ADIABATIC THEORY OF ELECTRON OSCILLATIONS AND ITS APPLICATION TO SIS100/SIS200

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Abstract

We consider an accumulation of the long-lived ionization electrons in the electron cloud, which appears in the storage ring around the bunched ion beam in presence of ion leakage in the gap. In the frame of a one-dimensional model, transverse electron motion is defined by the second order non-linear differential equation with periodic coefficients depending on the ion longitudinal density. For 'smooth' density distributions an approximate solution of the equation can be written in adiabatic form. Adiabaticity perturbations results in half-integer resonances with strengths defined by leakage factor and neutralization degree. The action of these resonances in presence of non-linearity limits the 'survival' region where electrons can be accumulated. Electron concentration in this region is defined by the balance between electron creation due to ionization and electron losses due to electron scattering on primary ions. An estimation of neutralization degrees for SIS100/SIS200 (the rings now under design in GSI) has shown that for reasonable leakage factors and nominal gas pressure the electron concentration is small.

1 INTRODUCTION

An interaction of the electron cloud with the circular ion beam can result in development of electron-ion dipole instability, which was forecast many years ago [1]–[3]. Recently this instability attracted significant attention due to its experimental observation in high-current proton beams (see, for example, Ref. [4]). The instability is especially dangerous for ions with high charge number due to large ionization cross-sections and large yield of electrons from ions hitting the wall of the vacuum chamber.

A new accelerator complex is currently under construction at GSI (Germany) [5]. This complex includes two synchrotrons/storage rings: SIS100 and SIS200. Four ion bunches (for example, ions $^{+28}U^{238}$) should be injected in SIS100 from synchrotron SIS18 with a time interval of 1/3 s. Then the ions are accelerated and injected in SIS200, which is used as a 'stretcher' for physics experiments. Parameters of both machines are given in Table 1.

The goal of this paper is to investigate the electron cloud accumulation in these accelerators. For chosen beam parameters the number of electrons born due to secondary emission (SEM) from the wall seems to be comparatively small. Thus we limit ourselves to investigation of long-lived electrons born inside the beam due to the ionization of the residual gas. Accumulation of such electrons is possible only if part of the ions escape from the bunch in the gap [6]. If the electron space charge density is less than the minimal ion density in the gap such ions provide the focusing and give to electrons a possibility to survive after a passage of many bunches.

Table 1: Parameters of SIS100 and SIS200

Circumference (m)	1080	1080
Energy (MeV/u)	100	1000
Process time (s)	1	1
Number of bunches	4	None (1)
Kind of ions	$^{238}U^{+28}$	$^{238}U^{+28}$
Number of ions in each bunch N_b	2.5×10^{11}	10^{12}
Bunch length L_b (m)	216	(864)
rms vertical bunch size a_v (m)	0.015	0.01
rms horizontal bunch size a_h (m)	0.015	0.01
Vacuum chamber radius (m)	0.05	0.05
Pressure (10^{-10} mbar, without beam)	0.05	0.1

One-dimensional (vertical) electron oscillations are described by a non-linear equation of the second order whose solution depends on longitudinal and transverse distributions of the ions in the ring (Section 2). In this Section it is shown that for 'smooth' longitudinal distributions (continuous with its derivative) the amplitude of the electron oscillations is defined by the adiabatic law.

In a frame of linear theory non-adiabaticity of oscillations results in a set of half-integer resonances whose strengths are expressed through the trace of a transfer matrix (TrM_T) (Section 3). Examination of these resonances for SIS100 has shown that their effect depends on the number of bunches in the ring (filling scheme) and longitudinal distribution of the ions in the bunch as well as on the values of the leakage factor and neutralization degree. The most dangerous case corresponds to a completely filled ring (four bunches) and a smooth (continuous with its derivative) distribution. If the resonances are crossed due to modulation of the electron bounce frequency (such modulation can appear due to longitudinal electron motion) then these resonances result in electron heating.

The action of non-linearity results in the appearance of a 'physical chamber aperture' where the electrons can survive for a very long time (Section 4). The value of this aperture depends on the values of the leakage factor and neutralization degree, as well as on the longitudinal distribution of the ions.

These results are applied to the calculation of the equilibrium neutralization degree (Section 5). The scheme is the following:

1) The main source of electrons is ionization of residual gas.

2) The rate of heating is defined by electron scattering on the ions of the primary beam.

3) An electron is lost when its adiabatic invariant corresponds to the 'physical aperture'.

The analysis results in the expression for an equilibrium neutralization degree similar to the expression derived earlier for coasting beams [7]. However, in a bunched beam the neutralization degree is decreased as the third power of the dimensionless (divided by the r.m.s. ion beam size) physical aperture of the electron oscillations.

Application of this theory to SIS100 and SIS200 (Section 6) has shown that for both machine and nominal (very low) pressure the expected values of neutralization degree are small. However, the pressure increase (for example due to desorption of the gas from the walls) can change the situation.

2 TRANSVERSE ELECTRON OSCILLATIONS AND ADIABATIC INVARIANT

A dimensionless equation of one-dimensional (vertical) electron oscillations can be written as follows:

$$y'' + (2\pi Q_0)^2 F(\tau) y \Phi(x,y) = 0 \ , \qquad (1)$$

where $y = Y/a_e, x = X/a_e$ (Y, X = vertical and horizontal electron deviations, a_e = r.m.s. transverse beam size), independent variable $\tau = t/T$ (t = time, T = period of the ion line density variation); Q_0 = 'average electron betatron tune', equal to the number of betatron oscillations on the bunch length for uniform ion density.

$$Q_0 = \sqrt{\frac{N_b r_e Z_i R}{\pi \beta^2 a^2 h}} \ ,$$

where r_e is the classical electron radius, N_b is the number of ions inside the bunch, β is the ion relativistic parameter, Z_i is the charge ion number, R is the ring radius, h the number of bunches; the function $\Phi(x,y)$ defines the transverse distribution of the gradient. For a round Gaussian beam $\Phi(x,y) = (1 - \exp[-(x^2+y^2)])/(x^2+y^2)$. In Eq. (1), the 'instantaneous tune' $\Omega(\tau) = 2\pi Q_0 \sqrt{F(\tau)\Phi(x,y)}$.

Longitudinal distribution of the charge density in the bunch $F(\tau) = [Z_i \lambda_i(\tau) - \lambda_e]/\langle [Z_i \lambda_i(\tau) - \lambda_e] \rangle$, where $\lambda_i(\tau)$ is an ion longitudinal density inside the bunch, λ_e is an electron longitudinal density (uniform), the sign $\langle\rangle$ means averaging on the bunch length.

We have considered four models (in all cases the density in the gap is uniform):

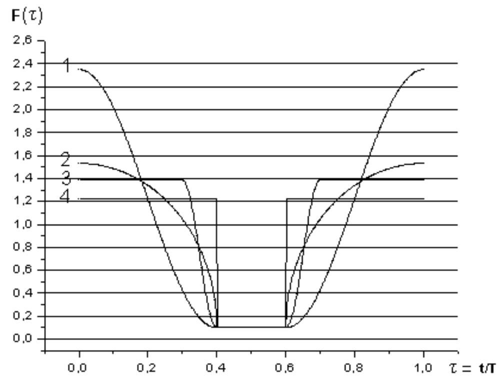

Figure 1: Different longitudinal distributions used in calculations (1 – smcos, 2 – elliptic, 3 – cosine, 4 – square).

1) The 'square' model with uniform density in the bunch.

2) The 'elliptical model' with elliptical density in the bunch.

3) The 'cosine model' with flat top of the bunch and cosine law in the bunch edge.

4) The smooth 'cosine model' with cosine density in the bunch.

These distributions are plotted in Fig. 1. Let us remark that the first model has breaks in the function and its derivative, the second one – only in derivative, the third and fourth functions are continuous with derivatives.

As is well known [8] for 'good' functions (positive and continuous with their derivatives) the 'adiabatic invariant' is approximately conserved. In our case the adiabatic invariant is

$$I(y_{\max}, \tau) = 4 \int_0^{y_{\max}} y' dy = 8\pi Q_0 \sqrt{F(\tau)} \int_0^{y_{\max}} \sqrt{H(y_{\max}) - H(y)} dy, H(y) = \int_0^y u \Phi(x,u) du \ . \qquad (2)$$

The maximal value of action corresponds to the gap centre ($\tau = 0.5$) and $y_{\max} = b$ (b is the ratio of the vacuum chamber aperture to the beam size a). In another point of the bunch $y_{\max}$ is defined by the equation $I(y_{\max}, \tau) = I(b, 0.5)$. Using this expression, we can find the dependence of $y_{\max}$ on τ for different values of the parameters χ, η (the 'gap density parameter' χ is equal to the ratio of ion density in the gap to ion density in the centre of the bunch, the 'neutralization degree' $\eta = N_e/Z_i N_i$ is the relation of the number of electrons in the ring N_e to the number of ions in the ring N_i). A typical dependence of the electron beam size on τ for different values of $\chi(\eta = 0)$ is given in Fig. 2.

Adiabaticity criterion:

$$K_{ad} = \frac{d\Omega(\tau)}{d\tau} T/\Omega(\tau) = \frac{dF(\tau)}{d\tau} T/F(\tau) \ll 1 \ . \qquad (3)$$

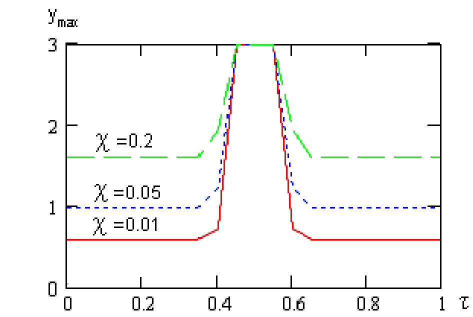

Figure 2: Dependence of normalized electron beam size $u = y_{\max}(s,\chi)$ on τ for different values of parameter $\chi(\eta = 0)$.

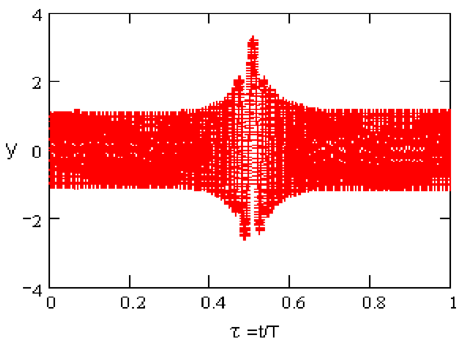

Figure 3: Trajectory for $\chi = 0.01$. Maximal deviation is equal to 3.19 (in accordance with adiabatic theory 3.16).

The adiabaticity criterion depends on the form of longitudinal distribution, as well as on the variables χ, η, τ; it reaches maximal value near the bunch edge. Let us remark that the adiabaticity confines even for large values of the adiabaticity criterion. For illustration let us see the example of trajectory shown in Fig. 3.

3 LINEAR OSCILLATIONS

An adiabatic solution in the linear case is:

$$\begin{aligned} y &= a\varphi(\tau) + CC, \varphi(\tau) \\ &= \exp[i2\pi Q_0 \int_0^\tau \sqrt{F(\tau_1)}d\tau_1]/\sqrt{2\pi Q_0\sqrt{F(\tau)}}\ . \end{aligned} \quad (4)$$

Here a is the complex amplitude, CC means complex conjugate number, $\varphi(\tau)$ is the 'adiabatic Floquet function'; the adiabatic invariant $I = 4|a|^2$. The adiabaticity perturbations result in amplitude perturbations. Using the method for the complex amplitude variation we obtain:

$$a' = -\frac{i}{2}\left\{a\varphi(\tau)\left[-\frac{\Omega''}{2\Omega} + \frac{3(\Omega')^2}{4\Omega^2}\right] + CC\right\}\varphi^*(\tau)\ . \quad (5)$$

Analysis of the equation shows that the adiabaticity perturbations produce a set of half-integer resonances with strength depending on $\Omega'(\tau), \Omega''(\tau)$. The resonance strengths can be calculated using standard matrix procedure. Eigenvalues of the transfer matrix $M_T \lambda_{1,2} = Tr(M_T)/2 \pm \sqrt{[Tr(M_T)/2]^2 - 1}$. If $|Tr(M_T)| < 2$, eigenvalues are imaginary and the motion is stable. In the opposite case a motion is unstable, and resonance strength

$$g = \ln\left[|Tr(M_T)|/2 + \sqrt{[Tr(M_T)/2]^2 - 1}\right]\ . \quad (6)$$

Owing to longitudinal motion the electrons cross these resonances. Using the theory of fast resonance crossing [9], we obtain the average rate of the invariant growth because of half-integer resonances

$$\langle\frac{dI}{d\tau}\rangle \approx \frac{\langle I\rangle}{8}\langle[(0.5TrM_T)^2 - 1]\rangle\ . \quad (7)$$

We see that in the frame of a linear model all electrons should be lost after some time interval. The rate of resonance heating strongly depends on the longitudinal density distribution.

As an example we have examined linear electron dynamics during the injection in SIS100 when five different schemes of bunch location are possible: 1) only one bunch in the ring; 2) two bunches in opposite separatrices; 3) two bunches in neighbouring separatrices; 4) one bunch is absent; 5) all four bunches are present.

The results of calculations have shown that stability strongly depends on the filling schemes and longitudinal distributions. The most unstable, of course, is the simple 'square bucket' model, which has breaks in function. In Fig. 4 we see the 'classical' picture: dependence of the 'focusing factor' $K^1_{foc} = Tr(M_T)/2$ on the leakage factor μ, which is equal to the ratio of the ion number in the gap to the ion number in the bunch. We see that for the 'smooth' model the focusing is much better, and oscillations become stable (i.e. adiabatical) for very small leakage factors.

At Fig. 5 is plotted a dependence of the focusing factor $K^2_{foc} = 0.5|TrM_T| - 1$ on the beam radius for $\mu = 0.1$, $\eta = 0$ (elliptical model). These pictures have a typical resonance character. The resonance strength is much higher for one bunch, then for four bunches.

Owing to random variations of tune the electrons cross the resonances. The heating rate is defined by

$$\langle\frac{dI}{dt}\rangle \approx \frac{\langle I\rangle}{8T}K^3_{foc},\ K^3_{foc} = \langle\frac{Tr(M_T)^2}{2}\rangle - 1\ . \quad (8)$$

At Fig. 6 is plotted a dependence of this factor on μ for $\eta = 0$ and the elliptical model. We see that the filling scheme with four bunches is much more dangerous than the last ones.

Equation (8) shows that in the linear approximation the adiabaticity perturbations result in diffusion which for a long enough time results in the loss of all particles. The situation is changed with field non-linearity.

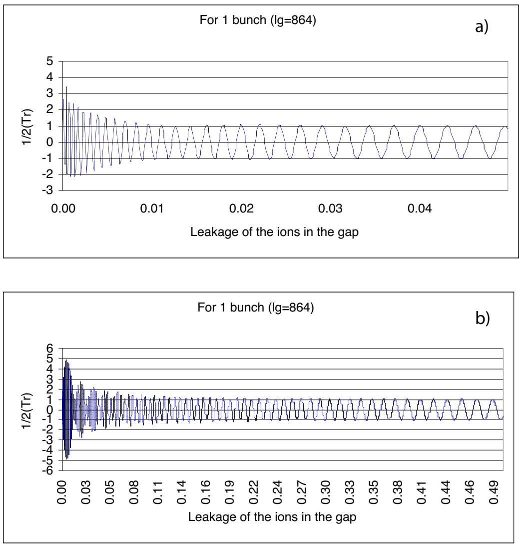

Figure 4: Dependence of K^1_{foc} on μ for single-bunch mode; (a) smooth cosine model, (b) elliptical model.

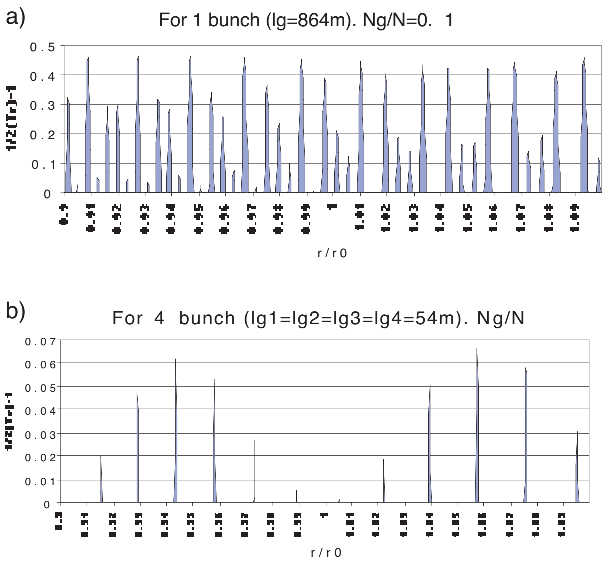

Figure 5: Dependence of K^2_{foc} on the beam size for the elliptical model and $\mu = 0.1$; (a) one bunch, (b) four bunches.

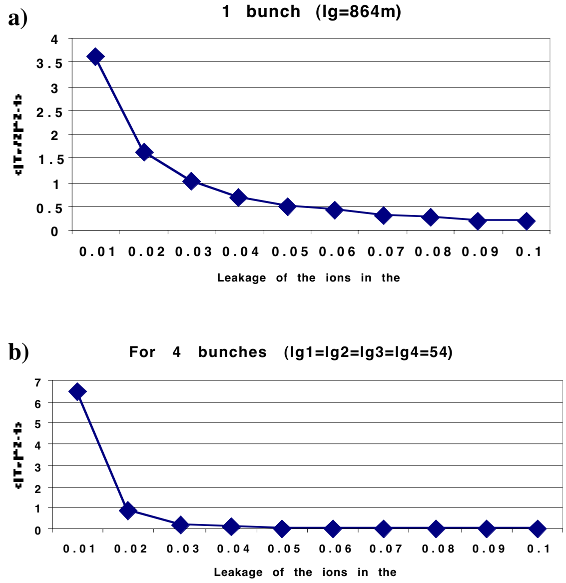

Figure 6: Dependence of K^3_{foc} on μ for the elliptical model; (a) one bunch, (b) four bunches.

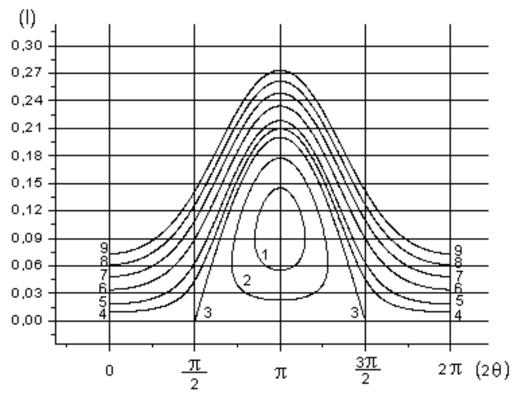

Figure 7: Phase diagram in action-phase plane for half-integer resonance in presence of cubic non-linearity ($k = 5$); curves: $1 - \mathrm{H} = -0.04$; $2 - \mathrm{H} = -0.02$; $3 - \mathrm{H} = 0.0$; $4 - \mathrm{H} = 0.01$; $5 - \mathrm{H} = 0.02$; $6 - \mathrm{H} = 0.04$; $7 - \mathrm{H} = 0.06$; $8 - \mathrm{H} = 0.08$; $9 - \mathrm{H} = 0.1$).

4 NON-LINEAR OSCILLATIONS

As is well known non-linearity stabilizes the oscillations. For illustration let us consider a half-integer resonance in presence of non-linearity. Then the normalized (divided on resonance strength) Hamiltonian $H = kI^2 + I\cos(2\theta)$. The corresponding phase diagram in the I, χ plane is plotted in Fig. 7.

The character of stability depends on the chamber aperture $I_{\max}$. From the diagram we see that if the chamber aperture $I_{\max} > 0.2$ (this value corresponds to the separatrix), for each initial phase there are particles which live infinitely long.

We have calculated the dependence of the electron maximal amplitude at the bunch centre on the time for different numbers of the ions in the beam, different values of 'gap factor' and neutralization $\eta = 0$ (SIS100, 4 bunches, 'cosine model').

We see from Fig. 8 that for high time intervals the amplitude of the surviving particles goes to some limit depending on the gap density factor $\chi(Y(\chi))$. Similar results are obtained for SIS200 (Fig. 9).

In the following text we use the term 'sharp border model': we assume that particles survive only if $Y < Y(\chi)$, (the parameter $Y(\chi)$ will be named 'physical chamber aperture' for the electrons).

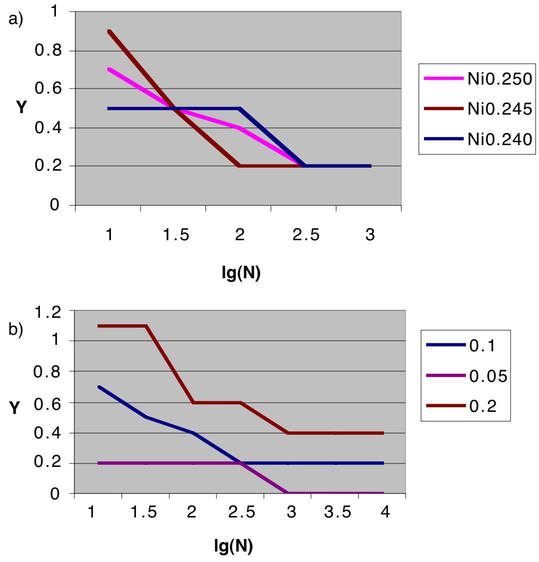

Figure 8: a) SIS100, four bunches, dependence of Y on $N = t/T$, for different values of the number of ions in the bunch in $10^{12} N_i$ (curves: $1 - N_i = 0.245$; $2 - N_i = 0.25$; $3 - N_i = 0.24$); $\chi = 0.1$; $\eta = 0$. b) SIS100, dependence of Y on $N = t/T$, for different values of χ (curves: $1 - \chi = 0.2$; $2 - \chi = 0.1$; $3 - \chi = 0.05$), $(N_i 0 = 0.25 \times 10^{12}), \eta = 0$.

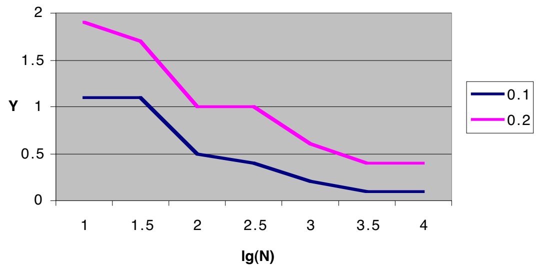

Figure 9: SIS200, dependence of $Y = Y_{AP}(\chi, \eta, N)$ on $N = t/T$, for different values of χ (curves: $1 - \chi = 0.2$; $2 - \chi = 0.1$), $(N_i 0 = 1 \times 10^{12}), \eta = 0$.

5 NEUTRALIZATION DEGREE

The ionization rate per ion may be written as follows

$$\frac{1}{T_{ion}} = \frac{1}{N_i}\frac{dN_e}{dt} = \beta c N_{Losch} P \langle \sigma_{ion} \rangle \; , \qquad (9)$$

where P is the residual gas pressure (in bar), $N_{Losch} = 2.7 \times 10^{19}$ cm^{-3} (Loschmidt number), $\langle \sigma_{ion} \rangle$ = ionization cross-section averaged on beam components; partial ionization cross-section is defined by [10]:

$$\begin{aligned} \sigma_{ion}^m &= Z_i^2 K \frac{\Omega_m(\beta)}{\beta^2}, \\ \Omega_m(\beta) &= C_m + M_m^2 \left(\ln \frac{\beta^2}{1-\beta^2} - \beta^2 \right) \end{aligned} \; . \qquad (10)$$

Here $K = 1.87 \times 10^{20}$ cm^2, the parameters C_m and K_m depend on the kind of gas.

Let us limit ourselves to a case of small neutralization degree. Then the rate of birth for ionization electrons in the ‘survival layer’ is

$$\left(\frac{dN_e}{dt} \right)_{surv} \approx \frac{N_i}{T_{ion}} Y(\chi) \; . \qquad (11)$$

The lifetime of these electrons is defined by Coulomb scattering of electrons on circulating ions. The heating rate is [7]:

$$\frac{dW_e}{dt} = E_0 \frac{4\pi c \rho_i r_e^2 Z_i^2}{\beta} L_{Coul} \; . \qquad (12)$$

The energy, corresponding to the ‘physical aperture’, for paraxial electrons is $W_e^{\lim} \approx E_0 2\pi Z_i \rho_i a^2 r_e Y(\chi)^2$, and the mean energy of born electrons $\langle W_e \rangle \approx W_e^{\lim}/2$; then we find the electron lifetime:

$$T_{life} \approx \frac{W_e^{lim}}{2} / \left(\frac{1}{2}\frac{dW_e}{dt} \right) = \frac{\beta a^2 Y(\chi)^2}{2 c r_e Z_i L_{Coul}} \; . \qquad (13)$$

Using Eqs. (9–13) we obtain the following equation for neutralization degree

$$\frac{d\eta}{dt} = \frac{Y(\chi)^2}{T_{neutr}^0} - \frac{\eta}{T_{life}^0 Y(\chi)} \; . \qquad (14)$$

Here $T_{neutr}^0 = T_{ion} Z_i$, $T_{life}^0 = T_{life}/Y(\chi)^2$. If $\eta_0 \ll \chi$ and $\tau \gg \tau_{life}$, an approximate solution for neutralization degree can be written analytically in the following form:

$$\eta^{eq} = \eta_0^{eq} Y(\chi_{\_})^3, \eta_0^{eq} = K_0 a^2 P \rangle \Omega_m(\beta) \rangle \qquad (15)$$

where a is in centimetres, P is in 10^{-10} mbar, and the constant

$$K_0 = \frac{10^{-13} N_{Losch} K}{2 r_e L_{Coul}} = \frac{0.0992}{L_{Coul}} \; .$$

An interesting feature of this expression is the weak dependence of the equilibrium neutralization degree on β and the independence from the ion charge Z_i.

Let us underline that in a frame of this simple model the equilibrium neutralization for coasting beam is defined by η_0^{eq}; the reduction of the electron population due to bunching is described by the multiplier $Y(\chi)^3$.

6 APPLICATION TO SIS100/SIS200

Estimations for SIS200 (coasting beam) have shown that for nominal vacuum pressure the neutralization degree is an

Table 2: Neutralization parameters for SIS100-SIS200: kind of ions $^{+28}\mathrm{U}^{238}$, in SIS100 $P = 5 \times 10^{-12}$ mbar, in SIS200 $P = 10 \times 10^{-12}$ mbar, gas composition coincides with measured gas composition in SIS-18 ($\mathrm{H_2} = 65\%,\ \mathrm{O/H_2O} = 17\%,\ \mathrm{CO/N_2} = 8\%,\ \mathrm{Ar} = 4\%,\ \mathrm{Cl} = 4\%,\ \mathrm{CO_2} = 1\%$).

Machine	SIS100 $\chi = 0.1$	SIS100 $\chi = 0.2$	SIS200 $\chi = 0.1$	SIS200 $\chi = 0.2$
$\langle\sigma_{ion}\rangle(10^{-16}\ \mathrm{cm}^2)$	10.3	10.3	7.25	0.05
τ^0_{neutr} (s)	15.6	15.6	13.1	13.1
τ^0_{life} (s)	0.102	0.102	0.092	0.092
$\eta_0^{eq} = \tau^0_{life}/\tau^0_{neutr}$	6.5×10^{-3}	6.5×10^{-3}	7.0×10^{-3}	7.0×10^{-3}
$Y(\chi)$	0.2	0.4	0.1	0.4
$\tau_{life}(\chi)$	0.0204	0.0408	0.0092	0.368
$\eta_0(\chi) = \eta_0^{eq} Y(\chi)^3$	5.2×10^{-5}	4.2×10^{-4}	7.0×10^{-6}	4.5×10^{-4}

order of 0.6–0.8%. However, the situation can become dangerous if the pressure increases sharply due to gas desorption. In this case the electron concentration can be diminished by beam bunching in one bunch (bunch length = 80% from the circumference).

The calculated values of equilibrium neutralization degree are given in Table 2. We see that these values are comparatively small (let us remark that the real neutralization degree will be less to an order of magnitude since typical system time is less than neutralization time to an order of magnitude).

7 RESULTS AND DISCUSSION

1) In the presence of non-linearity periodic variations of the electrical field result in the appearance of ‘physical aperture’, i.e. maximal amplitude of oscillations for ‘surviving’ electrons.

2) The degree of neutralization is determined by the balance between electron creation due to ionization and electron loss due to Coulomb collisions with circulating ions; bunching of the beam results in the reduction of the equilibrium neutralization degree as the third power of normalized (divided on r.m.s. beam size a) physical aperture.

3) The application of the model to SIS100/SIS200 has shown that for nominal vacuum pressure typical values of neutralization degree are small.

Further plans:

1) To check the model by comparison with more detailed numerical calculations.

2) To estimate the influence of other electron sources (SEM electrons and electrons born in walls due to ion–electron emission).

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