij} T, \qquad && [D, T] = -T \\ [D, h_i] &= m_i, && [D, m_i] = -k h_i - m_i \ ,\end{aligned} \tag{2.15}$$

where we have chosen the normalization $\alpha\beta = -\frac{1}{1-4k}$. It is clear that this decomposition of $\mathbf{g}$ is reductive. Because the second term in the left-hand-side of the last commutator is not zero, this decomposition of $\mathbf{g}$ is associated with a homogeneous space rather than a symmetric one. Choose the local section s as

$$s = e^{\frac{x^2}{2}T} e^{x^i h_i} e^{x^i m_i} e^{wD} e^{vT} \tag{2.16}$$

and write $e = e^T T + e^D D + e^i m_i$ and $\omega = \omega^i h_i$, where v, x^i, w are local coordinates. Then we find that

$$\begin{aligned}e^T &= e^w dv - \frac{k}{2} x^2 dw, \quad e^i = dx^i, \quad e^D = dw \\ \omega^i &= dx^i - k x^i dw \ .\end{aligned} \tag{2.17}$$

It is easy to verify that the structure equations for the homogeneous space are satisfied using (2.15). The curvature of the canonical connection is $\mathcal{R}^i = -kdx^i \wedge dw$. An H-invariant quadratic form on $\mathbf{m}$ can be constructed by setting

$$B(D,T) = 1, \qquad B(m_i, m_j) = \delta_{ij} \; . \tag{2.18}$$

The associated invariant metric on G/H is

$$ds^2 = 2e^D e^T + \delta_{ij} e^i e^j \tag{2.19}$$

or explicitly

$$ds^2 = dw(2e^w dv - kx^2 dw) + ds^2(\mathbb{E}^d) \; . \tag{2.20}$$

Changing the coordinates by setting $u = e^w$, we recover the metric (1.2). We conclude, therefore, that the space-time corresponding to the plane-wave metric (1.2) is a *homogeneous* space.

Next, consider the case of $k > \frac{1}{4}$. The Lie algebra $\mathbf{g}$ of the group G is again generated by $< T, D, X_i, \tilde{X}_i >$ satisfying the Lie bracket relations

$$\begin{aligned} \left[X_i, \tilde{X}_j\right] &= \gamma T \delta_{ij}, \qquad [D,T] = T \\ [D, X_i] &= -\frac{1}{2}X_i + \gamma \tilde{X}_i, \qquad \left[D, \tilde{X}_i\right] = -\frac{1}{2}\tilde{X}_i - \gamma X_i \; . \end{aligned} \tag{2.21}$$

i.e. those for the Killing vectors with a sign changed (where again $\gamma = \sqrt{k - \frac{1}{4}}$). Then we identify the generators h_i of the Lie algebra $\mathbf{h}$ of H as $h_i = \alpha X_i + \beta \tilde{X}_i$. Writing $\mathbf{g} = \mathbf{h} + \mathbf{m}$, the tangent space at the origin of the coset is then $\mathbf{m} =< T, D, m_i >$, where $m_i = -[h_i, D] = -(\frac{\alpha}{2} + \gamma\beta) X_i - (\frac{\beta}{2} - \alpha\gamma)\tilde{X}_i$. The Lie bracket relations of $\mathbf{g}$ corresponding to this decomposition can be rewritten as in (2.15) provided we choose $\gamma^2(\alpha^2 + \beta^2) = 1$ and allow $k > 1/4$. Because of the last commutator in (2.15), the space is again homogeneous rather than symmetric. Using the frame (2.17) and the quadratic form (2.18) and changing the coordinates as in the case $k < \frac{1}{4}$ above, we conclude that the space-time with metric (1.2) for $k > \frac{1}{4}$ is homogeneous. The canonical connection is $\omega^i = dx^i - kx^i dw$ and the canonical curvature is $\mathcal{R}^i = -kdx^i \wedge dw$.

It remains to investigate the case of $k = \frac{1}{4}$. The Lie algebra $\mathbf{g}$ of the group G is again generated by $< T, D, X_i, \tilde{X}_i >$ satisfying

$$\begin{aligned} \left[X_i, \tilde{X}_j\right] &= \delta_{ij} T, \qquad [D,T] = T \\ [D, X_i] &= -\frac{1}{2}X_i, \qquad \left[D, \tilde{X}_i\right] = -\frac{1}{2}\tilde{X}_i - X_i \; , \end{aligned} \tag{2.22}$$

i.e. the commutation relations of the Killing vectors with a sign changed. Then we identify the generators h_i of the Lie algebra $\mathbf{h}$ of H as $h_i = \alpha X_i + \beta \tilde{X}_i$. Writing $\mathbf{g} = \mathbf{h} + \mathbf{m}$, the tangent space at the origin of the coset is $\mathbf{m} =< T, D, m_i >$, where $m_i = -[h_i, D] = -(\frac{\alpha}{2} + \beta) X_i - \frac{\beta}{2}\tilde{X}_i$. The brackets of $\mathbf{g}$ according to this decomposition can be written as in in (2.15) provided we choose $\beta^2 = 1$ and set $k = 1/4$. Again, because of the last commutator in (2.15), the space is homogeneous rather than symmetric. Using the frame (2.17) and the quadratic form (2.18) and after changing coordinates as in the other two cases above, we conclude that the space-time with metric (1.2) for $k = \frac{1}{4}$ is homogeneous. The canonical connection is $\omega^i = dx^i - \frac{1}{4}x^i dw$ and the canonical curvature is $\mathcal{R}^i = -\frac{1}{4}dx^i \wedge dw$.

3 Global structure

3.1 Group–theoretic method

Homogeneous Lorentzian spaces are not always geodesically complete. The simplest example of that is two-dimensional Minkowski space-time with one light line removed. Since the structure of the homogeneous plane wave (1.2) is similar to the Minkowski space-time, we shall start with reviewing the Minkowski space example in some detail. Write the two-dimensional Minkowski metric in light-cone coordinates as $ds^2 = 2dudv$. Suppose we remove from the space the light line located at $u = 0$. The remaining space is topologically $\mathbb{R}^* \times \mathbb{R}$, where $\mathbb{R}^* = \mathbb{R} - \{0\}$. In addition $\mathbb{R}^* \times \mathbb{R}$ admits the following group action of isometries: $(u, v) \to (\ell u, \ell^{-1} v + a)$ where $\ell \in \mathbb{R}^*$ and $a \in \mathbb{R}$. The group is the semidirect product of $\mathbb{R}^*$ with the group of translations $\mathbb{R}$, i.e. $\mathbb{R} \bowtie \mathbb{R}^*$. It is easy to verify that this group action is transitive and the little group at every point is the identity. So $\mathbb{R}^* \bowtie \mathbb{R}$ is, in fact, a group but clearly incomplete with respect to the Minkowski metric. Of course we can add back the light line we have removed to recover the whole two-dimensional Minkowski space which is complete. This light line can be thought of as a special orbit $\mathbb{R} = (\mathbb{R}^* \bowtie \mathbb{R})/\mathbb{R}^*$ of the group $\mathbb{R} \bowtie \mathbb{R}^*$ acting now on the two-dimensional Minkowski space.

To describe the homogeneous structure of the plane wave (1.2) with $0 < k < \frac{1}{4}$ globally, we consider the group multiplication on $G^+ = \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$ as follows:

$$
\begin{aligned}
(\ell_1, x_1, y_1, v_1)(\ell_2, x_2, y_2, v_2) \quad &= \\
\Big(\ell_1\ell_2, \ell_1^{-\nu} x_2 + x_1, \ell_1^{-(1-\nu)} y_2 + y_1, \ell_1^{-1} v_2 \quad &+ \quad v_1 - \frac{2\nu - 1}{2}(\ell_1^{-(1-\nu)} x_1 y_2 - \ell_1^{-\nu} x_2 y_1)\Big)
\end{aligned}
$$

where $\mathbb{R}^+ = \{r \in \mathbb{R} : r > 0\}$ and $\nu = \frac{1}{2}(1 + \sqrt{1 - 4k})$. It is straightforward to see that the above multiplication is associative with identity $(1, 0, 0, 0)$ and inverse $(\ell, x, y, v)^{-1} = (\ell^{-1}, -\ell^{\nu} x, -\ell^{1-\nu} y, -\ell v)$. The Lie algebra of G^+ is $\mathbb{R} \oplus \mathbb{R}^d \oplus \mathbb{R}^d \oplus \mathbb{R}$. The left-invariant frame is

$$
\begin{aligned}
e^D &= \ell^{-1} d\ell \\
e^{X_i} &= \ell^{\nu} dx^i \\
e^{\tilde{X}_i} &= \ell^{1-\nu} dy^i \\
e^T &= \ell dv - \frac{2\nu - 1}{2} \ell (y_i dx^i - x_i dy^i)
\end{aligned}
\tag{3.1}
$$

We choose the subgroup $H = \mathbb{R}^d = \{(1, q_1(1-\nu)x, q_2 \nu x, 0) \in G^+\}$, where $q_1 q_2 = 1$. This normalization for q_1, q_2 has been chosen so that the left-invariant frame satisfies the Maurier-Cartan equations associated with the Lie algebra relations (2.14). The left invariant one-forms along the subgroup are $e^{h_i} = p(1-\nu)e^{X_i} + q\nu e^{\tilde{X}_i}$. A global section of the coset space G^+/H is $s = (\ell, q_1 x, q_2 x, v)$. The homogeneous metric associated with the quadratic form B (2.18) is

$$
ds^2 = 2d\ell dv + (q_1 \ell^{\nu} + q_2 \ell^{1-\nu})^2 dx^i dx^i \; . \tag{3.2}
$$

As we shall demonstrate in the next section, this is the form of the metric (1.2) in Rosen coordinates.

To describe the whole plane-wave space-time, we define the following group multiplication on $G^* = \mathbb{R}^* \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$:

$$(\ell_1, x_1, y_1, v_1)(\ell_2, x_2, y_2, v_2) = \Big(\ell_1\ell_2, |\ell_1|^{-\nu}x_2 + x_1, |\ell_1|^{-(1-\nu)}y_2 + y_1,$$
$$|\ell_1|^{-1}v_1 + v_2 - \frac{2\nu-1}{2}(|\ell|^{-(1-\nu)}x_1y_2 - |\ell|^{-\nu}x_2y_1)\Big) \ . \tag{3.3}$$

Clearly, the group G^* is disconnected. The left-invariant one-forms are

$$\begin{aligned} e^D &= \ell^{-1}d\ell \\ e^{X_i} &= |\ell|^\nu dx^i \\ e^{\tilde{X}_i} &= |\ell|^{1-\nu}dy^i \\ e^T &= |\ell|dv - \frac{2\nu-1}{2}|\ell|(y_i dx^i - x_i dy^i) \end{aligned} \tag{3.4}$$

We also identify the $H = \mathbb{R}^d$ subgroup of G^* as in the case G^+ above, and the plane wave space-time can be identified as G^*/H. This is a disconnected space. Topologically, it is to be identified with $\mathbb{R}^d \times \mathbb{R}^2$ after removing a null hyperplane located at $\ell = 0$. The homogeneous metric is

$$ds^2 = 2\ell^{-1}|\ell|d\ell dv + (q_1|\ell|^\nu + q_2|\ell|^{1-\nu})^2 dx_i dx_i \ . \tag{3.5}$$

It remains to investigate the possibility of gluing the null hyperplane back into the space-time. This can be modeled by taking G^* to act on $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}/H$ using (3.3), where H acts diagonally on the $\mathbb{R}^d$ subspaces with weights q_1, q_2. The hyperplane is the orbit $(0, x, y, v)H$ under G^*. This will extend the plane-wave space-time into the whole $\mathbb{R}^d \times \mathbb{R}^2$ space. The homogeneous metric (3.5) is, however, singular along the null hyperplane.

3.2 Rosen coordinates and conformal compactification

To conformally compactify the homogeneous plane-wave space (1.2), it is convenient to use Rosen coordinates as in (3.2). It will be useful later to describe the transformation of a more general class plane-wave metrics (2.1), i.e.

$$ds^2 = 2dudv - \lambda(u)x^2du^2 + dx^i dx^i \ . \tag{3.6}$$

from Brinkmann to Rosen coordinates. The required change of coordinates is $(u, v, x) \to (u, \mathrm{v}, \mathrm{x})$, where

$$v = \mathrm{v} + \frac{1}{2}h(u)\mathrm{x}^i\mathrm{x}^i \ , \qquad x^i = a(u)\mathrm{x}^i \ , \qquad h = -aa' \ , \qquad a''(u) = -\lambda(u)a(u), \tag{3.7}$$

which lead to

$$ds^2 = 2du d\mathrm{v} + a^2(u)d\mathrm{x}^i d\mathrm{x}^i \ . \tag{3.8}$$

Note that this metric and thus (3.6) are *conformally flat.*

In the special case of (1.2) with $\lambda = \frac{k}{u^2}$ and assuming that $u > 0$, we have $a'' = -\frac{k}{u^2}a$

$$a = q_1 u^\nu + q_2 u^{1-\nu} \ , \qquad \nu = \frac{1}{2}(1+\sqrt{1-4k}) \ , \qquad 0 < k < \frac{1}{4} \tag{3.9}$$

and

$$a = u^{1/2}(q_1 + q_2 \ln u) \ , \qquad k = \frac{1}{4} \ . \tag{3.10}$$

The singularity at $u = 0, x = 0$ in the Brinkmann coordinates is mapped to the hyperplane $u = 0$ in the Rosen coordinates (this can also be seen by looking at the location where timelike geodesics end, cf. Appendix C). For $k > \frac{1}{4}$ and $u > 0$, one gets an oscillating solution (cf. (2.9))

$$a = u^{1/2}[q_1 \cos(\gamma \ln u) + q_2 \sin(\gamma \ln u)] \ , \qquad \gamma = \sqrt{k - \tfrac{1}{4}} \ . \tag{3.11}$$

In what follows we shall focus in the case where $0 < k < \frac{1}{4}$, i.e. $\frac{1}{2} < \nu < 1$.

For the region $u < 0$ and assuming that $\lambda(u) = \lambda(-u)$, we set $\tilde{u} = -u$ and write (3.6) as

$$ds^2 = -2d\tilde{u}dv - \lambda(\tilde{u})x^2 d\tilde{u}^2 + dx^i dx^i \ . \tag{3.12}$$

Then performing the analog of the transformation (3.7) as

$$v = \mathrm{v} + \frac{1}{2}h(\tilde{u})\mathrm{x}^i\mathrm{x}^i \ , \qquad x^i = a(\tilde{u})\mathrm{x}^i \ , \qquad h = aa' \ , \qquad a''(\tilde{u}) = -\lambda(\tilde{u})a(\tilde{u}), \tag{3.13}$$

we find that

$$ds^2 = -2d\tilde{u}d\mathrm{v} + a^2(\tilde{u})d\mathrm{x}^i d\mathrm{x}^i \ . \tag{3.14}$$

In terms of u we thus have

$$ds^2 = 2dud\mathrm{v} + a^2(-u)d\mathrm{x}^i d\mathrm{x}^i \ . \tag{3.15}$$

Observe that for $k < \frac{1}{4}$ the metric for $-\infty < u < +\infty$ takes the form of (3.5) after substituting $\mathrm{v} \to -\mathrm{v}$ for the region $u < 0$. The transformation from Brinkmann to Rosen coordinates described above (3.7) and (3.13) can be inverted away from the region that $a(u)$ vanishes.

There is a freedom in the choice of a transformation from the Brinkmann to Rosen coordinates parameterized by two integration constants q_1 and q_2 because the equation for $a(u)$ in (3.7) is second order. For the homogeneous plane wave (1.2), the spacetime in Brinkmann coordinates is diffeomorphic to that in Rosen coordinates away from $u = 0$.

We shall use this freedom to patch the regions $u > 0$ and $u < 0$ in the context of a conformal compactification below. The conformal compactification to Einstein static universe is most easily described for the case $q_1 = 0$ and $q_2 \neq 0$, so in what follows we shall concentrate on this case. The rest of the cases will be presented in Appendix B. After a rescaling of the x coordinates we can set $q_2 = 1$ in which case we find

$$a(u) = u^{1-\nu} \ . \tag{3.16}$$

It is another remarkable property of the model (1.2) that it takes such a simple form in the Rosen coordinates.

We shall further express (3.8) and (3.15) in coordinates where the metric is explicitly conformal to Minkowski metric (we shall refer to them as "conformal" coordinates). In particular, we find

$$ds^2 = \Sigma(\mathrm{w})\ (2d\mathrm{w}d\mathrm{v} + d\mathrm{x}^i d\mathrm{x}^i)\ , \tag{3.17}$$

$$\Sigma(\mathrm{w}) = a^2(u)\ , \qquad d\mathrm{w} = \frac{du}{a^2(u)}\ . \tag{3.18}$$

In the case of the plane wave with $a(u)$ in (3.16) (i.e. for $q_1 = 0, q_2 = 1$) we find for $u > 0$

$$\mathrm{w} = \frac{1}{2\nu - 1} u^{2\nu-1}\ , \quad 0 < \mathrm{w} < \infty\ , \quad \Sigma(\mathrm{w}) = [(2\nu - 1)\mathrm{w}]^{\frac{2-2\nu}{2\nu-1}}\ . \tag{3.19}$$

Note that the plane wave metric (1.1) is conformally flat for all isotropic plane waves, i.e. having $A_{ij} = \lambda(u)\delta_{ij}$. In the present case of (1.2) with $\lambda(u) = k/u^2$, both $a(u)$ in (3.8) and $\Sigma(\mathrm{w})$ in (3.17) are simply powers of their arguments. This leads to an extra scaling symmetry – rescaling u or w combined with an appropriate rescaling of other coordinates is an isometry of the metric.

The analogous transformations for remaining values of (q_1, q_2) can be thought of as different conformal embeddings of the homogeneous plane wave into (d+2)-dimensional Minkowski spacetime, see Appendix B. In (3.19), the singularity at $u = 0$ in Rosen coordinates is mapped to $\mathrm{w} = 0$ in the Minkowski coordinates and the region $u = +\infty$ in Rosen coordinates is mapped to $\mathrm{w} = +\infty$ in conformal (Minkowski) coordinates, i.e. the image of the $u > 0$ spacetime in Rosen coordinates is the $\mathrm{w} > 0$ part of the Minkowski space with the singularity located at $\mathrm{w} = 0$.

For $u < 0$, we find

$$\mathrm{w} = -\frac{1}{2\nu - 1}(-u)^{2\nu-1}\ , \quad -\infty < \mathrm{w} < 0\ , \quad \Sigma(\mathrm{w}) = [-(2\nu - 1)\mathrm{w}]^{\frac{2-2\nu}{2\nu-1}}\ . \tag{3.20}$$

Again, the singularity at $u = 0$ in Rosen coordinates is mapped to $\mathrm{w} = 0$ and the $u = -\infty$ region – to $\mathrm{w} = -\infty$ region in conformal coordinates.

The "null cosmology" interpretation of the homogeneous plane wave spacetime with $-\infty < u < \infty$ (see Section 4.2) is thus of a universe that undergoes collapse to the singularity and then expands in light-cone time.

To summarize, the image of the homogeneous plane wave space with $-\infty < u < +\infty$ under the transformations (3.19) and (3.20) is *conformal to the Minkowski space with the hyperplane* $\mathrm{w} = 0$ *removed.* As a result, the Penrose diagram can be most easily constructed in these coordinates.

To do the conformal compactification of the homogeneous plane wave to the Einstein static universe we are to further compactify the Minkowski metric. For this we use (3.19) and the following change of coordinates. First, set $\mathrm{w} = t + y$ and $\mathrm{v} = \frac{1}{2}(-t + y)$ and then write the Euclidean space metric $ds^2(\mathbb{E}^{d+1}) = dy^2 + d\mathrm{x}^i d\mathrm{x}^i$ in angular coordinates $ds^2(\mathbb{E}^{d+1}) = dr^2 + r^2 ds^2(S^d)$, where $ds^2(S^d) = d\theta^2 + \sin^2\theta\ ds^2(S^{d-1})$. Next, write $v' = t + r$ and $w' = t - r$ and in addition $\tan\rho = v'$ and $\tan\sigma = w'$. After all these transformations, and setting $\varphi = \rho + \sigma,\ \psi = \rho - \sigma$, we find that

$$ds^2 = C(\varphi, \psi, \theta)[-d\varphi^2 + d\psi^2 + \sin^2\psi\ (d\theta^2 + \sin^2\theta\ ds^2(S^{d-1}))]\ , \tag{3.21}$$

where $0 \le \psi \le \pi$ and $0 \le \theta \le \pi$,

$$C(\varphi, \psi, \theta) = (4\nu - 2)^{\frac{2-2\nu}{2\nu-1}} \frac{(\sin\varphi + \sin\psi\cos\theta)^{\frac{2-2\nu}{2\nu-1}}}{(\cos\varphi + \cos\psi)^{\frac{2\nu}{2\nu-1}}} \ , \tag{3.22}$$

and we have used that

$$\mathrm{w} = 2\frac{\sin\varphi + \sin\psi\cos\theta}{\cos\varphi + \cos\psi} \ .$$

The conformal boundary is at the points where the conformal factor C is infinite, i.e. the equation for the conformal boundary is

$$\cos\varphi + \cos\psi = 0 \ . \tag{3.23}$$

The singularity at $\mathrm{w} = 0$ is where $C = 0$, which implies

$$\sin\varphi + \sin\psi\cos\theta = 0 \ . \tag{3.24}$$

The conformal boundary of the homogeneous plane wave is thus that of the Minkowski spacetime, i.e. it is generically $d+1$ dimensional but it has special points. If $\sin\psi = 0$, the conformal boundary collapses to points located at $\cos\varphi \pm 1 = 0$.

In general, the singularity is described by a (d+1)-dimensional subspace in the Einstein static universe and it is mostly spacelike. However, for $\cos\theta = \pm 1$, it becomes a null line. Since crossing the singularity changes the sign of w, we associate the region (II) above the singularity with $\mathrm{w} > 0$ and the region (I) below the singularity with $\mathrm{w} < 0$. The Penrose diagram of the homogeneous plane wave can be drawn in 3-dimensional space with coordinates φ, ψ, θ. A generic point in such a diagram is a $(d-1)$-sphere. It is more instructive though to draw the standard Penrose diagrams in two dimensions with coordinates (ψ, φ) parameterized with the angle θ. Again, a generic point in these diagrams is a $(d-1)$-sphere. There are infinitely many such diagrams, but it turns out that most of them have similar properties regarding the relative locations of the singularity and the conformal boundary. Plots of various Penrose diagrams are given in appendix D.

4 Homogeneous plane-wave metrics as string-theory backgrounds

4.1 The metric-dilaton model

To embed the metric (1.2) or, more generally, (2.1) into string theory we need to compensate its non-zero Ricci tensor $R_{uu} = \lambda(u)d$ by a contribution of other background fields. The simplest option is to include a u-dependent dilaton field. The resulting background may be viewed as a plane-wave analog of a metric-dilaton cosmological background (see below). The most symmetric ansatz is thus given by

$$ds^2 = 2dudv - \lambda(u)x^2du^2 + dx^idx^i \ , \qquad \phi = \phi(u) \ , \tag{4.1}$$

where $i = 1, ..., d$ and $d \le 8$. In what follows, we do not mention the additional free "spectator" coordinates which should complement x^i to 8 transverse bosonic coordinates because they do not affect our arguments. The (exact) conformal invariance condition $R_{\mu\nu} = -2D_\mu D_\nu \phi$, i.e. $R_{uu} = -\frac{1}{2}\partial_x^2 K = -2\partial_u^2 \phi$ with $K = -\lambda(u)x^2$ then implies (prime is derivative over u)

$$\phi''(u) = -\frac{d}{2}\ \lambda(u)\ . \tag{4.2}$$

In general, one simple solution is

$$\lambda = \lambda_0 = \text{const} > 0\ , \qquad \phi = \phi_0 - m^2 u^2\ , \qquad m^2 = \frac{1}{2} d\lambda_0\ . \tag{4.3}$$

A remarkable feature of the corresponding string model is that the effective string coupling $e^\phi = g_0 e^{-m^2u^2}$ is small everywhere if it small at $u = 0$, i.e. if $g_0 = e^{\phi_0} \ll 1$. As in the R-R 5-form model of [15] based on the BFHP solution [8] here the bosonic string modes with momentum $p^u \neq 0$ have mass $\sqrt{\lambda_0} p^u$ (the fermionic modes remain massless)[7] are thus confined to the small u region. As a result, the theory is always in the weakly-coupled regime: even the massless modes with $p^u = 0$ are weakly interacting.

Another special case – the one that we are primarily interested in here – is (1.2), i.e. $\lambda(u) = \frac{k}{u^2}$. We shall assume that $k > 0$ in order to have a positive mass term in the light cone gauge action as well as the vanishing string coupling at $u = 0$. In this case there is an additional scaling symmetry in u, v already mentioned above. It follows then from (4.2) that

$$\phi = \phi_0 - cu + \frac{1}{2} dk \ln u\ . \tag{4.4}$$

Thus the total string background is not invariant under the scaling symmetry. The linear dilaton term (with an arbitrary constant $c > 0$) ensures that the string coupling e^ϕ is regular not only at $u = 0$ but also at $u = \infty$.

Note the solution (4.4) for ϕ "spontaneously" breaks the $u \to -u$ symmetry of the equation (4.2). In writing (4.4) we assumed $u > 0$. To define the solution both at $u > 0$ and $u < 0$ we may replace $\ln u$ by $\frac{1}{2}\ln u^2$. However, the linear term represents a problem: if we keep it as $-cu$ at $u < 0$ then the string coupling blows up at $u = -\infty$; if we replace $-cu$ by $-c|u|$, that would mean introducing an additional ("domain-wall" or better shock-wave type) $\delta(u)$ term in $\lambda(u)$ supporting the solution of (4.2) for ϕ at $u = 0$. An alternative "regularization" of this model that allows one to maintain the symmetry $\phi(-u) = \phi(u)$ of the dilaton function and thus to smoothly continue the solution from $u > 0$ to $u < 0$ region will be discussed in section 4.2 below. An interesting feature of this model is that the issue of singularity of the metric at u is thus effectively connected to the behaviour of the dilaton at large $|u|$.

The string model we shall study below is defined by the following two-parameter (k, n) family of plane wave backgrounds

$$ds^2 = 2dudv - \frac{k}{u^2} x^2 du^2 + dx^i dx^i\ , \tag{4.5}$$

[7]The role of the dilaton $\sim u^2$ (i.e. $\sim \tau^2$ in the light cone gauge) will be to contribute a constant term to the stress tensor that will cancel the conformal anomaly coming from the bosonic mass terms (in the R-R model of [15] this anomaly was cancelled by the fermionic mass term contribution). Let us note that in general any sigma model satisfying $R_{\mu\nu} = D_\mu W_\nu + D_\nu W_\mu$ (for example, the one defined by the plane-wave metric (1.1)) is already scale-invariant (on-shell, or modulo field redefinition). The specific dilaton contribution ensures Weyl invariance of the resulting 2-d theory as required for a string background (see also section 6 below).

$$e^{2\phi} = u^{kd}\ e^{-2u}\ , \qquad u > 0\ . \tag{4.6}$$

Here we used that the constant c in (4.4) can be set to one by rescaling u, v and shifting the constant value of the dilaton. The value of the constant k cannot be changed by rescaling of u, v and is thus an important characteristic of the model. As it has been already noted in [19] and demonstrated in detail in section 2, the string-frame metric (4.5) describes a Lorentzian homogeneous space.

The special case of (4.5),(4.6) with $d = 8,\ k = \frac{3}{16}$ corresponds to the Penrose limit of the fundamental string background [19].[8] The metric (4.5) is also a Penrose limit of the (spatially-flat) cosmological FRW metric [19] (for a 4-dimensional FRW metric $d = 2$).[9]

Let us now mention some other related plane wave backgrounds with u-dependent dilaton. For example, we may generalize the BFHP solution [8] supported by 5-form flux by promoting the metric coefficient and the 5-form coefficient to functions of u and adding a u-dependent dilaton

$$ds^2 = 2dudv - \lambda(u) x^2 du^2 + dx^i dx^i,$$

$$F_5 = 2\mathrm{f}(u)\ (1 + *) du \wedge dx_1 \wedge ... \wedge dx_4\ , \qquad \phi = \phi(u)\ . \tag{4.7}$$

Then the conformal invariance condition $R_{\mu\nu} = -2D_\mu D_\nu \phi + \frac{1}{24} e^{2\phi} (F_5^2)_{\mu\nu}$ gives

$$\lambda = -\frac{1}{4}\phi'' + e^{2\phi} \mathrm{f}^2\ . \tag{4.8}$$

This is a generalization of (4.2) (here $d = 8$) to the case of a non-zero F_5-form, and a generalization of BFHP plane wave (where $\lambda, \mathrm{f} =$const) to the case of $\phi \neq$const.

One obvious solution has the same metric (4.5) with $\lambda = \frac{k}{u^2}$, trivial dilaton $\phi = 0$ and $\mathrm{f} = \frac{\sqrt{k}}{u}$, i.e. the same background as in [8] but with non-constant λ and f.

Another simple special solution is $\lambda = 0,\ \mathrm{f} = \mathrm{f}_0 = $ const, $e^{2\phi} = \frac{1}{2\mathrm{f}_0 u}$. Here the string-frame metric is flat, but the dilaton is non-constant. The corresponding superstring theory is exactly solvable (in light cone gauge on the cylinder) but the string coupling blows up near $u = 0$.

To avoid the strong-coupling singularity we may include also the $\lambda = \frac{k}{u^2}$ term in the metric. Then the solution near $u = 0$ will be as in the $\mathrm{f} = 0$ case (4.4), i.e. $\phi_{u\to 0} \to 4k \ln u$, $e^{2\phi} \to u^{8k} \to 0$. This model is thus a simple dilatonic deformation of the BFHP plane wave, or, alternatively, an F_5-deformation of the metric-dilaton model considered above. Here the supersymmetry preserved by the background is reduced from maximal (32 supercharges) to 1/2 (16 supercharges), as usual for a generic plane wave.

[8] Depending on how one takes the Penrose limit one may or may not [19] get a linear term in the dilaton (4.6). This linear term is, however, crucial to have string coupling small at large u, in agreement with the regularity of string coupling e^ϕ for the original asymptotically flat fundamental string solution one starts with.

[9] If the FRW scale factor is $a(t) = t^\beta,\ \beta = \frac{2}{3}(1+\gamma)^{-1}$ where $0 \le \gamma \le 1$ is the constant in the matter equation of state $p = \gamma\rho$, then $k = \frac{\beta}{(1+\beta)^2} = \frac{6(1+\gamma)}{(5+3\gamma)^2}$. The exotic matter case with $\gamma = -\frac{1}{3}$, i.e. $\beta = 1$, corresponds to $k = \frac{1}{4}$.

4.2 "Null cosmology" interpretation

As was discussed in the preceding section, the metric (4.5) takes a very simple form (3.8),(3.9) in the Rosen coordinates. We shall assume that $0 < k < \frac{1}{4}$, i.e. $1 > \nu > \frac{1}{2}$. With the choice of $q_1 = 0, q_2 = 1$ in (3.16) we have

$$ds^2 = 2du d\mathrm{v} + u^{2\mu} d\mathrm{x}^i d\mathrm{x}^i \ , \qquad \mu \equiv 1 - \nu \ . \tag{4.9}$$

In the flat space case $k = 0, \nu = 1, \mu = 0$. Another special case is $k = \frac{1}{4}$ when $\nu = \mu = \frac{1}{2}$. The corresponding metric $ds^2 = 2du d\mathrm{v} + u \ d\mathrm{x}^i d\mathrm{x}^i$ and dilaton $\phi = -cu + \frac{1}{4} \ln u$ (here we set $d = 2$) may be viewed as a flat spatial section analog of the 4-d string background with S^2-sections defined by the metric $ds^2 = 2du d\mathrm{v} + u \ ds^2(S^2)$ and the dilaton $\phi = v + \frac{1}{4} \ln u$ which was considered in [30].

The metric (4.5),(4.9) may be considered as a "plane-wave analog" of the simplest spatially flat cosmological metric[10]

$$ds^2 = -dt^2 + a^2(t) d\mathrm{x}^i d\mathrm{x}^i \ , \qquad a(t) = t^\mu \ . \tag{4.10}$$

While in the standard cosmological metric (4.10) the evolution of fields is described by second-order differential equations in time t, in the case of (4.9) the evolution effectively takes place in the "null time" u and is described, after the Fourier transform in v, by first order differential equations in u. It is thus natural to refer to the background defined by (4.9) as "null cosmology".

The key question is whether string theories defined on the backgrounds (4.9) and (4.10) are somehow related or have some common properties. The obvious difference is that while the "null cosmology" metric (4.9) supported by the dilaton (4.6) is an exact string solution, the metric (4.10) with $\mu = \frac{1}{\sqrt{d}}$, $d = D_{crit} - 1$ supplemented by the dilaton [31] $\phi = \phi_0 + \kappa \ln t$, $\kappa = 4\mu(1-\mu)d$ should receive corrections of all orders in α'. Furthermore, while the string 2-d equations can be solved for the sigma model defined by (4.9) and thus string theory can be quantized explicitly in the light cone gauge (as will be explained below), it is not clear how to do this directly for the cosmological model (4.10).

One relation between the $2 + d$ dimensional plane-wave metric (4.9) and $1 + d$ dimensional cosmological metric (4.10) is via a Penrose limit. The plane-wave metric (4.9) is the cosmological metric (4.10) "boosted to the speed of light" in one extra compact direction y. This is similar to the relation between the Schwarzschild metric and the Aichelburg-Sexl or shock wave metric.[11] Indeed, adding dy^2 term to (4.10), changing the coordinates

$$t = u, \qquad y = u + \epsilon^2 \mathrm{v}, \qquad x^i = \epsilon \mathrm{x}^i, \qquad \epsilon \to 0 \ , \tag{4.11}$$

and taking the limit $\epsilon \to 0$ we get the string model defined by (4.9) with rescaled string tension ($\alpha' \to \epsilon^2 \alpha'$).[12]

[10] In our case $0 < \mu < \frac{1}{2}$ so this is a subluminal expansion with $a'' < 0$.

[11] This is a different Penrose limit of the cosmological metric than the one considered in [19] where the y-direction was not added and the boost was in the radial spatial direction.

[12] This rescaling explains in particular why α' corrections present for the cosmological metric should disappear for its plane-wave descendant [32]. Let us note also that the Penrose limit of the Mueller solution [31] corresponds to the plane-wave metric with the special value of $k = \frac{1}{\sqrt{d}}(1 - \frac{1}{\sqrt{d}})$. The reason

The relationship demonstrated above is formal. In standard cosmology there is no natural reason for introduction of one extra dimension and study of the sector of states with large momentum along it, i.e. the states that probe the plane-wave geometry (4.9). Still, the simplicity and exact solvability of the model based on (4.9) makes it a natural starting point for an investigation of strings in *curved non-static* backgrounds.

In general, starting with a cosmological background

$$ds^2 = -dt^2 + g_{ij}(t)dx^i dx^j \ , \qquad \phi = \phi(t) \ , \tag{4.12}$$

adding a spectator dimension y and then taking the Penrose limit (4.11), we get the plane-wave metric

$$ds^2 = 2dud\mathrm{v} + g_{ij}(u)dx^i dx^j \ . \tag{4.13}$$

To ensure an embedding into string theory this metric should be supported by the dilaton $\phi(u)$ subject to one equation [13]

$$-\frac{1}{2}g^{ij}g''_{ij} + \frac{1}{4}g^{ij}g^{mn}g'_{im}g'_{jn} + 2\phi'' = 0 \ . \tag{4.14}$$

For example, starting with the inflationary (de Sitter) metric $g_{ij}(t) = e^{2mt}\delta_{ij}$ we find that the corresponding plane wave is supported by the dilaton $\phi = \phi_0 - cu + \frac{1}{4}dm^2u^2$. Here the string coupling grows at large u but this may be possible to change by adding extra background fields.[14]

More specifically, we may look for "null cosmology" analogs of pre-big bang cosmology backgrounds [2]. The suggestion in [2] is to start with the metric-dilaton system only, and assume that at $t = \pm\infty$ the cosmology is simple and nearly flat (and weakly-coupled, at least at $t = -\infty$) while in between (i.e. near $t = 0$) string α' corrections should smooth out the singularity usually present in all cosmological solutions of this type. The cosmological metric-dilaton system solves the leading-order string equations only for a specific scale factor $a(t)$ and dilaton $\phi(t)$ subject to two separate equations, with generic singularity at $t = 0$. At the same time, in the corresponding "null cosmology" set-up, there is only one (exact in α') equation relating the two functions $a(u)$ and $\phi(u)$ – eq.(4.14) with $g_{ij} = a^2(u)\delta_{ij}$, i.e. $-a''d + 2a\phi'' = 0$ (which is equivalent to (4.2) in view of (3.7)). Thus it is possible in principle to choose a solution so that to avoid the singularity at $u = 0$ keeping dilaton and thus the string coupling regular and small everywhere. That may produce a more regular solution than the one in (4.5),(4.6) (where we need to restrict u to be positive), but the down side will be the lack of explicit solvability at string-theory level.

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for this restriction is that before the Penrose limit the Einstein equations impose more constraints on the metric than after the limit. In particular, the 5-d plane wave associated with 4-d isotropic "critical" Mueller solution has $d = 4, \ k = \frac{1}{4}$.

[13] For completeness, let us mention that the flat models (null orbifold and null brane) considered in [24] were described by (4.13) with $d = 1, \ g_{11} = u^2$ and compact x_1, and with $d = 2, \ g_{11} = 1, \ g_{12} = R, \ g_{22} = R^2 + u^2$ with compact (2π periodic) x_1, x_2.

[14] The Brinkmann form of the corresponding plane-wave metric is $ds^2 = 2dudv + m^2x^2du^2 + dx^idx^i$. Here the coefficient of the second term here is u-independent but has a "wrong" sign, i.e. the mass term in the light cone gauge is "tachyonic". Thus here the fluctuations are not confined near $x = 0$ as in the model of [8] but rather are repelled to infinity (see [33, 28] for a discussion of similar examples). This is reminiscent of a distinction between the AdS and dS spaces.

Indeed, it is easy to find examples of such regular backgrounds. One is the direct 1-parameter generalization of the $\lambda(u) = \frac{k}{u^2}$ model (4.5)

$$\lambda(u) = \frac{k^2}{u^2 + s^2} \ , \tag{4.15}$$

where s is an arbitrary constant. Here the components of the curvature are regular at $u = 0$: $R_{iuju} = \frac{k^2}{u^2+s^2}\delta_{ij}$. The corresponding dilaton that solves $\phi'' = -\frac{dk}{2(u^2+s^2)}$ is (here we set the integration constant in the linear dilaton term to zero)

$$e^{2\phi} = e^{2\phi_0}(1 + \frac{u^2}{s^2})^{\frac{1}{2}dk} \exp(-\frac{dk}{2s}\ u \ \arctan\frac{u}{s}) \ . \tag{4.16}$$

For $u \to \pm\infty$ we have $e^{2\phi} \to 0$, so the string coupling is small everywhere. In the limit $s \to 0$ we indeed recover the expression (4.4) (with specific coefficient of the linear term $c = \frac{dk\pi}{4s}$). Note that here $\phi(u) = \phi(-u)$; this suggests that a natural continuation of the model (4.5),(4.6) to $u < 0$ region is indeed to replace u by $-u$ there. We may thus view the background (4.15), (4.16) as defining a regularized version of a string model for the solution (4.5),(4.6).

Another similar regular background corresponds to

$$\lambda(u) = \frac{k}{(u^2 + s^2)^2} \ . \tag{4.17}$$

Note that for $k = \frac{2s^3}{\pi}$ and $s \to 0$ this $\lambda(u)$ is a regularised δ-function. The dilaton here looks very simple

$$e^{2\phi} = e^{2\phi_0} \exp(-\frac{dk}{2s^3}\ u \ \arctan\frac{u}{s}) \ , \tag{4.18}$$

i.e. the string coupling can again be made small everywhere – from $u = -\infty$ to $u = +\infty$.

In what follows we shall concentrate on the model (4.5),(4.6) due to its explicit solvability. Note that its extra scaling symmetry is not shared by the above regular models.[15]

5 Scalar field theory in plane-wave background: point-particle quantization

Before proceeding to quantize string theory in the metric–dilaton background (4.5),(4.6), it is instructive to consider first the point-particle limit, i.e. the quantum theory of a scalar relativistic particle propagating in this background. This may be viewed as an infinite tension limit of the corresponding first-quantized string theory, with the particle representing the "lightest" point-like state of the string (a massless supergravity mode in the case of a superstring).

The standard covariant quantization of a relativistic particle leads to the Klein-Gordon equation in the corresponding curved background. Using the isometry of our plane-wave

[15]The regularized "$\delta(u)$"-model (4.17) may be possible to solve explicitly in the limit $s \to 0$. Similar shock wave model with $K = A_{ij}(u)x^i x^j = -\lambda(u)(x_1^2 - x_2^2)$ and thus no dilaton was discussed in [7], and its different "regularization" – in [29].

background generated by $T = \partial_v$, it is possible to relate the covariant quantization of this system to the quantization in the light-cone gauge. The light cone quantization of a relativistic particle on a plane wave metric (2.1),(4.1) reduces to a harmonic oscillator problem with a time-dependent frequency. Solution of similar models have been previously discussed in [7, 21].

5.1 Covariant Klein-Gordon equation

The leading-order equation for the space-time field Φ representing a massive scalar string mode in a metric-dilaton background is described by the standard action

$$S = \int d^D x \; e^{-2\phi}\sqrt{G}(G^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + m^2\Phi^2 + c_3\Phi^3 + ...) \; , \tag{5.1}$$

where $G_{\mu\nu}$ is the string-frame metric and we included a cubic interaction term. Defining the new field $\tilde{\Phi}$ as

$$\tilde{\Phi} = e^{-\phi}\Phi \; , \tag{5.2}$$

we can rewrite (5.1), using integration by parts, as

$$S = \int d^D x\sqrt{G}[G^{\mu\nu}\partial_\mu\tilde{\Phi}\partial_\nu\tilde{\Phi} + \tilde{m}^2(x)\tilde{\Phi}^2 + c_3 g_s(x)\tilde{\Phi}^3 + ...] \; , \tag{5.3}$$

with

$$\tilde{m}^2(x) = m^2 - \mathrm{D}^2\phi + G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi, \qquad g_s(x) = e^{\phi} \; , \tag{5.4}$$

where D is the covariant derivative with respect to the metric G. In the case of the plane-wave metric (4.1) and the dilaton depending only on u we have $D^2\phi = 0, (\partial\phi)^2 = 0$, and thus the redefined mass is the same as the original constant one: $\tilde{m}(x) = m$. This implies that the *free* field Klein-Gordon equation

$$[-\frac{1}{e^{-2\phi}\sqrt{G}}\partial_\mu(e^{-2\phi}\sqrt{G}G^{\mu\nu}\partial_\nu) + m^2]\Phi = 0 \tag{5.5}$$

expressed in terms of the redefined (5.2) field $\tilde{\Phi}$

$$[-\frac{1}{\sqrt{G}}\partial_\mu(\sqrt{G}G^{\mu\nu}\partial_\nu) + m^2]\tilde{\Phi} = 0 \tag{5.6}$$

will depend only on the string-frame metric but *not* on the dilaton.[16] However, the dilaton does influence the tree-level interaction terms through the effective string coupling factor g_s which depends on u. Thus for the background in (4.6) we are interested in the interactions of redefined $\tilde{\Phi}$-fields will be *suppressed* near $u = 0$.

In general, there is a question about potential strong back reaction on the geometry near the $u = 0$ singularity (as was the case in the null orbifold case [24]). As a step towards clarifying this issue let us solve the KG equation (5.6) explicitly using different choices of coordinates.

[16] Related remark is that while the original string field Φ would have the dilaton factor $e^{-2\phi}$ in the corresponding measure, the normalization condition for the redefined field $\tilde{\Phi}$ does *not* involve the dilaton.

As usual, the general real solution of the KG equation may be written as $\tilde{\Phi} = \sum_k[\alpha_k\varphi_k(x)+\alpha^*_k\varphi_k(x)]$ where $\{\varphi_k,\varphi^*_k\}$ is a complete set of special solutions normalized according to $\int d^{D-1}x\,\sqrt{-G}G^{0\mu}(\varphi^*_k\partial_\mu\varphi_{k'}-\varphi_{k'}\partial_\mu\varphi^*_k) = -i\delta_{kk'}$ which guarantees $[a_k, a^+_{k'}] = \delta_{kk'}$ (and thus particle interpretation) after the quantization. In the present case the role of time-like Killing vector is played by ∂_v and so $G^{0\mu}\partial_\mu \to G^{u\mu}\partial_\mu = \partial_v$. The explicit form of the basis $\{\varphi_k,\varphi^*_k\}$ will depend on a choice of coordinates (and boundary conditions).

5.1.1 Rosen coordinates

The Klein-Gordon (5.6) can be readily solved for the general plane-wave metric (1.1) written in Rosen coordinates (3.8),(4.13), i.e. $ds^2 = 2du d\mathrm{v} + g_{ij}(u)d\mathrm{x}^i d\mathrm{x}^j$, where it takes the form

$$\Big[\partial_{\mathrm{v}}\partial_u + \frac{1}{\sqrt{g(u)}}\partial_u(\sqrt{g(u)}\partial_{\mathrm{v}}) + g^{ij}(u)\partial_i\partial_j - m^2\Big]\tilde{\Phi} = 0\ . \qquad (5.7)$$

It is straightforward to show that [34, 35]

$$\tilde{\Phi}(u,\mathrm{v},\mathrm{x}^i) = \int dp_{\mathrm{v}}d^dp_i\ e^{ip_{\mathrm{v}}\mathrm{v}}e^{ip_i\mathrm{x}^i}\ \chi(u;p_{\mathrm{v}},p_i)\ , \qquad (5.8)$$

$$\chi(u;p_{\mathrm{v}},p_i) = \frac{1}{(\sqrt{g(u)}\ )^{1/2}}\ \exp(-\frac{i}{2p_{\mathrm{v}}}[m^2u + \mathrm{w}^{ij}(u)p_ip_j])\ F(p_{\mathrm{v}},p_i)\ , \qquad (5.9)$$

where

$$\mathrm{w}^{ij}(u) \equiv \int^u du' g^{ij}(u')\ , \qquad g(u) = \det\, g_{ij}\ . \qquad (5.10)$$

F is an arbitrary function of the $d+1$ conserved "momenta" p_{v} and p_i corresponding to the "linear" isometries of the plane-wave background. In general, this solution (more precisely, the original field $\Phi = e^{\phi}\tilde{\Phi}$ in (5.2)) may be used to determine the corresponding string vertex operator (cf. [29]).

For the scalar equation in plane-wave background it is natural to define the KG scalar product at null surface $u =$ const as [44, 28]

$$(\tilde{\Phi},\tilde{\Phi}') = i\int d\mathrm{v}\ d^d\mathrm{x}\ \sqrt{g(u)}\ (\tilde{\Phi}^*\ \partial_{\mathrm{v}}\tilde{\Phi}' - \partial_{\mathrm{v}}\tilde{\Phi}^*\ \tilde{\Phi}')\ . \qquad (5.11)$$

Then

$$\begin{aligned}(\tilde{\Phi},\tilde{\Phi}') &= \int dp_{\mathrm{v}}d^dp_i\ \sqrt{g(u)}\ 2p_{\mathrm{v}}\ \chi^*(u;p_{\mathrm{v}},p_i)\chi'(u;p_{\mathrm{v}},p_i)\\ &= \int dp_{\mathrm{v}}d^dp_i\ 2p_{\mathrm{v}}\ F^*(p_{\mathrm{v}},p_i)F'(p_{\mathrm{v}},p_i)\ ,\end{aligned} \qquad (5.12)$$

i.e. the scalar product does not indeed depend on u and "Fourier modes" in (5.8),(5.9) are δ-function normalized (with extra measure factor p_{v}).

In the present case of the metric (3.8) we have $g_{ij} = a^2(u)\delta_{ij}$, $g = a^{2d}$, where $a(u)$ is given by (3.9) or (for the simplest choice $q_1 = 0, q_2 = 1$) by (3.16),(4.9) with $u > 0$. We get $\mathrm{w}^{ij}(u) = \delta^{ij}\mathrm{w}(u)$ where $\mathrm{w}(u)$ was already found in (3.19). Thus [17]

$$\chi(u;p_{\mathrm{v}},p_i) = \frac{1}{u^{(1-\nu)d/2}}\ \exp(-\frac{i}{2p_{\mathrm{v}}}[m^2u + \frac{u^{2\nu-1}}{2\nu-1}p_i^2])\ F(p_{\mathrm{v}},p_i)\ . \qquad (5.13)$$

[17] The expression in the case of $q_1 = 1, q_2 = 0$ in (3.9) is obtained by replacing $\nu \to 1-\nu$.

As $u \to 0$ this function exhibits singular behaviour because of the overall factor $[g(u)]^{-1/4}$ $(0 < 1 - \nu < \frac{1}{2})$. The choice of $\nu = 1$ corresponds to the flat space case. Note that one can readily rewrite the solution (5.13) in Brinkmann coordinates using the transformation (3.7), i.e. (for $a = u^{1-\nu}$) $\ \mathrm{v} = v + \frac{1-\nu}{2u} x_i^2$, $\ \mathrm{x}^i = u^{\nu-1} x^i$.

One feature of the Rosen coordinates is that the metric (3.8) does not reduce to the flat one at $u \to \infty$, so that p_i, p_v do not have the interpretation of momenta of asymptotical plane-wave states at infinity.[18] This raises the question about the choice of the Fourier mode functions $F(p_{\mathrm{v}}, p_i)$ that should correspond to natural asymptotic states.

In order to try to see how the singular behavior of (5.13) shows up in gauge invariant physical quantities, let us compute the value of the n-point local vertex function in the action (5.1),(5.3) evaluated on a classical solution. To probe possible singular behavior at $u = 0$, it is enough to consider a wave packet moving in a direction orthogonal to the u direction, carrying some distribution of momentum p_{v} and having $p_i = 0$. Using (5.13) with $F(p_{\mathrm{v}}, p_i) = f(p_{\mathrm{v}}) \delta^{(d)}(p_i)$, and integrating over u, v and x^i we obtain for a n-point term in (5.3)

$$\begin{aligned} \langle \tilde{\Phi}^n \rangle \ &\equiv \ \int d^D x \, \sqrt{G} \ e^{(n-2)\phi} \ \tilde{\Phi}^n \\ &= \ \Gamma(\alpha) \int \prod_{r=1}^{n} [dp_{\mathrm{v}}^r \ f(p_{\mathrm{v}}^r)] \ \delta(\sum_{s=1}^{n} p_{\mathrm{v}}^s) \ [n - 2 + \frac{1}{2} i m^2 \sum_{s'=1}^{n} (p_{\mathrm{v}}^{s'})^{-1}]^{-\alpha} \ , \end{aligned} \tag{5.14}$$

where

$$\alpha = 1 - \frac{1}{2} d(n-2)\nu^2 \ . \tag{5.15}$$

We have used the expression for the dilaton in (4.6). For sufficiently large n and generic value of ν in the interval $\frac{1}{2} < \nu < 1$ the parameter α is negative. As a result, the integral over u in the first line of (5.14) is formally divergent at $u \to 0$ where the integrand behaves as u^{α}. In obtaining (5.14) we have adopted the analytic continuation prescription implied by the definition of the Γ-function. As a result, we got a regular expression. It remains to see whether this prescription is consistent at the level of the full quantum theory. We shall return to the issue of back reaction later in this section.

Let us note that the above Γ-function prescription could not be used in the null orbifold case [24]. The reason is that there the basic functions contain factors of $\frac{1}{\sqrt{u}}$, leading to the expression for n-point vertex with a Γ-function term whose argument is a negative integer. In addition, in our present example there is the $\exp(-u)$ term in the effective string coupling in (4.6) that regularizes the integral at infinity.

5.1.2 Conformally-flat coordinates

A simple, yet remarkable property of *massless* KG equation in a plane-wave background with a conformally-flat metric like the one under the discussion here is that it reduces to the KG equation in flat Minkowski space. The reason is simply the vanishing of the curvature scalar for the conformally-flat plane-wave metric.

To see this explicitly in the present context, we shall use (5.1) and the expression of the homogeneous plane wave metric in conformal coordinates given in (3.17) i.e. $ds^2 =$

[18] This problem would not appear if $a(u)$ in the Rosen metric (3.8) were approaching 1 at large u.

$\Sigma(\mathrm{w})(2d\mathrm{w}d\mathrm{v} + d\mathrm{x}^2)$, where Σ is given in (3.19). After the redefinition[19] $\tilde{\Phi}_0 = e^{-\phi}\Sigma^{\frac{d}{4}}\Phi_0$, the quadratic part of the action (5.1) becomes that of a free massless scalar field in flat space.

Since the homogeneous plane wave is conformal to the Minkowski space with either $\mathrm{w} > 0$ or the hyperplane $\mathrm{w} = 0$ removed, it is convenient to quantize the theory in light-cone coordinates instead of the standard Minkowski ones. For this we choose w as the light-cone time, and have to find an invariant measure on the constant w slices. We shall consider the case $\mathrm{w} > 0$. The presence of the $\mathrm{w} = 0$ hypersurface breaks the Poincaré group of the Minkowski space to a subgroup generated by the infinitesimal transformations (cf. (2.6))

$$\begin{aligned} T &= \partial_{\mathrm{v}} \ , \qquad X_i = \partial_i \ , \qquad D = \mathrm{w}\partial_{\mathrm{w}} - \mathrm{v}\partial_{\mathrm{v}} \ , \\ \tilde{X}_i &= \mathrm{x}_i\partial_{\mathrm{v}} - \mathrm{w}\partial_i \ , \qquad R_{ij} = \mathrm{x}_i\partial_j - \mathrm{x}_j\partial_i \ . \end{aligned} \tag{5.16}$$

The Poincaré group generators $\mathrm{x}_i\partial_{\mathrm{w}} - \mathrm{v}\partial_i$ and ∂_{w} do not appear because their orbits include the $\mathrm{w} < 0$ values. The measure on the light-cone time slices should be invariant under the above transformations; it can be chosen as

$$d\mu(p) = \frac{d^{d+2}p}{(2\pi)^{d+2}}\delta(p^2)\theta(-p_{\mathrm{w}}) \ . \tag{5.17}$$

The presence of the last factor is necessary for the light-cone energy to be positive. The most general solution of the Klein-Gordon equation for the field $\tilde{\Phi}$ is the standard flat-space one

$$\tilde{\Phi}_0 = \int d\mu(p)\Big[a(p)e^{i(p_{\mathrm{w}}\mathrm{w}+p_{\mathrm{v}}\mathrm{v}+\mathrm{p}_i\mathrm{x}^i)} + a^\dagger(p)e^{-i(p_{\mathrm{w}}\mathrm{w}+p_{\mathrm{v}}\mathrm{v}+\mathrm{p}_i\mathrm{x}^i)}\Big] \ . \tag{5.18}$$

The theory can be quantized in the standard way leading to the commutation relations $[a(p), a^\dagger(p')] = 2p_{\mathrm{v}}(2\pi)^{d+1}\delta(p_{\mathrm{v}} - p'_{\mathrm{v}})\delta^d(\mathrm{p}_i - \mathrm{p}'_i)$. The Fock space is constructed in the standard way by acting with creation operator $a^\dagger$ on vacuum $|0\rangle$ which is annihilated by the operators $a(p)$, i.e. $a(p)|0\rangle = 0$. After normal ordering the energy of the vacuum vanishes, i.e. $\langle 0|\int d\mathrm{v}d^d\mathrm{x}\ T_{\mathrm{vv}}|0\rangle = 0$.

5.1.3 Brinkmann coordinates

In contrast to Rosen coordinates, the Brinkmann coordinates are global and manifestly "asymptotically-flat". Below we shall solve the KG equation directly in the Brinkmann coordinate system, establishing also a connection with the light cone gauge framework which will be the only formalism available to us in the case of the string theory. For that reason, the discussion of the particle case in the light cone frame is a natural preparation for the study of the string case.

The basis of states that will be used below in the light cone treatment will be different from the one of the Fourier modes (5.13) in Rosen coordinates. The basis in the Hilbert space will not be labeled by continuous parameters p_i as in (5.8) but rather it will be constructed using oscillator-type creation operators, much like in for BFHP plane-wave

[19] We shall use subscript 0 to indicate a massless field.

case in [15]. This basis will not be smoothly related to plane-wave basis in flat space, and that seems to be a key feature of the present model: the massless particle (zero-mode) sector of the theory does not resemble the one in flat space, despite the fact that the metric (1.2) approaches flat space metric at large u. Different choices of bases (adapted to Rosen or conformal or Brinkmann coordinates) correspond to different definitions of observables in this time-dependent geometry.

Using Brinkmann coordinates, the explicit form of the massless Klein-Gordon equation for the background (4.5),(4.6) is

$$(2\partial_u\partial_v + \frac{k}{u^2}x^2\partial_v^2 + \delta^{ij}\partial_i\partial_j)\tilde{\Phi}_0 = 0 \ . \tag{5.19}$$

Performing the Fourier transformation with respect to v

$$\tilde{\Phi}_0(v,u,x) = \int dp_v \ e^{ip_v v}\psi(u,x;p_v) \tag{5.20}$$

we find

$$[2ip_v\partial_u - \frac{k}{u^2}x^2(p_v)^2 + \delta^{ij}\partial_i\partial_j] \ \psi(u,x;p_v) = 0 \ . \tag{5.21}$$

The dependence on the coordinate v drops out of the Klein-Gordon equation because of the isometry generated by the Killing vector $T = \partial_v$. Renaming the coordinate u as τ

$$u = p_v\tau \ , \tag{5.22}$$

we can rewrite the equation (5.21) as the standard Schrödinger equation

$$i\partial_\tau\psi(\tau,x;p_v) = \frac{1}{2}\big(- \delta^{ij}\partial_i\partial_j + \frac{k}{\tau^2}x^2\big) \ \psi(\tau,x;p_v) \tag{5.23}$$

for a non-relativistic harmonic oscillator with a *time-dependent* frequency. Note that eq. (5.23) does not explicitly depend on the value of the momentum p_v: this is due to the scaling symmetry (2.5) of our metric.

The solution of the Klein-Gordon equation (5.19) thus reduces to the solution of the time-dependent Schrödinger equation (5.23). It is easy to see that the latter is precisely the equation which can be derived by quantizing a massless relativistic particle in the light-cone gauge in the plane-wave metric (4.5). Indeed, using the capital letters (U, V, X^i) to denote the particle coordinates as functions of world-line time τ, the Lagrangian of the particle theory in the light-cone gauge $U = p^u\tau$ (cf. (5.22), $p^u = p_v$) can be written as

$$L = \frac{1}{2}(\partial_\tau X^i\partial_\tau X^j\delta_{ij} - \frac{k}{\tau^2}X^2) \ . \tag{5.24}$$

The corresponding Hamiltonian is then $H = \frac{1}{2}(P^2 + \frac{k}{\tau^2}X^2)$. After the quantization, the Schrödinger equation associated with this Hamiltonian is given by (5.23).

5.2 Light cone quantization

Let us now study the solution of the KG equation in Brinkmann coordinates in the "light-cone" representation (5.21) or (5.23). Before proceeding to solve the specific Schrödinger equation (5.23) let us make some general remarks on the quantum mechanical problem of a harmonic oscillator with time-dependent frequency,

$$i\partial_\tau|\Psi\rangle = \hat{H}|\Psi\rangle \ , \qquad \hat{H} = \frac{1}{2}[\hat{P}^2 + \omega^2(\tau)\hat{X}^2]. \tag{5.25}$$

5.2.1 Harmonic oscillator with time-dependent frequency

Eq. (5.25) is solvable by the method developed in [36] which was discussed in a similar context in [21]. The main point of the method is to construct a *basis* in the Hilbert space of the system such that the operator $i\partial_\tau - \hat{H}$ is *diagonal.* In fact, we shall show that it is proportional to a unit operator. Working in the Schrödinger picture, where the position and momentum operators are time-independent, we define the operators (here for simplicity we consider the case of a single oscillator coordinate)

$$\hat{A}(\tau) = i(\mathcal{X}^*\hat{P} - \partial_\tau\mathcal{X}^*\hat{X}), \qquad \hat{A}^\dagger(\tau) = -i(\mathcal{X}\hat{P} - \partial_\tau\mathcal{X}\hat{X}) \ , \tag{5.26}$$

where $\mathcal{X}(\tau)$ is a complex solution of the classical equations of motion satisfying the Wronskian condition

$$\mathcal{X}\partial_\tau\mathcal{X}^* - \mathcal{X}^*\partial_\tau\mathcal{X} = i \ . \tag{5.27}$$

For constant frequency $\mathcal{X} = \frac{1}{\sqrt{2\omega}}e^{-i\omega\tau}$. Using this condition and the canonical commutation relations, one can show that $\hat{A}, \hat{A}^\dagger$ satisfy

$$[\hat{A}, \hat{A}^\dagger] = 1 \ , \qquad i\partial_\tau\hat{A} = \left[\hat{H}, \hat{A}\right] \ , \qquad i\partial_\tau\hat{A}^\dagger = \left[\hat{H}, \hat{A}^\dagger\right] \ . \tag{5.28}$$

The required (normalised) basis in the Hilbert space is defined as the standard Fock space basis at given τ with $\hat{A}$ and $\hat{A}^\dagger$ interpreted as the annihilation and creation operators, i.e.

$$|\ell, \tau\rangle = \frac{1}{\sqrt{\ell!}}\left(\hat{A}^\dagger(\tau)\right)^\ell|0, \tau\rangle \ , \qquad \hat{A}(\tau)|0, \tau\rangle = 0. \tag{5.29}$$

It is easy to see that the operator $i\partial_\tau - \hat{H}$ is diagonal in the above basis. Indeed, for the $|0, \tau\rangle$ state we have

$$i\partial_\tau(\hat{A}(\tau)|0, \tau\rangle) = (i\partial_\tau\hat{A}(\tau))|0, \tau\rangle + \hat{A}(\tau)(i\partial_\tau|0, \tau\rangle) = 0 \ . \tag{5.30}$$

Using the equations (5.28) and the definition of the state $|0, \tau\rangle$, we find that

$$\hat{A}((i\partial_\tau - \hat{H})|0, \tau\rangle) = 0 \tag{5.31}$$

and therefore, provided that $|0, \tau\rangle$ is unique, the state $(i\partial_\tau - \hat{H})|0, \tau\rangle$ should be proportional to the state $|0, \tau\rangle$. We set

$$(i\partial_\tau - \hat{H})|0, \tau\rangle = \Lambda(\tau)|0, \tau\rangle \ , \tag{5.32}$$

for some function Λ of τ. We remark that the operator $(i\partial_\tau - \hat{H})$ is not necessarily self-adjoint and so Λ may be complex. We shall proceed to show that $(i\partial_\tau - \hat{H})$ is diagonal on the rest of the basis $\{|\ell,\tau\rangle\}$. We shall first show this for $\ell = 1$ and then use the induction argument. For $\ell = 1$, we have

$$\begin{aligned}(i\partial_\tau - \hat{H})|1,\tau\rangle &= (i\partial_\tau - \hat{H})\hat{A}^\dagger|0,\tau\rangle \\ &= (i\partial_\tau \hat{A}^\dagger)|0,\tau\rangle + \hat{A}^\dagger i\partial_\tau|0,\tau\rangle - \hat{H}\hat{A}^\dagger|0,\tau\rangle \\ &= \hat{A}^\dagger(i\partial_\tau - \hat{H})|0,\tau\rangle = \Lambda(\tau)|1,\tau\rangle \ . \end{aligned} \tag{5.33}$$

Suppose now that the same relation holds for the state $|k-1,\tau\rangle$, and let us show that it holds then for $|k,\tau\rangle = \frac{1}{\sqrt{k}}\hat{A}^\dagger|k-1,\tau\rangle$. Indeed,

$$\begin{aligned}(i\partial_\tau - \hat{H})|k,\tau\rangle &= \frac{1}{\sqrt{k}}(i\partial_\tau - \hat{H})\hat{A}^\dagger|k-1,\tau\rangle \\ &= \frac{1}{\sqrt{k}}(i\partial_\tau \hat{A}^\dagger)|k-1,\tau\rangle + \frac{1}{\sqrt{k}}\hat{A}^\dagger i\partial_\tau|k-1,\tau\rangle - \frac{1}{\sqrt{k}}\hat{H}\hat{A}^\dagger|k-1,\tau\rangle \\ &= \frac{1}{\sqrt{k}}\hat{A}^\dagger(i\partial_\tau - \hat{H})|k-1,\tau\rangle = \Lambda(\tau)|k,\tau\rangle \ . \end{aligned} \tag{5.34}$$

This demonstrates that the Schrödinger operator is indeed proportional to a unit operator when acting on the basis (5.29).

Using the completeness relation, the most general solution of the Schrödinger equation is then (c_ℓ=const)

$$|\Psi\rangle = \sum_\ell c_\ell |\Psi_\ell\rangle = \sum_\ell c_\ell e^{i\gamma_\ell(\tau)}|\ell,\tau\rangle \ , \tag{5.35}$$

$$\gamma_\ell(\tau) = \int^\tau ds\ \langle \ell, s| i\frac{\partial}{\partial s} - \hat{H}(s)|\ell,s\rangle = \int^\tau ds\ \Lambda(s) \ , \tag{5.36}$$

where the phase $\gamma_\ell(\tau)$ does not actually depend on ℓ.

Note that in coordinate space representation the basis (5.29) has the form (see, e.g., [37])

$$\langle x|\ell,\tau\rangle = (\sqrt{2\pi}2^\ell \ell! |\mathcal{X}(\tau)|)^{-1/2}\ (\frac{\mathcal{X}(\tau)}{\mathcal{X}^*(\tau)})^\ell\ \mathrm{H}_\ell(\frac{x}{\sqrt{2}|\mathcal{X}(\tau)|})\ \exp[\frac{i}{2}\frac{\partial_\tau \mathcal{X}^*(\tau)}{\mathcal{X}^*(\tau)}x^2] \ , \tag{5.37}$$

where H_ℓ is the Hermite polynomial.

The advantage of this basis is that various expectation values (which are the main observables in the time-dependent cases like the present one) can be readily computed. In particular, to compute the expectation values of the Hamiltonian operator in the basis $\{|\ell,\tau\rangle\}$, it is convenient to express it in terms of the operators $\hat{A}$ and $\hat{A}^\dagger$. We find

$$\hat{H} = \frac{1}{2}\left[(\partial_\tau \mathcal{X})^2 + \omega^2 \mathcal{X}^2\right]\hat{A}^2 + \frac{1}{2}\left[(\partial_\tau \mathcal{X}^*)^2 + \omega^2 \mathcal{X}^{*2}\right](\hat{A}^\dagger)^2 + (|\partial_\tau \mathcal{X}|^2 + \omega^2|\mathcal{X}|^2)(\hat{A}^\dagger\hat{A} + \frac{1}{2}) \ . \tag{5.38}$$

Expectation values of the Hamiltonian operator in the basis $\{|\Psi_\ell\rangle\}$ of solutions of the Schödinger equation can be expressed in terms of expectation values of $\hat{H}$ in the basis $\{|\ell,\tau\rangle\}$ as follows

$$\langle\Psi_\ell|\hat{H}|\Psi_m\rangle = \exp(-2\int^\tau ds\ \mathrm{Im}\ \Lambda(s))\ \langle\ell,s|\hat{H}|m,\tau\rangle\ . \tag{5.39}$$

Since $\hat{H}$ is quadratic in the creation and annihilation operators $\hat{A}$ and $\hat{A}^\dagger$, we conclude that

$$\begin{aligned}\langle\Psi_\ell|\hat{H}|\Psi_\ell\rangle &= \exp(-2\int^\tau ds\ \mathrm{Im}\Lambda(s))\ (|\partial_\tau\mathcal{X}|^2+\omega^2|\mathcal{X}|^2)(\ell+\frac{1}{2})\ ,\\ \langle\Psi_\ell|\hat{H}|\Psi_{\ell+2}\rangle &= \frac{1}{2}\sqrt{(\ell+2)(\ell+1)}\exp(-2\int^\tau ds\ \mathrm{Im}\ \Lambda(s))\ [(\partial_\tau\mathcal{X})^2+\omega^2\mathcal{X}^2]\ ,\end{aligned}\tag{5.40}$$

with $\langle\Psi_{\ell+2}|\hat{H}|\Psi_\ell\rangle = \langle\Psi_\ell|\hat{H}|\Psi_{\ell+2}\rangle^*$ and the remaining matrix elements being zero.

5.2.2 Solution of the Schrödinger equation

Let us now return to the problem of solution of the Schrödinger equation (5.23) corresponding to the special case of (5.25) with

$$\omega^2(\tau) = \frac{k}{\tau^2}\ .$$

We begin by constructing the classical solutions for the equations of motion $\partial_\tau^2 X^i + \frac{k}{\tau^2}X^i = 0$ corresponding to the Lagrangian (5.24). This is essentially the same problem that we have solved already above (in sections 2,3) in order to determine the Killing vectors and the Rosen form of the space-time metric (1.2). For $0<k<\frac{1}{4}$, the solutions can be expressed as (cf. (3.9))

$$X^i(\tau) = k^i\tau^\nu + \tilde{k}^i\tau^{1-\nu}\ ,\qquad \nu = \frac{1}{2}(1+\sqrt{1-4k})\ ,\qquad \tau>0\ , \tag{5.41}$$

where $k^i, \tilde{\kappa}^i$ are real constants. The momenta are $P^i(\tau) = \partial_\tau X^i$. We actually need to find a complex basis of the classical solutions which satisfy the Wronskian condition (5.27). To do this we observe that at $\tau=\tau_0$, the system is a collection of harmonic oscillators with frequency

$$\omega_0 = \omega(\tau_0) = \frac{\sqrt{k}}{\tau_0}\ . \tag{5.42}$$

Then, we follow the analogy with the creation/annihilation operators of a harmonic oscillator and define the complex constants

$$\alpha^i = \frac{\omega_0 X^i(\tau_0) + iP^i(\tau_0)}{\sqrt{2\omega_0}}\ ,\qquad \alpha^{*i} = \frac{\omega_0 X^i(\tau_0) - iP^i(\tau_0)}{\sqrt{2\omega_0}}\ . \tag{5.43}$$

Expressing β and $\tilde{\beta}$ in terms of α and α^* we can rewrite (5.41) as

$$X^i(\tau) = \mathcal{X}(\tau)\alpha^i + \mathcal{X}^*(\tau)\alpha^{i*}\ , \tag{5.44}$$

where

$$\mathcal{X}(\tau) = \frac{i}{\sqrt{2\omega_0}(2\nu-1)} \left[-(\sqrt{k} - i(1-\nu))(\frac{\tau}{\tau_0})^{\nu} + (\sqrt{k} - i\nu)(\frac{\tau}{\tau_0})^{1-\nu} \right] . \tag{5.45}$$

It is easy to check that $\mathcal{X}$ does satisfy the Wronskian condition (5.27). Using $\mathcal{X}$ and $\mathcal{X}^*$, we then define the operators $\hat{A}^i(\tau), \hat{A}^{i\dagger}$ as in (5.26), i.e.

$$\hat{A}^i(\tau) = i(\mathcal{X}^* \hat{P}^i - \partial_\tau \mathcal{X}^* \hat{X}^i) , \qquad \hat{A}^{i\dagger}(\tau) = -i(\mathcal{X} \hat{P}^i - \partial_\tau \mathcal{X} \hat{X}^i) \tag{5.46}$$

The Hamiltonian $\hat{H}$ expressed in terms of $\hat{A}^i$ and $\hat{A}^{i\dagger}$ is given by (5.38), i.e.

$$\hat{H} = c(\tau)\hat{A}^2 + c^*(\tau)\hat{A}^{\dagger 2} + b(\tau)(\hat{A}^\dagger \hat{A} + \frac{d}{2}) , \tag{5.47}$$

where we suppressed the summation over the index $i = 1, ..., d$ of oscillators with the same frequency and

$$c(\tau) = -\frac{1}{2(1-4k)k} \Big[\nu(\sqrt{k} - i(1-\nu))^2 (\frac{\tau}{\tau_0})^{2\nu-2} + (1-\nu)(\sqrt{k} - i\nu)^2 (\frac{\tau}{\tau_0})^{-2\nu} + i\ 4k\sqrt{k}(\frac{\tau}{\tau_0})^{-1} \Big] , \tag{5.48}$$

$$b(\tau) = \frac{\omega_0}{2(1-4k)} \left[(\frac{\tau}{\tau_0})^{2\nu-2} + (\frac{\tau}{\tau_0})^{-2\nu} - 8k(\frac{\tau}{\tau_0})^{-1} \right] . \tag{5.49}$$

Note that $b(\tau_0) = \omega_0$ and $c(\tau_0) = 0$ in agreement with our initial condition.

To find the explicit form of the general solution (5.36) of the Schrödinger equation, we shall first determine the state $|0, \tau\rangle$ defined by $\hat{A}^i|0, \tau\rangle = 0$. In the position space representation we have (cf. (5.37))

$$(\mathcal{X}^* \partial_i - i\partial_\tau \mathcal{X}^* x^i)\psi_0(x, \tau) = 0 , \qquad \psi_0(x, \tau) \equiv \langle x|0, \tau\rangle , \tag{5.50}$$

which leads to

$$\psi_0(x, \tau) = [\frac{\sqrt{k}(2\nu-1)^2}{\pi f(\tau)}]^{\frac{d}{4}} \exp\left[\frac{-(2\nu-1)^2\sqrt{k} + ik[(\frac{\tau}{\tau_0})^{2\nu-1} + (\frac{\tau}{\tau_0})^{1-2\nu} - 2]}{2f(\tau)}\ x^2 \right] \tag{5.51}$$

where

$$f(\tau) \equiv \nu(\frac{\tau}{\tau_0})^{2-2\nu} + (1-\nu)(\frac{\tau}{\tau_0})^{2\nu} . \tag{5.52}$$

The factor in front of the exponent arises from the normalization condition $\langle 0, \tau|0, \tau\rangle = 1$.

Other basis functions are given by the expressions in (5.37), so that the solution of the Schrödinger equation in coordinate representation then follows from (5.36). To find the phase factor there it remains to compute the function $\Lambda(\tau) = \langle 0, \tau|i\partial_\tau - \hat{H}|0, \tau\rangle$ in (5.32). Using that $A|0, \tau\rangle = 0$ we get

$$\langle 0, \tau|\hat{H}|0, \tau\rangle = \frac{d\omega_0}{4(1-4k)} \left[(\frac{\tau}{\tau_0})^{2\nu-2} + (\frac{\tau}{\tau_0})^{-2\nu} - 8k(\frac{\tau}{\tau_0})^{-1} \right] . \tag{5.53}$$

In addition, we have

$$\langle 0,\tau|i\partial_\tau|0,\tau\rangle$$

$$= -\frac{d\omega_0}{4(1-4k)f(\tau)}\Big[1-8k-(1-\nu)(\frac{\tau}{\tau_0})^{4\nu-2}-\nu(\frac{\tau}{\tau_0})^{2-4\nu}+4k(\frac{\tau}{\tau_0})^{1-2\nu}+4k(\frac{\tau}{\tau_0})^{2\nu-1}\Big] \ . \quad (5.54)$$

Thus we find

$$\Lambda(\tau) = \frac{d\omega_0}{2(1-4k)f(\tau)}\Big(4k-1+2k(2\nu-1)[(\frac{\tau}{\tau_0})^{1-2\nu}-(\frac{\tau}{\tau_0})^{2\nu-1}]\Big) \ . \quad (5.55)$$

Observe that $\mathrm{Im}\Lambda = 0$. Its integral is

$$\int^\tau ds\Lambda(s) = \frac{d\sqrt{k}}{2(1-4k)}\Big[\frac{4k-1}{(2\nu-1)\sqrt{\nu(1-\nu)}}\ \arctan(\sqrt{\nu^{-1}-1}\ (\frac{\tau}{\tau_0})^{2\nu-1})$$

$$+\frac{k}{\nu(\nu-1)}([2\nu(2\nu-1)+2]\ln\frac{\tau}{\tau_0}+\ln[(1-\nu)(\frac{\tau}{\tau_0})^{4\nu-2}+\nu])\Big] + \mathrm{const} \ , \quad (5.56)$$

where $0 < k < \frac{1}{4}$ ($\frac{1}{2} < \nu < 1$) and $\omega_0 = \sqrt{k}/\tau_0$. For example, for $k = 1/4$ we get $\nu = \frac{1}{2}$ and then $\int^\tau ds\Lambda(s) = -\frac{d}{4}\ln\frac{\tau}{\tau_0}$.

As in (5.40) we can also compute the expectation values of the Hamiltonian operator $\hat{H}$. Now the quantum number ℓ is a "vector" $\ell = (\ell_1,\ldots,\ell_i,\ldots,\ell_d)$ so that the basis of solutions in (5.35) is $|\Psi_\ell\rangle = |\Psi_{(\ell_1,\ldots,\ell_d)}\rangle$. In particular, we find that

$$\langle\Psi_\ell|\hat{H}|\Psi_\ell\rangle = (\sum_{i=1}^d \ell_i + \frac{d}{2})\ b(\tau) \ , \quad (5.57)$$

$$\langle\Psi_{(\ell_1,\ldots,\ell_i+2,\ldots\ell_d)}|\hat{H}|\Psi_\ell\rangle = \langle\Psi_\ell|\hat{H}|\Psi_{(\ell_1,\ldots,\ell_i+2,\ldots\ell_d)}\rangle^* = \sqrt{(\ell_i+2)(\ell_i+1)}\ c(\tau) \ . \quad (5.58)$$

where c and b are given in (5.48) and (5.49), respectively. We observe that the expectation values of the light-cone Hamiltonian operator diverge at $\tau \to 0$ as well as $\tau \to \infty$.

Let us now comment on an interpretation of these results. The set of functions $\psi_\ell(\tau,x) \equiv \langle x|\Psi_\ell(\tau)\rangle$ represents a basis (5.35) in the space of solutions of the Schrödinger equation (5.23), and thus of the original KG equation (5.19),(5.20). This basis (labelled by the natural quantum numbers ℓ_i) corresponds to a different set of physical states (different choice of boundary conditions) as compared to the Fourier mode basis (5.9) used in the solution in Rosen coordinates. The expectation value of the light cone Hamiltonian (5.57) or $E = \int d^dx\ \psi^*(\tau,x)H(\tau)\psi(\tau,x)$ may be related to the value of the free part of the scalar Lagrangian (5.1) or the target-space energy evaluated on the corresponding solution of the KG equation.

Qualitatively, this combination, multiplied by second power of the effective string coupling, i.e. by $e^{2\phi(u)}$, will appear as a source in the Einstein equations for the string-frame metric (note that E is built out "redefined" fields in (5.2)).[20] At large τ (large u)

[20] Let us note that sources terms produced by a bilinear of a given state in equations for other string modes (described by "redefined" fields in (5.2)) will be multiplied by a single power of e^ϕ (see (5.3)) and thus will be more singular at small τ.

the dilaton (4.6) decays exponentially and thus suppresses further the expectation value E. At small times we get $e^{2\phi} \sim \tau^{kd}$ and $E \sim \tau^{-2\nu}$ (see (5.57),(5.49)) so that the condition of small back reaction appears to be $kd \geq 2\nu$. Assuming $d = 8$ and $0 < k \leq \frac{1}{4}$ this is satisfied if $\frac{1}{4} > k \geq \frac{3}{16}$, i.e. $\frac{1}{2} < \nu \leq \frac{3}{4}$. [21]

Classically, a growth of the energy density near $u = 0$ may be attributed to the focusing of null and time-like geodesics near $u = 0$ in the plane wave background. This may potentially produce a large gravitational back reaction. As we have just seen, a large back reaction near the singularity $u = 0$ may be *suppressed* due to vanishing of the *string coupling* there. Nevertheless, this conclusion depends on the definition of observables and/or choice of physical states and deserves further clarification.

6 Solution of string-theory model

Our aim here is to solve the first-quantized superstring model corresponding to the background (4.5),(4.6) using light cone gauge. The light cone gauge action is quadratic in "transverse" bosonic and fermionic coordinates for any plane-wave metric (1.1), or, in particular, for any function $\lambda(u)$ in (4.1), and thus is formally solvable. By "solution" of first-quantized string model here we mean solving explicitly the classical equations, performing the canonical quantization and writing down the expression for the light cone Hamiltonian in terms of creation and annihilation operators, allowing one to study time evolution of expectation values.

What distinguishes the model (4.5) with $\lambda = \frac{k}{u^2}$ (in addition to its remarkable scale invariance (2.5)) is that the corresponding expressions are more explicit and analytically controllable than for other potentially interesting choices of $\lambda(u)$ (like the ones in (4.15),(4.17)).

We shall assume that the parameter k is restricted to $0 < k \leq \frac{1}{4}$. The parameter d equal to the number of "massive" scalars in the light cone action explicitly appears in the dilaton (4.6), but the dilaton coupling does not enter the classical string equations and its only role is to cancel the quantum conformal anomaly (see below).

We will not explicitly discuss the contributions of the fermion modes: in the light cone gauge the fermionic fields are the standard massless GS fermions, and their inclusion is straightforward. Indeed, as shown in [11], for any pp-wave background the fermion part of GS action in the light-cone gauge is always quadratic in the fermions. The only possible non-trivial coupling of fermions to the background in the light cone gauge is through the generalized covariant derivative. In the present case there are no p-form background fields and the gravitational connection term is trivial in the bosonic light cone gauge, i.e. the covariant derivative reduces to the flat one. Thus there are 8 massless GS fermion modes (left and right components).

Since the bosonic fields have u-dependent mass terms, it is clear that here there is no global world sheet supersymmetries.[22] The absence of world-sheet supersymmetries in GS

[21] Let us note for comparison that if we would construct E (i.e. energy or stress tensor) as a bilinear of the solution (5.13) in Rosen coordinates, then $E \sim u^{-8(1-\nu)}$ and thus the condition for finiteness of $e^{2\phi}E$ would be $k \geq 1 - \nu$, which is never satisfied.

[22] These typically arise when there are extra "supernumerary" supersymmetries [38] which are "orthog-

action indicates, in particular, that the number of unbroken space-time supersymmetries in the present background must be 16, as in a generic plane wave. This can be easily seen directly from the dilatino and gravitino transformation laws. The former leads to the condition $\Gamma^m \partial_m \phi \ \epsilon = 0$. Since in the present case $\phi = \phi(u)$, we find $\Gamma^u \epsilon = 0$, leaving 16 unbroken supersymmetries. The gravitino transformation law then implies that ϵ should be constant.

Before proceeding to the solution of this string theory on a cylinder in the light cone gauge let us make a brief comment on its 1-loop (torus) partition function Z_1. To define the partition function one should start with the standard covariant path integral representation for it. Then it is easy to argue that for any pp-wave model the value of Z_1 is the same as in the flat space case. Indeed, integrating over the v-coordinate in the path integral gives the delta-function constraint $\partial^a \partial_a u = 0$. If we formally define (by an analytic continuation) the pp-wave sigma model on a euclidean 2-torus, this constraint will imply (assuming u is a non-compact coordinate) that $u =$ const. Then the rest of the path integral becomes trivial. In the case of the Minkowski signature in the target space it is more natural [24, 39, 3] to define the sigma model on a Lorentzian 2-torus $ds^2 = (d\sigma_1 + \tau d\sigma_2)(d\sigma_1 + \bar{\tau} d\sigma_2)$, where σ_a are periodic, $\sigma_a \equiv \sigma_a + 1$, and the moduli parameters τ and $\bar{\tau}$ are real and independent. Here again the equation $\partial^a \partial_a u = 0$ has only $u =$ const as its solution. The same argument should apply also at higher loops, i.e. at higher genera. In the present case of supersymmetric plane wave, the vanishing of the partition function (and of the 1-point functions on a torus) follows also from the residual 16 supersymmetries preserved by the background.

6.1 Classical equations and canonical quantization

In this section we shall use the capital letters U, V, X_i to denote the 2-d scalar fields representing the bosonic string coordinates. The bosonic part of the string action in the light-cone gauge

$$U = 2\alpha' p^u \tau \tag{6.1}$$

is [23]

$$I = -\frac{1}{4\pi\alpha'} \int d\tau \int_0^\pi d\sigma (\partial^a X^i \partial_a X^j \delta_{ij} + \frac{k}{\tau^2} X_i^2) \ . \tag{6.2}$$

Note again the cancellation of the $p^u = p_v$ dependence as in the particle case (5.24). The equations of motion are

$$(\partial_\tau^2 - \partial_\sigma^2) X^i + \frac{k}{\tau^2} X^i = 0 \ . \tag{6.3}$$

onal" to the light-cone gauge condition, i.e. the Killing spinors obeying $\Gamma^u \epsilon \neq 0$.

[23] Here we assume that τ and σ are dimensionless while the string coordinates (and $\sqrt{\alpha'}$) have dimension of length. The 2-d metric is $\eta_{ab} = (-1, 1)$.

Expanding in Fourier modes in σ, we get an infinite collection of oscillators with time-dependent frequencies.[24] The general solution of (6.3) is given by[25]

$$\begin{aligned} X^i(\sigma,\tau) &= x_0^i(\tau) + \frac{i}{2}\sqrt{2\alpha'}\sum_{n=1}^{\infty}\frac{1}{n}\Big[Z(2n\tau)(\alpha_n^i e^{2in\sigma} + \tilde{\alpha}_n^i e^{-2in\sigma}) \\ &- Z^*(2n\tau)(\alpha_{-n}^i e^{-2in\sigma} + \tilde{\alpha}_{-n}^i e^{2in\sigma})\Big] , \end{aligned} \tag{6.4}$$

where

$$Z(2n\tau) \equiv e^{-i\frac{\pi}{2}\nu}\sqrt{\pi n\tau}\ [J_{\nu-\frac{1}{2}}(2n\tau) - iY_{\nu-\frac{1}{2}}(2n\tau)] , \qquad \nu \equiv \frac{1}{2}(1+\sqrt{1-4k}) , \tag{6.5}$$

$$x_0^i(\tau) = \frac{1}{\sqrt{2\nu-1}}(\tilde{x}^i\ \tau^{1-\nu} + 2\alpha'\tilde{p}^i\ \tau^{\nu}) , \quad k \neq \frac{1}{4} , \tag{6.6}$$

$$x_0^i(\tau) = \sqrt{\tau}(\tilde{x}^i + 2\alpha'\tilde{p}^i\ \log\tau) , \quad k = \frac{1}{4} , \tag{6.7}$$

$$\tilde{x}^i = \frac{\sqrt{\alpha'}}{\sqrt{2}}(a_0^i + a_0^{i\dagger}) , \quad \tilde{p}^i = \frac{1}{i\sqrt{2\alpha'}}(a_0^i - a_0^{i\dagger}) . \tag{6.8}$$

Here $J_{\nu-\frac{1}{2}}(z)$ and $Y_{\nu-\frac{1}{2}}(z)$ are the usual Bessel functions. Asymptotically,

$$Z(2n\tau) \cong e^{-2in\tau}[1 + O(\tau^{-1})] , \tag{6.9}$$

so that for large τ the oscillator part of (6.4) reduces to that of the flat-space theory.

This asymptotic "flatness" behaviour is not shared by the zero mode part of the string coordinate (6.4): it never reduces to $x^i_{0\ \text{flat}}(\tau) = \tilde{x}^i + 2\alpha'\tilde{p}^i\ \tau$, in any coordinate system.[26] This is a direct consequence of the *scale-invariance* of the equation (6.3) restricted to the zero-mode (σ-independent) part, i.e. the invariance under $\tau \to a\tau,\ \ a =$const.[27] As a result, the Fock-space vacuum for the zero-mode part is different from the flat-space zero-mode vacuum at all scales.

As was mentioned in section 2, the parameter k must be non-negative in order to have a regular string coupling at $u = 0$ (as well as a positive mass-squared term for X_i in the light cone gauge). There are two special cases:

a) $k = 0$: this is the flat space, and Z, Z^* reduce to plane waves $e^{\mp 2in\tau}$.

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[24] As was already mentioned earlier, similar equation appears in the study of scalar perturbations in cosmology, $\varphi_p'' + [p^2 - \frac{a''}{a}]\varphi_p = 0$, where p is spatial momentum and $a(t)$ is scale factor. For $a(t) = t^\mu$ we get the effective frequency (mass term) as $p^2 + \frac{k}{t^2}$, $k = \mu(1-\mu)$. We are grateful to G. Veneziano for a discussion of that point.

[25] One can also solve the equations in Rosen coordinates. In this case, one has to solve $(\partial_\tau^2 + \frac{\beta}{\tau}\partial_\tau - \partial_\sigma^2)X^i_{\text{rosen}} = 0$, with $\beta = 2(1-\nu)$. The general solution is the same as in (6.3) up to a factor $\tau^{-\beta/2}$. Note the resemblance to the equation of a damped harmonic oscillator (with friction coefficient proportional to β/τ_0 in a small interval of time around some τ_0).

[26] In Rosen coordinates (3.8) there is a translational mode $x^i_{0\ \text{rosen}}(\tau) = \tilde{x}^i + 2\alpha'\tilde{p}^i\ \tau^{2\nu-1}$ which grows for large τ.

[27] The same "non-flatness" property of the zero-mode solution is found also in non-scale-invariant cases of $\lambda(u) = \frac{k}{u^n}$ with $n < 2$.

b) $k = \frac{1}{4}$: this is a limiting value, where the solution depends on J_0, Y_0 Bessel functions. For higher values of k, the parameter ν becomes imaginary, and the Bessel functions have a singular (infinitely oscillatory) behavior at $\tau = 0$.[28] We will restrict our discussion to the case of $0 < k < \frac{1}{4}$, corresponding to $\frac{1}{2} < \nu < 1$.

The requirement that X^i are real functions implies

$$(\alpha_n^i)^\dagger = \alpha_{-n}^i \ , \qquad (\tilde{\alpha}_n^i)^\dagger = \tilde{\alpha}_{-n}^i \tag{6.10}$$

The canonical momenta Π^i and the total momentum carried by the string are given by

$$\Pi^i(\sigma,\tau) = \frac{1}{2\pi\alpha'}\partial_\tau X^i \ , \qquad p_0^i(\tau) = \int_0^\pi d\sigma\ \Pi^i = \frac{1}{2\alpha'}\ \dot{x}_0^i(\tau)\ . \tag{6.11}$$

Using the recursion relation for the Bessel functions, we get:

$$\partial_\tau Z(2n\tau) = \frac{\nu}{\tau}\ Z(2n\tau) - 2nW(2n\tau) \tag{6.12}$$

with

$$W(2n\tau) \equiv e^{-i\frac{\pi}{2}\nu}\sqrt{\pi n\tau}\ [J_{\nu+\frac{1}{2}}(2n\tau) - iY_{\nu+\frac{1}{2}}(2n\tau)]\ . \tag{6.13}$$

Thus

$$\Pi^i = \frac{1}{2\pi\alpha'}\frac{\nu}{\tau}(X^i - x_0^i(\tau)) + \hat{\Pi}^i\ , \tag{6.14}$$

where

$$\hat{\Pi}^i = \frac{\dot{x}_0^i(\tau)}{2\pi\alpha'} - \frac{i}{\pi\sqrt{2\alpha'}}\sum_{n=1}^\infty \Big[W(2n\tau)(\alpha_n^i e^{2in\sigma} + \tilde{\alpha}_n^i e^{-2in\sigma}) - W^*(2n\tau)(\alpha_{-n}^i e^{-2in\sigma} + \tilde{\alpha}_{-n}^i e^{2in\sigma})\Big]\ . \tag{6.15}$$

Next, we need to impose the canonical commutation relations

$$[\Pi^i(\sigma,\tau), X^j(\sigma',\tau)] = -i\delta^{ij}\delta(\sigma-\sigma')\ , \qquad [X^i(\sigma,\tau), X^j(\sigma',\tau)] = 0\ . \tag{6.16}$$

These are ensured by assuming the standard commutators for the modes

$$[\alpha_n^i, \alpha_m^j] = n\delta^{ij}\delta_{n+m}\ , \qquad [\tilde{\alpha}_n^i, \tilde{\alpha}_m^j] = n\delta^{ij}\delta_{n+m}\ , \qquad [\alpha_n^i, \tilde{\alpha}_m^j] = 0\ . \tag{6.17}$$

For the zero-mode part, we find

$$[a_0^i, a_0^{j\dagger}] = \delta^{ij}\ , \tag{6.18}$$

which implies

$$[\tilde{x}^i, \tilde{p}^j] = i\delta^{ij} \quad \text{and} \quad [x_0^i(\tau), p_0^j(\tau)] = i\delta^{ij}\ . \tag{6.19}$$

To check the above commutation relations of the mode operators, we note that

$$\begin{aligned}[\Pi^i, X^j] = [\hat{\Pi}^i, X^j] \ &= \ -\frac{i}{\pi}\delta^{ij} + \frac{1}{2\pi}\delta^{ij}\sum_{n=1}^\infty [Z(2n\tau)W^*(2n\tau) - Z^*(2n\tau)W(2n\tau)] \\ &\quad \times\ (e^{2in(\sigma-\sigma')} + e^{-2in(\sigma-\sigma')})\ .\end{aligned} \tag{6.20}$$

At first sight, this looks non-trivial and time-dependent. Note, however, that

$$ZW^* - Z^*W = 2i\pi n\tau[J_{\nu-\frac{1}{2}}(2n\tau)Y_{\nu+\frac{1}{2}}(2n\tau) - J_{\nu+\frac{1}{2}}(2n\tau)Y_{\nu-\frac{1}{2}}(2n\tau)] = -2i\ , \tag{6.21}$$

where we have used a standard relation for the Bessel functions. As a result, we verify the canonical commutation relations in (6.16).

[28] The distinction between models with $k < 1/4$ and $k > 1/4$ has the same origin as in the case of the k/r^2 potential in quantum mechanics.

6.2 Light-cone Hamiltonian

The light cone Hamiltonian of the model is given by ($p^v = p_u,\ p_v = p^u$)

$$H = -p_u = \frac{1}{8\pi\alpha'^2 p_v}\int_0^\pi d\sigma\Big(\partial_\tau X_i\partial_\tau X_i + \partial_\sigma X_i\partial_\sigma X_i + \frac{k}{\tau^2}X_i^2\Big) \ . \tag{6.22}$$

Inserting the expansion of X^i in terms of mode operators into H, we obtain

$$H = \frac{p_s^2}{2p_v} + \frac{1}{\alpha' p_v}\mathcal{H}\ , \tag{6.23}$$

where for generality we have included the contribution of $8-d$ spectator dimensions with zero-mode momenta p_s and

$$\mathcal{H} = \mathcal{H}_0(\tau) + \frac{1}{2}\sum_{n=1}^{\infty}[\Omega_n(\tau)(\alpha^i_{-n}\alpha^i_n + \tilde\alpha^i_{-n}\tilde\alpha^i_n) - B_n(\tau)\alpha^i_n\tilde\alpha^i_n - B^*_n(\tau)\alpha^i_{-n}\tilde\alpha^i_{-n}]\ . \tag{6.24}$$

Here

$$\Omega_n(\tau) = (1+\frac{\nu}{4\tau^2 n^2})|Z|^2 + |W|^2 - \frac{\nu}{2n\tau}(ZW^* + Z^*W)\ , \tag{6.25}$$

$$B_n(\tau) = (1+\frac{\nu}{4\tau^2 n^2})Z^2 + W^2 - \frac{\nu}{\tau n}ZW\ . \tag{6.26}$$

$Z(2n\tau)$ and $W(2n\tau)$ are defined in (6.5) and (6.13). It will be useful in what follows to know how the functions $\Omega_n(\tau), B_n(\tau)$ in (6.25),(6.26) behave at large and small τ. Using the asymptotic expansions of the Bessel functions, we obtain the following expressions for $\Omega_n(\tau)$ and $B_n(\tau)$ at large τ or $n\tau \gg 1$:

$$\Omega_n(\tau) = 2 + \frac{k}{4n^2\tau^2} - \frac{k^2}{64n^4\tau^4} + \frac{k^2(2+k)}{512n^6\tau^6} + ...\ , \tag{6.27}$$

$$B_n(\tau) = k\ e^{-4in\tau}\ \Big(-\frac{i}{8n^3\tau^3} + \frac{1}{32n^4\tau^4}(3+2k) + ...\Big)\ . \tag{6.28}$$

For $n\tau \ll 1$, one has

$$Z(2n\tau) \cong -ie^{-i\frac{\pi}{2}\nu}\ \frac{\sqrt{\pi}\ (n\tau)^{1-\nu}}{\cos(\pi\nu)\Gamma(\frac{3}{2}-\nu)}\ , \qquad B_n(\tau) \cong -e^{-i\pi\nu}\ \Omega_n(\tau)\ , \tag{6.29}$$

$$\Omega_n(\tau) \cong \frac{\pi}{(n\tau)^{2\nu}\cos^2(\pi\nu)}\ \Big(\frac{1}{[\Gamma(\frac{1}{2}-\nu)]^2} + \frac{\nu}{4[\Gamma(\frac{3}{2}-\nu)]^2} + \frac{\nu}{\Gamma(\frac{3}{2}-\nu)\Gamma(\frac{1}{2}-\nu)}\Big)\ . \tag{6.30}$$

The zero-mode part $\mathcal{H}_0(\tau)$ in (6.24) is the same as the point-particle Hamiltonian (5.47) in Section 5.2. In terms of $x^i_0(\tau)$ and $p_{0i}(\tau)$ in (6.6) it is given by

$$\mathcal{H}_0(\tau) = \frac{\alpha'}{2}\ [(p_{0i})^2 + \frac{k}{4\alpha'^2\tau^2}(x^i_0)^2]\ . \tag{6.31}$$

In parallel with the discussion of the particle case in section 5.2, where it was shown how the covariant Klein-Gordon equation reduces (in the sector with fixed p_v) to the time-dependent Schrödinger equation, here we will have the string wave functional (of light cone string field theory, see [40]) satisfying the time-dependent functional Schrödinger equation (cf. (5.23))

$$i\partial_\tau|\Psi(\tau;p_v)\rangle = H|\Psi(\tau;p_v)\rangle \ . \tag{6.32}$$

Expressed in terms of the modes $\alpha_n, \ \tilde\alpha_n$ and $a_0, \ a_0^\dagger$ in (6.8) the Hamiltonian (6.24) is non-diagonal. The treatment of the zero-mode part is again the same as in section 5.2. In the non-zero mode part, there are non-diagonal terms proportional to $B_n(\tau), \ B_n^*(\tau)$. The evolution of generic states made out of $\alpha_n^i, \ \tilde\alpha_n^i$ acting on the vacuum is thus non-trivial. In principle, it can be studied using time-dependent perturbation theory at large τ, where the light-cone Hamiltonian has the form $H = H_0 + \tilde H(\tau)$, with $\tilde H = O(\tau^{-2})$, but even this simpler problem is complicated by the non-diagonal form of the Hamiltonian.

Below we will find a new set of modes in terms of which the Hamiltonian becomes *diagonal*. Before doing that, let us note that the Hamiltonian (6.23) is diagonal in the special subspace of the full Fock space which is obtained by acting on the vacuum by products of $\alpha_n^i, \ \tilde\alpha_n^i$ which are antisymmetric under the exchange of the "right" and "left" moving modes, $\alpha_n^i \leftrightarrow \tilde\alpha_n^i$. These are the states of the oriented closed string whose quantum wave functional $\Psi(X^i(\sigma))$ is antisymmetric under $\sigma \to -\sigma$. Indeed, consider the expectation value of the non-diagonal term $\sum_n[B_n(\tau)\alpha_n^i\tilde\alpha_n^i - B_n^*(\tau)\alpha_{-n}^i\tilde\alpha_{-n}^i]$ in (6.24) between two of such states. We may commute $\delta_{ij}\alpha_n^i\tilde\alpha_n^j$ to the right until it gets to $|0,p_v\rangle$. What remains as a result of the commutation are terms with antisymmetric indices contracted with δ_{ij}, which vanish. Similarly, we may commute the term $\delta_{ij}\alpha_{-n}^i\tilde\alpha_{-n}^j$ to the left until it annihilates $\langle 0,p_v|$. The time evolution of such states is, however, non-trivial, because these states are not eigenstates of the Hamiltonian. For example, the state

$$|\Psi\rangle = \mathcal{N}\ b_{ij}\alpha_{-n}^i\tilde\alpha_{-n}^j|0,p_v\rangle \ , \qquad b_{ij} = -b_{ji} \ , \qquad \mathcal{N} = \frac{1}{n}(b_{ij}b^{ij})^{-1/2} \ , \qquad \langle\Psi|\Psi\rangle = 1 \tag{6.33}$$

has energy

$$\mathcal{E}_n(\tau) \equiv \langle\Psi|H(\tau)|\Psi\rangle = \frac{1}{\alpha' p_v}\ n\ \Omega_n(\tau) \ . \tag{6.34}$$

Returning to the problem of diagonalising the Hamiltonian, let us introduce a new set of *time-dependent* string modes $\mathcal{A}_n^i, \tilde{\mathcal{A}}_n^i$ defined by

$$\begin{aligned}
\frac{i}{n}(Z(2n\tau)\alpha_n^i - Z^*(2n\tau)\tilde\alpha_{-n}^i) &= \frac{i}{\sqrt{w_n}}(e^{-2iw_n\tau}\mathcal{A}_n^i(\tau) - e^{2iw_n\tau}\tilde{\mathcal{A}}_{-n}^i(\tau)) \ , \\
\frac{i}{n}(\dot Z(2n\tau)\alpha_n^i - \dot Z^*(2n\tau)\tilde\alpha_{-n}^i) &= 2\sqrt{w_n}(e^{-2iw_n\tau}\mathcal{A}_n^i(\tau) + e^{2iw_n\tau}\tilde{\mathcal{A}}_{-n}^i(\tau)) \ ,
\end{aligned} \tag{6.35}$$

where

$$w_n \equiv w_n(\tau) = \sqrt{n^2 + \frac{k}{4\tau^2}} \ , \tag{6.36}$$

with similar relations defining $\mathcal{A}_n^\dagger = \mathcal{A}_{-n}$ and $\tilde{\mathcal{A}}_n^\dagger = \tilde{\mathcal{A}}_{-n}$. It follows then that

$$\mathcal{A}_n^i(\tau) = \alpha_n^i\ f_n(\tau) + \tilde\alpha_{-n}^i\ g_n^*(\tau) \ , \qquad \mathcal{A}_n^{i\dagger}(\tau) = \alpha_{-n}^i\ f_n^*(\tau) + \tilde\alpha_n^i\ g_n(\tau) \ , \tag{6.37}$$

$$\tilde{\mathcal{A}}_n^i(\tau) = \alpha_{-n}^i\ g_n^*(\tau) + \tilde\alpha_n^i\ f_n(\tau) \ , \qquad \tilde{\mathcal{A}}_n^{i\dagger}(\tau) = \alpha_n^i\ g_n(\tau) + \tilde\alpha_{-n}^i\ f_n^*(\tau) \ , \tag{6.38}$$

where

$$f_n(\tau) = \tfrac{1}{2}e^{2iw_n\tau}[Z(2n\tau) + \frac{i}{2w_n}\dot{Z}(2n\tau)] \ , \quad g_n(\tau) = \tfrac{1}{2}e^{-2iw_n\tau}[-Z(2n\tau) + \frac{i}{2w_n}\dot{Z}(2n\tau)] \ . \tag{6.39}$$

Using the commutation relations (6.17) for $\alpha_n, \tilde{\alpha}_n$ (6.17) and the properties (6.12), (6.21) of the Bessel functions, we find

$$[\mathcal{A}^i_n(\tau), \mathcal{A}^{j\dagger}_m(\tau)] = \delta_{nm}\delta^{ij} \ , \quad [\tilde{\mathcal{A}}^i_n(\tau), \tilde{\mathcal{A}}^{j\dagger}_m(\tau)] = \delta_{nm}\delta^{ij} \ , \tag{6.40}$$

$$[\mathcal{A}^i_n(\tau), \tilde{\mathcal{A}}^{j\dagger}_m(\tau)] = 0 \ .$$

In terms of these variables, the mode expansion of $X^i(\tau)$ in (6.4) becomes

$$\begin{aligned} X^i(\sigma,\tau) \ &= \ x^i_0(\tau) + \frac{i}{2}\sqrt{2\alpha'}\sum_{n=1}^{\infty}\frac{1}{\sqrt{nw_n(\tau)}}\Big(e^{-2iw_n\tau}[\mathcal{A}^i_n(\tau)e^{2in\sigma} + \tilde{\mathcal{A}}^i_n(\tau)e^{-2in\sigma}] \\ &- \ e^{2iw_n\tau}[\mathcal{A}^i_{-n}(\tau)e^{-2in\sigma} + \tilde{\mathcal{A}}^i_{-n}(\tau)e^{2in\sigma}]\Big) \ , \end{aligned} \tag{6.41}$$

Inserting this expansion into the Hamiltonian (6.22), we find

$$\mathcal{H}(\tau) = \mathcal{H}_0(\tau) + \sum_{n=1}^{\infty} w_n(\tau)[\mathcal{A}^{i\dagger}_n(\tau)\mathcal{A}^i_n(\tau) + \tilde{\mathcal{A}}^{i\dagger}_n(\tau)\tilde{\mathcal{A}}^i_n(\tau)] + \mathrm{h}(\tau) \ , \tag{6.42}$$

where w_n is defined in (6.36) and $\mathrm{h}(\tau)$ is a "normal ordering" c-function discussed below in section 6.3.

Thus the Hamiltonian (6.23),(6.42) became diagonal. In terms of the new modes, its bosonic part looks like the Hamiltonian of a free massive 2-d field theory with a constant mass replaced by an effective time-dependent mass $\omega(\tau) = \frac{\sqrt{k}}{\tau}$. To obtain the full superstring Hamiltonian, one just needs to add to (6.42) the standard contribution of *massless* free GS fermionic modes.

As in the point-particle case in Section 5.2, constructing string states using $\mathcal{A}_n$ and $\mathcal{A}^\dagger_n$ as annihilation and creation operators, it is, in principle, straightforward to find the solutions of the corresponding infinite set of Schr'odinger equations (6.32). The wave functions will thus contain extra time-dependent phase factors, which, however, will cancel in simplest expectation values (cf. (5.57)).

6.3 UV finiteness and the role of dilaton

The light cone gauge model (6.2), (6.22) has an interesting feature of being scale-invariant but not 2-d Lorentz invariant. Indeed, despite the presence of the mass term for X_i, the Lagrangian is invariant under $\tau \to a\tau, \ \sigma \to a\sigma, \ \ a =$const. This classical symmetry is, however, expected to be broken by quantum corrections producing a logarithmic divergence proportional to the mass term, i.e. $\frac{k}{\tau^2}\ln\epsilon$. Indeed, the vacuum expectation value of the light cone Hamiltonian (6.42) given by the naive expression for the normal-ordering c-function h_0 in (6.42) is logarithmically divergent:

$$\mathrm{h}_0(\tau) = d\sum_{n=1}^{\infty}[w^{(B)}_n(\tau) - w^{(F)}_n] = d\sum_{n=1}^{\infty}(\sqrt{n^2 + \frac{k}{4\tau^2}} - n) = \frac{dk}{8\tau^2}\sum_{n=1}^{\infty}\frac{1}{n} + \dots \ . \tag{6.43}$$

Here $w_n^{(B)} = w_n$ in (6.36) and we included the free GS fermion contribution with $w_n^{(F)} = n$. As a result, there is a standard cancellation of the power divergence in the vacuum energy, but there is remaining logarithmic divergence.

In the case of the BFHP plane wave model this logarithmic divergence was cancelled against a similar one coming from the fermionic mass terms [15]. The latter originated from the non-zero 5-form background that ensured that the plane-wave background solved the string equations of motion. The same would happen again if we would consider the non-constant mass analog of the model of [15] corresponding to the solution of (4.8) with $\lambda = \mathrm{f}^2 = \frac{k}{u^2}$ and zero dilaton. The light cone Hamiltonian corresponding to this model is an obvious analog of (6.42) with both bosonic and fermionic oscillator terms multiplied by the same $w_n(\tau)$ functions and $\mathrm{h} = 0$.

The UV finiteness of the light cone 2-d theory should be of course correlated with the 2-d conformal (Weyl) invariance of the original covariant string sigma model. In the present case of (4.5),(4.6) the role of an additional background that cancels the conformal anomaly due to the non-vanishing Ricci tensor of the plane-wave metric (4.1) is played by the dilaton field. Since the dilaton is known to contribute to the expression for the 2-d stress tensor of the string model, this implies that the definition of the light cone Hamiltonian should be adjusted so that to cancel an apparent divergence in (6.43). More precisely, this divergent term is cancelled by a singular field redefinition while the role of the dilaton contribution is to cancel the finite Weyl anomaly term which acompanies this divergence (see below). Then the "dilaton-corrected" finite expression for $\mathrm{h}(\tau)$ in (6.42) should be

$$\mathrm{h}(\tau) = d\sum_{n=1}^{\infty} \Big(\sqrt{n^2 + \frac{k}{4\tau^2}} - n - \frac{k}{8n\tau^2}\Big) = d\sum_{r=2}^{\infty} \frac{\sqrt{\pi}\ \zeta(2r-1)}{2\ r!\ \Gamma(\frac{3}{2}-r)} \Big(\frac{k}{4\tau^2}\Big)^r , \qquad (6.44)$$

where we have used that $\sqrt{n^2 + \frac{k}{4\tau^2}} = \sum_{r=0}^{\infty} \frac{\sqrt{\pi}}{2r!\Gamma(\frac{3}{2}-r)} (\frac{k}{4\tau^2})^r \frac{1}{n^{2r-1}}$. It is possible to show that $\mathrm{h}(\tau)$ is negative-definite, and $\mathrm{h}(\tau) \to -\infty$ at $\tau \to 0$.

Let us elaborate a bit more on the cancellation of the logarithmic divergence and the role of the dilaton contribution. Any covariant 2-d sigma model with metric whose Ricci tensor of the form $R_{\mu\nu} = D_\mu W_\nu + D_\nu W_\mu$ is formally (1-loop) scale-invariant on a flat 2-d background: the logarithmic divergence $\sim \ln\epsilon\ R_{\mu\nu}\partial^a X^\mu \partial_a X^\nu$ vanishes on-shell or can be cancelled by a (singular) field redefinition, i.e. by a 2-d field renormalization[29] $X^\mu \to X^\mu + \alpha' \ln\epsilon\ W^\mu(X)$. In the present case of the plane-wave metric (4.1) we have only one non-zero component $R_{uu} = \ d\ \lambda(u)$, i.e. $W_u(U) = \frac{1}{2} d \int^U du \lambda(u)$. The required redefinition is then: $V \to V + \frac{1}{2} d\alpha' \ln\epsilon\ \int^U du\lambda(u)$, or, for $\lambda = \frac{k}{u^2}$, $V \to V - \frac{1}{2} d\alpha' \ln\epsilon\ \frac{k}{U}$. This redefinition of the V-coordinate should lead to the presence of a "counterterm" in the light cone Hamiltonian (6.42),(6.43) that ensures its finiteness (note that the redefinition applied to $\partial_\tau V \partial_\tau U$ produces, in the light cone gauge $U = \alpha' p^u \tau$, a term $\sim \ln\epsilon\ \frac{k}{\tau^2}$).

In the string-theory context, the condition on the sigma-model is not simply scale invariance but (in general, for non-compact or Lorentzian target spaces) stronger requirement of Weyl (conformal) invariance of a theory defined on a curved 2-d background.

[29] Such redefinitions should not of course change target-space diffeomorphism-invariant observables which should thus be finite (see [45] and refs. there for a discussion of field renormalizations in 2-d sigma models).

That implies a relation between the Ricci tensor and the dilaton, i.e. $W_\mu = \partial_\mu \phi$ (to 1-loop order in general, but to all orders in the present case; for details see, e.g., [41, 42, 43]). In general, the dilaton couples to the 2-d curvature as

$$I = -\frac{1}{4\pi\alpha'} \int d^2\xi \sqrt{-g}[G_{\mu\nu}(X)\partial^a X^\mu \partial_a X^\nu + \alpha' R^{(2)}\phi(X)] \ , \qquad (6.45)$$

and thus modifies the expression for the 2-d stress tensor by a finite term $\partial_a\partial_b\phi - \eta_{ab}\partial^2\phi$ whose role is to cancel the corresponding conformal anomaly coming from the sigma model part. In the present case where $\phi'' = -\frac{d}{2}\lambda(u)$ (see (4.2)), we get, in the light cone gauge, $(T_{\tau\tau})_\phi \sim \frac{k}{\tau^2}$, etc. The Weyl invariance implies also the finiteness of the expectation value of stress tensor components. In the conformal gauge $g_{ab} = e^{2\rho}\eta_{ab}$, and using a *covariant* 2-d world-sheet regularization ($e^{2\rho}|\Delta\xi|^2 > \epsilon^2$) the dependence on ρ and on the 2-d UV cutoff $\epsilon \to 0$ should be correlated (they should effectively appear in a combination $\epsilon\ e^{-\rho}$). Thus, the finite dilaton contribution cancelling the Weyl anomaly should be accompanied also by a divergent counter-term, cancelling the associated logarithmic 2-d UV divergence (the one which may be cancelled by a field renormalization as discussed above). The final result is the finite expression (6.42),(6.44) for the light cone Hamiltonian operator.

Apart from the role of the dilaton in the Weyl anomaly cancellation, it also determines the effective interactions of the properly normalized string modes (see (5.1)–(5.6)).

6.4 Evolution of a classical rotating string

To get a better understanding of dynamics of strings in the our plane wave background it is useful to consider first some simple examples of classical solutions.

In flat space, a rigid string rotating in a plane represents a state on the leading Regge trajectory with maximal angular momentum for a given energy. In the light-cone gauge, it is described by the solution

$$U = 2\alpha' p_v \tau, \qquad V = 2\alpha' p_u \tau \ , \qquad L^2 = -2\alpha'^2 p_u p_v \ , \qquad (6.46)$$

$$X \equiv X_1 + iX_2 = L\ e^{-2i\tau}\ \cos(2\sigma) \ , \qquad (6.47)$$

where X_1 and X_2 stand for Cartesian coordinates of a transverse 2-plane and we used the constraint to fix the value of L. The center of mass can be at rest ($p_u = p_v$) or moving on a line, and the string is rotating in this frame.

The analogue of this rotating string solution in the present plane-wave background is:

$$U = 2\alpha' p_v \tau \ , \qquad X = \ L\ Z(2\tau)\ \cos(2\sigma) \ , \qquad (6.48)$$

with V determined by solving the constraint[30] and Z defined in (3.17). At $\tau \to \infty$, the solution (6.48) reduces to the flat-case one (6.47). In general, it represents a rotating string whose effective length $L_{\rm eff} = L\ |Z(2\tau)|$ shrinks with time as the incoming front

[30]One gets $\partial_\tau V(\sigma,\tau) = \frac{L^2}{\alpha' p_v}\big[ZZ^*(\frac{\nu}{4\tau^2}\cos^2 2\sigma - \sin^2 2\sigma) - WW^*\cos^2 2\sigma) + \frac{\nu}{2\tau}(ZW^* + WZ^*)\cos^2 2\sigma\big]$, so that the string trajectories are non-trivial.

wave is approaching. For very small τ, the effective length rapidly goes to zero as $\tau^{1-\nu}$ before a period can be completed (see eq. (6.29)).[31]

The light-cone energy of the rotating string (6.48) can be computed by inserting the solution (6.48) into (6.22), or simply by using the values of $\alpha_1,\ \tilde{\alpha}_1,\ \alpha_{-1}$ and $\tilde{\alpha}_{-1}$ corresponding to the above solution in the eq. (6.24). We get

$$H = -p_u(\tau) = \frac{L^2}{4\alpha'^2 p_v}\ \Omega_1(\tau)\ , \tag{6.49}$$

where we used (6.12). Note that the non-diagonal term of (6.24) proportional to $B_1(\tau)$ and $B_1^*(\tau)$ has cancelled out from (6.49). The light-cone energy H of this state is thus time-dependent and is determined by eqs. (6.27), (6.30) with $n = 1$: it is approximately constant at $\tau \to \infty$, and it increases as $\tau^{-2\nu}$ as $\tau \to 0$.

As in flat space, the solution has two integrals of motion – p_v and the angular momentum J corresponding the symmetries of our metric under shift of v and rotations in transverse space, but p_u is no longer conserved.[32] The value of the angular momentum is

$$J = \frac{1}{2\pi\alpha'}\int_0^\pi d\sigma(X_1\dot{X}_2 - \dot{X}_1X_2) = \frac{L^2}{2\alpha'}\ , \tag{6.50}$$

where we used (6.12) and the relation (6.21).

It is interesting to note that we find the analogue of the standard leading Regge trajectory relation

$$-2\alpha' p_u p_v = \alpha'(E^2 - p_y^2) = 2J\ , \tag{6.51}$$

($E = p_0$) but with an "effective tension" function $T = \frac{1}{2\pi\alpha'}\cdot\frac{1}{2}\Omega_1(\tau)$, i.e.

$$-2\alpha'\ p_u(\tau)\ p_v\ \ =\ \ \Omega_1(\tau)J\ . \tag{6.52}$$

At $\tau \to \infty$, we have $\Omega_1 \to 2$, and so we recover the standard flat-space Regge relation. As we go back in time to the region of small τ, the energy of this physical state gradually grows until it diverges as $\tau \to 0$ (where the "effective Regge slope" goes to zero).

One may wish to compare this solution with a similar one in the case of BFHP plane-wave background where $\lambda(u) = \lambda_0$. Here the rotating string solution is given by

$$U = 2\alpha' p_v\tau\ , \qquad X = L\ e^{2i\omega\tau}\ \cos(2\sigma)\ , \qquad \omega = \sqrt{1+\mathrm{m}^2}\ , \quad \mathrm{m} = \alpha'\sqrt{\lambda_0}p_v\ , \tag{6.53}$$

with V again determined by solving the constraint. This gives $V = 2\alpha' p^v\tau\ ,\ \ 2\alpha'^2 p_v p^v = -L^2$. From the canonical momenta, for the above solution one has the relations, $p^u = p_v\ ,\ p^v = p_u + \frac{L^2\mathrm{m}^2}{2\alpha' p_v}$. Here p_v, p_u and J are all conserved and we find $-p_u = \frac{L^2}{2\alpha'^2 p_v}\ \omega^2$ and $J = \frac{L^2}{2\alpha'}\ \omega$, so that we get the direct analog of the flat-space Regge relation

$$-2\alpha' p_u p_v = \omega\cdot 2J\ , \tag{6.54}$$

with the "effective tension" is $T = \frac{\omega}{2\pi\alpha'}$ being increased compared to the flat case.

[31] It is interesting to note that in Rosen coordinates (obtained by multiplying (6.47) by $\tau^{\nu-1}$) the effective length is regular at $\tau = 0$. However, the solution in Rosen coordinates has the disadvantage of not reducing to flat-space solution at $\tau = \infty$.

[32] Note that the integral of motion corresponding to the invariance of the metric under the scaling symmetry, $u' = \ell u,\ v' = \ell^{-1}v$ is trivial (equal to zero) because of the constraint.

6.5 Quantum string mode creation

Let us now turn to a study of properties of quantum strings.

It is well known that in gravitational pp-wave backgrounds there is no particle creation [44]. Indeed, the existence of a covariantly constant null Killing vector guarantees a definition of a frequency which is conserved. Similarly, there cannot be string creation, so one can consistently describe string propagation in this background by using the usual first quantized formalism.

Nevertheless, as was shown in [7], and further studied in [46, 29], there can be string mode creation. In general, given a pp-wave background with asymptotically flat regions at $u = -\infty$ and $u = +\infty$, the time evolution of a string which starts in a given state at $u = -\infty$, may be such that the string ends up at $u = +\infty$ in a different state. In particular, the string could have extra internal excitations, produced by the interaction with the pp-wave background.

Passing through the singularity at $u = 0$ obviously requires some prescription. This will be discussed in the next subsection. Here we will consider a creation of string modes as seen by an observer in the "in" vacuum $|0, p_v\rangle_0$ at $u = \infty$. By the "in" vacuum we mean the Fock space state which is annihilated by the operators α_n^i, $\tilde{\alpha}_n^i$ in (6.17). We shall start with the string in the $|0, p_v\rangle_0$ state at $u = \infty$ and study how this state should evolve back to $u = 0$. Equivalently, one may reverse the orientation of time, given the symmetry of the metric under $u \to -u$, $v \to -v$, and interpret this as an evolution from the "in" vacuum at $u = -\infty$ to some excited state at later time.

Let us consider the expectation value of the "oscillator number" operator that appears in the Hamiltonian (6.42):

$$\bar{N}_n(\tau) \equiv {}_0\langle 0, p_v|(\mathcal{A}_n^{i\dagger}\mathcal{A}_n^i + \tilde{\mathcal{A}}_n^{i\dagger}\tilde{\mathcal{A}}_n^i)|0, p_v\rangle_0 = 2d\ n\ g_n^*(\tau)g_n(\tau)\ , \tag{6.55}$$

where d is again the range of index i, i.e. the number of "massive" 2-d coordinates X^i. We have used eq. (6.38). Inserting the definition (6.39) of $g_n(\tau)$ and using (6.21) we find

$$\bar{N}_n(\tau) = \frac{d\ n^2}{2w_n^2}\ [n\Omega_n(\tau) - 2w_n(\tau)]\ . \tag{6.56}$$

Here $\Omega_n(\tau)$ was defined in (6.25) and w_n – in (6.36). The total number of created oscillator modes is then

$$\bar{N}_T(\tau) = \sum_{n=1}^{\infty} \bar{N}_n(\tau) = d \sum_{n=1}^{\infty} \frac{n^2}{2w_n^2}\ [n\Omega_n(\tau) - 2w_n(\tau)]\ . \tag{6.57}$$

Note that for any, no matter how small, τ there will be an infinite number of modes for which $n\tau \gg 1$, which will thus behave essentially like massless modes (cf. (6.36)). For them, one can use the asymptotic form (6.27) of $\Omega_n(\tau)$ which applies for $n\tau \gg 1$. When $\tau \gg 1$, one can use the asymptotic form of (6.27) of $\Omega_n(\tau)$ for all modes, including the $n = 1$ one.

Inserting eq. (6.27) into (6.57), one finds a surprising cancellation between the first three terms in the expansion. Only the cancellation of the leading term of (6.27) (i.e. 2) is obvious: we know that $g_n(\tau)$ must vanish at large τ (by construction, since at $\tau \to \infty$

the definition (6.35) implies that the modes α_n, $\tilde{\alpha}_n$ and $\mathcal{A}_n$, $\tilde{\mathcal{A}}_n$ are the same, modulo normalization). The resulting expression at *large* τ is then

$$\bar{N}_T(\tau) \cong d \sum_{n=1}^{\infty} \frac{k^2}{512\ \tau^6}\ \frac{1}{n^5} = \frac{d\ k^2 \zeta(5)}{512\ \tau^6}\ , \tag{6.58}$$

i.e. is finite. This also proves that the series is converging for *any* τ: as was pointed out above, for any given τ, we can use the expansion (6.27) for all the modes with $n \gg \tau^{-1}$. This produces the convergent sum $\sum_{n=n_0}^{\infty} \frac{1}{n^5}$, with $n_0 \gg \tau^{-1}$.

Let us now investigate the behavior of $\bar{N}_T$ as we approach the *small* τ region, i.e. $\tau \ll 1$. In this case, for all $n \ll \tau^{-1}$ the function $\Omega_n(\tau)$ may be approximated by the expression in (6.30), so that the leading contribution is

$$\bar{N}_n(\tau) \sim \text{const} \cdot\ \tau^{1-\sqrt{1-4k}}\ . \tag{6.59}$$

Thus each individual contribution in the sum over modes vanishes for sufficiently small τ.

Nevertheless, the total number of created oscillating modes $\bar{N}_T(\tau)$ diverges as $\tau \to 0$. This can be seen as follows. Let $\tau = \epsilon$, $0 < \epsilon \ll 1$. The number of excitations $\bar{N}_n(\tau)$ of a given frequency n must have a maximum as a function of n, since it vanishes in the limits $n \ll \epsilon^{-1}$ (see eq. (6.59)) and $n \gg \epsilon^{-1}$ (as $1/\tau^6$, see (6.58)).[33] For small τ, the most important contributions to the sum come in fact from the modes with $n\tau = O(1)$. Since $\Omega_n(\tau)$ in (6.25) is a function of $n\tau = O(1)$, one has $n\Omega_n(\tau) = O(n)$ in (6.56), so that $\bar{N}_n(\tau) = O(n)$ for $n = O(\epsilon^{-1})$. Thus $\bar{N}_T(\tau)$ picks up the main contribution from the terms $\bar{N}_n(\tau)$ with $n = O(\epsilon^{-1})$. To compute $\bar{N}_T(\tau)$, one notes that the number of terms $\bar{N}_n$ with $n\tau = O(1)$ is also of order $n = O(\epsilon^{-1})$. Therefore, $\bar{N}_T \sim n^2 \sim \tau^{-2}$ for small τ.

In conclusion, we find that the total number of excited modes on a string state which started in the "in" vacuum $|0, p_v\rangle_0$ tends to infinity as we approach the $\tau = 0$ point. We stress that the origin of the divergence is in the creation of very high frequency n modes, with $n \to \infty$ as $\tau \to 0$, since, as explained above, the creation of modes with $n < \tau^{-1}$ is suppressed at small τ.

This singularity is, however, observer-dependent. The state $|0, p_v\rangle_0$ represents a vacuum only at $\tau = \infty$, since it is defined as being annihilated by *massless* 2-d scalar bosonic modes (the mass $\frac{\sqrt{k}}{\tau}$ vanishes at $\tau = \infty$). The singularity of $\bar{N}_T(\tau) \to \infty$ at $\tau \to 0$ is analogous to the creation of a divergent number of modes near a horizon of a black hole as seen by a Schwarzschild observer. At the same time, there is no string mode creation in the vacuum state annihilated by the operators $\mathcal{A}_n^i$, $\tilde{\mathcal{A}}_n^i$ in terms of which the Hamiltonian (6.42) is diagonal.

6.6 String transition through the $u = 0$ singularity

Let us now consider the evolution from a free string state in the asymptotically flat region at $u = -\infty$ to the asymptotically flat region at $u = +\infty$. We shall assume that the complete space-time can be obtained by patching together the regions with $u < 0$ and $u > 0$.

[33] One can check numerically that there is only one maximum.

Some of the paths (i.e. geodesics) of *point-like* particles cannot smoothly traverse the singular point at $u = 0$, and excising this point makes the space-time geodesically incomplete [7]. In [7], string mode creation in the evolution from $u = -\infty$ to $u = \infty$ was discussed for a generic $\lambda(u)$ which goes to zero at large $|u|$ and is non-trivial for finite u with singularity in the middle. In the case when $-\lambda(u)$ (i.e. an effective "potential" in the string equation) has a form of a potential well it was argued that the number of excitations typically increase with the depth of the well, but the case of $\lambda(u) = k/u^2$ was not explicitly considered.

Here we would like to argue that strings may pass through the singular point $u = 0$. We will adopt a natural prescription implied by an analytic continuation of the Bessel functions.

We will consider two regions I and II, corresponding to $u \sim \tau < 0$ and $u \sim \tau > 0$ respectively. In region II, the solution for $X^i(\sigma,\tau)$ is given by eq. (6.4), with α^i_n, $\tilde\alpha^i_n$, α^i_{-n} and $\tilde\alpha^i_{-n}$ representing the "out" modes. In region I, the general solution is given by

$$\begin{aligned} X^i(\sigma,\tau) &= x^i_0(\tau) + \frac{i}{2}\sqrt{2\alpha'}\sum_{n=1}^{\infty}\frac{1}{n}\Big[Z^*(-2n\tau)(\beta^i_n e^{2in\sigma} + \tilde\beta^i_n e^{-2in\sigma}) \\ &- Z(-2n\tau)(\beta^i_{-n}e^{-2in\sigma} + \tilde\beta^i_{-n}e^{2in\sigma})\Big] , \end{aligned} \tag{6.60}$$

where the "in" modes β^i_n, $\tilde\beta^i_n$, β^i_{-n}, $\tilde\beta^i_{-n}$ obey similar commutation relations as the α^i_n in (6.17). Here Z-functions are the same as in (6.5) and the zero mode part $x^i_0(\tau)$ is the same as in (6.4) with the replacement of the "out" modes a^i_0, $a^{i\dagger}_0$ by the "in" modes b^i_0, $b^{i\dagger}_0$.

The series expansion (6.60) is written so that asymptotically, at $\tau \to -\infty$, where (see (6.9))

$$Z^*(-2n\tau) \cong e^{-2in\tau}[1 + O(\tau^{-1})] , \tag{6.61}$$

it reduces to the standard free string theory mode expansion for $X^i(\sigma,\tau)$ (modulo a peculiarity of the zero-mode part asymptotics mentioned below (6.9)).

Since the evolution is governed by linear equations, the "in" $\beta_n, \tilde\beta_n$ and "out" $\alpha_n, \tilde\alpha_n$ modes will be related by a linear transformation:

$$\alpha_n = C_n\beta_n + D_n\tilde\beta_{-n} , \qquad \tilde\alpha_{-n} = \tilde C_n\beta_n + \tilde D_n\tilde\beta_{-n} . \tag{6.62}$$

The most natural way to go from the region I to the region II is by an analytic continuation of the Bessel functions in (6.5). The Bessel functions have a branch point at $\tau = 0$. To make the transition from $\tau < 0$ to $\tau > 0$ we choose a contour which goes below the $\tau = 0$ point (or, equivalently, we replace $\frac{k}{u^2}$ by $\frac{k}{(u-i\epsilon)^2}$). As we shall show below, an alternative choice of the contour which passes above the $\tau = 0$ point would lead to an unphysical result.

Using the formula [47]

$$H^{(2)}_\mu(e^{-i\pi}z) = -e^{i\pi\mu}H^{(1)}_\mu(z) , \qquad H^{(1,2)}_\mu(z) \equiv J_\mu(z) \pm iY_\mu(z) , \tag{6.63}$$

we get

$$Z(e^{-i\pi}2n\tau) = Z^*(2n\tau) , \qquad Z^*(e^{-i\pi}2n\tau) = Z(2n\tau) , \tag{6.64}$$

where we have taken into account the extra factor of $\sqrt{\tau}$ in the definition of $Z(2n\tau)$ in (6.5). It follows that

$$\alpha_n = \beta_n \ , \qquad \tilde{\alpha}_{-n} = \tilde{\beta}_{-n} \ , \tag{6.65}$$

and similar relations for the modes with $n \to -n$. It is then natural to assume the same conditions for the zero-mode part:

$$a_0^i = b_0^i \ , \qquad a_0^{i\dagger} = b_0^{i\dagger} \ ,$$

even though this produces a discontinuity in the time derivative.[34]

With the above prescription, the 2-d "S-matrix" is trivial, and there is no mode creation. In particular,

$$\langle 0_{\rm in}|\hat{N}_n^{\rm out}|0_{\rm in}\rangle = \langle 0_{\rm in}|(\alpha_{-n}^i\alpha_n^i + \tilde{\alpha}_{-n}^i\tilde{\alpha}_n^i)|0_{\rm in}\rangle = 0 \ , \tag{6.66}$$

where $|0_{\rm in}\rangle$ is the vacuum state for the "in" operators β_n^i, $\tilde{\beta}_n^i$.

If the contour passes the point $\tau = 0$ from above, one gets the relation

$$H_\mu^{(2)}(e^{i\pi}z) = 2\cos(\mu\pi)H_\mu^{(2)} + e^{i\pi\mu}H_\mu^{(1)}(z) \ . \tag{6.67}$$

This prescription would lead to a non-unitary 2-d "S-matrix" connecting $(\alpha_n, \tilde{\alpha}_{-n})$ with $(\beta_n, \tilde{\beta}_{-n})$ which does not conserve probabilities.

In conclusion, there is a natural way to identify oscillating modes which leads to trivial transition amplitudes in this sector. In the zero mode sector, there is a discontinuity in the time derivative. This may suggest that some source is needed at $u = 0$, which affects the string oscillations only in a mild way, but takes care of the zero-mode part. Needless to say, issues related to the singularity at $u = 0$ deserve further study.

7 Concluding remarks

In an attempt to address the issue of singularities of cosmological or time dependent backgrounds in string theory, here we have studied a simple solvable model which has the basic ingredients that are needed to mimic big-bang cosmology: a "null" time dependent scale factor and a singularity at $u = 0$, where tidal forces become infinite. This is a mild type of singularity because all curvature scalars vanish and some geodesics can be extended through it. Nevertheless, some other time-like geodesics of classical test particles are incomplete, making this simple spacetime a rather direct analog of a singular cosmology. Having a solvable string model, one may try to elucidate, in particular, the evolution of quantum string states as they approach and pass through the $u = 0$ region.

Understanding string theory in a time-dependent background involves some well known conceptual issues, such as how to define observables in a situation with a singularity. In

[34] One may try to find a relation between "in" and "out" zero modes by using the regularized background (4.15). Then the zero-mode part is given in terms of hypergeometric functions and extends from $\tau = -\infty$ to $\tau = \infty$. The matching at $\tau \to \pm\infty$ with $x_0^i(\tau)$ (given in terms of a_0^i, $a_0^{i\dagger}$, and b_0^i, $b_0^{i\dagger}$, respectively) leads to a non-trivial Bogoliubov transformation between a_0^i, $a_0^{i\dagger}$, and b_0^i, $b_0^{i\dagger}$. It depends on the "regularization parameter" s, and is singular as $s \to 0$, so it cannot be used to determine the relation between the two sets of modes.

particular, one may ask whether a consistent quantum theory can be defined by specifying initial conditions at the singularity and considering only the region $0 < u < \infty$, or one must include also the "pre - big-bang" $-\infty < u < 0$ part. Some related studies of string theory in simplest time-dependent dilatonic backgrounds seem to favour [3] the latter option. In the present case, the existence of special geodesics which can go through $u = 0$ smoothly from $u < 0$ to $u > 0$ (see Appendix B and also [48]) is another indication that the full quantum theory should be formulated in the extended region $-\infty < u < \infty$. Thus it is important to understand how to define a consistent string theory in this case. As we have shown in section 6, there is indeed a natural way to extend oscillating string modes from $u < 0$ to $u > 0$ region, using the analytic continuation property of the Bessel functions.

In the zero mode sector, however, there is a discontinuity in the time derivative, which suggests that some source or "domain wall" is needed at $u = 0$. This could resolve the problem of classical geodesic incompleteness in the following hypothetic way. A closed string coming from $u < 0$ and approaching $u = 0$ may be absorbed by the source (or "brane"), which then gets excited and re-emits the closed-string mode at $u > 0$. One possible approach is to try to add an open string sector with boundary condition $\partial_\tau X^\mu|_{\tau=0} = 0$. Introducing a D-brane at $u = 0$ seems natural, given the origin of the metric (1.2) with $\lambda = \frac{k}{u^2}$ as a Penrose limit of Dp-branes, with $u = 0$ corresponding to the original $r = 0$ singularity of the Dp-branes [19, 26] (see also Introduction). One technical problem with this idea is that then one cannot use the light-cone gauge (where $\partial_\tau U \neq 0$), while it is not clear how to solve the model directly in a covariant gauge.

As was recently discussed in [24, 49], formulating string theory in singular time-dependent backgrounds may lead to a number of potential problems, such as new divergences in loop diagrams and various sorts of instabilities. It would be interesting to study string scattering amplitudes for our present plane-wave background but this is much more complicated than in the flat (orbifold) case of [24] (computation of scattering amplitudes is a non-trivial problem already for the $\lambda(u) = \text{const}$ or BFHP plane wave). Simplest scalar massless vertex operator is related to the solution of the Klein-Gordon equation discussed in section 5.[35]

Comparing the present case to the null orbifold model [22, 24], where time dependence is introduced by a global identification and there is a static covering picture, the present plane wave "null cosmological" model is closer to a standard cosmology. In particular, the nature of the singularity at $u = 0$ here is different from the null orbifold case (despite similarity in the form of the metric in Rosen coordinates ($ds^2 = dudv + u^2 dx^2$): it is the compactness of x that causes a singularity at $u = 0$ in [24] and is related to multiplicity of images in Cartesian coordinates of the orbifold. In our case the singularity is related to the blow-up of the *curvature*, causing infinite tidal forces near $u = 0$. At the same time, as already mentioned above, this singularity is still of a different type compared to the standard cosmological one, as all curvature invariants vanish in the plane-wave case.

Another new feature of our model is the vanishing of the string coupling at the singularity, implying a reduction of back reaction effects.[36] As discussed in section 5, the

[35]It simplifies for special states with $p_i = 0$ moving along v-direction, so it may be interesting to try to study the tree-level amplitudes for such states.

[36]The back reaction problem in the null orbifold case is somewhat different, being related to multiple

fall-off in e^{ϕ} competes with the focussing effect of geodesics, which typically leads to configurations with infinite energy density near $u = 0$.

A basic problem, however, is which are the proper invariant quantities to "measure" back reaction effects (for example, one can always go to a frame where the energy discussed at the end of Section 5 is small). The situation may be analogous to the one when a small amount of matter is approaching the horizon of a black hole. In Schwarzchild coordinates, the energy would diverge near the horizon, but in an inertial frame the energy is small, so strong back reaction effects are not expected. More generally, there is a well-known ambiguity in choice of vacuum and observables in time-dependent backgrounds. All these questions await further study in the string-theory context.

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Appendix A A group manifold structure

The plane-wave space-time (1.2) admits a group manifold structure. We should note from the start that the metric (1.2) will be left-invariant only: the corresponding bi-invariant metric will be degenerate (and will not be directly related to (1.2)).

To determine the group structure, it is sufficient to find the frames that satisfy the Maurer-Cartan equations for a Lie group. For this it is convenient to work with the plane wave metric as given in (2.20). Define the frames

$$e^0 = dw \ , \qquad e^1 = e^w dv + \mu x^2 dw + \rho x_i dx^i \ , \qquad e^i = dx^i + \sigma x^i dw \ , \tag{A.1}$$

where μ, ρ and σ are real constants which are to be determined. It is natural to impose the following Maurer-Cartan equations

$$de^0 = 0, \qquad de^1 = e^0 \wedge e^1, \qquad de^i = \sigma e^i \wedge e^0 \ . \tag{A.2}$$

Then we find that $\sigma + \rho = 0$, $2\mu + \rho = 0$. For the metric

$$ds^2 = 2e^0 e^1 + \delta_{ij} e^i e^j \tag{A.3}$$

to be equal to the one given by (2.20), we also find that $2\mu + \sigma^2 = -k$. Thus we get $\rho^2 - \rho + k = 0$. This has real solutions if $k \leq \frac{1}{4}$. As a result, the metric (1.2) with $0 < k \leq \frac{1}{4}$ can be interpreted as left-invariant (but not bi-invariant) metric on a group manifold.

images of the fixed point in global coordinates: as pointed out by Horowitz and Polchinski [49], there an introduction of a single particle makes the spacetime to collapse.

Appendix B Brinkmann, Rosen and conformal coordinates

As we have explained there is freedom in the choice of coordinate transformation from Brinkmann to Rosen coordinates parameterized by the the two integration constants q_1 and q_2 because the equation for $a(u)$ in (3.7) is second order. The case where $q_1 = 0$ has already been presented in section 3.2. For the case $q_2 = 0$ after a rescaling of the x^i coordinates we find

$$a(u) = u^\nu \ . \tag{B.1}$$

Therefore the metric (1.2) in these Rosen coordinates is

$$ds^2 = 2du\mathrm{dv} + u^{2\nu} d\mathrm{x}^2 \ . \tag{B.2}$$

This metric can be easily be expressed in conformal coordinates (see (3.17),(C.9)) as

$$ds^2 = \ \Sigma(\mathrm{w})\ (2d\mathrm{w}d\mathrm{v} + d\mathrm{x}^i d\mathrm{x}^i)\ , \tag{B.3}$$

where for $u > 0$

$$\mathrm{w} = -\frac{c}{u^{2\nu-1}}\ , \qquad -\infty < \mathrm{w} < 0\ , \qquad \Sigma(\mathrm{w}) = (-\frac{\mathrm{w}}{c})^{-\frac{2\nu}{2\nu-1}}\ , \qquad q_1 = 1\ ,\ \ q_2 = 0\ , \tag{B.4}$$

and

$$c \equiv \frac{1}{2\nu - 1}\ .$$

Observe that the singularity at $u = 0$ in Rosen coordinates is mapped to $-\infty$ in the Minkowski coordinates and the region $u = +\infty$ – to $\mathrm{w} = 0$ hyperplane in Minkowski coordinates. This is unlike the case with $q_1 = 0$ which we have discussed in section 3.2. For $u < 0$, we find

$$\mathrm{w} = \frac{c}{(-u)^{2\nu-1}}\ , \qquad 0 < \mathrm{w} < \infty\ , \qquad \Sigma(\mathrm{w}) = (\frac{\mathrm{w}}{c})^{-\frac{2\nu}{2\nu-1}}\ , \qquad q_1 = 1\ ,\ \ q_2 = 0\ . \tag{B.5}$$

For $q_1 q_2 \neq 0$, we can normalize the constants q_1, q_2 so that $q_1 q_2 = 1$, which can be achieved by a further rescaling of the coordinates x^i. So we can set $q_1 = q$ and $q_2 = q^{-1}$. For $u > 0$

$$\mathrm{w} = \frac{cq^2 u^{2\nu-1}}{q^2 u^{2\nu-1} + 1}\ , \qquad 0 < \mathrm{w} < c\ ,\ \ 0 < u < +\infty\ , \tag{B.6}$$

where we have chosen the integration constant so that to have $\mathrm{w}(0) = 0$. Similarly, for $u < 0$

$$\mathrm{w} = -\frac{cq^2(-u)^{2\nu-1}}{q^2(-u)^{2\nu-1} + 1}\ , \qquad -c < \mathrm{w} < 0\ ,\ \ -\infty < u < 0\ . \tag{B.7}$$

Then the metric in the Rosen coordinates (3.14) written in conformally-flat form (3.17) for $0 < \mathrm{w} < \frac{1}{2\nu-1}$ has $\Sigma(\mathrm{w})$ given by

$$\Sigma(\mathrm{w}) = b\frac{\mathrm{w}^{\frac{2-2\nu}{2\nu-1}}}{(c - \mathrm{w})^{\frac{2\nu}{2\nu-1}}}\ , \qquad b \equiv (2\nu - 1)^{-2} q^{-\frac{4\nu}{2\nu-1}}\ , \tag{B.8}$$

while for $-\frac{1}{2\nu-1} < \mathrm{w} < 0$,

$$\Sigma(\mathrm{w}) = b\frac{(-\mathrm{w})^{\frac{2-2\nu}{2\nu-1}}}{(c+\mathrm{w})^{\frac{2\nu}{2\nu-1}}} \ . \tag{B.9}$$

As a result, the homogeneous plane wave in the above coordinates is conformal to a strip in $d+2$-dimensional Minkowski space. The two boundaries of the strip are the hypersurfaces located at the values $\mathrm{w} = c$ and $\mathrm{w} = -c$ of the light-cone coordinate w. The singularity is again a hypersurface located at $\mathrm{w} = 0$. We remark that the above form of the metric (3.17) suggests a generalization of the type $ds^2 = \frac{p(\mathrm{w})}{r(\mathrm{w})}(2d\mathrm{w}d\mathrm{v} + d\mathrm{x}^i d\mathrm{x}^i)$, where $p(\mathrm{w}), r(\mathrm{w})$ are polynomials without common factors. The singularities of the spacetime are at the roots of p while the conformal boundaries are at the roots of r.

Let us mention also that it is easy to construct the embedding of the homogeneous plane-wave space-time into the flat space $\mathbb{R}^{2,2+d}$ with the metric

$$ds^2(\mathbb{R}^{2,2+d}) = -dX_1^2 - dX_2^2 + \sum_{i=1}^{d+2} dY_i^2 \ . \tag{B.10}$$

For example, in the case $q_1 q_2 = 1$, the embedding equations are

$$\begin{aligned} \sum_i Y_i^2 - X_1^2 - X_2^2 &= 0 \ , \\ \Big[\frac{1}{4\nu-2}(X_2+Y_1) - (X_1+Y_2)\Big]^{2\nu} &= \frac{1}{q^{4\nu}(2\nu-1)^{4\nu-2}}(X_1+Y_2)^{2-2\nu} \ . \end{aligned} \tag{B.11}$$

Replacing Y_i and $X_{1,2}$ by the spherical coordinates, $Y_1 = r\cos\psi, Y_2 = r\sin\psi\cos\theta, \ldots$, and $X_1 = r\sin\varphi, X_2 = r\cos\varphi$, we recover the plane-wave metric (3.21) written in the static Einstein universe coordinates.

Appendix C Geodesics

The geodesics of the spacetime (1.2) in Brinkmann coordinates can be found explicitly. It is easy to see that the equation for v implies that the coordinate u should be linear in an affine parameter s. Thus we have $u = u_1 s + u_0$, where u_0, u_1 are constants. Now there are two cases to consider. If $u_1 = 0$, then the geodesics are as in flat space (for any k in (1.2))

$$u = u_0 \ , \qquad x^i = x_1^i s + x_0^i \ , \qquad v = v_1 s + v_0 \ , \tag{C.1}$$

where x_1^i, x_0^i, v_1 and v_0 are constants. These geodesics are spacelike or null depending on whether $x_1^i \neq 0$ or $x_1^i = 0$, respectively.

In the case where $u_1 \neq 0$, we find that

$$\begin{aligned} u(s) &= u_1 s + u_0 \ , \\ x^i(s) &= x_0^i u^\nu(s) + x_1^i u^{1-\nu}(s) \ , \qquad \frac{1}{2} < \nu < 1 \quad (0 < k < \frac{1}{4}) \ , \\ v &= \frac{\epsilon}{u_1}s - \frac{1}{2}[\nu x_0^2 u^{2\nu-1}(s) + (\nu-1)x_1^2 u^{1-2\nu}(s)] + v_0 \ , \end{aligned} \tag{C.2}$$

where x_0^i, x_1^i and v_0 are constants. For $\epsilon < 0$ the geodesics are spacelike, for $\epsilon > 0$ the geodesics are timelike and for $\epsilon = 0$ the geodesics are null. Since for the case when $u_1 = 0$ the geodesics are either spacelike or null, all *time-like* geodesics are of the type (C.2).

Observe that for $u \to 0$, $x^i \to 0$. Thus the geodesics, and, in particular, the time-like ones with both x_0 and x_1 non-vanishing, focus at the line $u = 0$, $x^i = 0$. As stressed in [48], the geodesics for which $x_1 = x_0 = 0$, i.e.

$$u = u_1 s + u_0 \ , \qquad v = \frac{\epsilon}{u_1} s + v_0 \ , \qquad x^i = 0 \ , \tag{C.3}$$

go smoothly through $u = 0$ and are defined for any value of the affine parameter $-\infty < s < +\infty$.

In the special case where $k = \frac{1}{4}$ and $u_1 \neq 0$, we find

$$\begin{aligned} u &= u_1 s + u_0 \ , \\ x^i &= x_0^i u^{\frac{1}{2}} \log u + x_1^i u^{\frac{1}{2}} \ , \qquad \nu = \frac{1}{2} \ \ (k = \frac{1}{4}) \ , \\ v &= \frac{\epsilon}{u_1^2} s - \frac{1}{8} x_0 \log u (x_0 \log u + 2x_1 + x_0) + v_0 \ . \end{aligned} \tag{C.4}$$

These geodesics are spacelike if $\epsilon > 0$, null if $\epsilon = 0$ or timelike if $\epsilon < 0$. As in the previous case, the geodesics, and in particular the time-like ones with x_1 and x_0 non-vanishing, focus at the line $u = 0$, $x^i = 0$ as $u \to 0$. If $u_1 = 0$ we get again (C.1). The geodesics with $x_1 = x_0 = 0$, i.e. $x^i = 0$, are the same as in (C.3), i.e. can be extended through $u = 0$.

One can also find the geodesics in Rosen (3.8) and conformal (3.17) coordinates where we have translational symmetry in x^i directions. The geodesics of the metric (3.8) $ds^2 = 2du d\mathrm{v} + a^2(u) d\mathrm{x}_i^2$ are given by

$$u = u_1 s + u_0 \ , \quad \mathrm{x}^i = \mathrm{x}_1^i \int^s ds' a^{-2}(s') + \mathrm{x}_0^i \ , \quad \mathrm{v} = \frac{\epsilon}{2u_1} s - \frac{\mathrm{x}_1^2}{2u_1} \int^s ds' a^{-2}(s') + \mathrm{v}_0 \tag{C.5}$$

for $u_1 \neq 0$. We shall consider the case of $a(u) = u^{1-\nu}$ and $0 < k < \frac{1}{4}$, i.e. $\frac{1}{2} < \nu < 1$. Then

$$u = u_1 s + u_0 \ , \quad \mathrm{x}^i = \frac{1}{u_1(2\nu - 1)} \mathrm{x}_1^i u^{2\nu-1} + \mathrm{x}_0^i \ , \quad \mathrm{v} = \frac{\epsilon}{2u_1} s - \frac{\mathrm{x}_1^2}{(4\nu - 2)u_1^2} u^{2\nu-1} + \mathrm{v}_0 \ . \tag{C.6}$$

The geodesics for $\epsilon < 0$ are time-like, for $\epsilon = 0$ are null and for $\epsilon > 0$ are space-like. As $u \to 0$ these geodesics can end at any x_0^i, unlike what happened in the Brinkmann coordinate case discussed above. The geodesics for which $\mathrm{x}_1 \neq 0$ can not be extended through $u = 0$. However, if $\mathrm{x}_1 = 0$, i.e. $\mathrm{x}^i = \text{const}$, then the geodesics are

$$u = u_1 s + u_0 \ , \qquad \mathrm{v} = \frac{\epsilon}{2u_1} s + \mathrm{v}_0 \ , \qquad \mathrm{x}^i = \mathrm{x}_0^i \ , \tag{C.7}$$

and can be extended smoothly through $u = 0$ as in the Brinkmann coordinate case above. For $u_1 = 0$, i.e. $u = \text{const}$, we get

$$u = u_0 \ , \qquad \mathrm{x}^i = \mathrm{x}_1^i s + \mathrm{x}_0^i \ , \qquad \mathrm{v} = -\frac{1}{2}(1-\nu) u_0^{1-2\nu} \mathrm{x}_1^2 s^2 + \mathrm{v}_1 s + \mathrm{v}_0 \ . \tag{C.8}$$

These are the geodesics parallel to the plane wave. These geodesics are null if $x_1 = 0$, otherwise they are space-like.

The geodesics of the metric in conformal coordinates (3.17) $ds^2 = \Sigma(\mathrm{w})(2du d\mathrm{w} + d\mathrm{x}_i^2)$ are directly related to the above by redefining the coordinate $u \to \mathrm{w}$. Explicitly,

$$\int^{\mathrm{w}(s)} d\mathrm{w}'\Sigma(\mathrm{w}') = \mathrm{w}_1 s + \mathrm{w}_0 \ , \qquad \mathrm{x}^i = \frac{\mathrm{x}_1^i}{\mathrm{w}_1}\mathrm{w}(s) + \mathrm{x}_0^i \ , \qquad \mathrm{v} = \frac{\epsilon}{2\mathrm{w}_1}s - \frac{x_1^2}{2\mathrm{w}_1^2}\mathrm{w}(s) + \mathrm{v}_0 \ , \quad \text{(C.9)}$$

for $\mathrm{w}_1 \neq 0$. In particular, for $\Sigma = (\frac{\mathrm{w}}{2\nu-1})^{\frac{2-2\nu}{2\nu-1}}$ (see (3.19)), we find

$$\mathrm{w} = \frac{1}{(2\nu-1)^{2\nu-2}}(\mathrm{w}_1 s + \mathrm{w}_0)^{2\nu-1} \ , \qquad \mathrm{x}^i = \frac{\mathrm{x}_1^i}{(2\nu-1)^{1-2\nu}}(\mathrm{w}_1 s + \mathrm{w}_0)^{2\nu-1} + \mathrm{x}_0^i \ , \quad \text{(C.10)}$$

$$\mathrm{v} = \frac{\epsilon}{2\mathrm{w}_1}s - \frac{\mathrm{x}_1^2}{2(2\nu-1)^{1-2\nu}\mathrm{w}_1^2}(\mathrm{w}_1 s + \mathrm{w}_0)^{2\nu-1} + \mathrm{v}_0 \ . \quad \text{(C.11)}$$

The geodesics for $\epsilon < 0$ are time-like, for $\epsilon = 0$ are null and for $\epsilon > 0$ are space-like. For $z_1 = 0$, we find

$$\mathrm{w} = \mathrm{w}_0 \ , \qquad \mathrm{x}^i = \mathrm{x}_1^i s + \mathrm{x}_0^i \ , \qquad \mathrm{v} = -\frac{1-\nu}{\mathrm{w}_0(4\nu-2)}s^2 + \mathrm{v}_1 s + \mathrm{v}_0 \ . \quad \text{(C.12)}$$

These are the geodesics parallel to the plane wave and they are null if $\mathrm{x}_1 = 0$; otherwise they are spacelike.

It appears that there are no geodesics that can go through the singularity in conformal coordinates. This is due to the fact that the coordinate transformation from Rosen to conformal coordinates is not C^1-differentiable at $u = 0$ and so the tangent vector of the geodesics at $u = 0$ is not defined. However, we know that for the geodesics (C.7) one can extend the affine parameter so that to reach the $u < 0$ values. Transformed from Rosen to conformal coordinates they are given by

$$\mathrm{w} = \frac{1}{2\nu-1}(u_1 s + u_0)^{2\nu-1} \ , \qquad \mathrm{v} = \frac{\epsilon}{2u_1}s + \mathrm{v}_0 \ , \qquad \mathrm{x}^i = \mathrm{x}_0^i \ . \quad \text{(C.13)}$$

Appendix D Penrose diagrams

Here we shall describe the Penrose diagram of homogeneous plane wave (1.2) in more detail. The equations for the conformal boundary and the singularity are (3.23) and (3.24), i.e. $\cos\varphi + \cos\psi = 0$ and $\sin\varphi + \sin\psi\cos\theta = 0$, where $0 < \psi, \theta < \pi$. As we have mentioned in section 3, the Penrose diagram of homogeneous plane wave spacetime is three dimensional but it is better described by two dimensional diagrams for the coordinates (ψ, φ) parameterized by the angle θ. In this description the only special cases arise when $\cos\theta = \pm 1$. If $\cos\theta \neq \pm 1$ in the (ψ, φ) coordinate system, the singularity is a curve that joins $(0, 0)$ with $(\pi, 0)$. For example, for $\theta = \frac{\pi}{2}$, it is the line joining $(0, 0)$ and $(\pi, 0)$. For $\cos\theta < 0$, the Penrose diagram is given in Fig. 1 and for $\cos\theta > 0$ the Penrose diagram is given in Fig. 2. In fact, solutions of the equation for the singularity equation join the points $(0, 2n\pi)$ with $(\pi, 2n\pi)$; here we describe the case $n = 0$. These curves are generically spacelike.

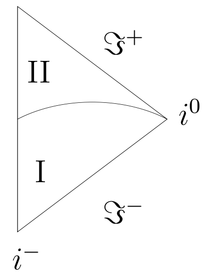

Figure 1:

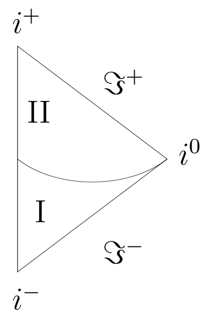

Figure 2:

In the special case $\cos\theta = -1$, i.e. $\psi = \varphi$, the singularity becomes the null line $\varphi = \psi$ (Fig. 3). In this case again the Penrose diagram is separated in two regions (II) and (I) by the singularity.

In the other special case $\cos\theta = 1$, i.e. $\theta = 0$, the singularity is a null line $\varphi = -\psi$ (Fig. 4). The Penrose diagram is separated in two regions (II) and (I) by the singularity.

In all the above cases, the conformal boundary is that of Minkowski spacetime, i.e. it is represented by the null-lines $\Im^+$ and $\Im^-$ joining the points $(0, \pi)$ with $(\pi, 0)$ and $(\pi, 0)$ with $(0, -\pi)$, respectively. For a general angle θ (Figs. 1,2), a generic point in the Penrose diagrams is a S^{d-1} sphere. In the two special cases (Figs. 3,4), the Penrose diagrams are two-dimensional. In all these Penrose diagrams, there are special points

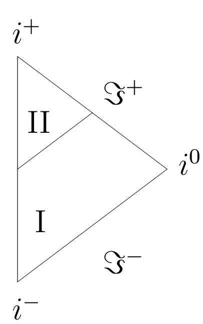

Figure 3:

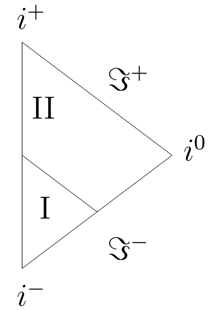

Figure 4:

$i^+ = (0, \pi)$, $i^- = (0, -\pi)$ and $i^0 = (\pi, 0)$, representing the time-like future infinity, the time-like past infinity and the spatial infinity, respectively. All the above diagrams are periodic in the angle φ with period 2π. For a general angle θ, the singularity intersects the conformal boundary at the point i^0. In the special case for which the singularity is $\psi = \varphi$, the conformal boundary intersects the singularity at $(\frac{\pi}{2}, \frac{\pi}{2})$. In the other special case the intersection of the singularity $\varphi = -\psi$ with the conformal boundary is at $(\frac{\pi}{2}, -\frac{\pi}{2})$.

In the present plane wave spacetime, some time-like geodesics begin at past infinity i^- and end at the singularity. Similarly, some time-like geodesics originating from the singularity end at future infinity i^+. The rest of time-like geodesics which go through the singularity will begin at i^- and end at i^+ but they will not be C^1 differentiable at the singularity in conformal coordinates. Some null geodesics, those parallel to the plane wave, can reach $\Im^+$ and $\Im^-$ without passing through the singularity. Only null geodesics end at either $\Im^-$ or $\Im^+$.

In the special case in which the singularity is $\psi = \varphi$, there are null geodesics in region (I), those parallel to the plane wave, which can begin at $\Im^-$ and end in part of $\Im^+$ without passing though the singularity. Similarly, in the other special case in which the singularity is $\theta = -\pi$ in region (II) there are null geodesics which begin in part of $\Im^-$ and end in $\Im^+$ without passing through the singularity.

Incidentally, the equation for the conformal boundary in the conformal compactification to the Einstein static universe for which $q_1 q_2 = 1$ is

$$\cos\varphi + \cos\psi = 2(2\nu - 1)(\sin\varphi \ + \ \sin\psi \ \cos\theta) \ . \tag{D.1}$$

The equation for the singularity is the same as above.

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