

Exact Chiral Symmetry on the Lattice: QCD Applications*

Leonardo Giusti ^aCERN]CERN, Theory Division, CH-1211 Geneva 23, Switzerland,^bMARSEILLE] Centre de Physique Théorique, CNRS Luminy, Case 907, F-13288 Marseille, France

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I review recent progress and results in lattice QCD obtained using fermions with exact chiral symmetry.

1. Introduction

In the last few years it has been understood that exact chiral symmetry can be realized at finite lattice spacing. A breakthrough started with the observation that massive Dirac fermions in $4 + 1$ dimensions reduce to chiral fermions in 4 dimensions under certain conditions [1–3]. The resulting effective operator D of light boundary fields [4,5] has the correct continuum limit, no doublers, and is local [6]. Most remarkably, it satisfies the Ginsparg–Wilson (GW) relation [7]

$$\gamma_5 D + D \gamma_5 = \bar{a} D \gamma_5 D, \quad (1)$$

which guarantees an exact chiral symmetry [8]

$$\delta q = \gamma_5 (1 - \bar{a} D) q, \quad \delta \bar{q} = \bar{q} \gamma_5 \quad (2)$$

of the fermion action at non-zero lattice spacing.

Within the perfect-action approach [9], a fermion operator that satisfies the GW relation can be defined [10], but no explicit construction has been found so far (see [11] for a recent review).

In QCD a chiral symmetric regularization entails many theoretical advantages: in the infrared it allows one to simulate massless quarks and it provides a natural definition of the topological charge; in the ultraviolet it simplifies the subtraction of divergences in composite operators.

Last year an important step forward was the demonstration that Neuberger’s fermions can be used for large-scale QCD computations, at least

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in the quenched approximation. As a result, very light quarks can be handled and legendary problems such as the $\Delta I = 1/2$ rule in $K \rightarrow \pi\pi$ decays are greatly simplified and can be attacked.

In this talk some of the phenomenological applications of Ginsparg–Wilson fermions are reviewed. Most of the numerical studies reported are exploratory. Please refer to the parallel session contributions for interesting theoretical developments which are not covered in the following.

2. Domain-wall-overlap fermions

The five-dimensional domain-wall Dirac operator can be defined as [3,12]

$$D = \frac{1}{2} \left[\gamma_5 (\partial_s^* + \partial_s) - a_s \partial_s^* \partial_s \right] + X, \quad (3)$$

where s labels the sites and a_s the lattice spacing in the fifth dimension. The operators ∂_s^* and ∂_s are the forward and backward derivatives,

$$X = D_W - \frac{1}{\bar{a}}, \quad \bar{a} = \frac{a}{\rho}, \quad (4)$$

with $0 < \rho < 2$. The massless four-dimensional Wilson operator is

$$D_W = \frac{1}{2} \left[\gamma_\mu (\nabla_\mu + \nabla_\mu^*) - a \nabla_\mu^* \nabla_\mu \right], \quad (5)$$

where ∇_μ and ∇_μ^* are the gauge-covariant forward and backward derivatives

$$\begin{aligned} \nabla_\mu q(x) &= \frac{1}{a} \left[U_\mu(x) q(x + a\hat{\mu}) - q(x) \right] \\ \nabla_\mu^* q(x) &= \frac{1}{a} \left[q(x) - U_\mu^\dagger(x - a\hat{\mu}) q(x - a\hat{\mu}) \right] \end{aligned} \quad (6)$$

and a is the lattice spacing in the physical four dimensions. The operator is supplemented with the boundary conditions

$$P_+q(0, x) = P_-q(a_s N_s + a_s, x) = 0 \quad (7)$$

where $P_\pm = \frac{1}{2}(1 \pm \gamma_5)$ and N_s is the extension in the fifth dimension. From a four-dimensional point of view, the system corresponds to QCD with many flavours mixed in a peculiar way.

By integrating out the heavy flavours, a four-dimensional effective action of light boundary fields is left [4,5]

$$\bar{a}D_{N_s} = 1 + \gamma_5 \frac{(1 + \tilde{H})^{N_s} - (1 - \tilde{H})^{N_s}}{(1 + \tilde{H})^{N_s} + (1 - \tilde{H})^{N_s}}, \quad (8)$$

where

$$\tilde{X} \equiv \frac{a_s X}{2 + a_s X}, \quad \tilde{H} \equiv \gamma_5 \tilde{X}. \quad (9)$$

For $N_s \rightarrow \infty$, a massless effective action

$$\bar{a}D_{DW} = \left(1 + \tilde{X} \frac{1}{\sqrt{\tilde{X}^\dagger \tilde{X}}}\right) \quad (10)$$

is derived and the Neuberger operator [5]

$$\bar{a}D_N = \left(1 + X \frac{1}{\sqrt{X^\dagger X}}\right) \quad (11)$$

is obtained if the limit $a_s \rightarrow 0$ is also taken. Most remarkably, the effective massless Dirac operators in Eqs. (10) and (11) satisfy the GW relation [13].

3. Exact chiral symmetry at finite a

For a given operator D that satisfies the GW relation, the QCD fermion action with N_f flavours can be written as

$$\frac{S_F}{a^4} = \sum_x \bar{\psi}(x) \left[(D + P_- \mathcal{M}^\dagger \hat{P}_- + P_+ \mathcal{M} \hat{P}_+) \psi \right] (x)$$

where

$$\hat{P}_\pm = \frac{1}{2}(1 \pm \hat{\gamma}_5), \quad \hat{\gamma}_5 = \gamma_5(1 - \bar{a}D), \quad (12)$$

$\mathcal{M} = \text{diag}(m_1, \dots, m_{N_f})$, $\bar{\psi} = (\bar{q}^1, \dots, \bar{q}^{N_f})$ and ψ is defined analogously. It is invariant under the $U(N_f)_L \times U(N_f)_R$ global transformations

$$\begin{aligned} \psi_L &\rightarrow V_L \psi_L & \bar{\psi}_L &\rightarrow \bar{\psi}_L V_L^\dagger \\ \psi_R &\rightarrow V_R \psi_R & \bar{\psi}_R &\rightarrow \bar{\psi}_R V_R^\dagger, \end{aligned} \quad (13)$$

where $V_{L,R} \in U(N_f)_{L,R}$ and

$$\psi_{R,L} = \hat{P}_\pm \psi \quad \bar{\psi}_{R,L} = \bar{\psi} P_\mp, \quad (14)$$

if also $\mathcal{M} \rightarrow V_L \mathcal{M} V_R^\dagger$. No additive quark mass renormalization is required. The action is $O(a)$ -improved, since no chiral invariant operators of dimension $d = 5$ can be constructed.

The global chiral anomaly is recovered à la Fujikawa [14,8]. The fermion integration measure is not invariant under $U(1)_A$ transformations, and the topological charge density from the corresponding Jacobian

$$a^4 Q(x) = \frac{\bar{a}}{2a} \text{Tr} \left[\gamma_5 D(x, x) \right] \quad (15)$$

satisfies [5,15]

$$n_- - n_+ = \text{index}(D) = a^4 \sum_x Q(x), \quad (16)$$

with n_\pm the number of right and left zero modes of the fermion operator.

Bilinear fermion operators with proper chiral transformations

$$\mathcal{O}_\Gamma^{\alpha\beta}(x) = \bar{q}^\alpha(x) \Gamma \tilde{q}^\beta(x), \quad \tilde{q}^\beta = \left(1 - \frac{\bar{a}}{2} D\right) q^\beta \quad (17)$$

are $O(a)$ -improved, but they do not transform in a simple way under CP [16,17]. However in correlation functions of local operators at non-zero physical distance, it holds

$$\mathcal{O}_\Gamma^{\alpha\beta}(x) = \frac{1}{(1 - \frac{\bar{a}}{2} m_\beta)} \bar{q}^\alpha(x) \Gamma q^\beta(x) \quad (18)$$

and a simple CP transformation $[\tilde{x} = (x_0, -\vec{x})]$

$$\mathcal{O}_\Gamma^{\alpha\beta}(x) \xrightarrow{\text{CP}} \frac{1 - \frac{\bar{a}}{2} m_\alpha}{1 - \frac{\bar{a}}{2} m_\beta} \mathcal{O}_\Gamma^{\beta\alpha}(\tilde{x}) \quad (19)$$

is recovered [16]. The generalization to four-fermion operators is straightforward.

Non-singlet local rotations lead to exact vector and axial Ward identities (WI), and the very same definition of the bare quark mass appears in the axial and vector WIs and in the quark propagator.

The conserved currents \mathcal{V}_μ^a and \mathcal{A}_μ^a can be constructed by extending the gauge group $SU(N_c) \rightarrow SU(N_c) \times U(1)$ [7,18]. By performing a local $U(1)$ flavor rotation

$$U_\mu(x) \rightarrow U_\mu^{(\alpha)}(x) = e^{i\alpha_\mu(x)} U_\mu(x), \quad (20)$$

the kernel

$$K_\mu = -i \frac{\delta D(U_\mu^{(\alpha)})}{\delta \alpha_\mu(x)} \Big|_{\alpha=0} \quad (21)$$

can be used to define the following conserved currents [18]

$$\mathcal{V}_\mu^\alpha = \bar{q} \left(P_- K_\mu \hat{P}_+ + P_+ K_\mu \hat{P}_- \right) T^a q \quad (22)$$

$$\mathcal{A}_\mu^\alpha = \bar{q} \left(P_- K_\mu \hat{P}_+ - P_+ K_\mu \hat{P}_- \right) T^a q \quad (23)$$

where T^a is a generator of the non-singlet transformations. It is interesting to note that (conserved) current-current correlators and their generalizations require the propagator from any point to any point [16].

In the chiral limit, the local anomalous flavour-singlet WIs read

$$\langle \partial_\mu^* \mathcal{A}_\mu^0(x) \hat{\mathcal{O}} \rangle = 2N_f \langle Q(x) \hat{\mathcal{O}} \rangle + \langle \delta_A^x \hat{\mathcal{O}} \rangle, \quad (24)$$

where $\delta_A^x \hat{\mathcal{O}}$ is the local variation of any finite (multi)local operator $\hat{\mathcal{O}}$, and $\mathcal{A}_\mu^0(x)$ is the singlet axial current. By assuming the absence of a $U_A(1)$ massless Goldstone boson, the corresponding integrated WIs read

$$0 = 2N_f a^4 \sum_x \langle Q(x) \hat{\mathcal{O}} \rangle + \langle \delta_A \hat{\mathcal{O}} \rangle. \quad (25)$$

Since the second term in the r.h.s. of Eq. (25) is finite, it follows that $a^4 \sum_x Q(x)$ is also finite, as it has finite insertions with any string of renormalized fundamental fields. Therefore $Q(x)$ can only mix with operators of dimension ≤ 4 and vanishing integral, hence only with $\partial_\mu^* \mathcal{A}_\mu^0(x)$. *No power-divergent subtractions* with lower dimensional operators (such as the pseudo-scalar quark density) have to be performed [19]. This is a very distinctive feature of GW fermions. Calling Z the mixing coefficient, one can define finite operators \hat{Q} and $\hat{\mathcal{A}}_\mu^0$ by writing

$$\hat{Q}(x) = Q(x) - \frac{Z}{2N_f} \partial_\mu^* \mathcal{A}_\mu^0(x) \quad (26)$$

$$\hat{\mathcal{A}}_\mu^0(x) = (1 - Z) \mathcal{A}_\mu^0(x), \quad (27)$$

and the renormalized singlet axial WIs are

$$\langle \partial_\mu^* \hat{\mathcal{A}}_\mu^0(x) \hat{\mathcal{O}} \rangle = 2N_f \langle \hat{Q}(x) \hat{\mathcal{O}} \rangle + \langle \delta_A^x \hat{\mathcal{O}} \rangle. \quad (28)$$

Renormalization constants of several composite operators have been studied at one loop in perturbation theory for domain-wall fermions [20–25] and overlap fermions [26–32]. Non-perturbative determinations have been obtained for the non-singlet local vector and axial currents [33–35], for the scalar and pseudoscalar densities [33,34,36], and for some four-fermion operators [37,38].

4. Meson spectroscopy

In the past year several collaborations have computed the meson spectrum and light quark masses using GW fermions, either with the overlap formulation [34–36,39] or with the perfect actions [40,41]. Simulated lattices have linear extensions $L = 1\text{--}3$ fm and lattice spacings $a = 0.08\text{--}0.2$ fm. Results for the pion mass squared in units of $1/r_0^2$ ($r_0 = 0.5$ fm) as a function of the quark mass normalized at the reference point $M_P^2 = 2M_K^2$ ($M_K = 495$ MeV), are shown in the first plot of Fig. 1. Data in the range $500 \lesssim M_P \lesssim 800$ MeV show a linear behaviour with a vanishing intercept within the statistical errors. The linearity manifests itself as a wide plateau in the second plot of Fig. 1 and it is in very good agreement with what was previously observed with Wilson-type fermions in the same range of masses (see for example [42–44]). For degenerate quarks, quenched chiral perturbation theory at the next-to-leading order (NLO) predicts [45,46]

$$\frac{M_P^2}{2m} = \frac{\Sigma}{F^2} \left[1 - \delta \left(1 + \log \left(\frac{M^2}{\mu_\chi^2} \right) \right) + \frac{\alpha M^2}{3(4\pi F)^2} \left(1 + 2 \log \left(\frac{M^2}{\mu_\chi^2} \right) \right) + (2\alpha_8 - \alpha_5) \frac{M^2}{(4\pi F)^2} \right] \quad (29)$$

where Σ , F , m_0 , α are the leading-order (LO) couplings of the quenched QCD chiral Lagrangian [47,48], α_5, α_8 are some of the NLO ones, $\mu_\chi = 4\pi F$, $M^2 = 2\Sigma m/F^2$ and $\delta = m_0^2/3(4\pi F)^2$. A comparison of Eq. (29) with the data in Fig. 1 indicates small corrections due to quenched chiral logs and/or higher order terms in this range.

Since GW fermions do not suffer from exceptional configurations, lighter pion masses can be simulated, provided the physical volume and the

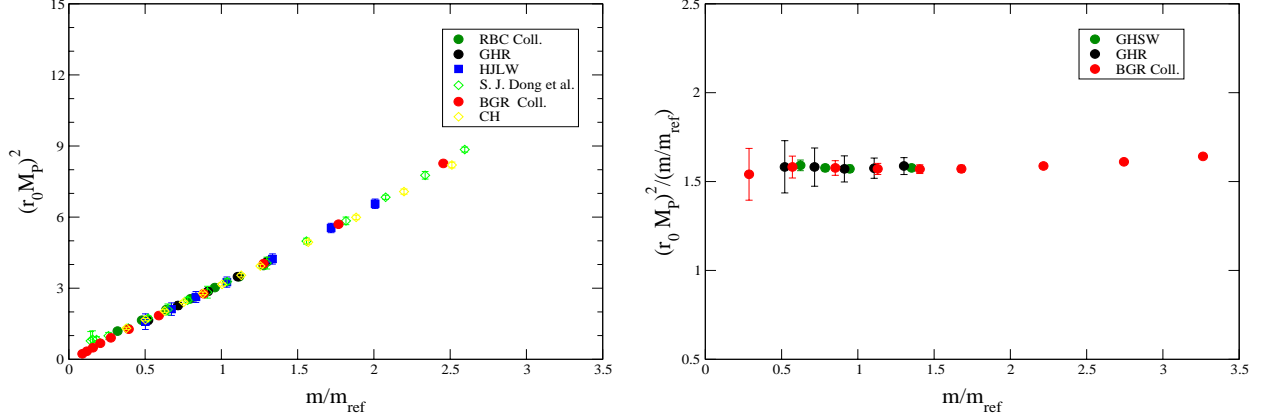


Figure 1. On the left, $(r_0 M_P)^2$ vs the quark mass normalized at the reference point $M_P^2 = 2M_K^2$: green circles [49], black circles [34], blue squares [36], green diamonds [35], red circles [41], yellow diamonds [39]. On the right, $(r_0 M_P)^2 / (m/m_{\text{ref}})$ vs (m/m_{ref}) : green circles [43], black circles [34], red circles [41].

cut-off are large enough. First exploratory studies in the region $200 \lesssim M_P \lesssim 400$ MeV have been reported at this conference [35,40,41].

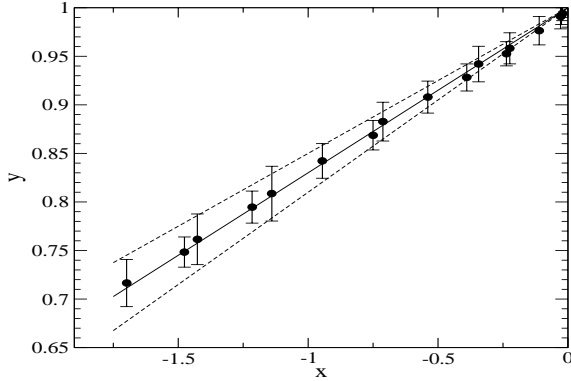


Figure 2. y vs x with FP action from Ref. [41].

The presence of quenched chiral logs can be tested in the double ratio of meson masses with non-degenerate quarks [47,44]

$$y = \frac{4m_1 m_2}{(m_1 + m_2)^2} \frac{M_{P,12}^2}{M_{P,11}^2} \frac{M_{P,12}^2}{M_{P,22}^2}. \quad (30)$$

At NLO

$$y = 1 + \delta x + \frac{\alpha}{3(4\pi F)^2} \frac{2\Sigma}{F^2} w + O(m_1^2, m_2^2), \quad (31)$$

where

$$x = 2 + \frac{m_1 + m_2}{m_1 - m_2} \log\left(\frac{m_2}{m_1}\right) \quad (32)$$

$$w = \left(\frac{2m_1 m_2}{m_2 - m_1} \log\left(\frac{m_2}{m_1}\right) - m_1 - m_2 \right) \quad (33)$$

and μ_χ appears explicitly in Eq. (31) at higher orders only.

Using the fixed-point (FP) and chirally improved (CI) actions, the Bern–Graz–Regensburg (BGR) collaboration computed meson masses with non-degenerate quarks [40,41]. In Fig. 2 data obtained with the FP operator on a lattice with $a \simeq 0.15$ fm and $L \simeq 2.4$ fm are shown. By neglecting $2\alpha\Sigma w/3F^2(4\pi F)^2$ and higher order terms in Eq. (31), they obtain $\delta = 0.17(2)$ and $\delta = 0.18(2)$ for the FP and CI actions respectively [41]. More studies are needed to properly assess the systematics due to finite volume effects and/or finite lattice spacing, and to remove the uncertainties due to the leftover explicit symmetry breaking of these actions. An extensive comparison of these results with those obtained in the past with standard actions can be found in [50].

5. The chiral condensate

The chiral condensate Σ has been computed with Wilson-type fermions by fitting $M_P^2/2m$ with the LO term on the r.h.s. of Eq. (29) [51,52]. The same analysis has been repeated with overlap fermions in Refs. [34,36] and the results are reported in Fig. 3 as well. Within the statistical errors, the results are compatible with the continuum extrapolated value obtained with non-perturbatively improved Wilson fermions [43]. This represents a further indication that $O(a^2)$ effects are moderate with overlap fermions. More studies with lighter quark masses and larger volumes are needed to properly assess the systematics of these encouraging results.

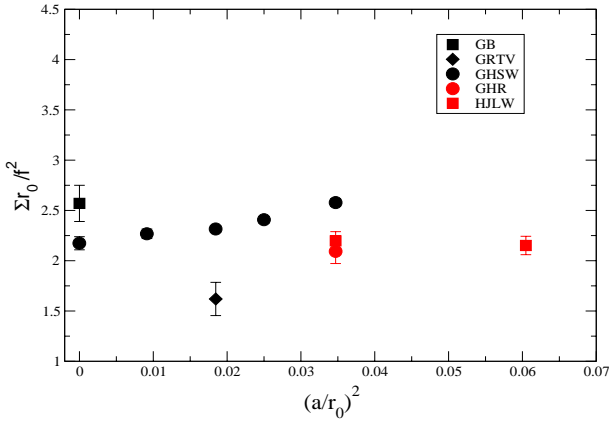


Figure 3. Chiral condensate vs a : black squares [51], black diamonds [52], black circles [43], red circles [34], red squares [36].

Properties of QCD Green's functions in a finite box of linear extension L and with very light quark masses can be studied within chiral perturbation theory [54]. If $2m\Sigma L^4 \sim 1$, $L \gg 1/(4\pi F)$ and $p^2 \sim 1/L^2$, the correlation functions can be expanded in powers of a parameter ϵ with

$$\frac{\sqrt{2\Sigma m}}{\Lambda_\chi^2} \sim \frac{p^2}{\Lambda_\chi^2} \sim O(\epsilon^2) \quad (34)$$

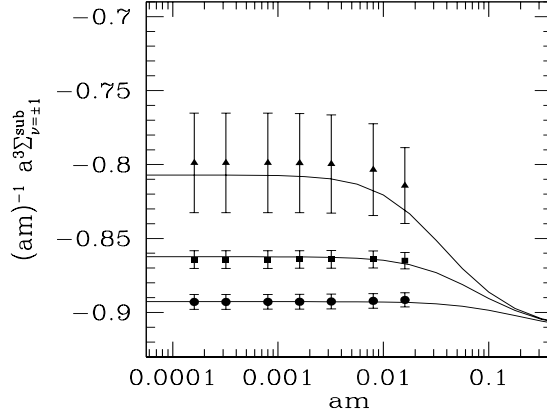


Figure 4. Quark mass dependence of the scalar condensate for three volumes: 8^4 (circles), 10^4 (squares), and 12^4 (triangles), from Ref. [53].

and Λ_χ being the cut-off of the effective theory [54]. At leading order the partition function is given by

$$Z(m, \theta) = \int_{\text{SU}(N_f)} dU_0 d\xi \times \exp \left[\frac{1}{2} \int d^4x \text{Tr} (\partial_\mu \xi \partial_\mu \xi) + z \text{ReTr} (e^{i\theta/N_f} U_0) \right] \quad (35)$$

where the pion field is factorized as $U(x) = U_0 \exp(i\sqrt{2}\xi(x)/F)$ and $z = m\Sigma V$. The integral over the global mode

$$\int_{\text{SU}(N_f)} dU_0 \exp \left[z \text{ReTr} (e^{i\theta/N_f} U_0) \right] \quad (36)$$

needs to be done exactly, while a reordered chiral perturbation theory applies to the non-zero integration mode $\xi(x)$. Partition functions $Z_\nu(m)$ and correlations can be defined in sectors of fixed topology by Fourier transforming in θ [55]. By comparing the chiral perturbation theory expectations for the correlation functions with the lattice data, the basic assumption of spontaneous symmetry breaking can in principle be verified and the low energy constants (LEC) of the chiral Lagrangian extracted. Properties of QCD in the infinite volume limit can then be recovered.

The ϵ -expansion has been extended to quenched QCD in Refs. [56,57]. At the leading order, the chiral condensate in a sector of fixed topology ν reads

$$\frac{\Sigma_\nu(z)}{\Sigma} = z \left(I_\nu(z) K_\nu(z) + I_{\nu+1}(z) K_{\nu-1}(z) \right) + \frac{\nu}{z} \quad (37)$$

where $I_\nu(z)$ and $K_\nu(z)$ are modified Bessel functions [57]. One-loop corrections give [58,59]

$$\Sigma_\nu^{1\text{-loop}}(z) = \frac{z'}{z} \Sigma_\nu(z') \quad (38)$$

where $z' = m \Sigma_{\text{eff}} V$,

$$\begin{aligned} \Sigma_{\text{eff}}(V) &= \Sigma \left[1 - \frac{m_0^2}{3(4\pi F)^2} \left(\tilde{\beta}_2 + \log \left(\frac{L_0^2}{L^2} \right) \right) \right. \\ &\quad \left. - \frac{\alpha}{3(4\pi FL)^2} \tilde{\beta}_1 \right], \end{aligned} \quad (39)$$

$1/L_0$ is the renormalization scale and $\tilde{\beta}_i$ are two universal ‘‘shape coefficients’’ [59]. It is interesting to note that the expansion parameter $\epsilon \sim 1/(4\pi LF)$ in Eq. (38) is comparable to $M_P^2/(4\pi F)^2$ in Eq. (29) for light pions ($M_P \sim 1/L$), while the prefactors turn out to be different.

Quenched QCD in the ϵ -regime has been explored on the lattice in Refs. [60,53]. In Fig. 4 an example of the results obtained in Ref. [53] with overlap fermions for the vacuum expectation value of the scalar density in the topological sector $\nu = \pm 1$ at different volumes and various masses is shown. Even with poor statistics and small volumes, a signal compatible with the expectations of chiral perturbation theory has been reported. Analogous analyses have been attempted in Ref. [61,40]. More studies at larger volumes and higher statistics are needed to confirm the indications of these exploratory investigations.

6. Topological susceptibility

In the chiral limit, the Fourier transform of the singlet axial WI in Eq. (28) for $\hat{\mathcal{O}} = \hat{Q}$ reads

$$\begin{aligned} \chi_t(p) &= a^4 \sum_x e^{-ipx} \langle \hat{Q}(x) \hat{Q}(0) \rangle + \text{CT}(p) \quad (40) \\ &= \frac{a^4}{2N_f} \sum_x e^{-ipx} \langle \partial_\mu^* \hat{A}_\mu^0(x) \hat{Q}(0) \rangle + \text{CT}(p) \end{aligned}$$

where the same contact term $\text{CT}(p)$ has been added to both sides of Eq. (40) to make them separately finite. $\text{CT}(p)$ is a fourth-degree polynomial, which can be chosen to vanish at $p = 0$ since in this case the second line of Eq. (40) is certainly finite in the absence of zero-mass particles in the singlet channel. As a consequence, the topological susceptibility vanishes in the chiral limit, i.e. $\chi_t(0) = 0$ [19].

For $M_P \rightarrow 0$, the leading chiral behaviour

$$\chi_t(0) = \frac{F^2 M_P^2}{2N_f} + O(M_P^4) \quad (41)$$

is recovered by comparing the integrated singlet axial WIs with $\hat{\mathcal{O}} = \hat{Q}$ and $\hat{\mathcal{O}} = \hat{P}^0$ [62].

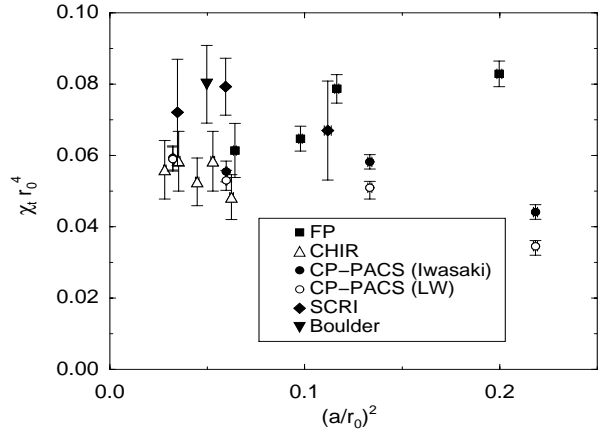


Figure 5. Summary for the quenched topological susceptibility from [40]: filled boxes [40], empty triangles [66], filled circles [67], empty circles [67], filled diamonds [68], filled triangles [69].

In the chiral limit, $\chi_t(p)$ satisfies a three-times-subtracted dispersion relation

$$\begin{aligned} \chi_t(p) &= b_1 + b_2 p^2 + b_3 (p^2)^2 \quad (42) \\ &\quad - \frac{R_{\eta'}^2}{p^2 + M_{\eta'}^2} + (p^2)^3 \int_{\text{cut}} \frac{\rho(t)}{(t + p^2)t^3} dt, \end{aligned}$$

where the contribution of the η' meson has been separated since it is expected to dominate the dispersive integral [63]. For $p^2 \rightarrow 0$, the ‘‘sum rule’’ $\chi_t(0) = 0$ implies

$$b_1 = \frac{F_{\eta'}^2 M_{\eta'}^2}{2N_f}, \quad (43)$$

where $R_{\eta'}^2 = F_{\eta'}^2 M_{\eta'}^4 / 2N_f$. Under the ‘‘smooth-quenching hypothesis’’, the Witten–Veneziano formula is obtained [64,65]

$$\begin{aligned} \frac{F^2 M_{\eta'}^2}{2N_f} \Big|_{\frac{N_f}{N_c}=0} &= a^4 \sum_x \langle Q(x)Q(0) \rangle_{\text{YM}} \\ &= \frac{\langle (n_- - n_+)^2 \rangle}{V}, \end{aligned} \quad (44)$$

where the quantum average has to be done in the pure Yang–Mills (YM) theory [19].

By using Wilson and staggered fermions it was argued that [70,71]

$$\chi_t(0) = \lim_{m \rightarrow 0} \left(\frac{2m}{2N_f} \right)^2 a^4 \sum_x \langle P^0(x)P^0(0) \rangle_{\text{Qven}}^{\text{ZV}} \quad (45)$$

where only Zweig-violating (ZV) diagrams are included. With GW fermions, this formula is the algebraic equivalent of Eq. (44) [62].

The topological susceptibility defined by using Neuberger’s operator has been computed for several lattice spacings and volumes for a pure SU(3) YM theory [68]. Analogous computations have been performed in the past year with overlap fermions [69], FP and CI actions [40,66]. A summary of the results obtained can be found in [40] and is reported in Fig. 5. The central value of χ_t obtained with GW fermions is quite stable as a function of the lattice spacing and is also compatible with the value obtained by other approaches [72,73]. More work is needed at larger volumes and smaller lattice spacings before these encouraging indications are fully confirmed and the magnitude of the systematic error is properly assessed.

Exploratory computations of χ_t for $N_c > 3$ have been performed in the past year [74–76]. The results are compatible with a smooth large- N_c limit and a non-zero χ_t in the limit $N_c \rightarrow \infty$.

7. $K \rightarrow \pi\pi$ decays

Non-leptonic $K \rightarrow \pi\pi$ amplitudes can be parametrized as

$$\begin{aligned} T[K^+ \rightarrow \pi^+\pi^0] &= \sqrt{\frac{3}{2}} A_2 e^{i\delta_2} \\ T[K^0 \rightarrow \pi^+\pi^-] &= \sqrt{\frac{2}{3}} A_0 e^{i\delta_0} + \sqrt{\frac{1}{3}} A_2 e^{i\delta_2} \\ T[K^0 \rightarrow \pi^0\pi^0] &= \sqrt{\frac{2}{3}} A_0 e^{i\delta_0} - 2\sqrt{\frac{1}{3}} A_2 e^{i\delta_2} \end{aligned} \quad (46)$$

where δ_I and A_I are the $\pi\pi$ phase shifts and the isospin amplitudes for $I = 0, 2$. Direct and indirect CP violation are parametrized by

$$\varepsilon' = \frac{1}{\sqrt{2}} e^{i\Phi} \frac{\text{Re}A_2}{\text{Re}A_0} \left(\frac{\text{Im}A_2}{\text{Re}A_2} - \frac{\text{Im}A_0}{\text{Re}A_0} \right) \quad (47)$$

and

$$\varepsilon = \frac{T[K_L \rightarrow (\pi\pi)_0]}{T[K_S \rightarrow (\pi\pi)_0]} \quad (48)$$

respectively. Experimental results reveal $\Phi = \pi/2 + \delta_2 - \delta_0 \approx \pi/4$, a $\Delta I = 1/2$ selection rule $|A_0/A_2| \simeq 22.2$ and the presence of direct and indirect CP violation in nature:

$$\begin{aligned} \text{Re}(\varepsilon'/\varepsilon) &= (16.6 \pm 1.6) \times 10^{-4} \quad [77] \\ |\varepsilon| &= (2.282 \pm 0.017) \times 10^{-3} \quad [78]. \end{aligned} \quad (49)$$

Phenomenological analyses of the unitarity triangle indicate that the Standard Model picture of indirect CP violation in the kaon system is consistent with that of B decays and oscillations [79,80].

The $\Delta I = 1/2$ rule and the value of ε'/ε can be explained within the Standard Model only if the strong interactions crucially affect these non-leptonic weak transitions (see [81–83] for recent reviews). In this case a more complicated blend of ultraviolet and infrared effects prevented reliable determinations of the relevant matrix elements. Power-divergent subtractions can be needed to construct the renormalized operators that enter the effective Hamiltonian [84–86,31]. In the infrared, the continuation of the theory to Euclidean space-time and the use of finite volumes in numerical simulations generate a non-simple relation between the physical amplitudes and those computed on the lattice [87–89].

7.1. The $\Delta I = 1/2$ rule

By using the operator product expansion (OPE), the CP-conserving $\Delta S = 1$ effective Hamiltonian above the charm threshold is given by

$$H_{\text{eff}}^{\Delta S=1} = \frac{G_F}{\sqrt{2}} \left[C_+(\mu) \widehat{\mathcal{O}}_+(\mu) + C_-(\mu) \widehat{\mathcal{O}}_-(\mu) \right],$$

where the Wilson coefficients $C_{\pm}(\mu)$ are known at the NLO [90,91] and the bare operators are

$$\begin{aligned} \mathcal{O}_{\pm} &= \left[(\bar{s}^a \gamma_{\mu} P_- \tilde{u}^b) (\tilde{u}^b \gamma_{\mu} P_- \tilde{d}^a) \right. \\ &\quad \left. \pm (\bar{s} \gamma_{\mu} P_- \tilde{u}) (\tilde{u} \gamma_{\mu} P_- \tilde{d}) \right] - (u \rightarrow c). \end{aligned} \quad (50)$$

The contributions that arise when the top quark is integrated out are heavily suppressed by CKM factors and can be neglected. \mathcal{O}_{\pm} belong to different chiral multiplets and are CPS-even. In correlation functions at non-zero physical distance, \mathcal{O}_{\pm} cannot mix between themselves or with other four-fermion operators, but only with the dimension-six operator [31,16]

$$\mathcal{Q}_m = (m_u^2 - m_c^2) \left[m_d (\bar{s} P_+ \tilde{d}) + m_s (\bar{s} P_- \tilde{d}) \right]. \quad (51)$$

The renormalized operators are

$$\widehat{\mathcal{O}}_{\pm}(\mu) = Z_{\pm}(\mu) \left[\mathcal{O}_{\pm} + b_{\pm}^m \mathcal{Q}_m \right], \quad (52)$$

where $Z_{\pm}(\mu)$ are logarithmic-divergent renormalization constants and b_{\pm}^m are suppressed by a factor α_s . *No power-divergent subtractions are needed to renormalize \mathcal{O}_{\pm} when fermions with an exact chiral symmetry are used [31].*

For $m_s \neq m_d$,

$$\begin{aligned} \mathcal{Q}_m &= (m_u^2 - m_c^2) \partial_{\mu}^* \left[\frac{m_d + m_s}{m_s - m_d} \gamma_{\mu}^{sd} \right. \\ &\quad \left. + \frac{m_d - m_s}{m_s + m_d} A_{\mu}^{sd} \right] \end{aligned} \quad (53)$$

and it does not contribute to matrix elements which preserve four-momentum [84,85].

If the charm is integrated out not only potentially large contributions of $O(\mu^2/m_c^2)$ are neglected, but ultraviolet power divergences can arise in the renormalization pattern of the relevant four-fermion operators. In this case the $\Delta S = 1$ effective Hamiltonian can be written as

$$H_{\text{eff}}^{\Delta S=1} = \frac{G_F}{\sqrt{2}} \sum_{i=1}^{10} C_i(\mu) \widehat{\mathcal{Q}}_i(\mu). \quad (54)$$

The so-called QCD-penguin operators are

$$\mathcal{Q}_{3,5} = (\bar{s} \gamma_{\mu} P_- \tilde{d}) \sum_{q=u,d,s} (\bar{q} \gamma_{\mu} P_{\mp} \tilde{q}) \quad (55)$$

$$\mathcal{Q}_{4,6} = (\bar{s}^a \gamma_{\mu} P_- \tilde{d}^b) \sum_{q=u,d,s} (\bar{q}^b \gamma_{\mu} P_{\mp} \tilde{q}^a) \quad (56)$$

(see Refs. [81,82] for definitions of the other operators). At non-zero physical distance, mixing with two lower-dimensional operators

$$\mathcal{Q}_p = m_d (\bar{s} P_+ \tilde{d}) + m_s (\bar{s} P_- \tilde{d}) \quad (57)$$

$$\mathcal{Q}_{\sigma} = m_d (\bar{s} F_{\mu\nu} \sigma_{\mu\nu} P_+ \tilde{d}) + m_s (\bar{s} F_{\mu\nu} \sigma_{\mu\nu} P_- \tilde{d})$$

can occur and power-divergent subtractions are needed even with Ginsparg–Wilson fermions.

With Wilson fermions, only CPS and flavour symmetry can be used to determine the renormalization pattern of \mathcal{O}^{\pm} . Even with an active charm, a quadratic divergent contribution needs to be subtracted in the parity-conserving sector

$$\begin{aligned} \widehat{\mathcal{O}}_{\pm}^{\text{PC}}(\mu) &= Z_{\pm}(\mu) \left[\mathcal{O}_{\pm}^{\text{PC}} + \sum_j b_{\pm}^j \mathcal{O}_j^{\pm} + \right. \\ &\quad \left. + b_{\pm}^{\tau} \mathcal{Q}_{\tau} + \frac{b_{\pm}^s}{a^2} \mathcal{Q}_s \right] \end{aligned} \quad (58)$$

where

$$\mathcal{Q}_s = (m_u - m_c) \bar{s} d \quad (59)$$

$$\mathcal{Q}_{\tau} = (m_u - m_c) \bar{s} \sigma_{\mu\nu} F_{\mu\nu} d. \quad (60)$$

and \mathcal{O}_j^{\pm} are four fermion operators with wrong chirality. In this case only the flavour part of the GIM mechanism survives due to the explicit breaking of chiral symmetry [85].

The LECs of the CP-conserving $\Delta S = 1$ electroweak chiral Lagrangian can be extracted from three-point correlation functions, thus avoiding the infrared problem that affects the direct computation of the $K \rightarrow \pi\pi$ matrix elements on the lattice [84]. No large cancellations among leading order terms are expected in the ratio $|A_0/A_2|$; an enhancement should therefore be visible already at this order. As a result, a combined use of fermions with an exact chiral symmetry and chiral perturbation theory can be the starting point to attack the $\Delta I = 1/2$ rule.

In the past years the RBC and the CP-PACS collaborations have studied the $\Delta I = 1/2$ rule

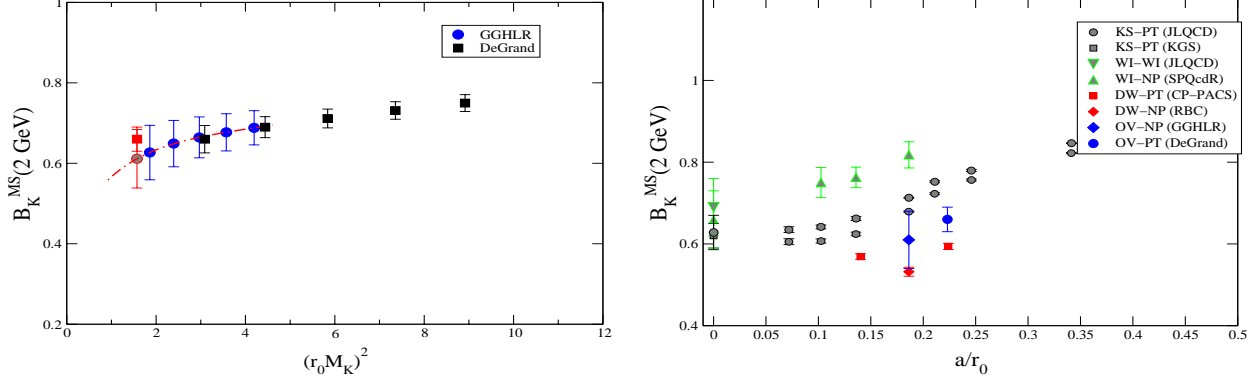


Figure 6. On the left $B_K^{\overline{\text{MS}}}(2 \text{ GeV})$ vs $(r_0 M_K)^2$ for plain (circles) [38] and NNC-HYP (squares) [96] overlap. On the right, $B_K^{\overline{\text{MS}}}(2 \text{ GeV})$ vs the lattice spacing from various actions: black circles [97], black squares [98], green triangles down [99], green triangles up [100], red squares [101], red diamonds [37], blue diamonds [38], blue circles [96].

with domain-wall fermions with a finite fifth dimension [37,92]. They have computed the $K \rightarrow \pi$ and $K \rightarrow 0$ matrix elements for the operators of the $\Delta S = 1$ effective Hamiltonian in Eq. (54) and used LO chiral perturbation theory to recover the physical amplitudes. Although with large statistical and systematic errors, both groups demonstrated that a controlled numerical signal can be obtained for these matrix elements. The systematics of these important results can be reduced by using fermions with an exact chiral symmetry, lighter quark masses and by considering the effective Hamiltonian with a dynamical charm.

A different avenue is being followed in Ref. [16]. It is conceivable that the LECs of the weak chiral Lagrangian can be extracted by studying the weak interactions in the ϵ -regime [16]. A numerical feasibility study of this approach on the lattice is under way. If feasible, it will be very interesting to compare the results of the LECs in the two regimes.

For direct CP violation, both the ultraviolet and the infrared problems are more severe. An active charm does not mitigate the ultraviolet renormalization, and divergent power subtractions are necessary. The cancellation between two large competing contributions from \mathcal{Q}_6 and \mathcal{Q}_8

renders ϵ'/ϵ very sensitive to higher order corrections in chiral perturbation theory. Leading-order terms may not be sufficient to reach a reliable prediction in the Standard Model [93,94].

7.2. $K^0 - \bar{K}^0$ mixing: ϵ

By using the OPE, the $\Delta S = 2$ effective Hamiltonian is given by

$$H_{\text{eff}}^{\Delta S=2} = \frac{G_F^2 M_W^2}{4\pi^2} \mathcal{C}_1(\mu) \hat{\mathcal{O}}_1(\mu) + \text{h.c.}, \quad (61)$$

where the expression of the Wilson coefficient $\mathcal{C}_1(\mu)$ is known at NLO [95] and the corresponding bare four-fermion operator

$$\mathcal{O}_1 = (\bar{s}\gamma_\mu P_- \tilde{d})(\bar{s}\gamma_\mu P_- \tilde{d}) \quad (62)$$

is multiplicatively renormalizable. The matrix element that encodes the long-distance QCD contributions to ϵ

$$\langle \bar{K}^0 | \hat{\mathcal{O}}_1(\mu) | K^0 \rangle \equiv \frac{4}{3} F_K^2 M_K^2 \hat{B}_K(\mu) \quad (63)$$

has been computed with overlap fermions in the last year [38,96]. A plain overlap action has been used in Ref. [38] for a lattice of linear extension $L \simeq 1.5 \text{ fm}$, with a spacing $a \simeq 0.093 \text{ fm}$ and for degenerate light quark masses in the

range $m_s/2 \lesssim m \lesssim m_s$. The RI/MOM non-perturbative renormalization procedure has been implemented to compute the logarithmic divergent renormalization constant. NNC-HYP overlap fermions have been used in Ref. [96] for a lattice of linear extension $L \simeq 1.5$ fm, with a spacing $a \simeq 0.125$ fm and for degenerate light quark masses in the range $m_s \lesssim m \lesssim 2.5m_s$. The operator has been renormalized using one-loop perturbation theory. The results of the two computations are in very good agreement in the common range of simulated masses, as shown in Fig. 6.

In Ref. [38] the results have been slightly extrapolated to the physical point by using the functional form

$$\begin{aligned} \widehat{B}_K^{\overline{\text{MS}}}(2 \text{ GeV}) &= B_0 \left(1 - 3 \left(\frac{M_K}{4\pi F} \right)^2 \log \left(\frac{M_K^2}{\Lambda_\chi^2} \right) \right) \\ &+ b \left(\frac{M_K}{4\pi F} \right)^4 \end{aligned} \quad (64)$$

with $F = F_\pi^{\text{phys}}$, while a linear extrapolation has been performed in [96]. Including the statistical errors only, the preliminary results

$$\widehat{B}_K^{\overline{\text{MS}}}(2 \text{ GeV}) = 0.61 \pm 0.07 \quad [38] \quad (65)$$

$$= 0.66 \pm 0.04 \quad [96] \quad (66)$$

are in very good agreement. More studies are needed to properly assess the magnitude of the various systematic errors.

A comparison with other determinations obtained with different regularizations is shown in the second plot of Fig. 6. Even if the statistical errors are large, the agreement with the continuum-limit world averages based on staggered results in Ref. [97] is very good. Results obtained with domain-wall fermions with a finite fifth dimension are below the overlap determinations, but still compatible within errors [37,101].

8. Conclusions

Studying QCD with an exact chiral symmetry at finite lattice spacing implies many theoretical advantages: in the ultraviolet it simplifies the subtraction of divergences in composite operators, in the infrared it allows one to simulate

massless quarks and it provides a natural definition for the topological charge and the topological susceptibility.

Large-scale QCD simulations with fermions with exact chiral symmetry are feasible with known algorithms and the present generation of computers, at least in the quenched approximation. A regime of light quark masses not accessible with standard fermions has already been reached.

First results for the meson spectrum, light quark masses, the chiral condensate and B_K are in good agreement with previous determinations and suggest moderate discretization errors for Neuberger's fermions.

Important long-standing problems such as the $\Delta I = 1/2$ rule in $K \rightarrow \pi\pi$ decays are greatly simplified and can be attacked. A combined use of chiral perturbation theory and fermions with exact chiral symmetry may lead to important new phenomenological informations in the next few years.

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