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# Low energy dynamics from deformed conformal symmetry in quantum 4D N = 2 SCFTs

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**Abstract**

We determine the one-loop deformation of the conformal symmetry of a general $\mathcal{N} = 2$ superconformally invariant Yang-Mills theory. The deformation is computed for several explicit examples which have a realization as world-volume theories on a stack of D3 branes. These include (i) $\mathcal{N} = 4$ SYM with gauge groups $SU(N)$, $USp(2N)$ and $SO(N)$; (ii) $USp(2N)$ gauge theory with one hypermultiplet in the traceless antisymmetric representation and four hypermultiplets in the fundamental; (iii) quiver gauge theory with gauge group $SU(N) \times SU(N)$ and two hypermultiplets in the bifundamental representations $(\mathbf{N}, \bar{\mathbf{N}})$ and $(\bar{\mathbf{N}}, \mathbf{N})$. The existence of quantum corrections to the conformal transformations imposes restrictions on the effective action which we study on a subset of the Coulomb branch corresponding to the separation of one brane from the stack. In the $\mathcal{N} = 4$ case, the one-loop corrected transformations provide a realization of the conformal algebra; this deformation is shown to be one-loop exact. For the other two models, higher-loop corrections are necessary to close the algebra. Requiring closure, we infer the two-loop conformal deformation.

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# 1 Introduction and summary

Four-dimensional conformal field theories have attracted much attention recently, mainly due to the possibility to study them via a dual supergravity description; for a review, see [1]. This was first proposed for maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills theory [2], which has long been the prime example of a four-dimensional conformally invariant quantum field theory. Here the dual supergravity theory is type IIB supergravity compactified on $AdS_5 \times S^5$, which is the near-horizon geometry of a stack of D3 branes. One simple argument in favour of the gauge theory – gravity correspondence is provided by comparison of their symmetries. The isometry group $SU(2,2) \times SO(6)$ of $AdS_5 \times S^5$ coincides with the conformal group of four-dimensional Minkowski space and the $\mathcal{R}$-symmetry group of $\mathcal{N} = 4$ SYM theory. This agreement can be extended to the full supergroup $SU(2,2|4)$, of which the above is the bosonic subgroup. From this comparison of symmetries it is clear that it is the $AdS_5$ factor of the compactification which is responsible for the conformal invariance of the dual gauge theory. The fact that the beta-function of the field theory vanishes is reflected by the constancy of the type IIB dilaton. Subsequently, many generalizations of the original proposal have been considered and dual supergravity descriptions of conformal and confining gauge theories have been constructed with various gauge groups and number of supercharges; see, e.g. [3].

In this paper we address the question to what extent one can infer the geometry of the dual gravity background – if any – from the conformal field theory. As to the latter, we restrict our attention to conformally invariant supersymmetric Yang-Mills theories. At the classical level they are invariant under conformal transformations, $\delta_{\rm c}\phi^I$, with respect to which the fields $\phi^I$ transform in the standard way, specified by their tensorial structure and their conformal weight. One may then ask whether this symmetry is still manifest (i.e. has the same functional form) in the effective action. The issue here is that quantization requires gauge fixing and the latter can be shown to necessarily break (part of) the conformal symmetry (see [4, 5] and references therein). The effective action will thus not be invariant under the same conformal transformations which were a symmetry of the classical action. The change in the gauge fixing condition under conformal transformations can be undone in the path integral by a compensating field-dependent gauge transformation. The invariance of the path integral under combined conformal and gauge transformations leads to modified conformal Ward identities for the effective action. In other words, the effective action is invariant under deformed conformal transformations,

$$\delta\phi^I \, \frac{\delta\Gamma[\phi]}{\delta\phi^I} = 0 \ , \qquad \delta\phi^I = \delta_{\rm c}\phi^I + \sum_{L=1}^{\infty} \hbar^L \, \delta_{(L)}\phi^I \ , \tag{1.1}$$

and this deformation can, in principle, be computed at each loop order. Of course, the deformation depends only on the parameters of those transformations which do not leave the gauge fixing condition invariant, and these transformations are the special conformal boosts. Having obtained the deformed conformal transformations, this imposes severe restrictions on the general form of the effective action. Indeed, if the effective action were invariant under the classical conformal transformations, each term in its loop expansion,

$$\Gamma[\phi] = S[\phi] + \sum_{L=1}^{\infty} \hbar^L \, \Gamma_{(L)}[\phi] \ , \tag{1.2}$$

would be conformally invariant. The deformed conformal transformations, however, mix different orders in the loop expansion of $\Gamma[\phi]$. In particular, it turns out that even the one-loop deformation, $\delta_{(1)}\phi^I$, contains some nontrivial information about the multi-loop structure of the effective action.

If we now consider the field theory as living on a D3 brane in a ten-dimensional space-time, the deviation of the position of the brane from a chosen reference position is parametrized by the vacuum expectation values of massless scalar fields. For the field theory this corresponds to going to the Coulomb branch of the vacuum manifold. If the ambient geometry is non-trivial, the effective action, which is constrained by requiring it to be invariant under deformed transformations, should provide information about this geometry. Analysis of the one-loop deformed conformal symmetry on the Coulomb branch of $\mathcal{N}=4$ $SU(N)$ super Yang-Mills theory has been carried out in [6, 7, 8].

When constructing SYM theories and their quantum-deformed symmetry properties we must, first of all, ensure that the conformal symmetry survives quantization. A necessary condition for this is the vanishing of the beta-function, which can be arranged by an appropriate choice of matter fields. In the $\mathcal{N}=4$ SYM theory there is no freedom and the theory is completely specified by a choice of gauge group. For $\mathcal{N}=2$ the beta-function is one-loop exact and conformal invariance imposes a single condition on the second Casimir invariants of the matter hypermultiplets [9]. For $\mathcal{N}=1$ the situation is more complicated and one generally finds lines of superconformal fixed points in a higher-dimensional moduli space of vacua [10] (this paper also includes an extensive list of references on finite $\mathcal{N}=1$ theories). In the present paper we concentrate on conformally invariant SYM theories with eight supercharges, i.e. $\mathcal{N}=2$. For these we will explicitly determine the one-loop deformation of the conformal transformation properties of the fields.

The remainder of the paper is organized as follows. In the next section we first recall some well-known facts about $\mathcal{N}=2$ SYM, including its background field quantization. Following [7, 8] we collect all necessary formulas which are needed to compute the deformed conformal transformation. In sect. 3 we do this, in full generality, for an arbitrary $\mathcal{N}=2$ SYM at one loop order. Sect. 4 is devoted to a discussion of the conformal deformation on the Coulomb branch with vanishing hypermultiplets. Here a non-renormalization theorem guarantees that the one-loop deformation is exact.[1] One test of this is that the deformed algebra closes without the need to add higher loop contributions. We then discuss to what extent invariance of the deformed transformation fixes the form of the effective action on the Coulomb branch. In particular, if we choose the background fields to correspond to moving one D3 brane away from a stack of $N$ D3 branes, we are probing the geometry produced by them. This geometry should then determine the general structure of the effective action.

In sect. 5 we specialize to $\mathcal{N}=4$ theories, which are equivalent to $\mathcal{N}=2$ SYM theories with one hypermultiplet in the adjoint representation of the gauge group. The brane constructions of these theories are known: in the case of a stack of $N$ D3 branes, one obtains the gauge group $SU(N)$; this is the case considered by Maldacena [2]. If one puts $N$ D3 branes on top of an orientifold D3 brane, one obtains, depending on the choice of the orientifold projection, either $SO(2N)$ or $USp(2N)$ gauge groups [11] (in the $SO$ case $N$ can be a half-integer). The near-horizon geometry of this brane configuration is $AdS_5\times\mathbf{RP}^5$. We compute the one-loop exact deformed conformal transformation for all these cases.

In sects. 6 and 7 we consider two examples of $\mathcal{N}=2$ theories. First, in sect. 6, we study the field theory which one obtains by placing $N$ D3 branes on top of four D7 branes which are coincident with an $O7$ plane. This gives a $USp(2N)$ gauge group and one hypermultiplet in the traceless antisymmetric representation and four hypermultiplets in the fundamental representation [12, 13, 14]. Here the near-horizon geometry is $AdS_5\times S^5/\mathbf{Z}_2$ where the $\mathbf{Z}_2$ action has a fixed $S^3\subset S^5$ [15]. Other examples of superconformal $\mathcal{N}=2$ theories are obtained by considering stacks of $N$ D3 branes at an $\mathbf{R}^4/\mathbf{Z}_k$ singularity. This leads to so-called quiver gauge theories [16, 17] with gauge group $SU(N)^k$ and matter

[1] This holds in 't Hooft gauge. Changing from this gauge to, say, $R_\xi$ gauge is equivalent to a nonlocal field redefinition in the effective action, which leads to a restructure of the loop expansion accompanied by a modification of the functional form of symmetries [8].

in the bifundamental representation of adjacent gauge groups in the quiver diagram, which is a $k$-gon in this simple case. The near-horizon geometry of this brane configuration is $S^5/\mathbf{Z}_k$ where the $\mathbf{Z}_k$ action leaves a $S^1 \subset S^5$ invariant [18]. In sect. 7 we consider the simplest case of such theories, namely $k = 2$. The generalization to other $k$'s is straightforward.

While the one-loop deformation in $\mathcal{N} = 4$ theories and on the pure Coulomb branch of $\mathcal{N} = 2$ theories is exact, this is no longer the case for the mixed branches of $\mathcal{N} = 2$ theories. One consequence of this is that the conformal algebra does not close as long as the gauge group has finite rank. Requiring closure one can, on the other hand, infer the form of the higher loop corrections to the conformal transformation of the various fields. We will determine them explicitly at two loop order.

# 2 $\mathcal{N} = 2$ SCFTs

We consider a four-dimensional $\mathcal{N} = 2$ superconformal field theory, which describes the coupling of an $\mathcal{N} = 2$ vector supermultiplet to a massless hypermultiplet in a (possibly reducible) representation R of the gauge group $G$. The finiteness condition (which coincides with the requirement of absence of one-loop divergences) [9] can be given in the form

$$\mathrm{tr}_{\mathrm{Ad}}\, W^2 = \mathrm{tr}_{\mathrm{R}}\, W^2 \ , \qquad (2.1)$$

where the subscript “Ad” denotes the adjoint representation, and $W$ is an arbitrary complex scalar field taking its values in the Lie algebra of the gauge group, $W = W_{\underline{a}} T_{\underline{a}}$, with $T_{\underline{a}} = (T_{\underline{a}})^\dagger$ the gauge group generators.

The $\mathcal{N} = 2$ vector multiplet is composed of a gauge field $V_m$, adjoint scalars $W$ and $\bar{W} = W^\dagger$, and adjoint spinors $\lambda^i_\alpha$ and $\bar{\lambda}_{\dot{\alpha} i} = (\lambda^i_\alpha)^\dagger$, where $i = 1, 2$. The hypermultiplet is described by R-representation scalars and spinors $(Q_i, \psi_\alpha, \bar{\mu}_{\dot{\alpha}})$ and their conjugates $(\bar{Q}^i, \bar{\psi}_{\dot{\alpha}}, \mu_\alpha)$, where $\bar{Q}^i = (Q_i)^\dagger$. The Lagrangian (with $g$ the coupling constant) is[2]

$$g^2\, L = -\mathrm{tr}_{\mathrm{F}} \Big\{ \frac{1}{4} F^{mn} F_{mn} + D^m \bar{W} D_m W + \frac{1}{2} [\bar{W}, W]^2$$
$$+\mathrm{i}\, \lambda^i \sigma^m D_m \bar{\lambda}_i - \frac{\mathrm{i}}{\sqrt{2}}\, \lambda^i [\bar{W}, \lambda_i] + \frac{\mathrm{i}}{\sqrt{2}}\, \bar{\lambda}_i [W, \bar{\lambda}^i] \Big\}$$

[2]Here and in the following, lower case Latin letters from the middle of the alphabet, $i, j, k$, are used to denote indices of the automorphism group of the $\mathcal{N} = 2$ supersymmetry algebra, or the $\mathcal{R}$-symmetry group $SU(2)_{\mathcal{R}}$. Such indices are raised and lowered by antisymmetric tensors $\varepsilon^{ij}$ and $\varepsilon_{ij}$, with $\varepsilon^{12} = \varepsilon_{21} = 1$, in the standard way: $Q^i = \varepsilon^{ij} Q_j$, $Q_i = \varepsilon_{ij} Q^j$, such that $(Q_i)^\dagger = \bar{Q}^i$, $(Q^i)^\dagger = -\bar{Q}_i$.

$$-D^m\bar{Q}^iD_mQ_i - \bar{Q}^i\{\bar{W},W\}Q_i - \bar{Q}^iT_{\underline{a}}Q_j\,\bar{Q}^jT_{\underline{a}}Q_i + \frac{1}{2}(\bar{Q}^iT_{\underline{a}}Q_i)^2$$
$$-\mathrm{i}\,\mu\sigma^mD_m\bar{\mu} - \mathrm{i}\,\bar{\psi}\tilde{\sigma}^mD_m\psi - \sqrt{2}\mathrm{i}\,\mu W\psi + \sqrt{2}\mathrm{i}\,\bar{\psi}\bar{W}\bar{\mu}$$
$$+\frac{\mathrm{i}}{\sqrt{2}}(\bar{Q}^i\lambda_i\psi + \bar{\psi}\bar{\lambda}^iQ_i) + \frac{\mathrm{i}}{\sqrt{2}}(\bar{Q}^i\bar{\lambda}_i\bar{\mu} - \mu\lambda^iQ_i)\ , \tag{2.2}$$

where $D_m = \partial_m + \mathrm{i}\,V_m$, and the generators are normalized such that $\mathrm{tr_F}(T_{\underline{a}}T_{\underline{b}}) = \delta_{\underline{ab}}$ in the fundamental representation. The $Q\bar{Q}$ self-interaction occurs after elimination of the auxiliary triplet, $X_{ij} = X_{(ij)}$, which belongs to the off-shell $\mathcal{N}=2$ vector multiplet. This self-interaction can be rewritten as[3]

$$-\bar{Q}^iT_{\underline{a}}Q_j\,\bar{Q}^jT_{\underline{a}}Q_i + \frac{1}{2}(\bar{Q}^iT_{\underline{a}}Q_i)^2 = \bar{Q}^{(i}T_{\underline{a}}Q^{j)}\,\bar{Q}_{(i}T_{\underline{a}}Q_{j)}\ . \tag{2.3}$$

The model (2.2) admits a manifestly $\mathcal{N}=2$ supersymmetric formulation with finitely many auxiliaries in terms of constrained superfields [19, 20] (in this approach, the hypermultiplet possesses an intrinsic off-shell central charge). It can also be formulated in terms of unconstrained superfields, which involve infinitely many auxiliary fields, in harmonic superspace [21]. In both superfield realizations, the $\mathcal{R}$-symmetry $SU(2)_{\mathcal{R}}$ is manifest. The superconformal symmetry $SU(2,2|2)$ is manifest in the harmonic superspace approach.

The moduli space of vacua of the theory under consideration is specified by the following conditions:

$$[\bar{\mathcal{W}},\mathcal{W}] = 0\ ,\qquad \mathcal{W}\mathcal{Q}_i = 0\ ,\qquad \bar{\mathcal{Q}}^{(i}T_{\underline{a}}\mathcal{Q}^{j)} = 0\ , \tag{2.4}$$

with $\bar{\mathcal{W}}\mathcal{Q}_i = 0$ a consequence of the first and second conditions. The solutions to the vacuum equations (2.4) can be classified according to the phase of the gauge theory they give rise to. In the pure Coulomb phase the rank of the gauge group is unreduced: generically it corresponds to $\mathcal{Q}_i = 0$ and $\mathcal{W} \neq 0$ and unbroken gauge group $U(1)^{\mathrm{rank}(G)}$. In the (pure) Higgs phase the gauge symmetry is completely broken; there are no massless gauge bosons. This requires $\mathcal{Q}_i \neq 0$. In the mixed phases there are some massless gauge bosons but the rank of the gauge group is reduced.

At tree level at energies below the symmetry breaking scale, we have free field massless dynamics if the $\mathcal{N}=2$ vector multiplet $(\mathcal{V}_m, \mathcal{W}, \ldots)$ and the hypermultiplet $(\mathcal{Q}_i, \ldots)$ are aligned along a particular direction in the moduli space of vacua. At the quantum level, however, exchanges of virtual massive particles produce corrections to the action of the

[3]By giving up manifest $SU(2)_{\mathcal{R}}$ invariance, this potential can be brought to a more familiar form. Defining $Q_1 \equiv Q$ and $Q_2 \equiv \tilde{Q}^\dagger$, and thus $\bar{Q}^1 = Q^\dagger$ and $\bar{Q}^2 = \tilde{Q}$, one obtains $\bar{Q}^{(i}T_{\underline{a}}Q^{j)}\bar{Q}_{(i}T_{\underline{a}}Q_{j)} = -2|\tilde{Q}T_{\underline{a}}Q|^2 - \frac{1}{2}D_{\underline{a}}D_{\underline{a}}$, where $D_{\underline{a}} = Q^\dagger T_{\underline{a}}Q - \tilde{Q}T_{\underline{a}}\tilde{Q}^\dagger$.

massless fields. The aim of this work is to determine restrictions on the structure of the low energy effective action which are implied by quantum conformal invariance of the theory.

We quantize the $\mathcal{N} = 2$ SYM theory (2.2) in the framework of the background field method (see [22, 23] and references therein), by splitting the dynamical variables $\Phi^I = (V_m, W, Q_i, \ldots)$ into the sum of *background* fields[4] $\phi^I = (\mathcal{V}_m, \mathcal{W}, \mathcal{Q}_i, \ldots)$ and *quantum* fields $\varphi^I = (v_m, w, q_i, \ldots)$. The classical action, $S[\Phi]$, is invariant under standard Yang-Mills gauge transformations

$$\delta V_m = -D_m \zeta \ , \quad \delta W = \mathrm{i}\,[\zeta, W] \ , \quad \delta Q_i = \mathrm{i}\,\zeta Q_i \ , \quad \ldots \tag{2.5}$$

which, in a condensed notation [22], read

$$\delta\Phi^I = R^I{}_{\underline{a}}[\Phi]\,\zeta_{\underline{a}} \ , \tag{2.6}$$

with $R^I{}_{\underline{a}}[\Phi]$ the gauge generators and $\zeta_{\underline{a}}$ infinitesimal gauge parameters. Upon background quantum splitting, the action $S[\phi+\varphi]$ is invariant under *background* gauge transformations

$$\delta\phi^I = R^I{}_{\underline{a}}[\phi]\,\zeta_{\underline{a}} \ , \qquad \delta\varphi^I = R^I{}_{\underline{a},J}\,\varphi^J\,\zeta_{\underline{a}} \ , \tag{2.7}$$

and *quantum* gauge transformations

$$\delta\phi^I = 0 \ , \qquad \delta\varphi^I = R^I{}_{\underline{a}}[\phi+\varphi]\,\zeta_{\underline{a}} \ . \tag{2.8}$$

The background field quantization procedure consists of fixing the quantum gauge freedom while keeping the background gauge invariance intact by means of background covariant gauge conditions. The effective action is given by the sum of all 1PI Feynman graphs which are vacuum with respect to the quantum fields.

The quantum gauge freedom will be fixed by choosing the following background covariant gauge conditions (often called 't Hooft gauge):

$$\chi_{\underline{a}} = \mathcal{D}\cdot v_{\underline{a}} + \mathrm{i}\,\bar{w}T_{\underline{a}}\mathcal{W} - \mathrm{i}\,\bar{\mathcal{W}}T_{\underline{a}}w + \mathrm{i}\,\bar{q}^iT_{\underline{a}}\mathcal{Q}_i - \mathrm{i}\,\bar{\mathcal{Q}}^iT_{\underline{a}}q_i \ , \tag{2.9}$$

and the gauge fixing functional[5]

$$S_{\mathrm{GF}}[\chi] = -\frac{1}{2g^2}\int \mathrm{d}^4x\,\chi^2 \ . \tag{2.10}$$

[4]For most of this section, the background fields are completely arbitrary. After eq. (2.21), they will be taken to be aligned along a particular direction in the moduli space of vacua.

[5]A note on notation: while e.g. in (2.4) $\mathcal{W}$ is matrix valued, it is an (adjoint) vector in (2.9). We will freely switch from one to the other form to simplify expressions.

In eq. (2.9), $\mathcal{D}$ denotes the background gauge covariant derivative, $\mathcal{D}_m = \partial_m + \mathrm{i}\,\mathcal{V}_m$. Introducing

$$Y = \begin{pmatrix} W \\ Q_i \end{pmatrix} = \mathcal{Y} + y \ , \qquad \bar{Y} = (\bar{W}, \ \ \bar{Q}^i) = \bar{\mathcal{Y}} + \bar{y} \ , \tag{2.11}$$

the gauge conditions can be rewritten in the abbreviated form

$$\chi_{\underline{a}} = \mathcal{D} \cdot v_{\underline{a}} + \mathrm{i}\,\bar{y} T_{\underline{a}} \mathcal{Y} - \mathrm{i}\,\bar{\mathcal{Y}} T_{\underline{a}} y \ . \tag{2.12}$$

Under the quantum gauge transformations, $\chi = \chi[\varphi, \phi]$ changes as

$$\delta \chi_{\underline{a}} = -(\mathcal{D}^m D_m \zeta)_{\underline{a}} + \bar{\mathcal{Y}} T_{\underline{a}} \zeta (y + \mathcal{Y}) + (\bar{y} + \bar{\mathcal{Y}}) \zeta T_{\underline{a}} \mathcal{Y} \equiv (\Delta \zeta)_{\underline{a}} \ , \tag{2.13}$$

with $\Delta = \Delta[\varphi, \phi]$ the Faddeev-Popov operator. To define the effective action, $\Gamma[\phi]$, let us introduce the generating functional of connected quantum Green's functions, $W[J, \phi]$,

$$\mathrm{e}^{\mathrm{i} W[J,\phi]} = N \int \mathcal{D}\varphi \, \mathrm{Det}(\Delta[\varphi, \phi]) \, \mathrm{e}^{\mathrm{i}(S[\phi+\varphi] + S_{\mathrm{GF}}[\chi[\varphi,\phi]] + J_I \varphi^I)} \ . \tag{2.14}$$

Its Legendre transform,

$$\Gamma[\langle\varphi\rangle, \phi] = W[J, \phi] - J_I \, \langle \varphi^I \rangle \ , \qquad \langle \varphi^I \rangle = \frac{\delta}{\delta J_I} W[J, \phi] \ , \tag{2.15}$$

is related to the effective action $\Gamma[\phi]$ as follows: $\Gamma[\phi] = \Gamma[\langle\varphi\rangle = 0, \phi]$. In other words, $\Gamma[\phi]$ coincides with $W[J, \phi]$ at its stationary point $J = J[\phi]$ defined by $\delta W[J, \phi]/\delta J = 0$. By construction, $\Gamma[\phi]$ is invariant under background gauge transformations.

The theory under consideration is conformally invariant, both at the classical and quantum levels. The classical action does not change under standard linear conformal transformations of the fields (these transformations differ in sign from those adopted in [7, 8]):

$$\begin{aligned} \delta_{\mathrm{c}} V_m &= \xi V_m + \omega_m{}^n V_n + \sigma V_m \ , \quad \delta_{\mathrm{c}} W = \xi W + \sigma W \ , \quad \delta_{\mathrm{c}} Q_i = \xi Q_i + \sigma Q_i \ , \\ \delta_{\mathrm{c}} \Psi &= \xi \Psi + \frac{1}{2} \omega^{mn} L_{mn} \Psi + \frac{3}{2} \sigma \Psi \ , \end{aligned} \tag{2.16}$$

where $\Psi$ denotes any spinor field, $L_{mn}$ the Lorentz generators in the spinor representation, and $\xi = \xi^m \partial_m$ an arbitrary conformal Killing vector field,

$$\partial_m \xi_n + \partial_n \xi_m = 2 \eta_{mn} \, \sigma \ , \qquad \sigma \equiv \frac{1}{4} \partial_m \xi^m \ , \qquad \omega_{mn} \equiv \frac{1}{2} (\partial_m \xi_n - \partial_n \xi_m) \ . \tag{2.17}$$

At the quantum level, conformal invariance is governed by the Ward identity [7, 8]

$$\delta_{\rm c}\phi^I[\phi]\,\frac{\delta\Gamma[\phi]}{\delta\phi^I} = \langle R^I{}_{\underline{a}}[\phi+\varphi]\,(\Delta^{-1}[\varphi,\phi]\,\rho[\varphi])_{\underline{a}}\rangle\,\frac{\delta\Gamma[\langle\varphi\rangle,\phi]}{\delta\langle\varphi^I\rangle}\Big|_{\langle\varphi\rangle=0}\;, \tag{2.18}$$

where $\rho[\varphi]$ denotes the inhomogeneous term[6] in the conformal transformation of $\chi[\varphi,\phi]$:

$$\delta_{\rm c}\chi = (\xi+2\sigma)\,\chi+\rho\;, \qquad \rho=-2\,(\partial^m\sigma)\,v_m\;. \tag{2.19}$$

In (2.18), the symbol $\langle\quad\rangle$ denotes the quantum average in the presence of sources,

$$\langle F[\varphi,\phi]\rangle \;=\; {\rm e}^{-{\rm i}W[J,\phi]}\,N\int\mathcal{D}\varphi\,F[\varphi,\phi]\,{\rm Det}(\Delta[\varphi,\phi])\,{\rm e}^{{\rm i}(S[\varphi+\phi]+S_{\rm GF}[\chi[\varphi,\phi]]+J_I\varphi^I)}\;. \tag{2.20}$$

Eq. (2.18) should be treated in conjunction with the identity [7, 8]

$$\begin{aligned}\delta\phi^I\,\frac{\delta\Gamma[\phi]}{\delta\phi^I} &= \Big\{\delta\phi^I+\langle R^I{}_{\underline{a}}[\phi+\varphi]\,(\Delta^{-1}[\varphi,\phi]\,\delta\chi[\varphi,\phi])_{\underline{a}}\rangle\Big\}\,\frac{\delta\Gamma[\langle\varphi\rangle,\phi]}{\delta\langle\varphi^I\rangle}\Big|_{\langle\varphi\rangle=0}\;,\\ \delta\chi[\varphi,\phi] &\equiv \chi[\varphi-\delta\phi,\phi+\delta\phi]-\chi[\varphi,\phi]\;,\end{aligned} \tag{2.21}$$

with $\delta\phi^I$ an arbitrary variation of the background fields. This identity allows one to express the functional derivative $\delta\Gamma[\langle\varphi\rangle,\phi]/\delta\langle\varphi\rangle$ at $\langle\varphi\rangle=0$ via $\delta\Gamma[\phi]/\delta\phi$.

It follows from eqs. (2.18) and (2.21) that the effective action $\Gamma[\phi]$ is invariant under quantum corrected conformal transformations, as described by eq. (1.1). In this paper we will evaluate the one-loop quantum deformation, $\delta_{(1)}\phi$, of conformal symmetry when the fields are aligned along a particular direction in the moduli space of vacua.

In the above discussion, the fields $\phi^I$ have been completely arbitrary. From now on, the background $\mathcal{N}=2$ vector multiplet and hypermultiplet will be chosen to be aligned along a fixed, but otherwise arbitrary, direction in the moduli space of vacua; in particular, their scalar fields should solve the equations (2.4). At later stages in this work, we will be forced to impose further restrictions on the fields, of the form

$$\mathcal{V}_m=\mathbf{V}_m(x)H\;,\quad \mathcal{W}=\mathbf{W}(x)H\;;\qquad \mathcal{Q}_i=\mathbf{Q}_i(x)\Upsilon\;, \tag{2.22}$$

corresponding to a separation of space-time and internal variables. Here $H$ is a fixed generator in the Cartan subalgebra, and $\Upsilon$ a fixed vector in the R-representation space of the gauge group, in which the hypermultiplet takes values, chosen so that $H\Upsilon=0$ and

[6]The general solution to the conformal Killing equation (2.17) is $\xi^m=a^m+\lambda x^m+K^m{}_nx^n+b^mx^2-2x^m(b\cdot x)$, where the parameters $a^m$ and $K^{mn}=-K^{nm}$ generate Poincaré transformations, $\lambda$ dilatations and $b^m$ special conformal boosts. By definition, $\sigma=\lambda-2(b\cdot x)$, hence $\partial_m\sigma=-2b_m$, and therefore the right-hand side in (2.18) is non-vanishing for the special conformal boosts only.

$\bar{\Upsilon} T_{\underline{a}} \Upsilon = 0$, cf. (2.4). The freedom in the choice of $H$ and $\Upsilon$ can be reduced by requiring the field configuration (2.22) to be invariant under a maximal unbroken gauge subgroup. Eq. (2.22) defines a single $U(1)$ vector multiplet and a single hypermultiplet which is neutral with respect to the $U(1)$ gauge subgroup generated by $H$. It worth noting that an Abelian vector field and six neutral scalars in four space-time dimensions is what we need to describe a (static gauge) D3 brane moving in a ten-dimensional space-time.

Consider background gauge transformations (2.7) with $\zeta = \zeta(x) H$ which will be referred to as $H$-gauge transformations. They leave all background fields (2.22) unchanged, except the Abelian gauge field $\mathbf{V}_m$. By construction, such transformations leave invariant the gauge-fixed action $S[\phi + \varphi] + S_{\rm GF}[\chi[\varphi, \phi]]$ and, in fact, each term in its Taylor expansion in powers of the quantum fields, since the background gauge transformations of the quantum fields are linear and homogeneous.

# 3 The one-loop deformation

For the purpose of loop calculations, we expand the action $S[\phi + \varphi]$ in powers of the quantum fields $\varphi$ and combine its quadratic part, $S_2$, with the gauge fixing functional, $S_{\rm GF}$. Modulo fermionic contributions, the quadratic action is

$$\begin{aligned} S_2 + S_{\rm GF} = \frac{1}{g^2} \int {\rm d}^4 x \Big\{ & -\frac{1}{2} v^m \tilde{\Delta} v_m + {\rm i}\, v^m \mathcal{F}_m{}^n v_n - \bar{w} \tilde{\Delta} w \\ & + \bar{q}^i \mathcal{D}^m \mathcal{D}_m q_i - \bar{q}^i \{ \bar{\mathcal{W}}, \mathcal{W} \} q_i + (\bar{q}^i T_{\underline{a}} \mathcal{Q}^j + \bar{\mathcal{Q}}^j T_{\underline{a}} q^i)\, (\bar{q}_i T_{\underline{a}} \mathcal{Q}_j + \bar{\mathcal{Q}}_j T_{\underline{a}} q_i) \\ & + 2{\rm i}\, (v^m (\mathcal{D}_m \bar{\mathcal{W}}) w - \bar{w} (\mathcal{D}_m \mathcal{W}) v^m) + 2{\rm i}\, (\bar{q}^i v^m \mathcal{D}_m \mathcal{Q}_i - (\mathcal{D}_m \bar{\mathcal{Q}}^i) v^m q_i) \Big\} , \end{aligned} \qquad (3.1)$$

where $\tilde{\Delta}$ is the Faddeev-Popov operator at $\varphi = 0$,

$$(\tilde{\Delta} \zeta)_{\underline{a}} = -(\mathcal{D}^m \mathcal{D}_m \zeta - \{ \bar{\mathcal{W}}, \mathcal{W} \} \zeta)_{\underline{a}} + \bar{\mathcal{Q}}^i \{ T_{\underline{a}}, T_{\underline{b}} \} \mathcal{Q}_i \, \zeta_{\underline{b}} = -(\mathcal{D}^m \mathcal{D}_m \zeta)_{\underline{a}} + \bar{\mathcal{Y}} \{ T_{\underline{a}}, \zeta \} \mathcal{Y} \, . \qquad (3.2)$$

It is assumed in (3.1) that the background $\mathcal{N} = 2$ vector multiplet and hypermultiplet are aligned along a particular direction in the moduli space of vacua such as to satisfy (2.4). The action (3.1) determines the background covariant propagators of quantum fields, $< \varphi^I(x) \, \varphi^J(x') >$, which are required to evaluate $\delta_{(L)} \phi$.

The one-loop deformation of the conformal transformations of the fields can be computed by minor modification of the method described in [7] for the case of $\mathcal{N} = 4$ SYM. The conformal Ward identity (2.18) can be rewritten in the form

$$0 = \delta_{\rm c} \mathcal{V}_{m\underline{a}} \frac{\delta \Gamma[\phi]}{\delta \mathcal{V}_{m\underline{a}}} - 2(\partial^n \sigma) \, \langle (D_m \Delta^{-1} \, v_n)_{\underline{a}} \rangle \, \frac{\delta \Gamma[\langle \varphi \rangle, \phi]}{\delta \langle v_{m\underline{a}} \rangle} \Big|_{\langle \varphi \rangle = 0}$$

$$+\delta_{\rm c}\mathcal{Y}\,\frac{\delta\Gamma[\phi]}{\delta\mathcal{Y}}\;+\;2\,{\rm i}\,(\partial^n\sigma)\,\langle(\Delta^{-1}\,v_n)_{\underline{a}}\,(T_{\underline{a}}y+T_{\underline{a}}\mathcal{Y})\rangle\;\frac{\delta\Gamma[\langle\varphi\rangle,\phi]}{\delta\langle y\rangle}\Big|_{\langle\varphi\rangle=0}$$

$$+\delta_{\rm c}\bar{\mathcal{Y}}\,\frac{\delta\Gamma[\phi]}{\delta\bar{\mathcal{Y}}}\;-\;2\,{\rm i}\,(\partial^n\sigma)\,\langle(\bar{y}T_{\underline{a}}+\bar{\mathcal{Y}}T_{\underline{a}})\,(\Delta^{-1}\,v_n)_{\underline{a}}\rangle\;\frac{\delta\Gamma[\langle\varphi\rangle,\phi]}{\delta\langle\bar{y}\rangle}\Big|_{\langle\varphi\rangle=0}\;. \qquad (3.3)$$

At the one-loop level,

$$-2(\partial^n\sigma)\,\langle(D_m\Delta^{-1}\,v_n)_{\underline{a}}\rangle\;=\;\Big(\delta_m{}^n\delta_{\underline{ab}}+(\mathcal{D}_m\tilde{\Delta}^{-1}\mathcal{D}^n)_{\underline{ab}}\Big)\;\delta_{(1)}\mathcal{V}_{n\underline{b}}$$
$$-\;{\rm i}\,(\mathcal{D}_m\tilde{\Delta}^{-1})_{\underline{ab}}\,\big(\bar{\mathcal{Y}}\,T_{\underline{b}}\,\delta_{(1)}\mathcal{Y}-\delta_{(1)}\bar{\mathcal{Y}}\,T_{\underline{b}}\,\mathcal{Y}\big) \qquad (3.4)$$

and

$$2\,{\rm i}\,(\partial^n\sigma)\langle(\Delta^{-1}\,v_n)_{\underline{a}}\,(T_{\underline{a}}y+T_{\underline{a}}\mathcal{Y})\rangle\;=\;\Big(1-T_{\underline{a}}\mathcal{Y}\,(\tilde{\Delta}^{-1})_{\underline{ab}}\,\bar{\mathcal{Y}}T_{\underline{b}}\Big)\;\delta_{(1)}\mathcal{Y}$$
$$+T_{\underline{a}}\mathcal{Y}\,(\tilde{\Delta}^{-1})_{\underline{ab}}\;\big(\delta_{(1)}\bar{\mathcal{Y}}\,T_{\underline{b}}\,\mathcal{Y}\big)-{\rm i}\,T_{\underline{a}}\mathcal{Y}\,(\tilde{\Delta}^{-1}\mathcal{D}^m)_{\underline{ab}}\,\delta_{(1)}\mathcal{V}_{m\underline{b}}\;, \qquad (3.5)$$

where

$$\delta_{(1)}\mathcal{V}_{m\underline{a}}\;=\;-2\,{\rm i}\,(\partial^n\sigma)\,(T_{\underline{c}}\,\tilde{\Delta}^{-1})_{\underline{ab}}\;<v_{n\underline{b}}(x)\,v_{m\underline{c}}(x')>|_{x'=x}\;, \qquad (3.6)$$

$$\delta_{(1)}\mathcal{Y}\;=\;2\,{\rm i}\,(\partial^n\sigma)\,(\tilde{\Delta}^{-1})_{\underline{ab}}\;<v_{n\underline{b}}(x)\,T_{\underline{a}}\,y(x')>|_{x'=x} \qquad (3.7)$$

and $\tilde{\Delta}$ is given by eq. (3.2). As in the $\mathcal{N}=4$ case [7], substitution of (3.4) and (3.5) into (3.3) yields the appropriate one-loop versions of the combination

$$\delta\phi^I+\langle R^I{}_{\underline{a}}[\phi+\varphi]\,(\Delta^{-1}[\varphi,\phi]\,\delta\chi[\varphi,\phi])_{\underline{a}}\rangle \qquad (3.8)$$

appearing in the identity (2.21) to allow conversion of derivatives of the effective action with respect to quantum fields into derivatives of the effective action with respect to background fields. The conformal Ward identity then takes the one-loop form

$$0\;=\;\big(\delta_{\rm c}\mathcal{V}_{m\underline{a}}+\delta_{(1)}\mathcal{V}_{m\underline{a}}\big)\;\frac{\delta\Gamma[\phi]}{\delta\mathcal{V}_{m\underline{a}}}+\big(\delta_{\rm c}\mathcal{Y}+\delta_{(1)}\mathcal{Y}\big)\;\frac{\delta\Gamma[\phi]}{\delta\mathcal{Y}}+\big(\delta_{\rm c}\bar{\mathcal{Y}}+\delta_{(1)}\bar{\mathcal{Y}}\big)\;\frac{\delta\Gamma[\phi]}{\delta\bar{\mathcal{Y}}}\;. \qquad (3.9)$$

The one-loop deformations of the conformal field transformations are therefore given by (3.6) and (3.7). The right-hand sides in (3.6) and (3.7) are nonlocal functionals of $\mathcal{V}_m$ and $\mathcal{Y}$, each of which can be represented as a sum of infinitely many local terms with increasing number of derivatives of the fields. We are interested in evaluating $\delta_{(1)}\mathcal{V}_m$ and $\delta_{(1)}\mathcal{Y}$ to first order in the derivative expansion.

To first order in the derivative expansion,

$$<v_{m\underline{b}}\,w_{\underline{c}}>=2g^2\Big(\tilde{\Delta}^{-1}\,(\mathcal{D}_m\mathcal{W})\,\tilde{\Delta}^{-1}\Big)_{\underline{bc}}=2g^2\Big(\tilde{\Delta}^{-2}\,(\mathcal{D}_m\mathcal{W})\Big)_{\underline{bc}}\;, \qquad (3.10)$$

where the last relation follows from the identity

$$(\mathcal{D}_m \mathcal{W})_{\underline{bc}}\, \bar{\mathcal{Y}}\{T_{\underline{c}}, T_{\underline{d}}\}\mathcal{Y} = \bar{\mathcal{Y}}\{T_{\underline{b}}, T_{\underline{c}}\}\mathcal{Y}\,(\mathcal{D}_m \mathcal{W})_{\underline{cd}}\ , \tag{3.11}$$

which is valid since $\mathcal{W}\,\mathcal{Y} = (\mathcal{D}_m\mathcal{W})\,\mathcal{Y} = 0$ due to (2.4). As a result, we get

$$\delta_{(1)}\mathcal{W}_{\underline{a}} = 4\mathrm{i}\, g^2\, (\partial^m \sigma)\, \mathrm{tr}_{\mathrm{Ad}} \Big( T_{\underline{a}}\, (\tilde{\Delta}^{-3}|)\, \mathcal{D}_m \mathcal{W} \Big) \tag{3.12}$$

where $\tilde{\Delta}^{-3}|$ denotes the kernel of $\tilde{\Delta}^{-3}$ at coincident space-time points. Let us now turn to the vector field variation. To first order in the derivative expansion, one similarly gets

$$< v_{m\underline{a}}\, v_{n\underline{b}} >= -\mathrm{i} g^2 (\tilde{\Delta}^{-1})_{\underline{ab}}\, \delta_{mn} + 2g^2 (\tilde{\Delta}^{-2}\, \mathcal{F}_{mn})_{\underline{ab}}\ . \tag{3.13}$$

When substituted into (3.6), the first term in the vector propagator yields a potentially divergent contribution. However, it vanishes on symmetry grounds, as

$$(T_{\underline{c}}\, \tilde{\Delta}^{-2})_{\underline{ac}} = -\mathrm{i}\, f_{\underline{cad}}\, (\tilde{\Delta}^{-2})_{\underline{dc}}, \tag{3.14}$$

and the mass matrix in $\tilde{\Delta}$ is symmetric under interchange of $\underline{c}$ and $\underline{d}$, while $f_{\underline{cad}}$ is totally antisymmetric. As a result, eq. (3.6) leads to

$$\delta_{(1)}\mathcal{V}_{m\underline{a}} = 4\mathrm{i}\, g^2\, (\partial^n \sigma)\, \mathrm{tr}_{\mathrm{Ad}} \Big( T_{\underline{a}}\, (\tilde{\Delta}^{-3}|)\, \mathcal{F}_{nm} \Big)\ . \tag{3.15}$$

Since the expressions (3.12) and (3.15) are already of first order in derivatives, we can approximate the operator $\tilde{\Delta}$ defined in (3.2) by

$$\tilde{\Delta}_{\underline{ab}} \approx -\partial^m \partial_m\, \delta_{\underline{ab}} + (\mathcal{M}^2_{\mathrm{v}})_{\underline{ab}}\ , \qquad (\mathcal{M}^2_{\mathrm{v}})_{\underline{ab}} \equiv \bar{\mathcal{Y}}\{T_{\underline{a}}, T_{\underline{b}}\}\mathcal{Y}\ , \tag{3.16}$$

with $\mathcal{Y}$ constant. Then, a direct evaluation of (3.12) and (3.15) gives

$$\delta_{(1)}\mathcal{W}_{\underline{a}} = -\frac{g^2}{8\pi^2}\, (\partial^n \sigma)\, \mathrm{tr}_{\mathrm{Ad}} \left( T_{\underline{a}}\, \mathcal{M}_{\mathrm{v}}^{-2} \mathcal{D}_n \mathcal{W} \right)\ , \tag{3.17}$$

$$\delta_{(1)}\mathcal{V}_{m\underline{a}} = -\frac{g^2}{8\pi^2}\, (\partial^n \sigma)\, \mathrm{tr}_{\mathrm{Ad}} \left( T_{\underline{a}}\, \mathcal{M}_{\mathrm{v}}^{-2} \mathcal{F}_{nm} \right)\ . \tag{3.18}$$

These relations determine the quantum modification to the conformal transformations of the bosonic fields of the $\mathcal{N} = 2$ vector multiplet.

It is worth noting that the mass matrix $\mathcal{M}^2_{\mathrm{v}}$ may possess some zero eigenvalues, and therefore $\mathcal{M}^{-2}_{\mathrm{v}}$ is not defined by itself. But in (3.18), for example, $\mathcal{M}^{-2}_{\mathrm{v}}$ occurs in the combination $\mathcal{M}^{-2}_{\mathrm{v}}\mathcal{F}_{nm}$ and the multiplier $\mathcal{F}$ projects out all zero eigenvalues of $\mathcal{M}^2_{\mathrm{v}}$.

Finally, let us analyse the hypermultiplet conformal deformation. We will use underlined Greek letters to denote hypermultiplet indices, i.e. $Q_i = (Q_{i\underline{\alpha}})$ and $\bar{Q}^i = (\bar{Q}^{i\underline{\alpha}})$. In accordance with (3.7), we have

$$\delta_{(1)}\mathcal{Q}_{i\underline{\alpha}} = 2\mathrm{i}\,(\partial^m\sigma)\,(\tilde{\Delta}^{-1})_{\underline{bc}}\; < v_{m\underline{c}}(x)\,q_{i\underline{\beta}}(x') > |_{x'=x}\,(T_{\underline{b}})_{\underline{\alpha}}{}^{\underline{\beta}}\,. \tag{3.19}$$

To first order in the derivative expansion, we can approximate $< v_m\, q_i >$ as

$$\begin{aligned} < v_{m\underline{c}}\, q_{i\underline{\beta}} >= 2\mathrm{i}\,(\tilde{\Delta}^{-1})_{\underline{cd}}\Big\{ & (T_{\underline{d}}\mathcal{D}_m\mathcal{Q}_j)_{\underline{\gamma}} < \bar{q}^{j\underline{\gamma}}\, q_{i\underline{\beta}} >_0 \\ & -(\mathcal{D}_m\bar{\mathcal{Q}}^j T_{\underline{d}})^{\underline{\gamma}} < q_{j\underline{\gamma}} q_{i\underline{\beta}} >_0 \Big\}\,. \end{aligned} \tag{3.20}$$

Here $< \dots >_0$ denotes the propagator corresponding to the free quadratic action

$$\begin{aligned} S_0 = \frac{1}{g^2}\int \mathrm{d}^4x\,\Big\{ & \bar{q}^i\partial^m\partial_m q_i - \bar{q}^i\{\bar{\mathcal{W}},\mathcal{W}\}q_i \\ & + (\bar{q}^i T_{\underline{a}}\mathcal{Q}^j + \bar{\mathcal{Q}}^j T_{\underline{a}} q^i)\,(\bar{q}_i T_{\underline{a}}\mathcal{Q}_j + \bar{\mathcal{Q}}_j T_{\underline{a}} q_i)\Big\}\,, \end{aligned} \tag{3.21}$$

with $\mathcal{W}$ and $\mathcal{Q}_i$ constant. Introducing

$$q = \begin{pmatrix} q_{j\underline{\beta}} \\ \bar{q}^{j\underline{\beta}} \end{pmatrix}\,, \qquad q^\dagger = (\bar{q}^{i\underline{\alpha}},\; q_{i\underline{\alpha}})\,, \tag{3.22}$$

the action can be rewritten in a more compact form

$$S_0 = \frac{1}{2g^2}\int \mathrm{d}^4x\,\Big\{q^\dagger\partial^m\partial_m q - q^\dagger\mathcal{M}_{\mathrm{h}}^2 q\Big\}\,, \tag{3.23}$$

where the hypermultiplet mass matrix is

$$\begin{aligned} \mathcal{M}_{\mathrm{h}}^2 &= \left(\begin{array}{c|c} \delta_i{}^j\,\Sigma_{\underline{\alpha}}{}^{\underline{\beta}} & -2\epsilon_{ij}\,(T_{\underline{a}}\mathcal{Q}^k)_{\underline{\alpha}}\,(T_{\underline{a}}\mathcal{Q}_k)_{\underline{\beta}} \\ \hline -2\epsilon^{ij}\,(\bar{\mathcal{Q}}_k T_{\underline{a}})^{\underline{\alpha}}\,(\bar{\mathcal{Q}}^k T_{\underline{a}})^{\underline{\beta}} & \delta^i{}_j\,\Sigma_{\underline{\beta}}{}^{\underline{\alpha}} \end{array}\right)\,, \\ \Sigma_{\underline{\alpha}}{}^{\underline{\beta}} &= \{\bar{\mathcal{W}},\mathcal{W}\}_{\underline{\alpha}}{}^{\underline{\beta}} + 2(T_{\underline{a}}\mathcal{Q}_k)_{\underline{\alpha}}\,(\bar{\mathcal{Q}}^k T_{\underline{a}})^{\underline{\beta}}\,. \end{aligned} \tag{3.24}$$

Eq. (3.19) becomes

$$\begin{aligned} \delta_{(1)}\mathcal{Q}_{i\underline{\alpha}} = -4(\partial^m\sigma)\,(T_{\underline{b}})_{\underline{\alpha}}{}^{\underline{\beta}}\,(\tilde{\Delta}^{-2})_{\underline{bd}}\Big\{ & (T_{\underline{d}}\mathcal{D}_m\mathcal{Q}_j)_{\underline{\gamma}} < \bar{q}^{j\underline{\gamma}}\, q_{i\underline{\beta}} >_0 \\ & -(\mathcal{D}_m\bar{\mathcal{Q}}^j T_{\underline{d}})^{\underline{\gamma}} < q_{j\underline{\gamma}}\, q_{i\underline{\beta}} >_0 \Big\}\,. \end{aligned} \tag{3.25}$$

Due to the off-diagonal terms in the hypermultiplet mass matrix, the propagator $< q_{j\underline{\gamma}}\, q_{i\underline{\beta}} >_0$ does not vanish in general. For the background field configurations we will consider below, it does, however, vanish.

Unlike eqs. (3.17) and (3.18), the variation $\delta_{(1)}\mathcal{Q}$ involves two different propagators with different mass matrices, $\mathcal{M}^2_{\rm v}$ and $\mathcal{M}^2_{\rm h}$. The transformation (3.25) for the hypermultiplets will have a form similar to (3.17) only if a special relationship exists between the mass matrices $\mathcal{M}^2_{\rm v}$ and $\mathcal{M}^2_{\rm h}$. The point is that $\delta_{(1)}\mathcal{Q}$ proves to be a linear combination of terms proportional to the following (Euclidean) momentum integral

$$\int \frac{{\rm d}^4 k}{(2\pi)^4}\,\frac{1}{(k^2+M_1{}^2)^2}\,\frac{1}{(k^2+M_2{}^2)} = \frac{M_2{}^2}{32\pi^2(M_1{}^2-M_2{}^2)^2}\,\ln\Big(\frac{M_2{}^2}{M_1{}^2}\Big) + \frac{1}{32\pi^2(M_1{}^2-M_2{}^2)}\;, \tag{3.26}$$

with $M_1{}^2$ and $M_2{}^2$ being some eigenvalues of $\mathcal{M}^2_{\rm v}$ and $\mathcal{M}^2_{\rm h}$ respectively. Only for $M_1 = M_2 = M$ is the mass dependence of the form $1/(32\pi^2 M^2)$, as in (3.17). This therefore raises the question: under what circumstances are there coinciding mass eigenvalues?

Let the background vector multiplet fields $(\mathcal{V}_m, \mathcal{W})$ be of the form (2.22), with $H$ a given generator in the Cartan subalgebra. It is useful to examine the implications of the $U(1)$ gauge symmetry (2.7) generated by $H$. We know that (i) the background scalar fields $\mathcal{W}$ and $\mathcal{Q}_i$ are invariant under the $H$-gauge transformations; (ii) the quadratic action (3.1) is invariant under the $H$-gauge transformations. These observations imply

$$[H, \mathcal{M}^2_{\rm v}] = 0\;, \qquad [H, \mathcal{M}^2_{\rm h}] = 0\;, \tag{3.27}$$

and therefore the charge operator $H$ and the mass matrices can be simultaneously diagonalized. One further observes that the derivative interaction terms in (3.1),

$$\frac{\rm i}{2}\,L^{(1)}_{\rm int} = \bar{w}(\mathcal{D}_m\mathcal{W})v^m - v^m(\mathcal{D}_m\bar{\mathcal{W}})w - \frac{1}{2}v^m\mathcal{F}_m{}^n v_n\;, \tag{3.28}$$

$$\frac{\rm i}{2}L^{(2)}_{\rm int} = (\mathcal{D}_m\bar{\mathcal{Q}}^i)v^m q_i - \bar{q}^i v^m \mathcal{D}_m\mathcal{Q}_i\;, \tag{3.29}$$

are invariant under the $H$-gauge transformations. The $H$-invariance of $L^{(1)}_{\rm int}$ implies that each term on the right-hand side of $L^{(1)}_{\rm int}$ involves two quantum fields of opposite $H$-charge; these fields have the same mass due to the first identity in (3.27). Similarly, the $H$-invariance of $L^{(2)}_{\rm int}$ implies that each term on the right-hand side of $L^{(2)}_{\rm int}$ involves two quantum fields of opposite $H$-charge. In order for these fields to have the same mass, it is sufficient to require (using the condensed notation (3.22))

$$(\mathcal{D}_m\mathcal{Q}^\dagger)\,v^m\,\mathcal{M}^2_{\rm h}\,q = (\mathcal{D}_m\mathcal{Q}^\dagger)\,(\mathcal{M}^2_{\rm v}v^m)\,q\;, \tag{3.30}$$

as a consequence of (3.27). The latter requirement is satisfied provided $\mathcal{Q}_i$ is of the form (2.22). In this case, the off-diagonal terms in the hypermultiplet mass matrix (3.24) also vanish.

Thus, when the conditions (2.22) are imposed, considerable simplification of the expression (3.25) occurs. The vector mass matrix $\mathcal{M}_{\rm v}^2$ can be diagonalized by an appropriate choice of basis for the Lie algebra of the gauge group. Each non-zero mass eigenvalue corresponds to a generator $T_{\hat{a}}$ which is broken, in the sense that $[T_{\hat{a}}, \mathcal{W}] \neq 0$ and/or $T_{\hat{a}} \mathcal{Q}_i \neq 0$. The deformation (3.25) can be put in the form

$$\delta_{(1)} \mathcal{Q}_i = -\frac{g^2}{8\pi^2} (\partial^m \sigma) \sum_I \frac{1}{M_I{}^2} {\sum_{\hat{a}}}' T_{\hat{a}} T_{\hat{a}} \mathcal{D}_m \mathcal{Q}_i \ . \tag{3.31}$$

Here, $I$ labels the different nonzero eigenvalues $M_I{}^2$ of the vector mass matrix $\mathcal{M}_{\rm v}^2$, and, for a given $I$, the sum ${\sum}'_{\hat{a}}$ is over all broken generators $T_{\hat{a}}$ corresponding to the mass eigenvalue $M_I{}^2$. To derive this result, one makes use of the fact that when the conditions (2.22) are imposed, the propagator $< q_{j\underline{\gamma}}\, q_{i\underline{\beta}} >_0$ in (3.20) vanishes, as there are no off-diagonal blocks in the hypermultiplet mass matrix. Further, as proven above, the interaction $(T_{\underline{d}} \mathcal{D}_m \mathcal{Q}_j)_{\underline{\gamma}}$ in the first term on the right-hand side of (3.20) only mixes hypermultiplets and vectors of the same mass. Thus the propagator (3.20) decomposes into a sum of terms, one for each non-zero mass eigenvalue of the vector mass matrix. For a given mass eigenvalue $M_I{}^2$, the operator $(\tilde{\Delta}^{-1})_{\hat{c}\hat{d}}$ takes the form $(-\partial^2 + M_I{}^2)^{-1}\, \delta_{\hat{c}\hat{d}}$, while

$$< \bar{q}^{j\underline{\gamma}}\, q_{i\underline{\beta}} >_0 = -{\rm i}\, g^2\, (-\partial^2 + M_I{}^2)^{-1}\, \delta^j{}_i\, \delta^{\underline{\gamma}}{}_{\underline{\beta}} \ . \tag{3.32}$$

Substituting the resulting expression (3.20) into (3.19), the deformed hypermultiplet transformation is

$$\delta_{(1)} \mathcal{Q}_i = 4\,{\rm i}\, g^2\ (\partial^m \sigma) \sum_I (-\partial^2 + M_I{}^2)^{-3} | {\sum_{\hat{a}}}' T_{\hat{a}} T_{\hat{a}} \mathcal{D}_m \mathcal{Q}_i \ . \tag{3.33}$$

In momentum space, this yields integrals of the form (3.26) with $M_1{}^2 = M_2{}^2 = M_I{}^2$.

An analogous result applies for the transformations (3.17), which can be cast in the form

$$\delta_{(1)} \mathcal{W} = -\frac{g^2}{8\pi^2} (\partial^m \sigma) \sum_I \frac{1}{M_I{}^2} {\sum_{\hat{a}}}' [T_{\hat{a}},\, [T_{\hat{a}}, \mathcal{D}_m \mathcal{W}]] \ , \tag{3.34}$$

and similarly for the vector transformation (3.18). Of course, the covariant derivatives in (3.31) and (3.34) coincide with partial derivatives under the (gauge) choice (2.22).

# 4 Conformal deformation on the Coulomb branch

Relations (3.17) and (3.18) allow us to evaluate the one-loop deformation of conformal symmetry on the Coulomb branch, where we restrict the analysis in this section to the case where the scalars in the hypermultiplets are zero[7], i.e. we consider solutions to the equations

$$[\bar{\mathcal{W}}, \mathcal{W}] = 0 \ , \qquad \mathcal{Q}_i = 0 \ . \tag{4.1}$$

In accordance with (3.16), the mass matrix $\mathcal{M}^2_{\rm v}$ now becomes

$$\frac{1}{2}\mathcal{M}^2_{\rm v}\,\zeta = [\bar{\mathcal{W}}, [\mathcal{W}, \zeta]] = [\mathcal{W}, [\bar{\mathcal{W}}, \zeta]] \ . \tag{4.2}$$

For definiteness, the variations (3.17) and (3.18) will be evaluated for a special superconformal theory – $\mathcal{N} = 2$ super Yang-Mills with gauge group $SU(N)$ and $2N$ hypermultiplets in the fundamental [9]. This example captures the general features which we wish to highlight; in particular the analysis of this section is directly applicable to the models considered in sect. 6 and 7 after setting the hypermultiplets to zero. It is worth noting that the finiteness condition (2.1) for the $SU(N)$ model follows from the following identity for the group $SU(N)$: $\mathrm{tr}_{\rm Ad} = 2N\,\mathrm{tr}_{\rm F}$.

First we briefly recall some well known facts concerning the explicit structure of the Coulomb branch of the theory under consideration (see, for instance, [24]). Up to a gauge transformation, the general solution to the first equation in (4.1) is

$$\mathcal{W} = \mathrm{diag}(W_1, \ldots, W_N) \ , \qquad \sum_\alpha W_\alpha = 0 \ . \tag{4.3}$$

Generically, when all eigenvalues of $\mathcal{W}$ are different, the gauge symmetry $SU(N)$ is broken to $U(1)^{N-1}$; then, $\mathcal{Q} = 0$ is the only solution to the second and third equations in (2.4). If $k$ eigenvalues of $\mathcal{W}$ coincide, the gauge group $SU(N)$ is broken to $SU(k) \times U(1)^{N-k}$; the second and third equations in (2.4) have non-trivial solutions when some eigenvalues of $\mathcal{W}$ vanish.

Here we will be interested in a maximally symmetric field configuration which contains a single Abelian $\mathcal{N} = 2$ vector multiplet,

$$\mathcal{W} = \mathbf{W}\,H \ , \quad \mathcal{V}_m = \mathbf{V}_m\,H \ , \quad H = \frac{1}{\sqrt{N(N-1)}}\,\mathrm{diag}\,(N-1, -1, \cdots, -1) \ , \tag{4.4}$$

and leaves the subgroup $SU(N-1) \times U(1) \subset SU(N)$ unbroken. With this choice for $\mathcal{W}$ the only solution to the second equation in (2.4) is $\mathcal{Q} = 0$.

For the $\mathcal{N} = 2$ vector multiplet (4.4), the explicit structure of the variations (3.17) and (3.18) can be read off from similar results for the $\mathcal{N} = 4$ super Yang-Mills theory [7],

[7]This is the generic case, but e.g. the model discussed in sect. 6 also allows for non-vanishing $\mathcal{Q}_i$ without reducing the rank of the gauge group.

see also sect. 5. One obtains

$$\delta_{(1)}\mathbf{W} = -\frac{g^2(N-1)}{8\pi^2}\,\frac{(\partial^n\sigma)\,\partial_n\mathbf{W}}{\bar{\mathbf{W}}\mathbf{W}}\;, \qquad \delta_{(1)}\mathbf{V}_m = -\frac{g^2(N-1)}{8\pi^2}\,\frac{(\partial^n\sigma)\,\mathbf{F}_{nm}}{\bar{\mathbf{W}}\mathbf{W}}\;. \tag{4.5}$$

This result is universal: modulo an overall common factor, which is the number of broken generators, the same deformation of conformal symmetry occurs for any $\mathcal{N}=2$ superconformal theory in the Coulomb phase described by a single $U(1)$ vector multiplet.

The deformed conformal transformations (1.1), with the quantum corrections from all loops and all orders in the derivative expansion taken into account, should realize the conformal algebra, as can be justified by a direct generalization of the considerations given in [25, 5]. On the Coulomb branch, the one-loop quantum deformation to first order in derivatives,

$$\delta\mathbf{V}_m = \delta_{\rm c}\mathbf{V}_m - R^4\,\frac{(\partial^n\sigma)\,\mathbf{F}_{nm}}{4\bar{\mathbf{W}}\mathbf{W}}\;, \tag{4.6}$$

$$\delta\mathbf{W} = \delta_{\rm c}\mathbf{W} - R^4\,\frac{(\partial^n\sigma)\,\partial_n\mathbf{W}}{4\bar{\mathbf{W}}\mathbf{W}}\;, \tag{4.7}$$

with $R^4 = g^2(N-1)/2\pi^2$, realizes the conformal algebra (up to a pure gauge transformation) without the need to take any higher loop corrections into account. Using the non-renormalization theorem of Dine and Seiberg [26], we will show that the above conformal deformation is one-loop exact.

It should first be pointed out that the transformations (4.7) and (4.6) coincide with the $AdS_5$ isometries of a (static gauge) D3 brane embedded in $AdS_5\times S^1$:

$$S = -\frac{1}{g^2}\int {\rm d}^4x\left(\sqrt{-\det\Big(\frac{\mathbf{X}^2}{R^2}\,\eta_{mn} + \frac{R^2}{\mathbf{X}^2}\,\partial_m\mathbf{X}_\mu\partial_n\mathbf{X}_\mu + \mathbf{F}_{mn}\Big)} - \frac{\mathbf{X}^4}{R^4}\right)\;, \tag{4.8}$$

$$\mathbf{X}^2 = \mathbf{X}_\mu\mathbf{X}_\mu\;,$$

where $\mathbf{X}^2 = 2\,\bar{\mathbf{W}}\mathbf{W}$ and $\mathbf{X}_\mu = (\mathbf{X}_1,\mathbf{X}_2)$ are defined by $\sqrt{2}\,\mathbf{W} = \mathbf{X}_1 + {\rm i}\,\mathbf{X}_2$. Of course, the conformal invariance (4.7) and (4.6) does not uniquely fix the D3 brane action. But in conjunction with $\mathcal{N}=2$ supersymmetric non-renormalization theorems, it severely restricts the functional form of invariant actions.

Let us demonstrate, by analogy with Maldacena's analysis [2], that the conformal symmetry (4.7) and simple $\mathcal{N}=2$ non-renormalization theorems allow one to restore part of the low energy effective action of the form

$$\Gamma[\mathbf{W},\bar{\mathbf{W}}] = \int {\rm d}^4x\, L_{\rm eff}(\mathbf{W},\bar{\mathbf{W}},\partial\mathbf{W},\partial\bar{\mathbf{W}})\;. \tag{4.9}$$

It is useful to introduce radial $\mathbf{X}$ and angular $\varphi$ variables, $\sqrt{2}\,\mathbf{W} = \mathbf{X}\exp(\mathrm{i}\,\varphi)$. It follows from (4.7) and (4.6) that $\ln \mathbf{X}$ can be interpreted as a Goldstone field for partial symmetry breaking of the conformal group $SU(2,2)$ to $SO(4,1)$. Similarly, $\varphi$ can be treated as a Goldstone field corresponding to spontaneous breakdown of the $U(1)_{\mathcal{R}}$ factor in the $\mathcal{N}=2$ $\mathcal{R}$-symmetry group $U(1)_{\mathcal{R}} \times SU(2)_{\mathcal{R}}$, of which the $SU(2)_{\mathcal{R}}$ factor leaves $W$ invariant. As follows, for instance, from the techniques of nonlinear realizations, the general form for $L_{\rm eff}$ is

$$L_{\rm eff} = \gamma\,\mathbf{X}^4 + \sqrt{-G}\,\sum_{k=0}^{\infty} c_{2k}\,(G^{mn}\,\partial_m\varphi\partial_n\varphi)^k\;, \qquad G = \det(G_{mn})\;, \tag{4.10}$$

with $\gamma$ and $c_k$ constant parameters, and $G_{mn}$ the induced metric on the D3-brane,

$$G_{mn} = \frac{\mathbf{X}^2}{R^2}\,\eta_{mn} + \frac{R^2}{\mathbf{X}^2}\,\partial_m\mathbf{X}_\mu\partial_n\mathbf{X}_\mu = \frac{\mathbf{X}^2}{R^2}\,\eta_{mn} + \frac{R^2}{\mathbf{X}^2}\,\partial_m\mathbf{X}\partial_n\mathbf{X} + R^2\,\partial_m\varphi\partial_n\varphi\;. \tag{4.11}$$

Of course, the numerical coefficient of the angular term, $\partial_m\varphi\partial_n\varphi$, in $G_{mn}$ can be changed at the expense of modifying the infinite series in (4.10). The $\mathbf{F}$-independent part of the D3 brane Lagrangian in (4.8) is of the form (4.10) with $c_0 = -R^4\,\gamma = -1/g^2$ and $c_{2k} = 0$ for $k \geq 1$.

The theory we are considering is $\mathcal{N}=2$ superconformal, and thus neither the kinetic term nor the scalar potential receive quantum corrections. These non-renormalization properties and the fact that the field configuration (4.4) corresponds to a flat direction imply that the part of $L_{\rm eff}$ with at most two derivatives must coincide with the tree-level Lagrangian, and hence

$$L_{\rm eff} = -\frac{1}{g^2}\,\partial^m\bar{\mathbf{W}}\partial_m\mathbf{W} \;+\; O(\partial^4)\;, \tag{4.12}$$

implying that $c_0 = -R^4\,\gamma = -1/g^2$ and $c_2 = 0$. We see that the radial part of $L_{\rm eff}$ is completely fixed by the conformal symmetry (4.7) and $\mathcal{N}=2$ non-renormalization theorems [2]. In addition, the Dine-Seiberg theorem [26] turns out to imply $c_4 = 0$. Let $W(x,\theta)$ be the $\mathcal{N}=2$ chiral superfield strength with $\mathbf{W}(x)$ being its leading ($\theta$-independent) component. The non-renormalization theorem of Dine and Seiberg states that, in the Coulomb branch of $\mathcal{N}=2$ superconformal theories, the following manifestly $\mathcal{N}=2$ supersymmetric quantum correction to the low energy effective action

$$c\int \mathrm{d}^4x\,\mathrm{d}^4\bar\theta\mathrm{d}^4\theta\;\ln\bar{W}\ln W \tag{4.13}$$

is one-loop exact. In components, this functional contains two special bosonic structures with four derivatives (with $\mathbf{F}_{\alpha\beta}$ and $\bar{\mathbf{F}}_{\dot\alpha\dot\beta}$ the helicity $+1$ and $-1$ components of the field strength $\mathbf{F}_{mn}$):

$$\frac{\mathbf{F}^{\alpha\beta}\mathbf{F}_{\alpha\beta}\,\bar{\mathbf{F}}^{\dot\alpha\dot\beta}\bar{\mathbf{F}}_{\dot\alpha\dot\beta}}{(\bar{\mathbf{W}}\mathbf{W})^2}\,, \qquad \frac{(\partial\bar{\mathbf{W}})^2\,(\partial\mathbf{W})^2}{(\bar{\mathbf{W}}\mathbf{W})^2}\,, \tag{4.14}$$

which precisely match the four-derivative structures appearing in the expansion of the Born-Infeld action (4.8), including the relative coefficient [27]. The explicit value of the overall coefficient $c$ in (4.13) for the theory under consideration was computed in [27, 28, 29] to be $c = (N-1)/(4\pi)^2$. This confirms that $c_4 = 0$. The coefficients $c_6, c_8, \dots$, in (4.10) may have non-vanishing values, in particular $c_6 \neq 0$. The point is that the $\mathbf{F}^6$ quantum correction in the low energy effective action of the $SU(N)$ gauge theory with $2N$ fundamental hypermultiplets is known to be different from the one coming from the Born-Infeld action (4.8) [30], and the leading part of the $c_6$ term in (4.10) should occur in the same superfield functional which contains those $\mathbf{F}^6$ terms which are not present in the Born-Infeld action.

In summary, the terms in $L_{\rm eff}$ containing two and four derivatives of the fields coincide with the corresponding terms in the D3-brane action (4.8), and the four-derivative term in $L_{\rm eff}$ is one-loop exact. This information suffices to argue that the parameter $R^4$ in (4.7) and (4.6) cannot receive two- and higher-loop quantum corrections. Indeed, the transformation (4.7) mixes the two-derivative and four-derivative terms in $L_{\rm eff}$, of which the former is tree-level exact and the latter is one-loop exact.

# 5 $\mathcal{N} = 4$ SYM

In the case of $\mathcal{N} = 4$ SYM, which is just a special $\mathcal{N} = 2$ SYM theory, the hypermultiplet sector is composed of a single hypermultiplet in the adjoint representation of the gauge group, and hence $Q_{i\underline{\alpha}} = Q_{i\underline{a}}$ and $\bar{Q}^{i\underline{\alpha}} = \bar{Q}^i_{\underline{a}}$. With the use of eq. (2.4), one can now show that the off-diagonal elements of the hypermultiplet mass matrix $\mathcal{M}^2_{\rm h}$ vanish,

$$(T_{\underline{c}}\mathcal{Q}^k)_{\underline{a}}\,(T_{\underline{c}}\mathcal{Q}_k)_{\underline{b}} = 0\;, \tag{5.1}$$

while the block diagonal pieces of $\mathcal{M}^2_{\rm h}$ coincide with the vector multiplet mass matrix, $\mathcal{M}^2_{\rm v}$,

$$\{\bar{\mathcal{W}},\mathcal{W}\}_{\underline{ab}} + 2(T_{\underline{c}}\mathcal{Q}_k)_{\underline{a}}\,(\bar{\mathcal{Q}}^k T_{\underline{c}})_{\underline{b}} = \bar{\mathcal{W}}\{T_{\underline{a}},T_{\underline{b}}\}\mathcal{W} + \bar{\mathcal{Q}}^i\{T_{\underline{a}},T_{\underline{b}}\}\mathcal{Q}_i = (\mathcal{M}^2_{\rm v})_{\underline{ab}}\;. \tag{5.2}$$

As a consequence, eq. (3.25) becomes

$$\delta_{(1)}\,\mathcal{Q}_{i\underline{a}} = 4{\rm i}\,g^2\,(\partial^m\sigma)\,{\rm tr}_{\rm Ad}\Big(T_{\underline{a}}\,(\tilde\Delta^{-3}|)\,\mathcal{D}_m\mathcal{Q}_i\Big)\;, \tag{5.3}$$

which is of the same form as (3.12) and (3.15).

Evaluating (5.3), the result can be combined with eqs. (3.17) and (3.18) to give the final expression for the one-loop deformation of conformal symmetry:

$$
\begin{aligned}
\delta_{(1)}\mathcal{V}_{m\underline{a}} &= -\frac{g^2}{8\pi^2}\,(\partial^n\sigma)\,\mathrm{tr}_{\mathrm{Ad}}\left(T_{\underline{a}}\,\mathcal{M}_{\mathrm{v}}^{-2}\mathcal{F}_{nm}\right)\,, \\
\delta_{(1)}\mathcal{W}_{\underline{a}} &= -\frac{g^2}{8\pi^2}\,(\partial^n\sigma)\,\mathrm{tr}_{\mathrm{Ad}}\left(T_{\underline{a}}\,\mathcal{M}_{\mathrm{v}}^{-2}\mathcal{D}_n\mathcal{W}\right)\,, \\
\delta_{(1)}\mathcal{Q}_{i\underline{a}} &= -\frac{g^2}{8\pi^2}\,(\partial^n\sigma)\,\mathrm{tr}_{\mathrm{Ad}}\left(T_{\underline{a}}\,\mathcal{M}_{\mathrm{v}}^{-2}\mathcal{D}_n\mathcal{Q}_i\right)\,.
\end{aligned}
\tag{5.4}
$$

By construction, the mass matrix $\mathcal{M}_{\mathrm{v}}^2$ is invariant under the $\mathcal{R}$-symmetry group $SO(6)_{\mathcal{R}}$ which rotates the six scalars $\mathcal{W}$, $\bar{\mathcal{W}}$, $\mathcal{Q}_i$ and $\bar{\mathcal{Q}}^i$. It is clear that the one-loop deformation (5.4) respects the $SO(6)_{\mathcal{R}}$ symmetry.

We want to evaluate (5.4) for a general semi-simple gauge group $G$. For this purpose it is convenient to use a complex basis for the charged gauge bosons and their scalar partners. We choose them in one-to-one correspondence with the roots of the Lie-algebra. That is $V = \sum_{\underline{a}} V^{\underline{a}} T_{\underline{a}} = \sum_\alpha V^\alpha E_\alpha + \sum_i V^i H_i$ and likewise for the adjoint scalars. Here $E_\alpha$ is the generator corresponding to the root $\alpha$ normalized as $\mathrm{tr}(E_\alpha E_\beta^\dagger) = \mathrm{tr}(E_\alpha E_{-\beta}) = \delta_{\alpha,\beta}$, and $H_i$ are the rank$(G)$ Cartan subalgebra generators. They satisfy the commutation relations $[H_i, E_\alpha] = \alpha(H_i)E_\alpha$. With all background fields aligned along an arbitrary element $H$ of the Cartan subalgebra (cf. (2.22) with $\Upsilon = H$), the vector-multiplet mass matrix becomes

$$
(\mathcal{M}_{\mathrm{v}}^2)_{\alpha\beta} = \mathbf{X}^2\,\mathrm{tr}([H, E_\alpha^\dagger][E_\beta, H]) = \mathbf{X}^2\,(\alpha(H))^2\delta_{\alpha,\beta}
\tag{5.5}
$$

with $\mathbf{X}^2 = 2(\bar{\mathbf{W}}\mathbf{W} + \bar{\mathbf{Q}}^i\mathbf{Q}_i)$. All components of $\mathcal{M}_{\mathrm{v}}^2$ along the Cartan subalgebra directions vanish: the neutral gauge bosons stay massless (adjoint breaking does not reduce the rank of the gauge group) and they do not mix with the charged vector bosons at the quadratic level.

It is now straightforward to compute, say, $\delta_{(1)}\mathbf{W}$, for which we get from (5.4) $\delta_{(1)}\mathbf{W} \propto \mathrm{tr}_{\mathrm{Ad}}(H^2\mathcal{M}_{\mathrm{v}}^{-2})$. From the commutation relation $[H, E_\alpha] = \alpha(H)E_\alpha$ we read off $H$ in the adjoint representation. Then the factors $\alpha(H)$ cancel and the trace produces a numerical coefficient equal to the number of roots with $\alpha(H) \neq 0$, i.e. the number of broken generators or, in other words, the number of massive gauge bosons.

To be specific, we consider the classical groups with the background chosen such that one obtains the breaking patterns $SU(N) \to SU(N-1) \times U(1)$, $Sp(2N) \to Sp(2N-2) \times U(1)$, $SO(N) \to SO(N-2) \times U(1)$. These regular subgroups are obtained by removing an extremal node from the Dynkin diagram of the gauge group. In the brane realization of these theories this corresponds to moving one of the D3 branes away from the orientifold O3 plane. This implies a particular Cartan subalgebra generator $H$ (for $SU(N)$ it has been explicitly given in (4.4)) for which $\alpha(H) = \alpha\cdot\mu$, where $\mu$ is the fundamental weight corresponding to the simple root associated with the removed node of the Dynkin diagram. We then find,

$$\delta_{(1)}\mathbf{W} = -\frac{\lambda g^2}{8\pi^2\mathbf{X}^2}(\partial^n\sigma)\partial_n\mathbf{W}\,, \quad \text{with} \quad \lambda = \begin{cases} 2(N-1) & \text{for } SU(N), \\ 4N-2 & \text{for } Sp(2N) \text{ and } SO(2N+1), \\ 4N-1 & \text{for } SO(2N), \end{cases} \tag{5.6}$$

and likewise for $\delta_{(1)}\mathbf{Q}_i$ and $\delta_{(1)}\mathbf{V}_m$. As mentioned above, $\lambda$ is simply the number of broken generators. In the $SU(N)$ case, the deformation $\delta_{(1)}\mathbf{W}$ was derived in [6, 7].

What remains to be shown is that the other components of the background fields do not receive any one-loop deformation, i.e. that the deformation vanishes for $T_{\underline{a}}$ such that $\mathrm{tr}(T_{\underline{a}}H) = 0$. Indeed, for any direction $\underline{a}$ the deformation is proportional to $\sum_\alpha \frac{1}{\alpha(H)}(T_{\underline{a}})_{\alpha\alpha}$, where the sum is restricted to those roots for which $\alpha(H) \neq 0$. This vanishes if $T_{\underline{a}}$ is one of the $E_\alpha$ (since $[E_\alpha, E_\beta]$ is never proportional to $E_\alpha$). If $T_{\underline{a}} = H'$ with $\mathrm{tr}(HH') = 0$ the deformation is proportional to $\sum_\alpha \frac{\alpha(H')}{\alpha(H)}$. Using the roots and fundamental weights for the classical Lie algebras (see e.g. the Appendix of [31]) this can be explicitly shown to vanish.

The results of this section easily generalize to $\mathcal{N} = 2$ theories on the pure Coulomb branch by simply setting $\mathbf{X}^2 = 2\bar{\mathbf{W}}\mathbf{W}$.

It is worth discussing the results obtained in this section. To first order in the derivative expansion, the one-loop deformed conformal transformations are:

$$\begin{aligned} \delta\mathbf{V}_m &= \delta_c\mathbf{V}_m - R^4\,\frac{(\partial^n\sigma)\,\mathbf{F}_{nm}}{4(\bar{\mathbf{W}}\mathbf{W} + \bar{\mathbf{Q}}^j\mathbf{Q}_j)}\,, \\ \delta\mathbf{W} &= \delta_c\mathbf{W} - R^4\,\frac{(\partial^n\sigma)\,\partial_n\mathbf{W}}{4(\bar{\mathbf{W}}\mathbf{W} + \bar{\mathbf{Q}}^j\mathbf{Q}_j)}\,; \\ \delta\mathbf{Q}_i &= \delta_c\mathbf{Q}_i - R^4\,\frac{(\partial^n\sigma)\,\partial_n\mathbf{Q}_i}{4(\bar{\mathbf{W}}\mathbf{W} + \bar{\mathbf{Q}}^j\mathbf{Q}_j)}\,, \end{aligned} \tag{5.7}$$

with $R^4 = \lambda g^2/4\pi^2$ and $\lambda$ defined as above for the classical groups. These transformations leave invariant the D3 brane action (4.8), which now involves six scalars $\mathbf{X}_\mu$, where $\mu = 1,\ldots,6$, defined by $\sqrt{2}\,\mathbf{W} = \mathbf{X}_1 + \mathrm{i}\,\mathbf{X}_2$ and $\sqrt{2}\,\mathbf{Q}_i = \mathbf{X}_{i+1} + \mathrm{i}\,\mathbf{X}_{i+2}$. The transformations (5.7) realize the conformal algebra (up to a pure gauge transformation) without the need to take into account any higher loop quantum corrections. The consideration of sect. 4 implies that the parameter $R^4$ in (5.7) is one-loop exact. Unlike the situation with generic

$\mathcal{N}=2$ superconformal theories on the Coulomb branch, which we discussed in sect. 4, the low energy effective action in the $\mathcal{N}=4$ SYM theory is expected to be of the Born-Infeld form (4.8), at least in the large $N$ limit (see [32, 33] and references therein). For this to hold,[8] there should exist a host of (yet unknown) non-renormalization theorems in $\mathcal{N}=4$ SYM (see, e.g., the discussion in [30]).

# 6 $USp(2N)$ SYM with fundamental and traceless antisymmetric hypermultiplets

Here we consider the $\mathcal{N}=2$ superconformal Yang-Mills theory introduced in [12, 13], which is known to have a supergravity dual on $AdS_5 \times S^5/\mathbf{Z}^2$ [15]. It will be shown that the structure of the quantum conformal deformation differs significantly from the $\mathcal{N}=4$ SYM case.

The gauge group is $USp(2N) = Sp(2N,\mathbf{C}) \bigcap U(2N)$, and the theory contains hypermultiplets in two representations of the gauge group: four hypermultiplets[9] $Q_{\rm F}$ in the fundamental and one hypermultiplet $Q_{\rm A}$ in the antisymmetric traceless representation of $USp(2N)$. The Lie algebra $usp(2N)$ is spanned by $2N \times 2N$ matrices $({\rm i}\,\zeta)$ satisfying the constraints

$$\zeta^{\rm T} J + J\zeta = 0 , \qquad \zeta^\dagger = \zeta , \qquad J = -J^{\rm T} = \begin{pmatrix} 0 & \mathbf{1}_N \\ -\mathbf{1}_N & 0 \end{pmatrix} . \tag{6.1}$$

We will use Greek letters to denote the components of $USp(2N)$ spinors, $Q_{\rm F} = (Q_\alpha)$, and also make use of the symplectic metric $J = (J^{\alpha\beta}) = (J_{\alpha\beta})$ for raising and lowering $USp(2N)$ spinor indices; for example, $\Psi^\alpha \equiv J^{\alpha\beta}\Psi_\beta$ and $\Psi_\alpha \equiv -J_{\alpha\beta}\Psi^\beta$. The antisymmetric traceless representation of $USp(2N)$ is realized by second rank tensors of the form

$$Q_{\rm A} = ({Q_\alpha}^\beta) , \qquad {\rm tr}\, Q_{\rm A} = 0 , \qquad Q_{\alpha\beta} = -Q_{\beta\alpha} . \tag{6.2}$$

The hypermultiplets $Q_{\rm F}$ and $Q_{\rm A}$ transform under $USp(2N)$ as

$$\delta Q_{\rm F} = {\rm i}\,\zeta\, Q_{\rm F} , \qquad \delta Q_{\rm A} = {\rm i}\,[\zeta, Q_{\rm A}] . \tag{6.3}$$

[8]The claim [7, 8] that the deformed conformal symmetry (5.7), $SO(6)_{\mathcal{R}}$ invariance and the *known* $\mathcal{N}=4$ non-renormalization theorems (which we discussed in sect. 4 of the present paper) uniquely fix the scalar part, $L_{\rm eff}(\mathbf{X}_\mu, \partial_m \mathbf{X}_\nu)$, of the low energy effective Lagrangian in $\mathcal{N}=4$ SYM, is incorrect.

[9]The $SU(2)_{\mathcal{R}}$ indices of the hypermultiplets are suppressed.

It is worth noting that the symmetric representation of $USp(2N)$ can be identified with the Lie algebra $sp(2N, \mathbf{C})$, i.e. $\zeta = (\zeta_\alpha{}^\beta) \in sp(2N, \mathbf{C})$ iff $\zeta_{\alpha\beta} = \zeta_{\beta\alpha}$. We also note that the finiteness condition (2.1) is met due to the following property of $USp(2N)$ representations:

$$\mathrm{tr}_{\mathrm{Ad}} W^2 - 4\,\mathrm{tr}_{\mathrm{F}} W^2 - \mathrm{tr}_{\mathrm{A}} W^2 = 0 \; . \tag{6.4}$$

The background fields $\mathcal{W}$, $\mathcal{Q}_{\mathrm{F}}$ and $\mathcal{Q}_{\mathrm{A}}$ are chosen to solve the equations (2.4) defining the classical moduli space. Up to a gauge transformation, the general solution to the first equation in (2.4) is $\mathcal{W} = \mathrm{diag}\,(W_1, W_2, \dots, W_N, -W_1, -W_2, \dots, -W_N)$, with the $W$'s arbitrary complex numbers. Further analysis is restricted to the choice $W_1 \neq 0$ and $W_2 = \ldots = W_N = 0$; in this case, the unbroken gauge subgroup, $USp(2N-2) \times U(1)$, is maximal. We will also impose the requirement that $\mathcal{Q}_{\mathrm{F}}$ and $\mathcal{Q}_{\mathrm{A}}$, which must be solutions to the second and third equations in (2.4), be invariant under the unbroken group $USp(2N-2) \times U(1)$. This leads to[10]

$$\mathcal{W} = \frac{\mathbf{W}}{\sqrt{2}}\, \mathrm{diag}\,(1, \underbrace{0, \dots, 0}_{N-1}, -1, \underbrace{0, \dots, 0}_{N-1}) \; , \tag{6.5}$$

$$\mathcal{Q}_{\mathrm{F}} = 0 \; , \qquad \mathcal{Q}_{\mathrm{A}} = \frac{\mathbf{Q}}{\sqrt{2N(N-1)}}\, \mathrm{diag}\,(N-1, \underbrace{-1, \dots, -1}_{N-1}, N-1, \underbrace{-1, \dots, -1}_{N-1}) \; .$$

The nonvanishing background fields constitute the bosonic sector of an Abelian $\mathcal{N} = 2$ vector multiplet, $(\mathbf{V}_m, \mathbf{W}, \bar{\mathbf{W}})$, and a single neutral hypermultiplet, $(\mathbf{Q}_i, \bar{\mathbf{Q}}^j)$. Since the background is of the form (2.22), the formulas (3.31) and (3.34) can be applied to determine one-loop deformations of the conformal transformations once the mass matrices are known.

The mass matrix $\mathcal{M}^2_{\mathrm{v}}$ for quantum fields in the adjoint representation has $4(N-1)$ eigenvectors with the eigenvalue $\bar{\mathbf{W}}\mathbf{W} + \frac{N}{N-1}\bar{\mathbf{Q}}^j\mathbf{Q}_j$ and two eigenvectors with the eigenvalue $4\bar{\mathbf{W}}\mathbf{W}$. The massive degrees of freedom in the quantum adjoint scalars $w$ are parametrized in the form

$$w = \frac{1}{\sqrt{2}} \begin{pmatrix} Z & R \\ S & -Z^{\mathrm{T}} \end{pmatrix} \tag{6.6}$$

with

$$Z = \begin{pmatrix} 0 & \vec{z}_2^{\,\mathrm{T}} \\ \vec{z}_1 & \mathbf{0}_{N-1} \end{pmatrix} \; , \quad R = \begin{pmatrix} \sqrt{2}\,\alpha & \vec{r}^{\,\mathrm{T}} \\ \vec{r} & \mathbf{0}_{N-1} \end{pmatrix} \; , \quad S = \begin{pmatrix} \sqrt{2}\,\beta & \vec{s}^{\,\mathrm{T}} \\ \vec{s} & \mathbf{0}_{N-1} \end{pmatrix} \; , \tag{6.7}$$

[10] The background fields are chosen to yield correctly normalized kinetic terms in the classical action, $-\frac{1}{g^2}\int \mathrm{d}^4x\, (\partial^m \bar{\mathbf{W}} \partial_m \mathbf{W} + \partial^m \bar{\mathbf{Q}}^i \partial_m \mathbf{Q}_i)$.

where $\vec{z}_1$, $\vec{z}_2$, $\vec{r}$ and $\vec{s}$ are $(N-1)$-vectors. The fields which form the components of $\vec{z}_1$, $\vec{z}_2$, $\vec{r}$ and $\vec{s}$ are eigenvectors of $\mathcal{M}_{\mathrm{v}}^2$ with the eigenvalue $\bar{\mathbf{W}}\mathbf{W} + \frac{N}{N-1}\bar{\mathbf{Q}}^j\mathbf{Q}_j$, while the scalars $\alpha$ and $\beta$ are eigenvectors of $\mathcal{M}_{\mathrm{v}}^2$ with the eigenvalue $4\bar{\mathbf{W}}\mathbf{W}$.

The off-diagonal elements of the mass matrix $\mathcal{M}_{\mathrm{h}}^2$ (3.24) for the hypermultiplet in the antisymmetric representation vanish, since $\mathbf{Q}^i\,\mathbf{Q}_i = 0$. There are $4(N-1)$ eigenvectors of $\mathcal{M}_{\mathrm{h}}^2$ with the eigenvalue $\bar{\mathbf{W}}\mathbf{W} + \frac{N}{N-1}\bar{\mathbf{Q}}^j\mathbf{Q}_j$; the remaining eigenvalues vanish. The corresponding massive quantum degrees of freedom can be parametrized in the form

$$q = \frac{1}{\sqrt{2}} \begin{pmatrix} A & B \\ C & A^{\mathrm{T}} \end{pmatrix}, \tag{6.8}$$

where

$$A = \begin{pmatrix} 0 & \vec{a}_2^{\,\mathrm{T}} \\ \vec{a}_1 & \mathbf{0}_{N-1} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -\vec{b}^{\,\mathrm{T}} \\ \vec{b} & \mathbf{0}_{N-1} \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -\vec{c}^{\,\mathrm{T}} \\ \vec{c} & \mathbf{0}_{N-1} \end{pmatrix}. \tag{6.9}$$

Again, $\vec{a}_1$, $\vec{a}_2$, $\vec{b}$ and $\vec{c}$ are $(N-1)$-vectors.

Although some of the hypermultiplet degrees of freedom in the fundamental representation are massive in the chosen background, they decouple.

The deformed conformal transformations (3.31) and (3.34) take the form

$$\begin{aligned} \delta_{(1)}\mathbf{V}_m &= -\frac{g^2}{4\pi^2}\,(\partial^n\sigma)\,\mathbf{F}_{nm}\left(\frac{N-1}{\bar{\mathbf{W}}\mathbf{W} + \frac{N}{N-1}\bar{\mathbf{Q}}^j\mathbf{Q}_j} + \frac{1}{2\bar{\mathbf{W}}\mathbf{W}}\right), \\ \delta_{(1)}\mathbf{W} &= -\frac{g^2}{4\pi^2}\,(\partial^n\sigma)\,(\partial_n\mathbf{W})\left(\frac{N-1}{\bar{\mathbf{W}}\mathbf{W} + \frac{N}{N-1}\bar{\mathbf{Q}}^j\mathbf{Q}_j} + \frac{1}{2\bar{\mathbf{W}}\mathbf{W}}\right); \end{aligned} \tag{6.10}$$

$$\delta_{(1)}\mathbf{Q}_i = -\frac{g^2}{4\pi^2}\,(\partial^n\sigma)\,(\partial_n\mathbf{Q}_i)\,\frac{N}{\bar{\mathbf{W}}\mathbf{W} + \frac{N}{N-1}\bar{\mathbf{Q}}^j\mathbf{Q}_j}\,. \tag{6.11}$$

The vector multiplet and the hypermultiplet transformations differ due to eigenvectors with eigenvalue $4\bar{\mathbf{W}}\mathbf{W}$, which are present in $\mathcal{M}_{\mathrm{v}}^2$ but absent in $\mathcal{M}_{\mathrm{h}}^2$.

Unlike the situation in the $\mathcal{N}=4$ SYM theory, the one-loop deformed transformations $(\delta_{\mathrm{c}} + \delta_{(1)})\,\Phi$ specified in (6.10) and (6.11) do not realize the conformal algebra for finite values of $N$, but only in a large $N$ limit. To realize the conformal algebra for finite $N$, it is necessary to include two and higher-loop deformations to the conformal transformations of the fields; these can be determined order by order from the requirement of closure of the conformal algebra. In particular, the two-loop deformation can be shown by this means to have the form

$$\delta_{(2)}\mathbf{V}_m = -\frac{1}{N}\left(\frac{Ng^2}{4\pi^2}\right)^2 (\partial^n\sigma)\,\mathbf{F}_{nm}\,(\partial^p\bar{\mathbf{Q}}^j)(\partial_p\mathbf{Q}_j)\,\Omega(\mathbf{W},\mathbf{Q})\,,$$

$$\delta_{(2)}\mathbf{W} = -\frac{1}{N}\left(\frac{Ng^2}{4\pi^2}\right)^2 (\partial^n\sigma)(\partial_n\mathbf{W})(\partial^p\bar{\mathbf{Q}}^j)(\partial_p\mathbf{Q}_j)\,\Omega(\mathbf{W},\mathbf{Q})\ , \tag{6.12}$$

$$\delta_{(2)}\mathbf{Q}_i = \frac{1}{N}\left(\frac{Ng^2}{4\pi^2}\right)^2 (\partial^n\sigma)(\partial_n\mathbf{Q}_i)(\partial^p\bar{\mathbf{W}})(\partial_p\mathbf{W})\,\Omega(\mathbf{W},\mathbf{Q})\ ,$$

where

$$\Omega(\mathbf{W},\mathbf{Q}) = \frac{1}{(\bar{\mathbf{W}}\mathbf{W} + \frac{N}{N-1}\bar{\mathbf{Q}}^j\mathbf{Q}_j)^2}\left\{\frac{1}{2\bar{\mathbf{W}}\mathbf{W}} - \frac{1}{\bar{\mathbf{W}}\mathbf{W} + \frac{N}{N-1}\bar{\mathbf{Q}}^j\mathbf{Q}_j}\right\}\ . \tag{6.13}$$

The field theory under consideration possesses the global symmetries $SU(2)_{\mathcal{R}}\times U(1)_{\mathcal{R}}\times SO(8)\times SU(2)$ of which the first two factors denote the $\mathcal{R}$-symmetry. The group $SO(8)$ rotates the four fundamental hypermultiplets, while $SU(2)$ acts on the antisymmetric hypermultiplet.[11] Since $\mathcal{Q}_{\rm F}=0$, we cannot probe $SO(8)$ for the background chosen. But the symmetries $SU(2)_{\mathcal{R}}\times U(1)_{\mathcal{R}}\times SU(2)$ should be manifestly realized in the low energy effective action. It is not difficult to see that both the one-loop and two-loop conformal deformations respect these symmetries.

In the limit $N\to\infty$, the one-loop deformations (6.10) and (6.11) lead to conformal transformations of the form (5.7), which are symmetries of the Born-Infeld action (4.8). On symmetry grounds, when $N$ is finite, the low energy effective action of the field theory under consideration cannot have the Born-Infeld form considered in sects. 4 and 5.

# 7 Kachru-Silverstein model

Here we consider the simplest quiver gauge theory [16, 17, 18] – $\mathcal{N}=2$ super Yang-Mills theory with gauge group $SU(N)\times SU(N)\equiv SU(N)_L\times SU(N)_R$ and two hypermultiplets, $H_i$ and $\tilde{H}_i$, in the representations $(\mathbf{N},\bar{\mathbf{N}})$ and $(\bar{\mathbf{N}},\mathbf{N})$ of the gauge group. Both $H_i$ and $\tilde{H}_i$ carry an index of the automorphism group of the $\mathcal{N}=2$ supersymmetry algebra. The hypermultiplets transform under $SU(N)_L\times SU(N)_R$ as

$$\delta H = {\rm i}\,\zeta_L\,H - {\rm i}\,H\,\zeta_R\ , \qquad \delta\tilde{H} = {\rm i}\,\zeta_R\,\tilde{H} - {\rm i}\,\tilde{H}\,\zeta_L\ , \tag{7.1}$$

where $\zeta = \zeta_a t_{\underline{a}}$, with $t_{\underline{a}}$ the generators of $SU(N)$. For simplicity we take the gauge coupling constants in the two $SU(N)$ factors to be equal.

Eq. (2.4) specifies the flat directions in massless $\mathcal{N}=2$ super Yang-Mills theories. We are interested in those solutions of (2.4) in the Kachru-Silverstein model which allow

[11] In the harmonic superspace approach [21], the symmetries $SO(8)$ and $SU(2)$ can be realized as (a combination of flavour and) Pauli-Gürsey transformations of $q^+$ hypermultiplets.

for non-vanishing hypermultiplet components. Notationally, we now have $W = W_L \otimes \mathbf{1} + \mathbf{1} \otimes W_R$ and $Q_i = (H_i,\ \tilde{H}_i)$.

Up to a gauge transformation, the general solution to the first equation in (2.4), $[\mathcal{W}, \bar{\mathcal{W}}] = 0$, is $\mathcal{W} = \mathcal{W}_L \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{W}_R$, with $\mathcal{W}_L$ and $\mathcal{W}_R$ diagonal traceless matrices. The second equation in (2.4), $\mathcal{W}\mathcal{Q}_i = 0$, becomes

$$\mathcal{W}_L\,\mathcal{H} - \mathcal{H}\,\mathcal{W}_R = 0\ , \qquad \mathcal{W}_R\,\tilde{\mathcal{H}} - \tilde{\mathcal{H}}\,\mathcal{W}_L = 0\ . \tag{7.2}$$

These equations are obviously solved by any diagonal matrices $\mathcal{W}_L = \mathcal{W}_R$, $\mathcal{H}$ and $\tilde{\mathcal{H}}$. The third equation in (2.4), $\bar{\mathcal{Q}}_{(i} T_{\underline{a}} \mathcal{Q}_{j)} = 0$, is now

$$\mathrm{tr}\left(\overline{\mathcal{H}}_{(i}\, t_{\underline{a}}\, \mathcal{H}_{j)}\right) - \mathrm{tr}\left(\tilde{\mathcal{H}}_{(i}\, t_{\underline{a}}\, \overline{\tilde{\mathcal{H}}}_{j)}\right) = 0\ , \quad \mathrm{tr}\left(\mathcal{H}_{(i}\, t_{\underline{a}}\, \overline{\mathcal{H}}_{j)}\right) - \mathrm{tr}\left(\overline{\tilde{\mathcal{H}}}_{(i}\, t_{\underline{a}}\, \tilde{\mathcal{H}}_{j)}\right) = 0\ . \tag{7.3}$$

The moduli space of vacua of the model thus includes the field configuration[12]

$$\begin{aligned} \mathcal{W}_L = \mathcal{W}_R &= \frac{\mathbf{W}}{N\sqrt{2(N-1)}}\,\mathrm{diag}(N-1, -1, \ldots, -1)\ , \\ \mathcal{H}_i = \tilde{\mathcal{H}}_i &= \frac{\mathbf{Q}_i}{\sqrt{2}}\,\mathrm{diag}(1, 0, \ldots, 0)\ . \end{aligned} \tag{7.4}$$

which preserves an unbroken gauge group $SU(N-1) \times SU(N-1)$ together with the diagonal $U(1)$ subgroup in $SU(N)_L \times SU(N)_R$ associated with the $\mathcal{W}$ chosen. In such a background, the hypermultiplet mass matrix has $4(N-1)$ eigenvalues $\frac{1}{N-1}\bar{\mathbf{W}}\mathbf{W} + \frac{1}{N}\bar{\mathbf{Q}}^j\mathbf{Q}_j$ and one eigenvalue $\frac{4(N-1)}{N^2}\,\bar{\mathbf{Q}}^j\mathbf{Q}_j$. The massive degrees of freedom can be parametrized in the form

$$h_i = \begin{pmatrix} \alpha_i & \vec{s}_i^{\,\mathrm{T}} \\ \vec{r}_i & \mathbf{0}_{N-1} \end{pmatrix}\ , \quad \tilde{h}_i = \begin{pmatrix} \tilde{\alpha}_i & \vec{\tilde{s}}_i^{\,\mathrm{T}} \\ \vec{\tilde{r}}_i & \mathbf{0}_{N-1} \end{pmatrix}. \tag{7.5}$$

Here, all of the $(N-1)$-vectors $\vec{r}_i$, $\vec{\tilde{r}}_i$, $\vec{s}_i$ and $\vec{\tilde{s}}_i$ are eigenvectors of $\mathcal{M}^2_{\mathrm{h}}$ with the eigenvalue $\frac{1}{N-1}\bar{\mathbf{W}}\mathbf{W} + \frac{1}{N}\bar{\mathbf{Q}}^j\mathbf{Q}_j$, while the eigenvector with eigenvalue $\frac{4(N-1)}{N^2}\,\bar{\mathbf{Q}}^j\mathbf{Q}_j$ is $(\alpha_i - \tilde{\alpha}_i)/\sqrt{2}$. This linear combination is orthogonal to the massless eigenvector $(\alpha_i + \tilde{\alpha}_i)/\sqrt{2}$ corresponding to an unbroken linear combination of $U(1)$ generators from $SU(N)_L$ and $SU(N)_R$.

The mass matrix for the adjoint scalars and the gauge bosons also has $4(N-1)$ eigenvalues $\frac{1}{N-1}\bar{\mathbf{W}}\mathbf{W} + \frac{1}{N}\bar{\mathbf{Q}}^j\mathbf{Q}_j$ and one eigenvalue $\frac{4(N-1)}{N^2}\,\bar{\mathbf{Q}}^j\mathbf{Q}_j$. The adjoint massive degrees of freedom can be parametrized in the form

$$w_L = \frac{1}{\sqrt{2N}} \begin{pmatrix} \sqrt{\frac{2(N-1)}{N}}\,\alpha_L & \vec{r}_L^{\,\mathrm{T}} - \mathrm{i}\vec{s}_L^{\,\mathrm{T}} \\ \vec{r}_L + \mathrm{i}\vec{s}_L & -\sqrt{\frac{2}{N(N-1)}}\,\alpha_L\,\mathbf{1}_{N-1} \end{pmatrix}\ ,$$

[12]The choice $\overline{\tilde{\mathcal{H}}}_i = \mathcal{H}_i$, which appears to solve (7.3) at first sight, is in fact not a solution. The conjugation rules $(Q_i)^\dagger = \overline{Q}^i \iff (H_i)^\dagger = \overline{H}^i$, $(\tilde{H}_i)^\dagger = \overline{\tilde{H}}^i$ imply $\left(\overline{\tilde{\mathcal{H}}}_i\right)^\dagger = -\tilde{\mathcal{H}}^i$, and hence $\tilde{\mathcal{H}}_i = -\overline{\mathcal{H}}_i$.

$$w_R = \frac{1}{\sqrt{2N}} \begin{pmatrix} \sqrt{\frac{2(N-1)}{N}}\,\alpha_R & \vec{r}_R^{\,\mathrm{T}} - \mathrm{i}\vec{s}_R^{\,\mathrm{T}} \\ \vec{r}_R + \mathrm{i}\vec{s}_R & -\sqrt{\frac{2}{N(N-1)}}\,\alpha_R\,\mathbf{1}_{N-1} \end{pmatrix} . \tag{7.6}$$

The $(N-1)$-vectors $\vec{r}_L$, $\vec{s}_R$, $\vec{r}_L$ and $\vec{s}_R$ are eigenvectors of $\mathcal{M}_{\mathrm{v}}^2$ with the eigenvalue $\frac{1}{N-1}\bar{\mathbf{W}}\mathbf{W} + \frac{1}{N}\bar{\mathbf{Q}}^j\mathbf{Q}_j$, while the eigenvector with eigenvalue $\frac{4(N-1)}{N^2}\,\bar{\mathbf{Q}}^j\mathbf{Q}_j$ is $(\alpha_L - \alpha_R)/\sqrt{2}$.

The scalars (both adjoint and hypermultiplet) with mass $\frac{1}{N-1}\bar{\mathbf{W}}\mathbf{W} + \frac{1}{N}\bar{\mathbf{Q}}^j\mathbf{Q}_j$ couple to the gauge bosons of the same mass via a derivative of the corresponding background scalar. However, for the adjoint scalars, the linear combination $(\alpha_L - \alpha_R)/\sqrt{2}$ does not couple to the gauge bosons, and so there is no contribution from this mass eigenstate to the quantum corrected transformations. This is to be contrasted with the hypermultiplet mass eigenstate $(\alpha_i - \tilde{\alpha}_i)/\sqrt{2}$ with the mass $\frac{4(N-1)}{N^2}\,\bar{\mathbf{Q}}^j\mathbf{Q}_j$, which couples to the gauge boson with mass $\frac{4(N-1)}{N^2}\,\bar{\mathbf{Q}}^j\mathbf{Q}_j$, giving rise to an additional term in the one-loop hypermultiplet transformation compared with the adjoint scalar transformation. This is the "reverse" of the situation encountered in the $USp(2N)$ example. The one-loop conformal deformations (3.6) and (3.7) are

$$\begin{aligned} \delta_{(1)}\mathbf{V}_m &= -\frac{g^2}{4\pi^2}\,(\partial^n\sigma)\,\mathbf{F}_{nm}\,\frac{N}{\frac{N}{N-1}\bar{\mathbf{W}}\mathbf{W} + \bar{\mathbf{Q}}^j\mathbf{Q}_j}\ , \\ \delta_{(1)}\mathbf{W} &= -\frac{g^2}{4\pi^2}\,(\partial^n\sigma)\,(\partial_n\mathbf{W})\,\frac{N}{\frac{N}{N-1}\bar{\mathbf{W}}\mathbf{W} + \bar{\mathbf{Q}}^j\mathbf{Q}_j}\ ; \end{aligned} \tag{7.7}$$

$$\delta_{(1)}\mathbf{Q}_i = -\frac{g^2}{4\pi^2}\,(\partial^n\sigma)\,(\partial_n\mathbf{Q}_i)\left(\frac{N-1}{\frac{N}{N-1}\bar{\mathbf{W}}\mathbf{W} + \bar{\mathbf{Q}}^j\mathbf{Q}_j} + \frac{1}{4\bar{\mathbf{Q}}^j\mathbf{Q}_j}\right) . \tag{7.8}$$

Again, the one-loop deformed transformations $(\delta_{\mathrm{c}} + \delta_{(1)})\,\Phi$ specified by (7.7) and (7.8) only provide a realization of the conformal algebra in a large $N$ limit. For finite values of $N$, it is necessary to include higher-loop corrections, which can again be determined order by order in $g^2N$ by requiring closure of the algebra of conformal transformations of the fields. Using this procedure, the two-loop conformal deformation is found to be

$$\begin{aligned} \delta_{(2)}\mathbf{V}_m &= \frac{1}{N}\left(\frac{Ng^2}{4\pi^2}\right)^2 (\partial^n\sigma)\,\mathbf{F}_{nm}\,(\partial^p\bar{\mathbf{Q}}^j)(\partial_p\mathbf{Q}_j)\ \Omega(\mathbf{W},\mathbf{Q})\ , \\ \delta_{(2)}\mathbf{W} &= \frac{1}{N}\left(\frac{Ng^2}{4\pi^2}\right)^2 (\partial^n\sigma)(\partial_n W)(\partial^p\bar{\mathbf{Q}}^j)(\partial_p\mathbf{Q}_j)\ \Omega(\mathbf{W},\mathbf{Q})\ , \\ \delta_{(2)}\mathbf{Q}_i &= -\frac{1}{N}\left(\frac{Ng^2}{4\pi^2}\right)^2 (\partial^n\sigma)(\partial_n\mathbf{Q}_i)(\partial^p\bar{\mathbf{W}})(\partial_p\mathbf{W})\ \Omega(\mathbf{W},\mathbf{Q})\ , \end{aligned} \tag{7.9}$$

where

$$\Omega(\mathbf{W},\mathbf{Q}) = \frac{1}{(\frac{N}{N-1}\bar{\mathbf{W}}\mathbf{W} + \bar{\mathbf{Q}}^j\mathbf{Q}_j)^2}\left\{\frac{1}{4\bar{\mathbf{Q}}^j\mathbf{Q}_j} - \frac{1}{\frac{N}{N-1}\bar{\mathbf{W}}\mathbf{W} + \bar{\mathbf{Q}}^j\mathbf{Q}_j}\right\} . \tag{7.10}$$

The field theory under consideration possesses the global symmetries $SU(2)_{\mathcal{R}} \times U(1)_{\mathcal{R}} \times SU(2)$, where the group $SU(2)$ mixes the hypermultiplets $H$ and $\overline{\widetilde{H}}$. It is not difficult to see that both the one-loop and two-loop conformal deformations respect these symmetries.

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