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Beam-Beam Stability in Electron-Positron Storage Rings

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At the interaction point of a storage ring each beam is subject to perturbations due to the electromagnetic field of the counter-rotating beam. These perturbations lead to a limit of the achievable luminosity in the storage ring. We investigate this limit in the framework of the strong-strong picture. Motion is considered only in the vertical direction and the beams are presumed to be one-dimensional. Based on this model [1] we try to find stability criteria for the beam in an electron-positron storage ring taking into account the damping of the betatron oscillations by synchrotron radiation but neglecting the discrete nature of the radiation process and presuming a Gaussian distribution. We analyze the instabilities by solving a linearized "Fokker-Planck equation" without the quantum excitation term.

I. BEAM EVOLUTION

Aside of dipole induced bending the beam is subject to focusing, damping by synchrotron radiation and beam-beam kicks at the interaction point. The beam-beam kick from the first (second) beam on the second (first) one is given by

$$\Delta y'_{1,2} = -\frac{4\pi N r_e}{L_x \gamma} \int_{-\infty}^{\infty} d\overline{y}\, \mathrm{sgn}(y-\overline{y}) \int_{-\infty}^{\infty} d\overline{y}' \psi_{2,1}(\overline{y},\overline{y}') \tag{1}$$

where N is the number of particles in a bunch and r_e the classical radius of the electron. The distributions of the beams ψ_1 and ψ_2 are normalized to unity, i.e.

$$\int_{-\infty}^{\infty} d\overline{y} \int_{-\infty}^{\infty} d\overline{y}' \psi_{1,2}(\overline{y},\overline{y}') = 1 \tag{2}$$

and are assumed to be one-dimensional with horizontal width L_x. Motion is considered only in the vertical direction. Using the small-angle approximation we obtain for synchrotron radiation and for focusing, respectively

$$\frac{dy'_K}{ds} = -K(s)y \tag{3}$$

$$\Delta y'_{rad} = -\frac{dy}{dt}\frac{p_\gamma}{dx}\frac{dt}{dx}\frac{dt}{m} = -y'\frac{p_\gamma}{p_0} \equiv -\lambda y' \tag{4}$$

where p_0 is the momentum of the particle and p_γ the total momentum change after each turn. We neglect the discrete nature of the radiation process. The dipoles have no influence on the betatron oscillations. In a damped system the phase space density increases. The equations describing the motion of $\psi_{1,2}$ are given by

$$\frac{\partial \psi_{1,2}}{\partial s} + \frac{dy}{ds}\frac{\partial \psi_{1,2}}{\partial y} + \frac{dy'}{ds}\frac{\partial \psi_{1,2}}{\partial y'} = \frac{\lambda}{L}\psi_{1,2} \tag{5}$$

They differ from the Fokker-Planck equations by a missing quantum excitation term and from the Vlasov equations by the fact that the phase space density is not a constant anymore. This gives us

$$\frac{\partial \psi_{1,2}}{\partial s} + y'\frac{\partial \psi_{1,2}}{\partial y} - \left(\frac{\lambda}{L}y' + K(s)y + \frac{4\pi N r_e}{L_x\gamma}\delta_p(s) \int_{-\infty}^{\infty} d\overline{y}\, \mathrm{sgn}(y-\overline{y}) \int_{-\infty}^{\infty} d\overline{y}' \psi_{2,1}(\overline{y},\overline{y}',s)\right)\frac{\partial \psi_{1,2}}{\partial y'} = \frac{\lambda}{L}\psi_{1,2} \tag{6}$$

and a similar equation with the first and second beam being interchanged where $\delta_p(s)$ denotes a period delta function that has singularities at all interaction points. Because the system is damped an equilibrium does not exist. We define the distribution $\psi_0(s)$ such that it satisfies

$$\frac{\partial\psi_0}{\partial s}+y'\frac{\partial\psi_0}{\partial y}-\left(\frac{\lambda}{L}y'+K(s)y+\frac{4\pi Nr_e}{L_x\gamma}\delta_p(s)\int_{-\infty}^{\infty}d\overline{y}\,\mathrm{sgn}(y-\overline{y})\int_{-\infty}^{\infty}d\overline{y}'\psi_0(\overline{y},\overline{y}',s)\right)\frac{\partial\psi_0}{\partial y'}=\frac{\lambda}{L}\psi_0 \tag{7}$$

with ψ being a function of s if damping is present. We can think of $\psi_0(s)$ being a "temporary equilibrium distribution" if the characteristic damping time is much longer than the period of betatron oscillations or perturbations. We are not interested in finding the actual distributions. All we want to know is whether the beam gets unstable or not. Thus, we choose a perturbative ansatz

$$\psi_{1,2}=\psi_0\pm\Delta\psi_{1,2} \tag{8}$$

Substituting eqn. 8 into eqn. 6, subtracting eqn. 7 and neglecting the term made up of two perturbations we find

$$\begin{aligned}&\frac{\partial\Delta\psi_{1,2}}{\partial s}+y'\frac{\partial\Delta\psi_{1,2}}{\partial y}-\frac{\partial\Delta\psi_{1,2}}{\partial y'}\left(\frac{\lambda}{L}y'+F(y,s)\right)\\ -\frac{4\pi Nr_e}{L_x\gamma}\delta_p(s)\frac{\partial\psi_0}{\partial y'}&\int_{-\infty}^{\infty}d\overline{y}\,\mathrm{sgn}(y-\overline{y})\int_{-\infty}^{\infty}d\overline{y}'\Delta\psi_{2,1}(\overline{y},\overline{y}',s)=\frac{\lambda}{L}\Delta\psi_{1,2}\end{aligned} \tag{9}$$

where

$$F(y,s)=K(s)y+\frac{4\pi Nr_e}{L_x\gamma}\delta_p(s)\int_{-\infty}^{\infty}d\overline{y}\,\mathrm{sgn}(y-\overline{y})\int_{-\infty}^{\infty}d\overline{y}'\psi_0(\overline{y},\overline{y}') \tag{10}$$

With the approximation

$$F(y,s)\approx F(s)y \tag{11}$$

we can treat the perturbation as a part of the perturbed focusing function $F(s)$. The drawback is that this step makes the study of phenomena like Landau damping impossible. In the next step we transform eqn. 9 to action-angle coordinates

$$y=\sqrt{2\beta J}\cos\phi \tag{12}$$

$$y'=-\sqrt{\frac{2J}{\beta}}\left(\sin\phi-\frac{\beta'}{2}\cos\phi\right) \tag{13}$$

Only the phase of the two beams at the interaction point is of importance for our problem, but not the particular lattice design. Therefore, we can set $\beta'=0$. This gives us an explicit representation of J

$$J=\frac{1}{2}y'^2\beta+\frac{1}{2}\frac{y^2}{\beta} \tag{14}$$

where β denotes the perturbed instead of the unperturbed betatron function now. Forming the linear combinations

$$f_\pm=\Delta\psi_1\pm\Delta\psi_2 \tag{15}$$

and using the envelope equation eqn. 9 can be rewritten in action-angle coordinates as

$$\begin{aligned}&\frac{\partial f_\pm}{\partial s}+\frac{1}{\beta}\frac{\partial f_\pm}{\partial\phi}-\frac{2\lambda J}{\beta L}\sin^2\phi\frac{\partial f_\pm}{\partial J}+\frac{\lambda}{L}\cos\phi\sin\phi\frac{\partial f_\pm}{\partial\phi}\\ \pm\sqrt{2\beta J}\sin\phi\frac{4\pi Nr_e}{L_x\gamma}\delta_p(s)\frac{\partial\psi_0}{\partial J}&\int_{-\infty}^{\infty}d\overline{y}\,\mathrm{sgn}(y-\overline{y})\int_{-\infty}^{\infty}d\overline{y}'f_\pm(\overline{y},\overline{y}',s)=\frac{\lambda}{L}f_\pm\end{aligned} \tag{16}$$

assuming that $\psi_0=\psi_0(J)$. In the following discussion we omit the label $\pm$.

II. SOLVING THE EQUATIONS OF MOTION

The distribution for electron bunches is approximately Gaussian.

$$\psi_0(J) = \frac{1}{\sqrt{2\pi}\sigma\pi} e^{-\frac{J^2}{2\sigma^2}} \tag{17}$$

Note that the Jacobi determinant of the transformation 13 is 1. Although our ψ_0 as it is used in eqn. 7 depends on s we choose an s-independent ψ_0. This can be done if the damping after each turn is small and if we keep adjusting σ or the beam-beam parameter ξ, respectively, which will be introduced later. In the final result we simply have to consider ξ in an appropriate range. Considering the problem on a turn-by-turn basis also justifies the usage of 13 where J is a constant of motion. f must be periodic in ϕ. Thus, we choose the ansatz

$$f(J,\phi,s) = Je^{-\frac{J^2}{2\sigma^2}} \sum_{l=-\infty}^{\infty} g_l(s) e^{il\phi} \tag{18}$$

Substituting eqn. 17 and eqn. 18 into eqn. 16 and making use of the symmetry of the integrand we obtain

$$\frac{\partial g_l}{\partial s} + \frac{il}{\beta} g_l \mp \frac{2Nr_e\sqrt{\beta}}{L_x\gamma\pi^{3/2}\sigma^{5/2}}\delta_p(s)$$
$$\times \sum_{k=-\infty}^{\infty} g_k(-i) \int_0^{2\pi} d\phi \int_0^{2\pi} d\bar{\phi} \int_0^{\infty} d\bar{J} \sin(\phi)\sin(l\phi)\cos(k\bar{\phi})\bar{J}e^{-\frac{\bar{J}^2}{2\sigma^2}} \operatorname{sgn}(\sqrt{\sigma}\cos(\phi) - \sqrt{\bar{J}}\cos(\bar{\phi})) = \frac{\lambda}{L} g_l \tag{19}$$

having averaged the $\sin\phi\cos\phi$-term due to synchrotron radiation over betatron oscillations. This term represents a small alternating phase advance which we absorb by our assumptions about β and the radiation process. We evaluate the equation at $J = \sigma$ where we expect the perturbations to be largest since $J\psi_0(J)$ has a maximum at $J = \sigma$. However, doing so removes all radial modes. The $d\bar{J}$ - integration can be evaluated by integrating by parts. The Heaviside step function is denoted by H_0.

$$\int_0^{\infty} \bar{J}e^{-\frac{\bar{J}^2}{2\sigma^2}} \operatorname{sgn}(\sqrt{\sigma}\cos\phi - \sqrt{\bar{J}}\cos\bar{\phi}) = \sigma^2 \operatorname{sgn}(\cos\phi) - 2\sigma^2 \operatorname{sgn}(\cos\bar{\phi}) H_0\left(\frac{\cos\phi}{\cos\bar{\phi}}\right) e^{-\frac{\cos^4\phi}{\cos^4\bar{\phi}}} \tag{20}$$

Thus, the equation determining the g_l's becomes

$$\frac{\partial g_l}{\partial s} + \frac{il}{\beta} g_l \mp \frac{1}{\sqrt{\pi}} \frac{4Nr_e}{\pi L_x\gamma} \sqrt{\frac{\beta}{\sigma}} \delta_p(s) \sum_{k=-\infty}^{\infty} g_k M_{lk} = \frac{\lambda}{L} g_l \tag{21}$$

where

$$M_{lk} = i \int_0^{2\pi} d\phi \int_0^{2\pi} d\bar{\phi} \operatorname{sgn}(\cos\bar{\phi}) H_0\left(\frac{\cos\phi}{\cos\bar{\phi}}\right) \sin\phi \sin l\phi \cos k\bar{\phi} e^{-\frac{\cos^4\phi}{\cos^4\bar{\phi}}} \tag{22}$$

The matrix M has the following properties

$$\begin{aligned} M_{lk} &= -2i \int_{\pi/2}^{3\pi/2} \int_{\pi/2}^{3\pi/2} d\phi d\bar{\phi} \sin\phi \sin l\phi \cos k\bar{\phi} e^{-\frac{\cos^4\phi}{\cos^4\bar{\phi}}} \text{ for } l+k = \text{even} \\ M_{lk} &= 0 \text{ for } l+k = \text{odd} \\ M_{l,-k} &= M_{lk} \\ M_{-l,k} &= -M_{lk} \end{aligned} \tag{23}$$

No attempt is made to solve the remaining double integral analytically. Introducing a beam-beam strength parameter

$$\xi \equiv \frac{4Nr_e}{\pi L_x \gamma}\sqrt{\frac{\beta^*}{\sigma}} \tag{24}$$

where β^* denotes the beta function at the interaction point one obtains the following relation for the g_l's immediately before and immediately after the interaction point by integrating through the interaction point.

$$g_l(0^+) - g_l(0^-) = \pm\frac{\xi}{\sqrt{\pi}}\sum_{k=-\infty}^{\infty} M_{lk} g_k(0^-) \tag{25}$$

There is no coupling among different Fourier components between collisions. In this case eqn. 21 simplifies to

$$\frac{\partial g_l}{\partial s} + \frac{il}{\beta(s)} g_l = \frac{\lambda}{L} g_l \tag{26}$$

which is solved by

$$g_l(L^-) = g_l(0^+) e^{\lambda - il\int_0^L \frac{1}{\beta(s)} ds} = g_l(0^+) e^{\lambda - il\phi} \tag{27}$$

III. DYNAMIC TUNE

We calculate the tune ν in terms of the unperturbed tune ν_0 which allows us to express our final result in terms of ν_0. This is more convenient since we usually neglect beam-beam effects when doing the lattice design. The tune shift due to synchrotron radiation is many orders of magnitude lower and is neglected. Eqn. 28 expresses the perturbed betatron tune ν in terms of the unperturbed tune ν_0 and the difference in the focusing structure.

$$\cos 2\pi\nu = \cos 2\pi\nu_0 - \frac{1}{2}\sin 2\pi\nu_0 \oint \beta(s)(F(s) - K(s))ds \tag{28}$$

Evaluating the integral in eqn. 10 gives

$$\frac{1}{\sqrt{2\pi}\sigma\pi}\int_{-\infty}^{\infty} d\bar{y}\,\mathrm{sgn}(y-\bar{y})\int_{-\infty}^{\infty} d\bar{y}' e^{-\frac{\bar{y}^2}{2\sigma^2}} = \frac{2\sqrt{2}}{2\beta\sqrt{2\pi}\sigma\pi}\int_0^y d\bar{y} e^{-\frac{\bar{y}^4}{16\beta^2\sigma^2}} K_{1/4}\left(\frac{\bar{y}^4}{16\beta^2\sigma^2}\right)\bar{y} = \frac{2^{3/4}}{\sqrt{\pi\beta\sigma}\Gamma(\frac{3}{4})} y + O(y^2) \tag{29}$$

where we have used the linearization eqn. 11 and expanded the integrand keeping only the constant term. $K_\nu(x)$ denotes the modified Bessel function. Thus,

$$\cos 2\pi\nu = \cos 2\pi\nu_0 - \frac{2^{3/4}\pi^{3/2}}{2\Gamma(\frac{3}{4})}\xi \sin 2\pi\nu_0 \tag{30}$$

IV. COHERENT BEAM-BEAM INSTABILITY

We can summarize the solution of the equations of motion by introducing a matrix that acts on a coloumn vector G which contains all g_l's. Therefore,

$$G(L) = TG(0) \tag{31}$$

where

$$T = R\left(\mathbf{1} \pm \frac{\xi}{\sqrt{\pi}} M\right) \tag{32}$$

and R is a diagonal matrix which has entries $e^{-2\pi il\nu}$ (with $\phi = 2\pi\nu$) for all $l \in \mathbf{Z}$ on its diagonal. Beam-beam instability occurs if one of the eigenvalues of T has eigenvalues whose absolute value is larger than 1.

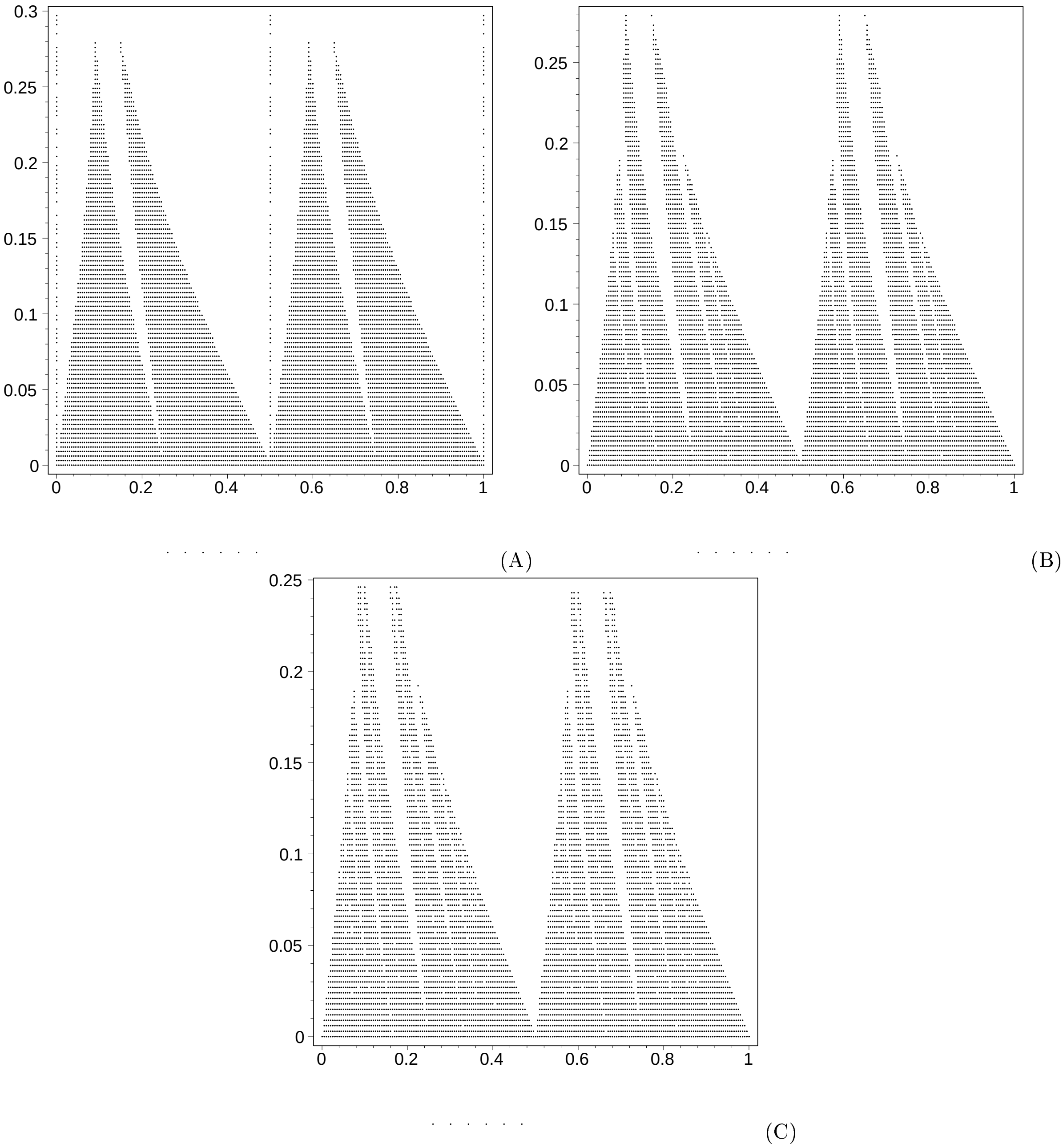

FIG. 1: Stability diagrams for $\lambda = 0.00$. Resonances up to second (A), fourth (B) and sixth order (C), respectively, have been included. The horizontal axis refers to ν_0 and the vertical axis refers to ξ.

V. RESULTS

In the following plots we have calculated the matrix to the indicated order for both signs and drawn a point at (ν_0, ξ) if all eigenvalues of T have an absolute value smaller or equal 1.

VI. DISCUSSION

The coarse structure of all plots is shown in fig. 4 where we plotted the region in which the dynamic tune becomes complex. In this region the accelerator cannot maintain a stable beam. All plots have this basic structure in common. Including resonances up to higher and higher orders the plots get more and more complicated. Small "joints" enter

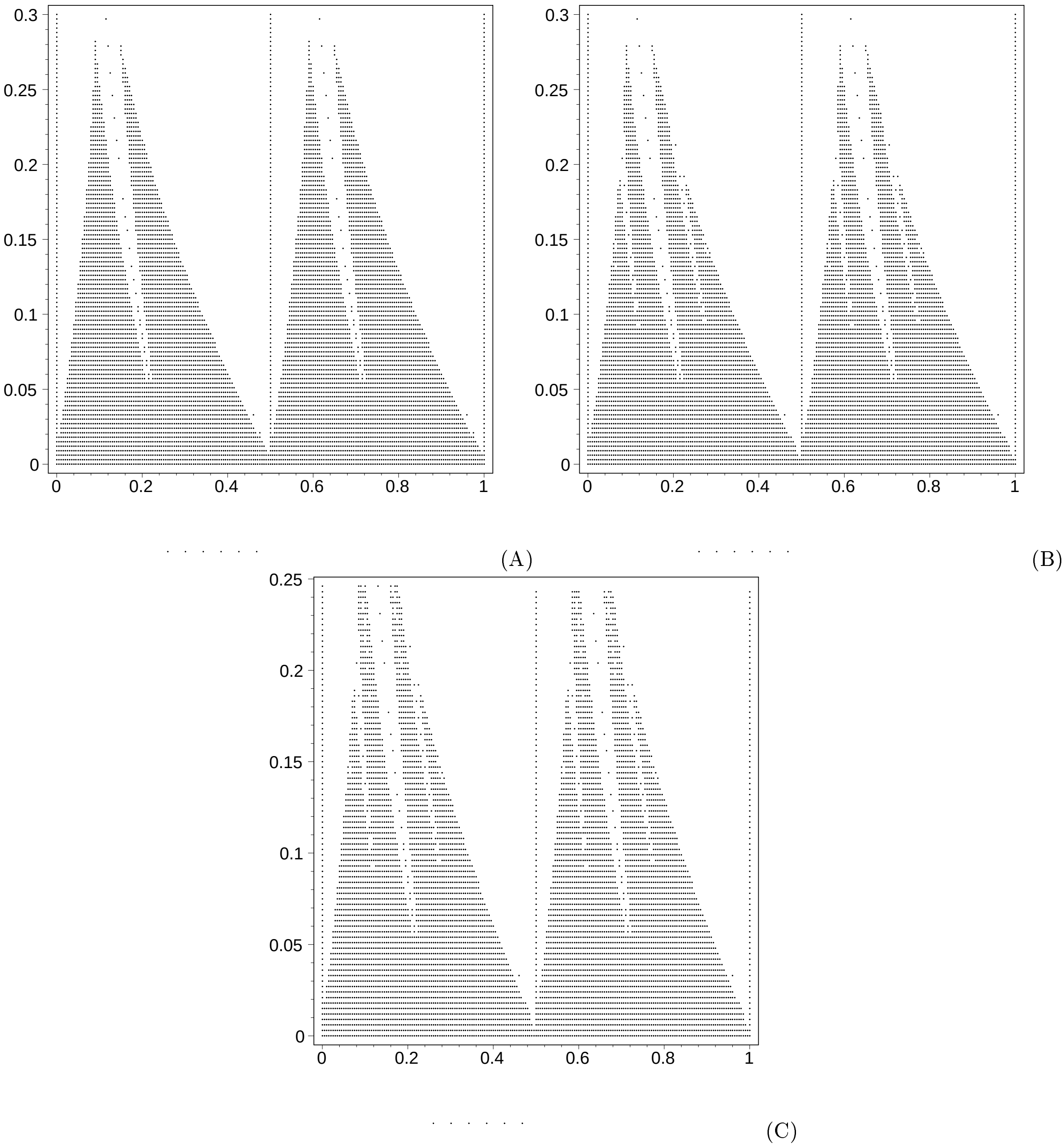

FIG. 2: Stability diagrams for $\lambda = 0.05$. Resonances up to second (A), fourth (B) and sixth order (C), respectively, have been included. The horizontal axis refers to ν_0 and the vertical axis refers to ξ.

the diagrams which disappear again when synchrotron radiation damping is increased. As a rule of thumb the beam is stable if the operating point lies in the shaded area of fig. 4 and satisfies $\xi < \lambda$. However, this rule is a bit too restrictive.

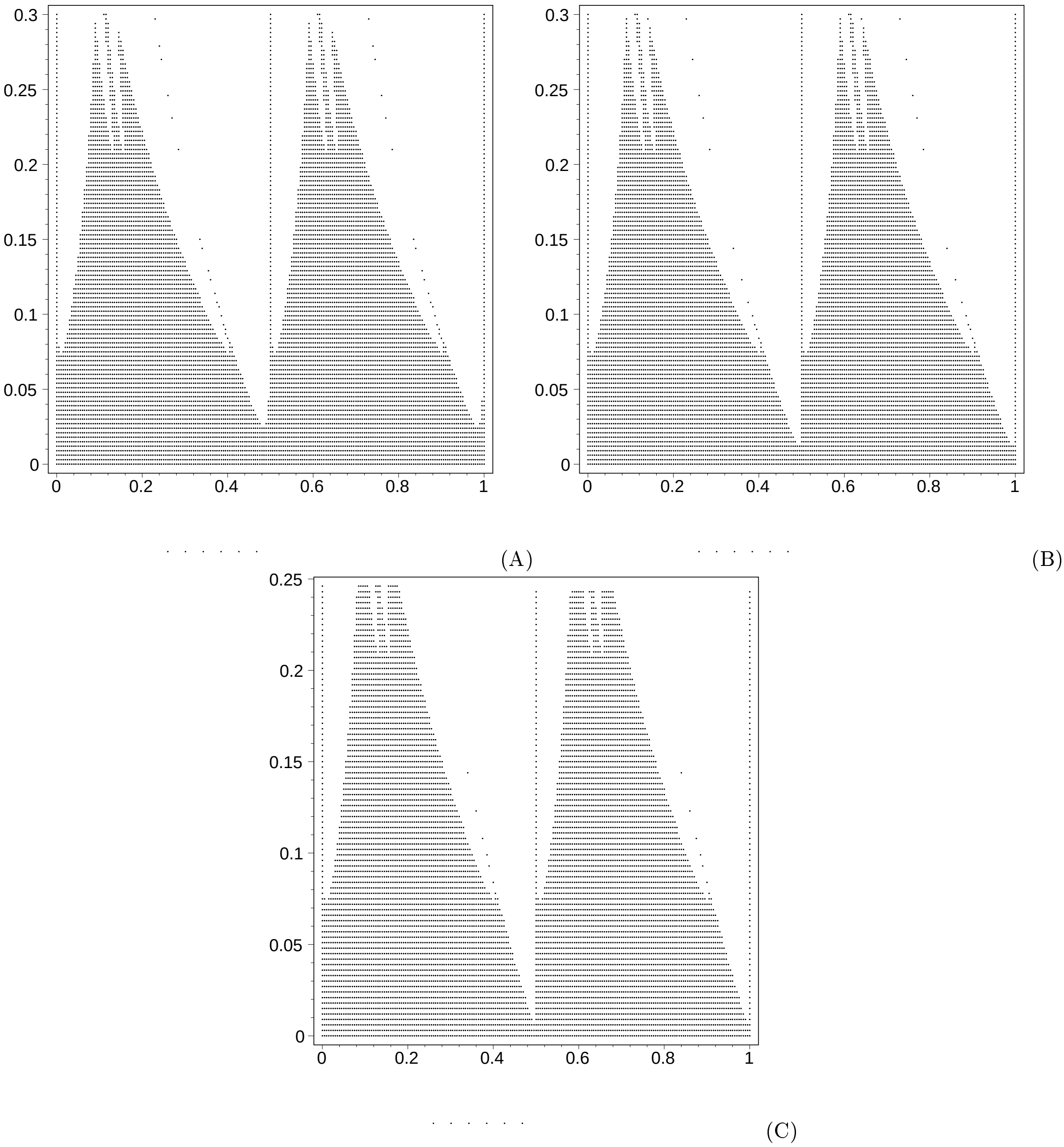

FIG. 3: Stability diagrams for $\lambda = 0.20$. Resonances up to second (A), fourth (B) and sixth order (C), respectively, have been included. The horizontal axis refers to ν_0 and the vertical axis refers to ξ.

VII. ACKNOWLEDGMENTS

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[1] A. W. Chao, R. D. Ruth, "Coherent Beam-Beam Instability in Colliding-Beam Storage Rings", Particle Accelerators, 1985, Vol. 16, pp. 201-216, Gordon and Breach

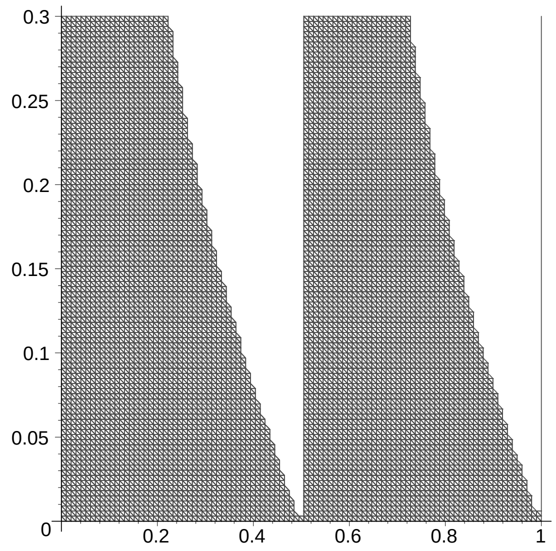

FIG. 4: Area in which the dynamic tune is real. The horizontal axis refers to ν_0 and the vertical axis refers to ξ.