Relativistic Turbulence: A Long Way from Preheating to Equilibrium

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We study, both numerically and analytically, the development of equilibrium after preheating. We show that the process is characterised by the appearance of Kolmogorov spectra and the evolution towards thermal equilibrium follows self-similar dynamics. Simplified kinetic theory gives values for all characteristic exponents which are close to what is observed in lattice simulations. The resulting time for thermalization is long, and temperature at thermalization is low, $T \sim 100$ eV in the simple $\lambda\Phi^4$ inflationary model. Our results allow a straightforward generalization to realistic models.

Introduction. The dynamics of equilibration and thermalization of field theories is of interest for various reasons. In high-energy physics understanding of these processes is crucial for applications to heavy ion collisions and to reheating of the early universe after inflation. Inflation solves the flatness and the horizon problems of the standard big bang cosmology and provides a calculable mechanism by which initial density perturbations were generated [1]. At the end of inflation the Universe was in a vacuum-like state. In the process of decay of this state and subsequent thermalization (reheating) the matter content of the universe is created. It was realized recently that the initial stage of reheating, dubbed preheating [2], is a fast, explosive process. This initial stage by now is well understood [3–7]. Strong and fast amplification of fluctuation fields at low momenta may lead to various interesting physical effects, like non-thermal phase transitions [8], peculiar baryogenesis [9], generation of high-frequency gravitational waves [10], etc.

Understanding of the subsequent stages of reheating and thermalization processes and calculation of the final equilibrium temperature is important for various applications, most notably baryogenesis and the problem of over-abundant gravitino production in supergravity models [11]. Thermalization of field theories was discussed already, see e.g. Refs. [12]. However, at present the process of thermalization after preheating is still far away from being well understood and developed. The problem is that at the preheating stage the occupation numbers are very large, of order of the inverse coupling constant. In addition, in many models the zero mode does not decay completely. Therefore, a simple kinetic approach is not applicable.

Fortunately, the description in terms of classical field theory is valid in this situation [3], and the process of preheating, as well as subsequent thermalization, can be studied on a lattice. In this paper we adopt this approach. Our goal is to integrate the system on a lattice sufficiently accurately and sufficiently far in time to be able to see generic features, and possibly to the stage, at which the kinetic description becomes a good approximation scheme. Lattice studies of thermalization, similar to ours, were done in Ref. [13]. Several generic rules of thermalization were formulated, like the early equipartition of energy between coupled fields. However, the problem is very complicated and there are other unanswered important questions like what is the final thermalization temperature, at what stage the kinetic description becomes valid, what is the functional form of particle distributions during the thermalization stage, etc.

For our study we use a higher accuracy, improved version of the LATTICEEASY code [14]. We show that the distribution functions follow a *self-similar* evolution related to the turbulent transport of wave energy. This property enables us to estimate the physical reheating temperature, which turns out to be very low. The concept should be rather model independent since typical ranges of particle momenta at preheating and in thermal equilibrium are widely separated. However, in this letter we will restrict our numerical integration (but not the discussion) to the "minimal" inflationary model, the massless $\lambda\Phi^4$-theory.

The Model. With conformal coupling to gravity and after a rescaling of the field, $\varphi \equiv \Phi a$, where $a(t)$ is the cosmological scale factor, the equation of motion in comoving coordinates describes a φ^4-theory in Minkowski space-time,

$$\Box\varphi + \lambda\varphi^3 = 0. \tag{1}$$

At the end of inflation the field is homogeneous, $\varphi = \varphi_0(t)$. Later on fluctuations develop, but the homogeneous component of the field, which corresponds to the zero momentum in the Fourier decomposition, may be dynamically important and is referred to as the "zero-mode." In such situations it is convenient to make a further rescaling of the field, $\phi \equiv \varphi/\varphi_0(t_0)$, and of the space-time coordinates, $x^\mu \to \sqrt{\lambda}\varphi_0(t_0)x^\mu$, which transforms the equation of motion (1) into dimensionless and parameter free form,

$$\Box\phi + \phi^3 = 0\,. \tag{2}$$

Here t_0 corresponds to the initial moment of time (end of inflation), and in what follows we denote dimensionless time as τ. With this rescaling the initial condition for the zero-mode oscillations is $\phi_0(\tau_0) = 1$. All model dependence on the coupling constant λ and on the initial amplitude of the field oscillations now is encoded in the initial conditions for the small (vacuum) fluctuations of the field with non-zero momenta [3]. The physical normalization of the inflationary model corresponds to a dimensionful initial amplitude of $\varphi_0(t_0) \approx 0.3M_{\rm Pl}$ and a coupling constant $\lambda \sim 10^{-13}$ [1]. The re-parametrization property of the system allows to chose a larger value of λ for numerical simulations. We have used $\lambda = 10^{-8}$.

Numerical Procedure and Results. We use a 3-D cubic lattice with periodic boundary conditions. The finite-differences scheme that was used is 2nd order in time and 4-th order in space. The results displayed here are taken from a simulation with 256^3 sites and a physical box size $L = 14\pi$. With this box size the infrared modes which belong to the resonance band are still well represented, while the ultraviolet lattice cut-off is sufficiently far away from the occupied modes, such that the particle spectra are not distorted even at late times. We have studied the dependence of our results on the lattice- and the box size to avoid lattice artifacts. Various quantities are measured and monitored both in configuration space (zero mode, $\phi_0 \equiv \langle\phi\rangle$, and the variance, $var(\phi) \equiv \langle\phi^2\rangle - \phi_0^2$) and in the Fourier space. Using fourier transformed fields we first define the wave amplitudes (which correspond to annihilation operators in the quantum problem),

$$a(\vec{k}) \equiv \frac{\omega_k\phi_{\vec{k}} + i\dot{\phi}_{\vec{k}}}{(2\pi)^{3/2}\sqrt{2\omega_k}}\,. \tag{3}$$

The effective frequency $\omega_k \equiv \sqrt{k_{\rm D}^2 + m_{eff}^2}$ is determined by the effective mass $m_{eff}^2 = 3\lambda\langle\phi^2\rangle$ and the inverse $k_{\rm D}^2$ of the lattice Laplacian. In our numerical scheme the latter is given by

$$k_{\rm D}^2 = b^{-2}\sum_{i\in\{1,2,3\}}\left(\frac{5}{2} - \frac{8}{3}\cos(bk_i) + \frac{1}{6}\cos(2bk_i)\right). \tag{4}$$

Here $b = 2\pi/L = 1/7$ is the lattice constant. Making use of a_k, we calculate various correlators, $n(k) \equiv \langle a^\dagger a\rangle$, $\sigma(k) \equiv \langle aa\rangle$, $\langle a^\dagger a^\dagger aa\rangle$, etc. The first one, which corresponds to the particle occupation numbers, is of prime interest.

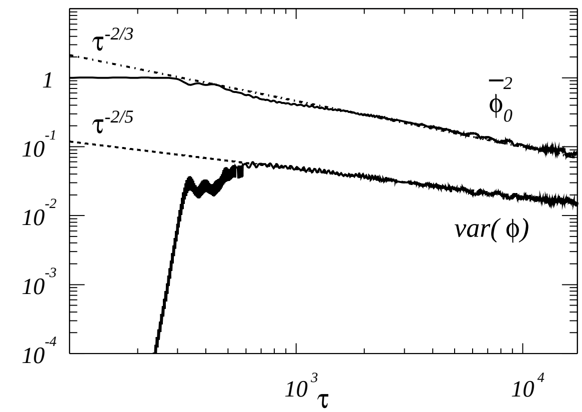

FIG. 1. Amplitude of the zero-mode oscillations, $\overline{\phi}_0^2$, and variance of the field fluctuations as functions of time τ.

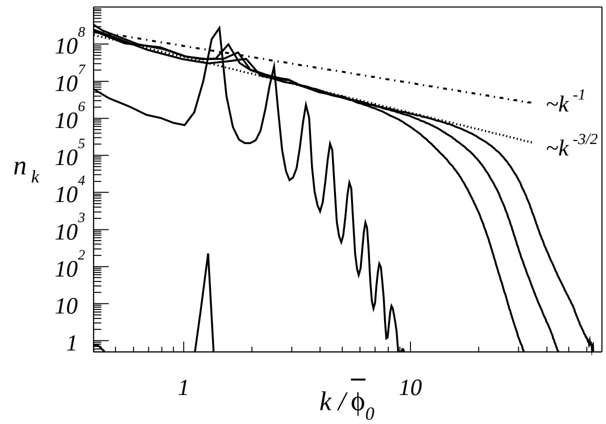

FIG. 2. Occupation numbers as function of $k\overline{\phi}_0^{-1}$ at $\tau = 100, 400, 2500, 5000, 10000$.

We begin the discussion of our numerical results with the evolution of the zero-mode and the variance of the field, which are shown in Fig. 1. Initially we see an exponentially fast transfer of the zero-mode energy into fluctuations during preheating (up to $\tau \sim 300$). It is followed by a long and slow relaxation phase. In this regime ($\tau > 1500$) the amplitude of the zero mode oscillations decreases according to $\sim \tau^{-1/3}$, the variance of the field (averaged over high-frequency oscillations) drops according to $\sim \tau^{-2/5}$. This is consistent with previous results [3]. In addition we find that in this regime the zero-mode is in a non-trivial dynamical equilibrium with the bath of highly occupied modes: when the zero-mode is artificially removed, it is recreated on a short time-scale (Bose condensation).

At early times the distribution functions of particles over momenta, see Fig. 2, have peaky structure. The first peak which corresponds to the parametric resonance is initially at the theoretically predicted value of $k \sim 1.27$ [5]. Later ($\tau > 1500$) the spectra become smooth and at small k approach a power-law, $n_k \sim k^{-s}$, where s fluctuates in the range of $1.5 - 1.7$, depending on time and the range of k where it is fitted. This power law clearly differs from the the classical thermal equilibrium, $n_k \sim \omega_k^{-1}$. It is followed by the exponential cut-off, whose position monotenously shifts with time towards higher k. Pumping of energy from the zero mode stays effective all the times (note a small bump in the particle distributions in Fig. 2 at $k \sim 1$). It corresponds to the annihilation of four condensed particles into two quanta. Rescattering of two particles into two particles is also effective. One of the two can belong to the zero-mode condensate either in initial or in the final two particle state. We also can see in Fig. 2 that in the power-law region n_k is a function of $k/\overline{\phi}_0$ only, where $\overline{\phi}_0(\tau)$ represents the amplitude of the zero-mode at time τ. This effect can be related to the above described dynamical equilibrium between zero-mode and the bath of particles. Indeed, going from Eq. (1) to Eq. (2) we can rescale by the current amplitude of the zero-mode.

The picture presented in Fig. 2 at late times resembles stationary Kolmogorov turbulence. It appears as such due to rescaling of momenta by the amplitude of the zero-mode, but in fact in the present model the turbulence can not be stationary because the amplitude of the zero-mode (i.e. the strength of the source of turbulence) decreases. Further examination of Fig 2 suggests that the evolution of particle spectra may be self-similar. We have tried therefore the following anzatz

$$n(k,\tau) = \tau^{-q} n_0(k\tau^{-p})\,. \tag{5}$$

Spectra rescaled at several moments of time by the relation inverse to Eq. (5) are shown in Fig. 3. We have found that the evolution is indeed self-similar with $q \approx 3.5p$ and $p \approx 1/5$.

Discussion. Here we discuss the question whether a simple kinetic theory gives predictions for turbulence and self-similarity exponents in agreement with the lattice calculations. Our lattice study of higher order correlators, like $\langle a^\dagger a^\dagger a a\rangle$, shows that the field distribution is very close to Gaussian, see also [12,13]. This facilitates the use of the kinetic approach. On the other hand we have found that the magnitude of $\sigma(k)$ is of order of a few percent compared to $n(k)$, and it is even larger in the region of resonant momenta. This means that the strict kinetic approach should include $\sigma(k)$. Nevertheless we neglect these effects and write a kinetic equation in a simple form $\dot{n_k} = I_k$, where the collision integral for a m-particle interaction is given by

$$I_k = \int d\Omega_k\, U_k\, F[n]\,. \tag{6}$$

In d spatial dimensions the integration measure $d\Omega_k$ is given by $m-1$ integrations over d-dimensional Fourier space. We include in it the energy-momentum conservation δ-functions. But we do not include there the relativistic $1/\omega(k_i)$ "on-shell" factors, which instead appear in the "matrix element" of the corresponding process, U_k. This will make the discussion of relativistic and non-relativistic cases uniform. The function $F[n]$ is a sum of products of the type $n_{k_j}^{-1}\prod_{i=1}^{m} n_{k_i}$, where $j \in \{1,\ldots,m\}$ with appropriate signs and permutations of indices for incoming and outgoing particles. All dynamical aspects of turbulence follow from the scaling properties of the system [15]. Let ω_k, n_k and U_k have defined weights under a ξ-rescaling of Fourier-space,

$$\begin{aligned} \omega(\xi k_i) &= \xi^{\alpha}\omega(k_i)\,, \\ U(\xi k_1,\ldots,\xi k_m) &= \xi^{\beta} U(k_1,\ldots,k_m)\,, \\ n(\xi k_i) &= \xi^{\gamma} n(k_i)\,. \end{aligned} \tag{7}$$

The weight of the full collision integral under this reparametrization is

$$I_{\xi k} = \xi^{d(m-2)-\alpha+\beta+(m-1)\gamma} I_k\,. \tag{8}$$

It follows that the stationary turbulence with constant energy flux over momentum space is characterised by a power-law distribution function, $n_k \sim k^{-s}$, where $s = d + \beta/(m-1)$. The scaling properties also give the exponents of the self-similar distribution, Eq. (5). Assuming energy conservation in particles and with $\xi = \tau^{-p}$ we find $q = 4p$ and $p = 1/((m-1)\alpha - \beta)$. For stationary turbulence we find that p should be $(m-1)$ times larger.

For a $\lambda\phi^4$-theory in three spatial dimensions and four-particle interaction we have $m = 4$, $\beta = -4\alpha$ and $\alpha = 1$. In this case $s = 5/3$ and $p = 1/7$. For three-particle interaction (the fourth particle belongs to the condensate in this case and the matrix element contains an additional factor of $\overline{\phi}_0^2$) we find $s = 3/2$ and a smaller value for p

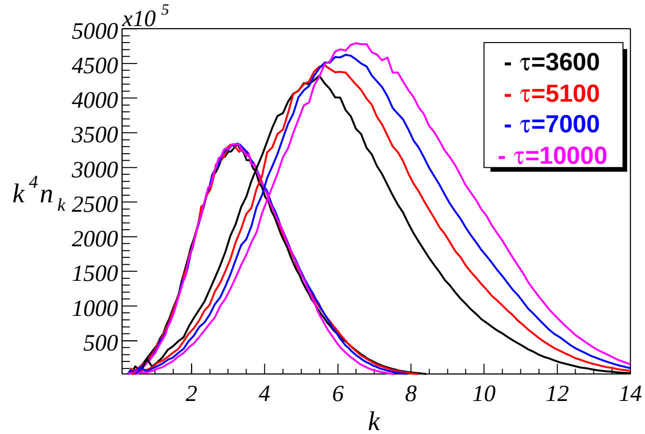

FIG. 3. On the right hand side we plot the wave energy per decade found in lattice integration. On the left hand side are the same graphs transformed according to the relation inverse to Eq. (5).

compared to the previous case. We can not distinguish between 5/3 and 3/2 for s in our numerical integrations, s rather fluctuates between these two numbers, while 1/7 for p gives a fit to the data not as good as displayed in Fig. 3. However, during the integration time the energy in particles is neither conserving, nor there is a stationary source of energy. Namely, starting from the time at which the solution becomes self-similar, $\tau \sim 3000$, to the end of our integration, the energy influx from the zero mode to particles is about 20 %. Correcting for this energy influx we find $q \approx 3.5p$ and $p \approx 1/6$. This should be considered as satisfactory agreement given the simplifications which were made.

Equilibration time and temperature. At late times the influence of the zero-mode should become negligible, but we still may expect the self-similar character of the evolution. Solution Eq. (5) with $p = 1/7$ should be valid in this case. This allows us to find the time needed to reach equilibrium. Indeed, the classical evolution will continue until the occupation numbers in the region of the peak in Fig. 3 will become of order one. At this time quantum effects become important and the distribution relaxes to thermal. Values of momenta were this happens are $k_{\rm max} \sim \lambda^{1/4}\varphi_0(\tau_0)$. On the other hand the initial distribution is centred around $k_0 \sim \lambda^{1/2}\varphi_0(\tau_0)$ and moves to ultraviolet according to Eq. (5) as $\propto k_0\tau^p$. It follows that the time to reach equilibrium is $\tau \sim \lambda^{-7/4} \sim 10^{23}$, where in the second equality we assumed the normalization to the inflationary model. For the reheating temperature we find, rotating back from the conformal reference frame, $T_R \sim k_{max}/a(\tau) \sim \lambda^2\varphi_0(t_0) \sim 10^{-26}M_{\rm Pl} \sim 100$ eV, where for the conformal scale factor we have used $a(\tau) = \tau$.

Conclusions. Reheating after preheating appears to be a rather slow process. Although the "effective temperature" measured at low momentum modes during preheating may be high, in the model we have considered the resulting true temperature is parametrically the same as what could have been obtained in "naive" perturbation theory. Namely, equating the rate of scattering in thermal equilibrium to the Hubble expansion rate one obtains $T \sim \lambda^2 M_{\rm Pl}$ in this model. We anticipate this result should be applicable to more realistic models of inflation. Note that realistic models involve many fields and interactions and larger coupling constants will determine the true temperature.

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