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Particle production in time-dependent gravitational fields: the expanding mass shell

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Abstract. We compute the production of particles from the gravitational field of an expanding mass shell. Contrary to the situation of Hawking radiation and the production of cosmological perturbations during cosmological inflation, the example of an expanding mass shell has no horizon and no singularity. We apply the method of 'ray-tracing', first introduced by Hawking, and calculate the energy spectrum of the produced particles. The result depends on three parameters: the expansion velocity of the mass shell, its radius, and its mass. Contrary to the situation of a collapsing mass shell, the energy spectrum is non-thermal. Invoking time reversal we reproduce Hawking's thermal spectrum in a certain limit.

1. Introduction

Gradients in a gravitational field may cause the production of particles out of the vacuum; this is similar to the situation in quantum electrodynamics (QED). In the latter case a virtual pair of particles is ripped apart by the electric field. In contrast to QED, the gravitational 'charge'—the mass—is always positive; in this case, the tidal forces are thus responsible for the separation of the particle pair. But there is also another difference. The energy that is needed for the production of particles in a charged vacuum is taken out of the field energy. Since the energy of the gravitational field cannot be localised, the energy gain must be explained otherwise. One possibility is the existence of a horizon. Another possibility is the time dependence of the gravitational field.

We compute the particle production in a time-dependent gravitational field induced by an expanding mass shell. This is done at a semiclassical level. The gravitational field, which acts through curvature of space-time, is treated as a classical background field in which a quantum field propagates. We consider a massless scalar field. So it is quantum field theory in curved space that we need to apply.

Perhaps the best known result in this field is the thermal radiation emitted by a black hole. This effect, discovered by Hawking [1], shows a deep connection between gravity, thermodynamics and quantum physics. Hawking's result was reproduced in several ways, but also other problems such as Unruh's accelerated observer [2] or the gravitational analogue to the Casimir effect [3] were studied.

In cosmological models, the production of particles plays a central role in explaining the origin of the structures observed in the Universe. During an epoch of cosmological inflation, quantum fluctuations of space-time and matter are amplified by the accelerated expansion of the Universe to become the seeds of large-scale structures [4].

All mentioned examples involve either a horizon, a boundary, or a singularity. To our knowledge, there exists no example in which particle production in an infinite space-time without horizon and without singularity has been calculated. The situation of an expanding mass shell represents such an example. We do not expect this work to have a direct astrophysical application, but the situation under study might be of interest when black holes evaporate. One could imagine that Hawking radiation is emitted from an evaporating black hole in a final burst, which we could then describe as a thin shell, carrying a mass somewhat larger than the Planck mass. If the emitted radiation consists of massive particles, this mass shell moves at a speed slightly below the speed of light and our calculation might be applicable. Since the expanding mass shell itself leads to gravitational particle production, this work might be useful for an estimate of backreaction effects in the final stages of black hole evaporation. We will not study this aspect here, the purpose of this paper being just to calculate the energy spectrum of the particles produced by an expanding mass shell.

There are three main procedures that are used to deal with quantum fields in curved space-time: Hawking's classical ray-tracing [1], renormalization of the energy-momentum tensor (e.g. by point-splitting) [5, 6], particle production in the reference frame of a uniformly accelerated observer [2, 7], from which we will choose the first one. A useful overview on computational matters can be found in [8, 9].

The procedure of ray-tracing in a semiclassical limit is appropriate if the energy carried by the quantum field is small with respect to the energy of the source of the gravitational field. Furthermore we have to stay on scales on which possible effects of quantum gravity do not occur. As Wheeler showed in [10] the characteristic length for this to happen is the Planck length $l_{\rm P} = \sqrt{\frac{G\hbar}{c^3}} \approx 10^{-33}$ cm.

So we have to restrict our results to cases in which the effects of interest take place on scales $\gg l_{\rm P}$ and $\gg t_{\rm P}$. However, it seems that there is no reason to restrict the calculation to masses $M > m_{\rm P}$, as long as the mass shell starts to expand at a radius $\gg l_{\rm P}$. We set $G = c = \hbar = 1$ in the following.

This paper is organized as follows. First we will need to find an observable of particle density in curved space, which will be done in Section 2. After this we will introduce our model of the expanding mass shell and discuss its properties. The computation of the number density operator is presented in Section 4. The results are discussed in section 5, which contains plots of the spectral energy flux and total energy flux through a 2-dimensional surface. We conclude with a short summary.

2. Quantum fields in curved space-time

Let us first recall some of the basics of quantum field theory in curved space-time [8, 6, 5]. Physics in flat space-time is invariant under Poincaré transformations. This singles out a set of (Fourier) modes and a vacuum state, which are invariant under Poincaré transformations. In curved space-time there is no global Poincaré symmetry.

In the following we consider a minimally coupled scalar field φ. Its Fourier modes in flat space are attached to the energy ω by

$$\mathrm{i}\partial_t u_k = \omega u_k \quad \text{with} \quad \omega > 0 \; . \tag{1}$$

This uniqueness is lost when the coordiates are chosen arbitrarily. In curved space-time we have no distinguished time coordinate, thus no distinguished energy belonging to it, and no distinguished set of modes that can be related to the notion of particles. In general we have to deal with an arbitrary set of modes (u_i) and ask for its relation to a different set (v_i). The quantum field $\hat{\varphi}$ can then be expanded in both sets:

$$\hat{\varphi} = \sum_i (u_i \hat{a}_i + u_i^* \hat{a}_i{}^\dagger) \; , \quad \hat{\varphi} = \sum_j (v_j \hat{b}_j + v_j^* \hat{b}_j{}^\dagger) \; . \tag{2}$$

The relations between these sets are given by the so-called Bogoliubov transformations [11] and have the form:

$$v_j = \sum_i (\alpha_{ji} u_i + \beta_{ji} u_i^*) \; , \quad u_i = \sum_j (\alpha_{ji}^* v_j - \beta_{ji} v_j^*) \tag{3}$$

with

$$\alpha_{jk} = (v_j, u_k) \; , \quad \beta_{jk} = -(v_j, u_k^*) \; , \tag{4}$$

and the scalar product given by

$$(u_i, v_j) = -\mathrm{i} \int_\Sigma \left[u_i (\partial_\mu v_j^*) - (\partial_\mu u_i) v_j^* \right] \sqrt{-g_\Sigma} \, \mathrm{d}\Sigma^\mu \; . \tag{5}$$

Here Σ is a space-like hypersurface with volume element $\mathrm{d}\Sigma$, $\mathrm{d}\Sigma^\mu = n^\mu \mathrm{d}\Sigma$, and n^μ is a normal vector of Σ with norm $n^\mu n_\mu = 1$.

The creation and annihilation operators (2) depend on the modes, and so does the vacuum state defined with respect to them. The vacuum state $|0_b\rangle$ belongs to $\hat{b}_i$ and is defined by $\hat{b}_i|0_b\rangle = 0$. In general, $|0_b\rangle$ is not identical with $|0_a\rangle$, for which we have $\hat{a}_i|0_a\rangle = 0$. It follows that the vacuum expectation value of the number density operator $\hat{a}_i^\dagger \hat{a}_i$ must not necessarily vanish when applied to the vacuum $|0_b\rangle$,

$$\langle 0_b | \hat{a}_i^\dagger \hat{a}_i | 0_b \rangle = \sum_j |\beta_{ji}|^2 \; . \tag{6}$$

This operator has a well defined meaning as the number (mode) density operator only if the modes used for the expansion can be put in a suitable relation to the known modes of Minkowski space-time to render an interpretation possible. This is possible if we are interested in the change of vacuum between two asymptotically flat regions of space-time.

If the eigenvalues are not discrete ($i \to \vec{k}$), we have to deal with an integral that may have divergences. These divergences caused by the limit from periodic boundary conditions with discrete spectrum to an infinite volume $\mathbf{V} = \langle 0|0\rangle$ can be cured by normalization. This may be accomplished by dividing through the same divergence:

$$\langle 0_b|\hat{a}^\dagger_{\vec{k}}\hat{a}_{\vec{k}}|0_b\rangle_{ren} = \frac{1}{\mathbf{V}}\langle 0_b|\hat{a}^\dagger_{\vec{k}}\hat{a}_{\vec{k}}|0_b\rangle = \frac{1}{\mathbf{V}}\int |\beta_{\vec{k}'\vec{k}}|^2 \, \mathrm{d}^3k' \ . \tag{7}$$

If $\langle 0_b|\hat{a}^\dagger_{\vec{k}}\hat{a}_{\tilde{\vec{k}}}|0_b\rangle$ has the form $\delta(\vec{k}-\tilde{\vec{k}})f(\vec{k},\tilde{\vec{k}})$, one finds

$$\langle 0_b|\hat{a}^\dagger_{\vec{k}}\hat{a}_{\vec{k}}|0_b\rangle_{ren} = \frac{1}{\mathbf{V}}\lim_{\vec{k}\to\tilde{\vec{k}}}\langle 0_b|\hat{a}^\dagger_{\vec{k}}\hat{a}_{\tilde{\vec{k}}}|0_b\rangle = f(\vec{k}) \ , \tag{8}$$

which will be useful later on. Now $\langle 0_b|\hat{a}^\dagger_{\vec{k}}\hat{a}_{\vec{k}}|0_b\rangle_{ren}$ is the well defined spectral number (mode) density.

3. The expanding mass shell

We consider a spherical shell of mass M and radius R. Outside of the shell, space-time is described by the Schwarzschild metric; inside it space-time is flat. We assume that the thickness of the shell in negligible with respect to its radius (thin shell). Originally the shell's radius is constant, $R = R_0$. The shell is 'ignited' at some time $\tau = 0$ (as measured in the rest frame of the total mass inside the shell). Afterwards the shell expands as $R = R(\tau)$. The situation is illustrated in Fig. 1.

Our first task is to write down the metric of the space-time describing an expanding mass shell. The surface of the shell is the same whether it is measured from the inside or from the outside. We can thus choose the same radial coordinate r on both sides of the shell and we obviously can choose the same angular coordinates θ and ϕ. Outside the shell we denote the time coordinate by t, which is the time measured by an asymptotic observer, who is at rest with respect to the centre of mass of the shell. The relation between the time coordinates t and τ remains to be fixed by the boundary conditions on the mass shell. We take null coordinates and ignore the $r^2 d\Omega^2$ part of the line element for the moment. Inside the shell these coordinates are denoted by (U, V), and by (u, v) outside, where the line element is given by

$$\mathrm{d}s^2 = \gamma(r)\,\mathrm{d}u\,\mathrm{d}v \ , \quad \gamma(r) = 1 - \frac{2M}{r} \ , \tag{9}$$

with

$$u + u_0 := t - \int_{R_0}^{r} \frac{\mathrm{d}r'}{\gamma(r')} \ , \quad v + v_0 := t + \int_{R_0}^{r} \frac{\mathrm{d}r'}{\gamma(r')} \ . \tag{10}$$

Inside we simply have

$$\mathrm{d}s^2 = \mathrm{d}U\,\mathrm{d}V \ , \tag{11}$$

with

$$V + V_0 := \tau + r - R_0 \ , \quad U + U_0 := \tau - r + R_0 \ . \tag{12}$$

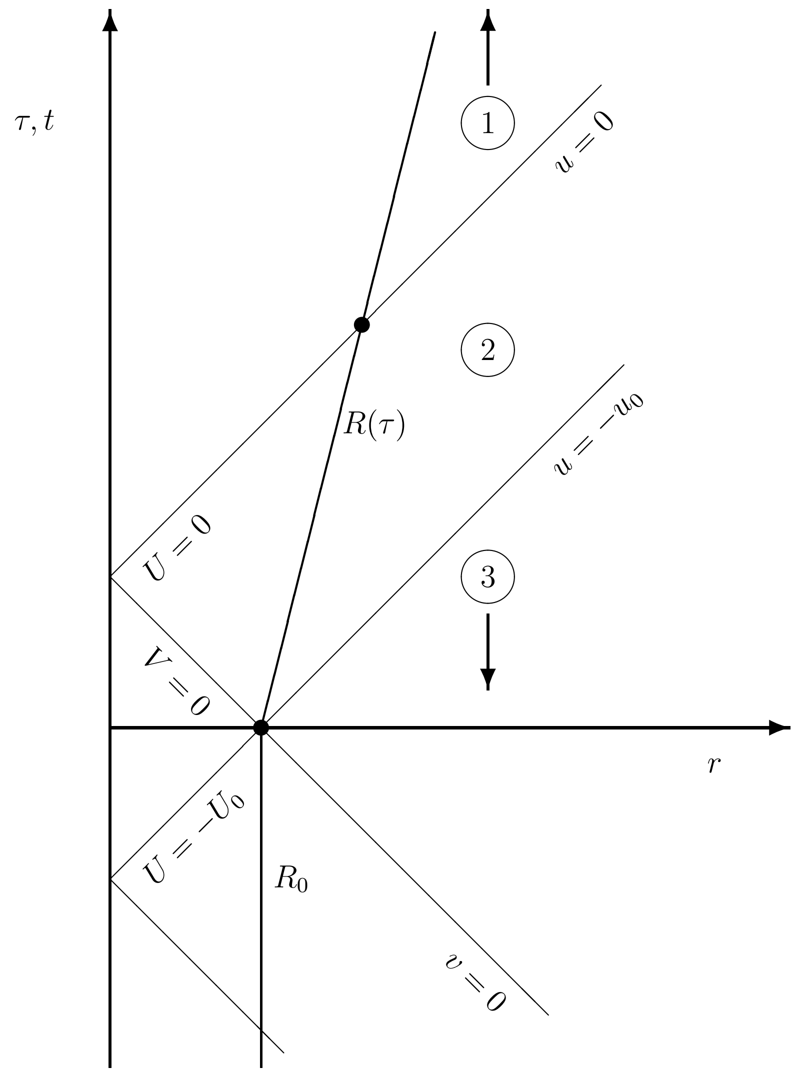

Figure 1. The expanding mass shell (thick line). The thin lines represent the rays at the border lines between the three cases of ray-tracing.

The outgoing light rays run on world lines with constant u and U. We fix the origin of the null coordinates by the convention that the incoming ray $v = V = 0$ goes through the point $(\tau = 0, R_0)$ and that $U = u = 0$ denotes the very same ray on its way out (see Fig. 1). Thus $v_0 = V_0 = 0$ and at $r = 0$ we have $U + U_0 = V + 2R_0$. With our choice for U we finally have $U_0 = 2R_0$.

We define the expansion velocity of the shell by

$$\nu := \frac{\mathrm{d}R}{\mathrm{d}\tau}\,, \tag{13}$$

which is actually the coordinate velocity as seen from the inside of the shell. In principle ν is a function of time, but we will see below that it is a good approximation to work with a constant expansion velocity. In that case the world line of a point on the shell is given by

$$R(\tau) = \begin{cases} R_0 + \frac{\nu}{1+\nu} V = R_0 + \frac{\nu}{1-\nu}(U + U_0) & \text{when} \quad \tau > 0 \\ R_0 & \text{when} \quad \tau < 0 \end{cases} . \tag{14}$$

Our result will finally depend on three parameters: R_0, M and ν.

From the coordinate transformations (10) and (12) and from the definition of the expansion velocity (13) we find

$$\left.\frac{\mathrm{d}u}{\mathrm{d}U}\right|_{r=R} = \begin{cases} \left(\frac{\mathrm{d}t}{\mathrm{d}\tau} - \frac{\nu}{\gamma(R)}\right)\Big/(1-\nu) & \text{when} \quad U > -U_0 \\ 1\big/\sqrt{\gamma_0} & \text{when} \quad U < -U_0 \end{cases} , \tag{15}$$

$$\left.\frac{\mathrm{d}v}{\mathrm{d}V}\right|_{r=R} = \begin{cases} \left(\frac{\mathrm{d}t}{\mathrm{d}\tau} + \frac{\nu}{\gamma(R)}\right)\Big/(1+\nu) & \text{when} \quad V > 0 \\ 1\big/\sqrt{\gamma_0} & \text{when} \quad V < 0 \end{cases} , \tag{16}$$

where

$$\gamma_0 := \gamma(R_0) = 1 - \frac{2M}{R_0} . \tag{17}$$

The matching of the inside and outside space-time demands the continuity of the line element

$$(\gamma \; \mathrm{d}u \; \mathrm{d}v)|_{r=R} = (\mathrm{d}U \; \mathrm{d}V)|_{r=R} , \tag{18}$$

which leads, with (15) and (16), to

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\gamma(R)}\sqrt{\gamma(R)(1-\nu^2)+\nu^2} , \tag{19}$$

which describes the relation between t and τ, and finally fixes the space-time geometry. Note that Eq. (19) describes nothing but the gravitational redshift z for a photon that is emitted inside the shell and crosses the shell at time τ, i.e. $\mathrm{d}t/\mathrm{d}\tau = 1 + z$.

To trace a radial light ray through space-time, we need to find the transition functions between the two coordinate patches on the surface of the mass shell. The setting is illustrated in Fig. 1. There are three different cases for a ray that arrives at infinity. Let us start with the trivial case (3). Everything happens before the onset of expansion. We do not expect any effect then. For the second case (2) the rays enter the shell before the start of the expansion and the shell is 'ignited' while the ray is crossing the interior. Only a final intervall in u of length u_0 is involved. For case (1), entrance and exit of the ray occur while the expansion is taking place.

We obtain a collapsing shell by inverting the setting ($u \to -u, v \to -v, \tau \to -\tau$, etc.). To reproduce a collapse to a black hole, we need to take $R_0 \to 2M$ and $\nu \to 1$.‡ This procedure will allow us to reproduce Hawking's result from the setting of the expanding shell.

The transition functions, which follow from (15) and (16) by integration, are too complicated for an exact analytic treatment of the problem. We therefore have to find

‡ After this change of the time direction the in-vacuum is defined in a different region, but the vacuum expectation value of the number density operator is still given by the same formula.

a reasonable approximation. Let us start from the inverse of Eqs. (15) and (16), which reads

$$\left.\frac{\mathrm{d}U}{\mathrm{d}u}\right|_{U>-U_0} = \frac{[\gamma(1-\nu^2)+\nu^2]^{\frac{1}{2}}+\nu}{(1+\nu)} \;, \tag{20}$$

$$\left.\frac{\mathrm{d}V}{\mathrm{d}v}\right|_{V>0} = \frac{[\gamma(1-\nu^2)+\nu^2]^{\frac{1}{2}}-\nu}{(1-\nu)} \;. \tag{21}$$

As a first approximation we assume that the expansion velocity ν is constant. Secondly, we expect the biggest effect when the space-time curvature is high, thus in the vicinity of the 'explosion'. Consequently we expand the transition functions in the Newtonian potential difference $-2M(1/R-1/R_0)$, which is a small quantity at the beginning of the expansion. Case (3) is trivial. For case (2) we approximate $\mathrm{d}U/\mathrm{d}u$ around the 'ignition', $R = R_0$. For case (1) we have to consider two points at which the ray crosses the shell. The effect from the entrance of the ray into the shell will be larger than the effect from the exit of the ray. Thus we need to approximate $\mathrm{d}V/\mathrm{d}v$ around γ_0 and $\mathrm{d}U/\mathrm{d}u$ around $\gamma(R(U=0)) =: \gamma_1$. Around γ_0 we include terms up to linear order, whereas around γ_1 we keep only the leading term. These expansions yield

$$\left.\frac{\mathrm{d}U}{\mathrm{d}u}\right|_{U>-U_0} \approx c_0 - c_0'\frac{2M}{R(U)} \quad \text{around} \quad \gamma_0 \tag{22}$$

$$\left.\frac{\mathrm{d}U}{\mathrm{d}u}\right|_{U>-U_0} \approx c_1 \quad \text{around} \quad \gamma_1 \tag{23}$$

$$\left.\frac{\mathrm{d}V}{\mathrm{d}v}\right|_{V>0} \approx d_0 - d_0'\frac{2M}{R(V)} \quad \text{around} \quad \gamma_0 \;, \tag{24}$$

where

$$c_0 := \left.\frac{\mathrm{d}U}{\mathrm{d}u}\right|_{\gamma_0} + c_0'(1-\gamma_0) \;, \qquad c_0' := \frac{\partial}{\partial\gamma}\left.\frac{\mathrm{d}U}{\mathrm{d}u}\right|_{\gamma_0} \;, \tag{25}$$

$$c_1 := \left.\frac{\mathrm{d}U}{\mathrm{d}u}\right|_{\gamma_1} \;, \tag{26}$$

$$d_0 := \left.\frac{\mathrm{d}V}{\mathrm{d}v}\right|_{\gamma_0} + d_0'(1-\gamma_0) \;, \qquad d_0' := \frac{\partial}{\partial\gamma}\left.\frac{\mathrm{d}V}{\mathrm{d}v}\right|_{\gamma_0} \;. \tag{27}$$

As a consequence

$$\frac{\mathrm{d}u}{\mathrm{d}U} \approx \frac{R_0(1-\nu)+\nu(U+U_0)}{c_0[R_0(1-\nu)+\nu(U+U_0)] - c_0' 2M(1-\nu)} \tag{28}$$

$$\frac{\mathrm{d}v}{\mathrm{d}V} \approx \frac{R_0(1+\nu)+\nu V}{d_0[R_0(1+\nu)+\nu V] - d_0' 2M(1+\nu)} \;. \tag{29}$$

We denote the approximate transition functions around γ_0 at the entrance with $\xi_0(V)$, around γ_0 at the exit with $\eta_0(u)$, and around γ_1 at the exit with $\eta_1(u)$. The inverse functions of the last two are written as $\eta_0^{-1}(U)$ and $\eta_1^{-1}(U)$, respectively. Integration of (28), (26) and (29) provides the functions that will be relevant to our calculation

$$\eta_0^{-1}(U) = \frac{U}{c_0} + \frac{c_0' 2M(1-\nu)}{c_0^2\nu}\ln\left[1+\frac{c_0\nu U}{(c_0R_0 - c_0'2M)(1-\nu)+c_0\nu U_0}\right], \tag{30}$$

$$\eta_1^{-1}(U) = \frac{U}{c_1}, \tag{31}$$

$$\xi_0(V) \;\; = \frac{V}{d_0} + \frac{d_0' 2M(1+\nu)}{d_0^2 \nu} \ln\left[1 + \frac{d_0 \nu V}{(d_0 R_0 - d_0' 2M)(1+\nu)}\right] . \tag{32}$$

4. Particle production

To compute the particle production caused by the moving mass shell, we consider an observer who defines vacuum before the onset of expansion (at $\mathscr{I}^-$) and ask how this vacuum is seen by an observer at $\mathscr{I}^+$, i.e. we have to calculate the Bogoliubov transformations between the vacua at $\mathscr{I}^-$ and $\mathscr{I}^+$. We can do this by following the track of the light-rays.

We restrict our attention to a massless, minimally coupled scalar field, which obeys the equation of motion $\varphi = 0$. We expect a damping of the amplitude with r^{-1} because of the spherical symmetry. Furthermore we split off an angular component in the form of spherical harmonics $Y_{lm}(\theta, \phi)$:

$$\varphi(r, t, \theta, \phi) = \sum_{l,m} \frac{1}{r} Y_{lm}(\theta, \phi) \Psi_l(t, r) \;\; . \tag{33}$$

With this ansatz, the wave equation in null coordinates becomes

$$\frac{\partial^2 \Psi_l}{\partial u \partial v} = \gamma(r) \left(\frac{2M}{r^3} + \frac{l(l+1)}{r^2} \right) \Psi_l \;\; . \tag{34}$$

The right-hand side has a minimum outside the Schwarzschild radius and acts like a potential well. Because of this, incoming rays may be reflected at the gravitational potential. To a good approximation one may neglect this effect of backscattering by simply dropping the right-hand side of (34). In the same way we neglect potential terms in the inside. The index l is left out in the following, since Ψ no longer depends on l if the centrifugal potential is neglected.

We are searching for solutions of the equations

$$\frac{\partial^2 \Psi}{\partial u \partial v} = 0 \quad \text{outside,} \quad \frac{\partial^2 \Psi}{\partial U \partial V} = 0 \quad \text{inside.} \tag{35}$$

Any function that does not mix the two variables u, v or U, V respectively is a solution. We are looking for an incoming mode which turns into an outgoing one in a smooth manner and passes through $r = 0$. These solutions must behave like spherically symmetric Minkowski modes at $\mathscr{I}^-$. At $r \to \infty$ the phase of an incoming wave is given by

$$e^{-\mathrm{i}\omega v} \;\; . \tag{36}$$

Inside of the mass shell, the corresponding mode travels on a curve with constant V. At $r = 0$ we have $U = V$ where the incoming ray $\exp(-\mathrm{i}\tilde{\omega} V)$ turns smoothly into the outgoing ray $\exp(-\mathrm{i}\tilde{\omega} U)$. At $\mathscr{I}^+$, outgoing modes are proportional to

$$e^{-\mathrm{i}\omega' u} \;\; . \tag{37}$$

With the help of the transition functions (30) we can now glue curves of constant v to curves of constant V and curves of constant U to curves of constant u. Figure 2 gives an overview of this procedure. Here we already inserted $U_0 = 2R_0$.

$$
\begin{tikzcd}
e^{-\mathrm{i}\omega v} \arrow[r, "v>0"] \arrow[dr, "v<0"'] & e^{-\mathrm{i}\omega\xi(V)} \arrow[r] & e^{-\mathrm{i}\omega\xi(U)} \arrow[r] & e^{-\mathrm{i}\omega\xi(\eta(u))} & \text{case (1)} \\
 & e^{-\mathrm{i}\omega V/\sqrt{\gamma_0}} \arrow[r] & e^{-\mathrm{i}\omega U/\sqrt{\gamma_0}} \arrow[r, "U>-2R_0"] \arrow[dr, "U<-2R_0"'] & e^{-\mathrm{i}\omega \eta(u)/\sqrt{\gamma_0}} & \text{case (2)} \\
 & & & e^{-\mathrm{i}\omega(u+u_0-2R_0/\sqrt{\gamma_0})} & \text{case (3)}
\end{tikzcd}
$$

Figure 2. Tracing of the incoming ray.

In the following we restrict our analysis to the most interesting case (1), which provides the result for the extended expansion phase of the mass shell. Case (2) is restricted to a short time after the ignition of the shell, and thus is only characteristic of the ignition phase of our problem. Its importance depends on the velocity of the shell. So we now calculate the coefficients $\beta^1_{\omega'\omega}$. We expand the outgoing modes into plane waves. The outgoing modes are functions that depend on u only, and we obtain

$$\Psi \simeq \frac{1}{4\pi\sqrt{2\omega}} e^{-\mathrm{i}\omega\xi_0(\eta_1(u))} = \int_{-\infty}^{\infty} A^1_\omega(\omega') \frac{1}{4\pi\sqrt{2\omega'}} e^{-\mathrm{i}\omega' u}\, \mathrm{d}\omega' \quad \text{for} \quad u > 0 \ , \tag{38}$$

with

$$A^1_\omega(\omega') = \left\{ \begin{array}{cc} \alpha^{1*}_{\omega'\omega} & \omega' > 0 \\ \mathrm{i}\beta^1_{-\omega'\omega} & \omega' < 0 \end{array} \right. \quad . \tag{39}$$

This is a Fourier transformation, which allows us to compute the Bogoliubov coefficients without evaluatiing the scalar product (5). A Fourier transformation of the outgoing modes gives

$$I := \sqrt{\frac{\omega}{\omega'}} A^1_\omega(\omega') = \frac{1}{2\pi} \int_0^\infty e^{-\mathrm{i}\omega\xi_0(\eta_1(u))} e^{\mathrm{i}\omega' u}\, \mathrm{d}u \ . \tag{40}$$

Using the approximations for $\xi_0(V)$ and $\eta_1(u)$, and introducing the notation $\lambda := d_0' 2M(1+\nu)/(d_0^2\nu)$, we find

$$I = \frac{1}{2\pi} \int_0^\infty e^{-\mathrm{i}(\omega\frac{c_1}{d_0} - \omega')u} \left[\frac{(R_0 - d_0' 2M/d_0)(1+\nu)/\nu + c_1 u}{(R_0 - d_0' 2M/d_0)(1+\nu)/\nu} \right]^{-\mathrm{i}\omega\lambda} \mathrm{d}u \ . \tag{41}$$

We further define $x - x_0 := c_1 u/d_0$ and $x_0 := (R_0 - d_0' 2M/d_0)(1+\nu)/(\nu d_0)$. The transition coefficients then become

$$\beta^1_{\omega'\omega} = -\frac{1}{2\pi} \sqrt{\frac{\omega'}{\omega}} e^{\mathrm{i}\omega_+ x_0} x_0^{\mathrm{i}\omega\lambda} \frac{d_0}{c_1} \int_{x_0}^\infty x^{-\mathrm{i}\omega\lambda} e^{-\mathrm{i}\omega_+ x}\, \mathrm{d}x \ , \tag{42}$$

where $\omega_+ := \omega + \omega' \frac{d_0}{c_1}$.

The vacuum expectation value of the number density operator is given by

$$\int_0^\infty |\beta^1_{\omega'\omega}|^2\, \mathrm{d}\omega' = \frac{d_0^2}{4\pi^2 c_1^2} \int_0^\infty \frac{\omega'}{\omega} \left| \int_{x_0}^\infty e^{-\mathrm{i}\omega_+ x} x^{-\mathrm{i}\omega\lambda}\, \mathrm{d}x \right|^2 \mathrm{d}\omega' \ . \tag{43}$$

This quantity is divergent and has to be regularized. First we will shift the ω'-integration into the complex plane for some ϵ. We do this because, for $\mathrm{Im}\,\omega_+ < 0$, the inner integrations are well defined and may be expressed as incomplete Γ-functions [12]. Further we regularize the number density with the help of a 'frequency split'

$$|\beta^1_{\omega'\omega}|^2 = \lim_{\omega\to\tilde\omega} \beta^1_{\omega'\omega}\beta^1_{\tilde\omega\omega}{}^* \ . \tag{44}$$

Finally, we take the difference to the limit $x_0 \to 0$, which allows us to subtract the divergent part in a well defined manner. The point is that (43) reduces in this limit to the expression for the number density that shows up in the Hawking effect [1],

$$\lim_{x_0\to 0}\int_0^\infty |\beta^1_{\omega'\omega}|^2 \,\mathrm{d}\omega' = \lim_{\tilde\omega\to\omega}\frac{\delta(\tilde\omega-\omega)}{e^{2\pi\omega\lambda}-1} \ . \tag{45}$$

The only difference is that λ replaces the inverse of the surface gravity, $1/\kappa = 4M$, that would show up in the situation of a collapsing mass shell.

As a first step we rotate the integration variable x in the complex plane and obtain

$$\int_0^\infty |\beta^1_{\omega'\omega}|^2 \,\mathrm{d}\omega' = \lim_{\epsilon\to 0}\frac{d_0^2}{4\pi^2 c_1^2}\int_{0-\mathrm{i}\epsilon}^{\infty-\mathrm{i}\epsilon}\frac{\omega'}{\omega}\left|\int_{x_0}^{x_0-\mathrm{i}\infty} e^{-\mathrm{i}\omega_+ x}x^{-\mathrm{i}\omega\lambda}\,\mathrm{d}x\right|^2 \mathrm{d}\omega' \ . \tag{46}$$

Introducing $\tilde\omega_+ = \tilde\omega + \omega'\frac{d_0}{c_1}$ and $\tilde\omega := \omega + \varepsilon$, we regularize the product of the inner integrations by a frequency split,

$$\int_0^\infty |\beta^1_{\omega'\omega}|^2 \,\mathrm{d}\omega' = \lim_{\substack{\epsilon\to 0\\ \varepsilon\to 0}}\frac{1}{4\pi^2 c_1^2}\times$$

$$\int_{0-\mathrm{i}\epsilon}^{\infty-\mathrm{i}\epsilon}\frac{\omega'}{\omega}\int_{x_0}^{x_0-\mathrm{i}\infty} e^{-\mathrm{i}\omega_+ x}x^{-\mathrm{i}\omega\lambda}\,\mathrm{d}x\int_{x_0}^{x_0+\mathrm{i}\infty} e^{\mathrm{i}\tilde\omega_+^*\tilde x}\tilde x^{\mathrm{i}\tilde\omega\lambda}\,\mathrm{d}\tilde x\,\mathrm{d}\omega' \ , \tag{47}$$

where we took the limit partially. In the first of the inner integrals we further substitute $\mathrm{i}\omega_+ x =: y$ and in the second $-\mathrm{i}\tilde\omega_+^*\tilde x =: \tilde y$,

$$\int_0^\infty |\beta^1_{\omega'\omega}|^2 \,\mathrm{d}\omega' = \lim_{\substack{\epsilon\to 0\\ \varepsilon\to 0}}\frac{1}{4\pi^2 c_1^2\omega}e^{-\pi\omega\lambda}\times \tag{48}$$

$$\int_{0-\mathrm{i}\epsilon}^{\infty-\mathrm{i}\epsilon}\omega'\frac{\omega_+^{\mathrm{i}\lambda\omega}}{\omega_+}\frac{(\tilde\omega_+^*)^{-\mathrm{i}\lambda\tilde\omega}}{\tilde\omega_+^*}\int_{\mathrm{i}\omega_+x_0}^{\mathrm{i}\omega_+x_0+\infty} e^{-y}y^{-\mathrm{i}\omega\lambda}\,\mathrm{d}y\int_{-\mathrm{i}\tilde\omega_+^*x_0}^{-\mathrm{i}\tilde\omega_+^*x_0+\infty} e^{-\tilde y}\tilde y^{\mathrm{i}\tilde\omega\lambda}\,\mathrm{d}\tilde y\,\mathrm{d}\omega' \ .$$

Since

$$(\omega_+^*+\varepsilon)^{-\mathrm{i}\tilde\omega\lambda} = (\omega_+^*)^{-\mathrm{i}\tilde\omega\lambda} + O(\varepsilon) \ , \tag{49}$$

and ignoring terms $\mathcal{O}(\varepsilon)$, we can write

$$\int_0^\infty |\beta^1_{\omega'\omega}|^2 \,\mathrm{d}\omega' = \lim_{\substack{\epsilon\to 0\\ \varepsilon\to 0}}\frac{d_0^2}{4\pi^2 c_1^2\omega}e^{-\pi\omega\lambda}\times$$

$$\int_{0-\mathrm{i}\epsilon}^{\infty-\mathrm{i}\epsilon}\omega'\omega_+^{-\mathrm{i}\lambda\varepsilon-2}\int_{\mathrm{i}\omega_+x_0}^{\mathrm{i}\omega_+x_0+\infty} e^{-y}y^{-\mathrm{i}\omega\lambda}\,\mathrm{d}y\int_{-\mathrm{i}\omega_+^*x_0}^{-\mathrm{i}\omega_+^*x_0+\infty} e^{-\tilde y}\tilde y^{\mathrm{i}\omega\lambda}\,\mathrm{d}\tilde y\,\mathrm{d}\omega' \ . \tag{50}$$

Let us now take the difference to the $x_0 \to 0$ limit (45),

$$\int_0^\infty |\beta^1_{\omega'\omega}|^2 \,\mathrm{d}\omega' - \lim_{x_0\to 0}\int_0^\infty |\beta^1_{\omega'\omega}|^2 \,\mathrm{d}\omega' = \lim_{\substack{\epsilon\to 0\\ \varepsilon\to 0}}\frac{d_0^2}{4\pi^2 c_1^2\omega}e^{-\pi\omega\lambda}\times$$

$$\int_{0-\mathrm{i}\epsilon}^{\infty-\mathrm{i}\epsilon}\omega'\omega_+^{-\mathrm{i}\lambda\varepsilon-2}\left[\left|\int_{\mathrm{i}\omega_+x_0}^{\mathrm{i}\omega_+x_0+\infty} y^{-\mathrm{i}\omega\lambda}e^{-y}\mathrm{d}y\right|^2 - |\Gamma(1-\mathrm{i}\omega\lambda)|^2\right]\mathrm{d}\omega' \ .$$

Note that the expression inside the square bracket is well defined for all ω'. We now replace the integration variable ω' by ω_+, using its definition from above, and obtain (we omit ϵ and ε in the discussion of the following step)

$$\frac{1}{4\pi^2\omega} e^{-\pi\omega\lambda} \left\{ \int_\omega^\infty \omega_+^{-1} \left[\left| \int_{\mathrm{i}\omega_+ x_0}^{\mathrm{i}\omega_+ x_0 + \infty} y^{-\mathrm{i}\omega\lambda} e^{-y} \mathrm{d}y \right|^2 - |\Gamma(1 - \mathrm{i}\omega\lambda)|^2 \right] \mathrm{d}\omega_+ \right.$$
$$\left. -\omega \int_\omega^\infty \omega_+^{-2} \left[\left| \int_{\mathrm{i}\omega_+ x_0}^{\mathrm{i}\omega_+ x_0 + \infty} y^{-\mathrm{i}\omega\lambda} e^{-y} \mathrm{d}y \right|^2 - |\Gamma(1 - \mathrm{i}\omega\lambda)|^2 \right] \mathrm{d}\omega_+ \right\} .$$

The second term is finite (see Appendix) and, since we expect a divergent result, we can neglect it. The first term can be written as the integral from 0 to ∞ minus the integral from 0 to ω. The latter integration is finite as well, since the difference of the Γ-functions vanishes as $\omega_+ \to 0$. We are left with

$$\int_0^\infty |\beta^1_{\omega'\omega}|^2 \, \mathrm{d}\omega' - \lim_{x_0 \to 0} \int_0^\infty |\beta^1_{\omega'\omega}|^2 \, \mathrm{d}\omega' = \text{finite} + \frac{1}{4\pi^2\omega} e^{-\pi\omega\lambda} \times$$
$$\int_0^\infty \omega_+^{-1} \left[\left| \int_{\mathrm{i}\omega_+ x_0}^{\mathrm{i}\omega_+ x_0 + \infty} y^{-\mathrm{i}\omega\lambda} e^{-y} \mathrm{d}y \right|^2 - |\Gamma(1 - \mathrm{i}\omega\lambda)|^2 \right] \mathrm{d}\omega_+ \, .$$

Now we neglect all finite terms and use the fact that the second term inside the square bracket is again (up to finite terms) the result of the $x_0 \to 0$ limit (45). We thus see that the divergent part of (50) can be written as

$$\int_0^\infty |\beta^1_{\omega'\omega}|^2 \, \mathrm{d}\omega' = \lim_{\substack{\epsilon \to 0 \\ \varepsilon \to 0}} \frac{1}{4\pi^2\omega} e^{-\pi\omega\lambda} \times$$
$$\int_{0-\mathrm{i}\epsilon}^{\infty-\mathrm{i}\epsilon} \omega_+^{-\mathrm{i}\lambda\varepsilon - 1} \int_{\mathrm{i}\omega_+ x_0}^{\mathrm{i}\omega_+ x_0 + \infty} e^{-y} y^{-\mathrm{i}\omega\lambda} \, \mathrm{d}y \int_{-\mathrm{i}\omega_+^* x_0}^{-\mathrm{i}\omega_+^* x_0 + \infty} e^{-\tilde{y}} \tilde{y}^{-\mathrm{i}\omega\lambda} \, \mathrm{d}\tilde{y} \, \mathrm{d}\omega_+ \, . \quad (51)$$

For positive ε we can replace the contour parallel to the positive real axis by an integration along the negative imaginary axis. Then we substitute $e^z = \mathrm{i}\frac{\omega_+}{\omega}$ and have

$$\int_0^\infty |\beta^1_{\omega'\omega}|^2 \, \mathrm{d}\omega' = \lim_{\substack{\epsilon \to 0 \\ \varepsilon \to 0}} \frac{1}{4\pi^2\omega} e^{-\pi\omega\lambda} \times$$
$$\int_{\ln\left(\frac{\epsilon}{\omega}\right)}^\infty e^{-\mathrm{i}\lambda\varepsilon z} \int_{x_0\omega e^z}^{x_0\omega e^z + \infty} y^{-\mathrm{i}\omega\lambda} e^{-y} \mathrm{d}y \int_{x_0\omega e^{z*}}^{x_0\omega e^{z*} + \infty} \tilde{y}^{\mathrm{i}\omega\lambda} e^{-\tilde{y}} \mathrm{d}\tilde{y} \, \mathrm{d}z \, . \quad (52)$$

We can now take the limit $\epsilon \to 0$ and see that, since the integration is along the real axis, our integral can be written as

$$\int_0^\infty |\beta^1_{\omega'\omega}|^2 \, \mathrm{d}\omega' = \lim_{\varepsilon \to 0} \int_{-\infty}^\infty e^{-\mathrm{i}\lambda\varepsilon z} h(z) \, \mathrm{d}z \, , \quad (53)$$

with

$$h(z) := \frac{e^{-\pi\omega\lambda}}{4\pi^2\omega} \int_{x_0\omega e^z}^\infty y^{-\mathrm{i}\omega\lambda} e^{-y} \mathrm{d}y \int_{x_0\omega e^z}^\infty \tilde{y}^{\mathrm{i}\omega\lambda} e^{-\tilde{y}} \mathrm{d}\tilde{y} \, , \quad (54)$$

for any complex z. For real z, as we integrate over on the r.h.s. of Eq. (53), this yields the quantity we search for; $h(z)$ is identical to the product of the inner integrations in

(52) for real z. This function is holomorphic in z. It has an upper bound B for any real z in $[-R, R]$. Thus

$$\left|\int_{-R}^{R} h(z)\mathrm{d}z\right| \leq 2RB \quad , \tag{55}$$

and, because the function is holomorphic,

$$\left|\int_{\gamma_R^u} h(z)\mathrm{d}z\right| \leq 2RB \quad \text{and} \quad \left|\int_{\gamma_R^d} h(z)\mathrm{d}z\right| \leq 2RB \quad , \tag{56}$$

where the path γ_R^d is the half-circle of radius R in the lower complex plane and γ_R^u the half-circle of radius R in the upper complex plane. From this we find

$$\lim_{R\to\infty} |\int_{\gamma_R^u} \frac{h(z)}{z^2}\mathrm{d}z| = 0 \tag{57}$$

$$\lim_{R\to\infty} |\int_{\gamma_R^d} \frac{h(z)}{z^2}\mathrm{d}z| = 0 \ . \tag{58}$$

Our integral of interest can now be expressed as

$$\int_{-\infty}^{\infty} e^{-\mathrm{i}\lambda\varepsilon z} h(z)\ \mathrm{d}z = -\frac{1}{\lambda^2}\partial_\varepsilon^2 \int_{-\infty}^{\infty} e^{-\mathrm{i}\lambda\varepsilon z}\frac{h(z)}{z^2}\ \mathrm{d}z \quad , \tag{59}$$

which allows us to make use of the residue theorem. Now for $\varepsilon > 0$ we may close the contour in the lower plane. To compute the integral we shift the pole into the lower plane. This yields

$$\int_{-\infty}^{\infty} e^{-\mathrm{i}\lambda\varepsilon z}\frac{h(z)}{z^2}\ \mathrm{d}z = -2\pi\left(\lambda\varepsilon h + \mathrm{i}\partial_z h\right)\Theta(\varepsilon) \ . \tag{60}$$

With this we now get

$$\int_0^{\infty} |\beta^1_{\omega'\omega}|^2\ \mathrm{d}\omega' = \lim_{\varepsilon\to 0}\frac{2\pi}{\lambda}h(0)\delta(\varepsilon) \tag{61}$$

$$= \lim_{\varepsilon\to 0}\frac{e^{-\pi\omega\lambda}}{2\pi\omega\lambda}\delta(\varepsilon)\int_{x_0\omega}^{\infty} y^{-\mathrm{i}\omega\lambda}e^{-y}\mathrm{d}y \int_{x_0\omega}^{\infty}\tilde{y}^{\mathrm{i}\omega\lambda}e^{-\tilde{y}}\mathrm{d}\tilde{y} \ , \tag{62}$$

with the differentiation on the δ-function interpreted as 0.§ If we now put all pieces together, we finally find for the vacuum expectation value of the number density operator $N(\omega)$ [in case (1)]:

$$\begin{aligned} \int_0^{\infty} |\beta^1_{\omega'\omega}|^2\ \mathrm{d}\omega' &= \frac{e^{-\pi\omega\lambda}}{2\pi\lambda\omega}\delta(\varepsilon)|\Gamma(1-\mathrm{i}\omega\lambda, x_0\omega)|^2 \quad + \text{finite} \\ \Rightarrow \frac{N(\omega)}{\mathbf{V}} &= \frac{e^{-\pi\omega\lambda}}{2\pi\lambda\omega}|\Gamma(1-\mathrm{i}\omega\lambda, x_0\omega)|^2 \quad . \end{aligned} \tag{63}$$

The spectral energy flux through a 2-dimensional surface $\rho(\omega)$ then reads

$$\rho(\omega) = \frac{e^{-\pi\omega\lambda}\omega^2}{4\pi^3\lambda}|\Gamma(1+\mathrm{i}\omega\lambda, x_0\omega)|^2 \ . \tag{64}$$

We have made several assumptions, but some of them may be relaxed. First, we assumed that the velocity ν of the mass shell is constant. In a realistic scenario we would

§ It is an even function.

expect a high velocity at the beginning, decreasing towards a constant value (inertial motion), in contrast to a collapse where the velocity would increase. Secondly, we approximated around the space-time points that cause the maximal amount of particle production. Therefore, our result is only correct in the vicinity of $u = 0$. To obtain a result for later times, we would have to approximate around other points. This gives rise to different values of the parameters λ and x_0, which actually become functions of R and $\nu(R)$. These functions then are [cf. (27)]:

$$\begin{aligned}
\lambda(R) &= \frac{2(1+\nu)Md'}{\nu d^2} \quad \text{and} \\
x(R) &:= \frac{(1+\nu)R}{\nu d} - \lambda\ , \quad \text{with} \\
d'(R) &:= \frac{\partial}{\partial\gamma(R)} \frac{\mathrm{d}V}{\mathrm{d}v}\Big|_{\gamma(R)} \\
d(R) &:= \frac{\mathrm{d}V}{\mathrm{d}v}\Big|_{\gamma(R)} + d'(R)[1-\gamma(R)]\ ,
\end{aligned} \tag{65}$$

from which we obtain a spectral energy flux that depends on the actual radius and the actual velocity of the mass shell:

$$\rho(\omega, R) = \frac{e^{-\pi\omega\lambda(R)}\omega^2}{4\pi^3\lambda(R)} |\Gamma[1 + \mathrm{i}\omega\lambda(R), x(R)]|^2\ . \tag{66}$$

The quality of this generalization mainly depends on the assumption that the change of velocity is small, so that $\ddot{R}$-contributions can be neglected. Should the acceleration of the mass shell get too high, the patches in which the velocity can be treated as nearly constant would get too small and effects from the boundaries would grow important.

5. Discussion

The particle production of an expanding mass shell depends on the velocity of the mass shell ν, its mass M, and its radius R. In the spectral energy flux, these three quantities enter only through the functions $\lambda(\nu, M, R)$ and $x(\nu, M, R)$. To get a feeling for the behaviour of the function $\rho(\omega)$ we will first turn to a discussion of these quantities. In Table 1 we provide the values of λ and x for some limits. The limit $\nu \to 1$ together with $R \to 2M$ reproduces Hawking's result.

In the Hawking limit we have $\lambda \to \kappa$, the so-called surface gravity. Additionally $x \to 0$, and we obtain exactly the expected Planck spectrum (cf. (66)). Figures 3 and 4 show $1/\lambda$ as a function of R for three different values of R and x as a function of ν for different values of R. We normalize all quantities to the Schwarzschild radius $R_{\mathrm{S}} = 2M$; $1/\lambda$ decreases rapidly with decreasing velocity. As is shown below, this implies a decrease of the spectral energy flux and a shift of the maximum towards smaller frequencies as the velocity decreases; λ is less sensitive to changes of the radius at high velocities. Particle production ceases, as expected, at large radii, because x increases with large R and small velocity.

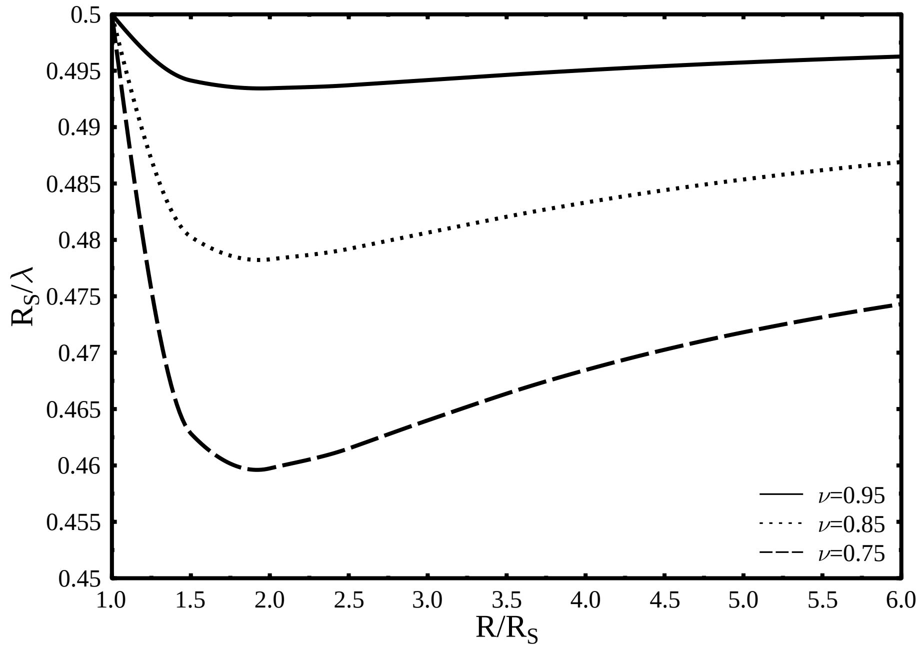

Figure 3. The function $R_\mathrm{S}/\lambda(R)$ for different velocities ν.

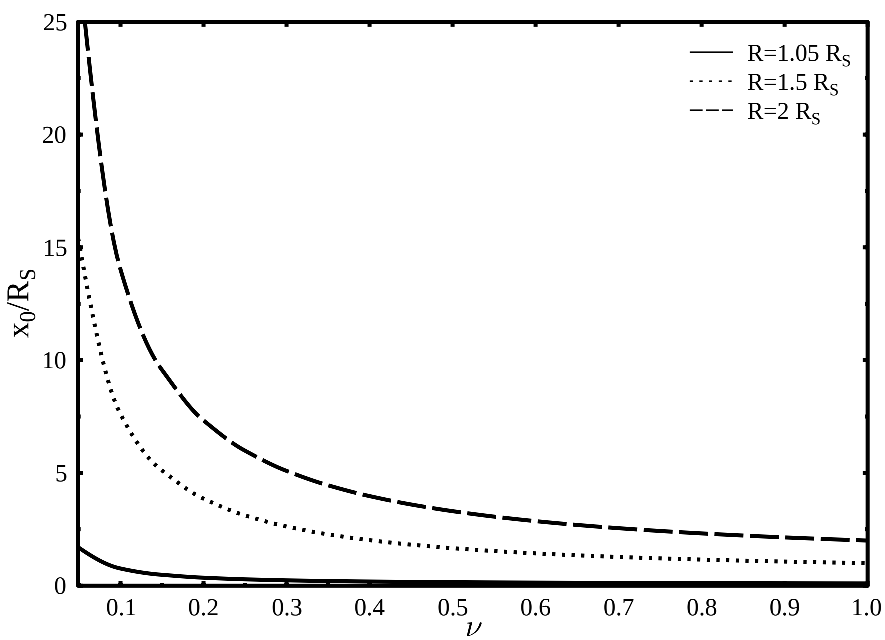

Figure 4. The function $x(\nu)/R_\mathrm{S}$ for different values of the radius R.

Limit	λ	x
$\nu \to 1$	$4M$	$2(R-2M)$
$\nu \to 0$	∞	∞
$\nu \to 1 \wedge R \to 2M$	$4M$	0
$R \to \infty$	$\frac{(1+\nu)^2}{\nu}M$	∞
$M \to 0$	0	$\frac{(1+\nu)}{\nu}R$
$R \to 2M$	$4M$	0

Table 1. Some limiting cases for the functions that enter the expression for the spectral energy flux through a 2-dimensional surface.

5.1. The spectral energy flux

We now study the behaviour of the spectral energy flux (64) as a function of λ/R_{S} and x/R_{S}. Figures 5 and 6 show $\rho(\omega)$ for different values of x/R_{S}, and for different values of λ/R_{S} with the second parameter fixed at a representative value each time. Let us first turn to Fig. 5. When $x = 0$ we find a Planck spectrum with temperature $T = (2\pi k_B \lambda)^{-1}$. For $x/R_{\mathrm{S}} > 0$ the spectrum is non-thermal. It is interesting to note that in the non-equilibrium situation more hard particles and less soft particles are produced at a fixed value of λ.

In Fig. 6 we see, as expected, that an increase in λ/R_{S} causes a shift of the maximum towards smaller frequencies. Furthermore the whole curve flattens quickly for increasing λ/R_{S}.

Remember that, according to the considerations presented in the last section, an increasing radius (which changes λ and x) dominates the evolution of the spectral energy flux during the expansion. This can be seen in Fig. 7.

5.2. The energy flux

Finally we come to the emitted energy flux through a 2-dimensional surface, $\varepsilon \equiv \int \rho(\omega)\mathrm{d}\omega$.

In Fig. 8 we see the energy flux as a function of radius for different values of ν. At large values of R/R_S no particles are produced, since the mass shell moves inertially (the tidal forces go to zero). However, more interesting is the maximum of the curve that appears for ultra-relativistic velocities ν. This reflects the fact that two effects are working against each other during the expansion. The probability of particle production depends on the strength of the tidal forces acting on the vacuum locally. The tidal forces decrease monotonically as the mass shell expands. Once particles have been created, they have to climb out of the gravitational potential well of the mass shell itself (gravitational redshift). This effect reduces the energy of the produced particles. At the Schwarzschild radius the energy would be reduced to zero. For an ultrarelativistic mass shell the tidal forces change very quickly and thus the particle production is large. Close

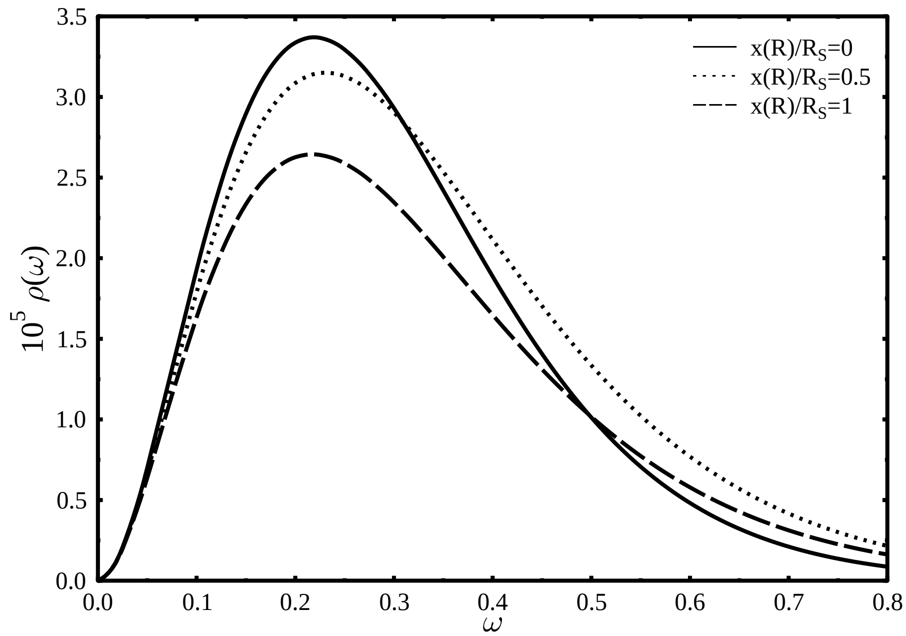

Figure 5. The spectral energy flux through a 2-dimensional surface $\rho(\omega)$ for different values of x/R_{S} at fixed $\lambda/R_{\mathrm{S}} = 2.05$.

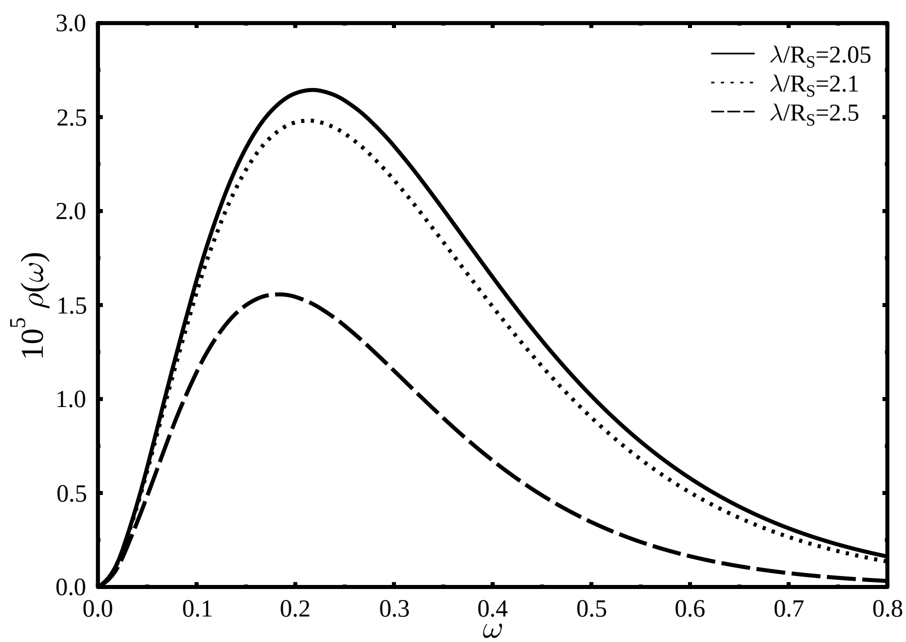

Figure 6. The spectral energy flux through a 2-dimensional surface $\rho(\omega)$ for different values of λ/R_{S} at fixed $x/R_{\mathrm{S}} = 1$.

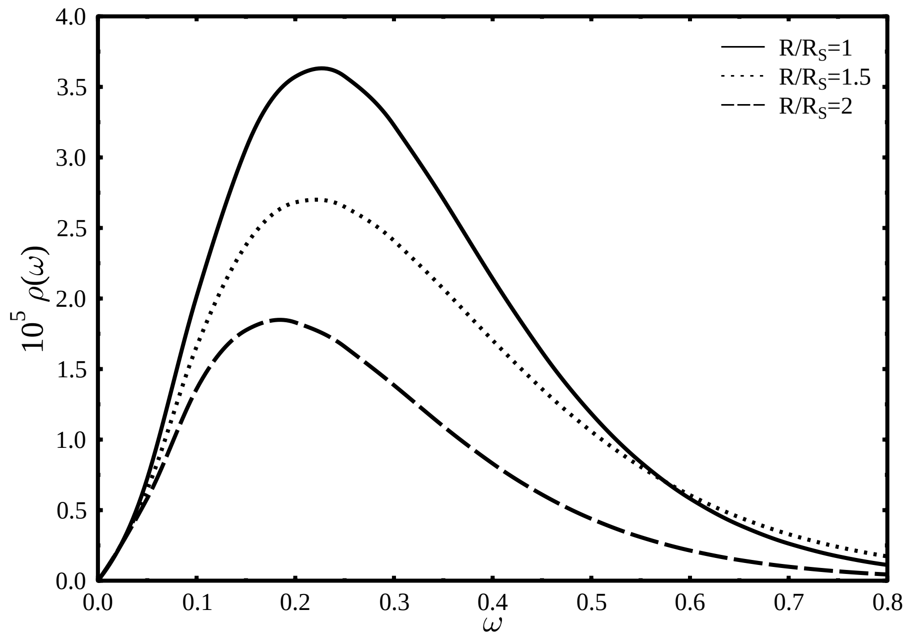

Figure 7. The spectral energy flux through a 2-dimensional surface $\rho(\omega)$ at different moments of time, respresented by three positions $R/R_{\rm S}$, for the constant velocity $\nu = 0.95$.

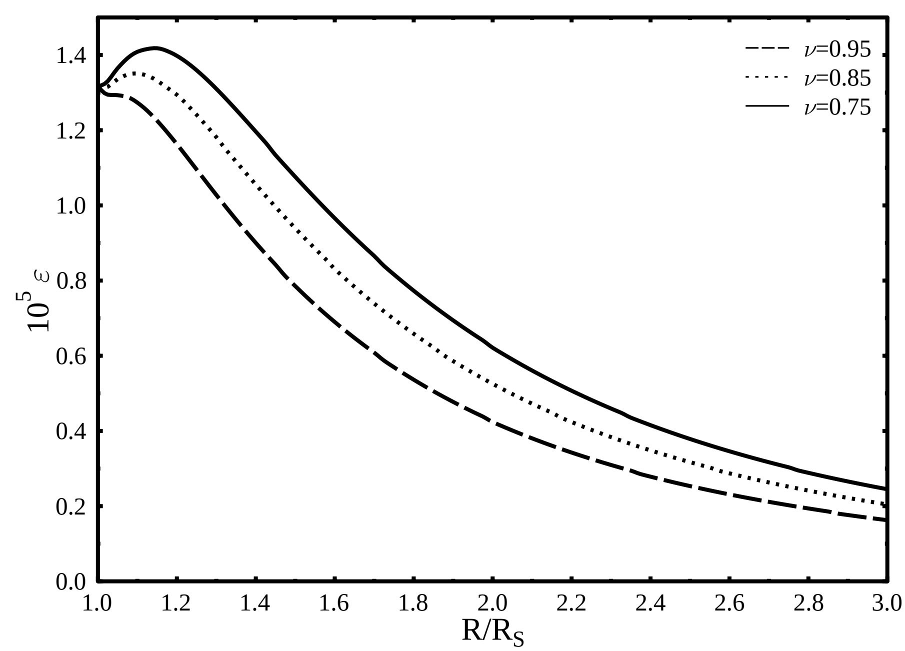

Figure 8. The energy flux through a 2-dimensional surface for different values of the velocity ν.

to the Schwarzschild radius however, this is counteracted by the gravitational redshift, giving rise to a maximum slightly away from $R_{\rm S}$.

6. Summary

We computed the spectrum of particles produced by the time-dependent gravitational field of an expanding mass shell. This system is—as long as no horizon forms—invariant under time-reversal, so that we can describe an expanding shell as well as a collapsing shell before the formation of a horizon. If the horizon is formed, the system gets quasi-static for the asymptotic observer and the effect is "frozen". In this way we can reproduce the result of Hawking radiation.

In the situation of an expanding mass shell the spectrum is non-thermal. Our result might be of relevance for the discussion of evaporating black holes. In the situation of a collapsing mass shell, we obtain the deviations from the Planck spectrum of Hawking radiation, which is produced just before the horizon forms.

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Appendix A.

To show that

$$\int_{\omega}^{\infty} \omega_+^{-2} \left[\int_{\mathrm{i}x_0\omega_+}^{\mathrm{i}\omega_+ x_0 + \infty} y^{\mathrm{i}\omega\lambda} e^{-y} \mathrm{d}y \int_{-\mathrm{i}x_0\omega_+}^{-\mathrm{i}\omega_+ x_0 + \infty} \tilde{y}^{-\mathrm{i}\omega\lambda} e^{-\tilde{y}} \mathrm{d}\tilde{y} - |\Gamma(1+\mathrm{i}\omega\lambda)|^2 \right] \mathrm{d}\omega_+ \tag{A.1}$$

is finite (for $\omega > 0$), it is sufficient to show that

$$\left| \int_{\mathrm{i}x_0\omega_+}^{\mathrm{i}\omega_+ x_0 + \infty} y^{\mathrm{i}\omega\lambda} e^{-y} \mathrm{d}y \right| \leq \text{const.}, \tag{A.2}$$

which means that it has an upper bound for $\omega_+ \to \infty$ since the second term in (A.1) does not depend on ω_+ anyhow. We are then left with an integral over ω_+^{-2} that is finite. If we succeed in showing that (A.2) is true, we know that the first term does not increase faster for large ω_+ and therefore is finite too. It is

$$\left| \int_{\mathrm{i}x_0\omega_+}^{\mathrm{i}\omega_+ x_0 + \infty} y^{\mathrm{i}\omega\lambda} e^{-y} \mathrm{d}y \right| = \left| \int_{\mathrm{i}x_0\omega_+}^{\mathrm{i}\omega_+ x_0 + \infty} e^{\mathrm{i}\omega\lambda \ln|y| - \omega\lambda \arg(y)} e^{-y} \mathrm{d}y \right| \tag{A.3}$$

$$\leq \sup_{y\in\gamma} \left| e^{\mathrm{i}\omega\lambda \ln|y| - \omega\lambda \arg(y)} \right| \cdot \left| \int_{x_0\omega_+}^{-\mathrm{i}\omega_+ + \infty} e^{-y} \mathrm{d}y \right| \;, \tag{A.4}$$

γ being the path of integration in the upper right quadrant of the complex plane and therefore is $\arg(y) \in \left[\frac{3}{2}\pi, 2\pi\right]$ on γ. This yields

$$\left| \int_{\mathrm{i}x_0\omega_+}^{\mathrm{i}\omega_+ x_0 + \infty} y^{\mathrm{i}\omega\lambda} e^{-y} \mathrm{d}y \right| \leq e^{2\pi\omega\lambda} \left| \int_{\mathrm{i}x_0\omega_+}^{\mathrm{i}\omega_+ x_0 + \infty} e^{-y} \mathrm{d}y \right| \tag{A.5}$$

$$= e^{2\pi\omega\lambda} \left| e^{-\mathrm{i}x_0\omega_+} \right| \;. \tag{A.6}$$

Putting this together we have

$$\left| \int_{\mathrm{i}x_0\omega_+}^{-\mathrm{i}\omega_+\infty} y^{\mathrm{i}\omega\lambda} e^{-y} \mathrm{d}y \right| \leq e^{2\pi\omega\lambda} \ , \tag{A.7}$$

which shows that (A.1) is finite, as claimed.

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