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A CORE-ENVELOPE MODEL OF COMPACT STARS

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We present a core envelope model of compact stars. The core of a compact star is described by anistropic fluid which is surrounded by an isotropic envelope. Assuming an ansatz that describe the spheroidal geometry of the space inside the the star we found solutions of the core and that of the envelope. The parameter which determines the order of spheroidicity of the space (λ) is found to play here an important role and to obtain core-envelope model which is found to have a lower bound ($\lambda > 2$). However, in the case of relativistic fluid sphere with perfect fluid distribution and without core envelope model the bound is lowered to the limit $\lambda > \frac{3}{17}$.
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1. INTRODUCTION

The work of Ruderman [1] and Canuto [2] on compact star having matter distributions with densities much greater than the nuclear regime indicate that the superdense stars are likely to develop anisotropic pressure. According to these views in such massive stellar objects the radial pressure differs from the tangential pressure inside the core. The origin of anisotropy in fluid pressure could be due to a number of physical processes that may take place inside the star. For example, it may be due to the existence of a solid core [3] or the presence of type P superfluid or boson star [4], different kinds of exotic phase transitions due to gravitational collapse [5], pion condensation [6] or other physical phenomena [7]. Maharaj and Maartens [8] obtained a solution for anisotropic fluid sphere with uniform energy density. Later, Gokhroo and Mehra [9] extended their work for variable density distribution. A definite information about the behavior of matter in superdense stars is not known. However, a compact object with such high energy matter in it may be studied from different approaches. The study of relativistic core-envelope model is an attempt in this direction. In this approach interior of the superdense star is considered to be

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comprising of two regions : (i) core region and (ii) envelope region. It is important to look for a compact object with the anisotropic pressure region to be the core and surrounding the core contains the fluid distribution different from that of it, in fact one may assume perfect fluid distribution to study the envelope region. Iyer *et al.* [10] studied a core-envelope model in which it is shown that the core-envelope approach leads to information about bounds on the various parameters of ultra compact objects in general relativity (e.g., neutron star) such as mass, size and their ratio. In this paper we present a core-envelope model of a compact star in which the core region is described by matter with anistropic pressure surrounded by a distribution of a fluid with isotropic pressure in the envelope regions. We follow here a different approch for the solution of the envelope region previously adopted by Mukherjee *et al.* [11]. The conventional approach for stellar models is to prescribe an equation of state for the fluid forming the interior of a star for solving the Einstein equation. In view of the non-linearity of equations as well as hydrodynamical complexity, one has in realistic situations always resort to numerical manipulations. In the case of superdense compact objects like neutron stars, the equation of state is uncertain and not well understood. In such a situation, apart from the conventional physical approach, it may be worthwhile to explore an alternative approach in which one assumes a simple geometry for the 3-space so as to make the Einstein equation tractable. This will lead to an equation of state which may be useful and physically acceptable. We consider spacetime geometry that was given by Vaidya and Tikekar [12] for a superdense star by proposing an ansatz for the geometry of the 3-surface embedded in a 4-Euclidean space. The ansatz prescribes a spheroidal geometry for the 3-surface, described by two parameters λ and R; $\lambda = 0$ gives spherical while $\lambda = -1$ corresponds to flat space.

2. THE FIELD EQUATIONS

We begin with a static spherically symmetric spacetime described by the metric,

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\mu(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{1}$$

with an ansatz

$$e^{2\mu(r)} = \frac{1 + \lambda r^2/R^2}{1 - r^2/R^2}. \tag{2}$$

Note that the $t = const.$ hypersurface has the geometry of a 3-spheroidal space immersed in a 4-Euclidean space and is characterised by the two curvature parameters λ and R. The suitability of the above metric has already been investigated by one of us [13].

The Einstein field equation is

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi G T_{\mu\nu} \tag{3}$$

where $g_{\mu\nu}$, $R_{\mu\nu}$, R are the metric temsor, Ricci tensor and scalar curvature respectively and $T_{\mu\nu}$ is the energy momentum tensor. For an anisotropic fluid distribution following Maharaj and Maartens [8] we consider the energy momentum tensor given by

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu} - \pi_{\mu\nu} \tag{4}$$

where ρ, p are the energy density and isotropic pressure and u_μ denotes unit four velocity field of matter, $\pi_{\mu\nu}$ is the anisotropic stress tensor. The anisotropic stress tensor is given by

$$\pi^{\mu\nu} = \sqrt{3}\mathbf{S}\left[C^\mu C^\nu - \frac{1}{3}\left(u^\mu u^\nu - g^{\mu\nu}\right)\right] \tag{5}$$

For radially symmeteric anisotropic fluid distribution of matter, $\mathbf{S} = \mathbf{S}(r)$ denotes the magnitude of the anisotropic stress tensor and $C^\mu = (0, e^{-\lambda}, 0, 0)$, which is a radial vector. The energy momentum tensor corresponding to the expression (4) has the following non-vanishing components

$$T_o^o = \rho,\ T_1^1 = -\left(p + \frac{2\mathbf{S}}{\sqrt{3}}\right),\ T_2^2 = T_3^3 = -\left(p - \frac{\mathbf{S}}{\sqrt{3}}\right).$$

The pressure along the radial and tengential direction are not same which we denote here by

$$P_r = T_1^1 = -\left(p + \frac{2\mathbf{S}}{\sqrt{3}}\right) \tag{6}$$

$$P_\perp = T_2^2 = -\left(p - \frac{\mathbf{S}}{\sqrt{3}}\right) \tag{7}$$

The difference between the radial pressure and the transverse pressure is

$$\mathbf{S} = \frac{P_r - P_\perp}{\sqrt{3}} \tag{8}$$

which in fact represents the measure of the anisotropy of the fluid distribution.

The field equation (3) corresponding to the metric (1) using ansatz (2) is given by a set of three equations (we choose $8\pi G = c = 1$)

$$\rho = \frac{(1+\lambda)(3 + \lambda r^2/R^2)}{R^2(1+\lambda r^2/R^2)^2}, \tag{9}$$

$$P_r = -\frac{\left[2\frac{\nu'}{r}(1 - r^2/R^2) - \frac{\lambda+1}{R^2}\right]}{(1+\lambda r^2/R^2)} \tag{10}$$

$$\sqrt{3}\mathbf{S} = -\left[\nu'' + \nu'^2 - \frac{\nu'}{r} - \frac{1+\lambda}{R^2(1 - r^2/R^2)(1+\lambda r^2/R^2)}\right] - \frac{1+\lambda}{R^2(1+\lambda r^2/R^2)} \tag{11}$$

We consider a star with an anisotropic core having radial pressure (P_r) different from the transverse pressure ($P_\perp$). However at the boundary of the core, let us say at radius $r = a$ the two pressures coincide and the fluid distribution in the envelope region is described by isotropic fluid distribution. The pressure decreases in the envelope region and it becomes zero at the surface (say at $r = b$, where b is the radius of the star under consideration). In the next section we describe a compact star with core described by an anistropic fluid distribution and the envelope of the star is described by a perfect fluid distribution. The equation of state of a compact object whose core is described by an anistropic fluid distribution and outside the core it is described by an istropic fluid distribution are also evaluated. We describe the core up to the radius where $\mathbf{S}(r = a) = 0$. We choose a star of size $r = b$ and divide it into two parts :

I : $0 \leq r \leq a$ as the CORE of the star described by an anisotropic fluid distribution.

II : $a \leq r \leq b$ as the outer ENVELOPE of the CORE which can be described by an isotropic fluid distribution.

2.1. *CORE OF THE STAR*

The solution of the field equation (9)-(11) is obtained here by introducing a new variable

$$x = \sqrt{1 - r^2/R^2}, \quad \phi = \frac{e^{\nu}}{(1 + \lambda - \lambda x^2)^{1/4}} \tag{12}$$

where one obtains

$$\frac{d^2\phi}{dx^2} + \left[\frac{2\lambda(\lambda+1)(2\lambda+1) - (4\lambda+7)\lambda^2 x^2}{4(1+\lambda-\lambda x^2)^2} - \frac{\sqrt{3}\mathbf{S}(1+\lambda-\lambda x^2)R^2}{1-x^2} \right] \phi = 0. \tag{13}$$

On prescribing the anisotropy parameter as

$$\mathbf{S} = -\left[\frac{(1-x^2)(2\lambda(\lambda+1)(2\lambda+1) + (4\lambda+7)\lambda^2 x^2)}{\sqrt{3}R^2(1+\lambda-\lambda x^2)^3} \right] \tag{14}$$

the second term in the second derivative differential equation (13) vanishes and the resulting equation permits a simple general solution given by

$$\phi = Cx + D \tag{15}$$

with C and D as arbitrary constants of integration leading to a simple solution

$$e^{\nu(r)} = (1 + \lambda r^2/R^2)^{1/4}(C\sqrt{1 - r^2/R^2} + D) \tag{16}$$

where the anistropy is described by eq. (14).

Thus the space-time metric of the core of a compact star is described by

$$ds^2 = -(1 + \lambda r^2/R^2)^{1/2}(C\sqrt{1 - r^2/R^2} + D)^2 \, dt^2 + \frac{1 + \lambda r^2/R^2}{1 - r^2/R^2} \, dr^2$$

$$+ r^2 \left(d\theta^2 + r^2 \sin^2\theta d\phi^2 \right) \tag{17}$$

The radial pressure and the anisotropy parameter $\mathbf{S}(r)$ are now given by

$$P_r = \frac{C\sqrt{1 - r^2/R^2}[3 + \lambda(\lambda+4)r^2/R^2] + D[1 + \lambda(\lambda+2)r^2/R^2]}{R^2(1 + \lambda r^2/R^2)^2(C\sqrt{1 - r^2/R^2} + D)}, \tag{18}$$

$$\mathbf{S} = -\frac{r^2}{4\sqrt{3}R^4(1 + \lambda r^2/R^2)^3} \left[-\lambda^2 + 2\lambda + \frac{r^2}{R^2}(4\lambda^3 + 7\lambda^2) \right] \tag{19}$$

The variation of anisotropy parameter for various λ are shown in fig. 1.

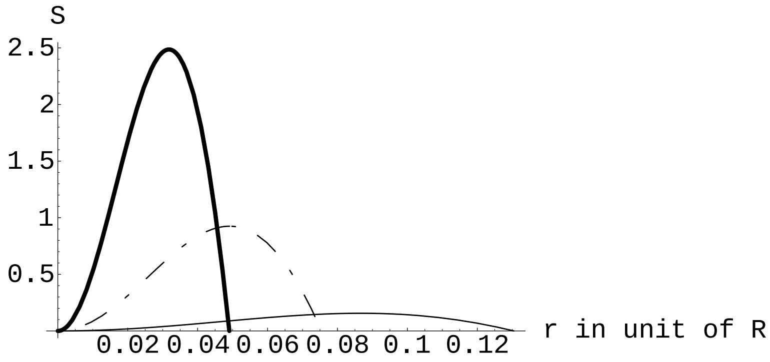

Figure 1: Variation of anisotropy with core size for $\lambda = 100, 40, 10$ are represented by dark, broken and thin lines respectively

It is evident that the anisotropy vanishes at different core sizes for different values of λ, as one increases λ the core size decreases. Here we find that the core size is determined by λ. One obtains that the anisotropy vanishes at

$$r = \sqrt{\frac{\lambda - 2}{\lambda(4\lambda + 7)}}\, R \tag{20}$$

which represents the size of the core and we denote it by $r = a$. For a positive λ we note that our solution is valid for $\lambda > 2$. Thus a core is found to exist with anisotropic fluid distribution decided by λ satisfying the lower limit. Thus the size of the core is given by $a = \sqrt{\frac{\lambda-2}{\lambda(4\lambda+7)}}\, R$ at which $P_r(r = a) = P_\perp(r = a)$ i.e., both the radial pressure and the transverse pressure converge to the same value. The radial pressure at $r = a$ is given by

$$P_{r=a} = C\, \frac{\sqrt{(4\lambda + 7)(4\lambda^2 + 6\lambda + 2)/\lambda}(\lambda^2 + 14\lambda + 13) + D(\lambda^2 + 4\lambda + 3)(4\lambda + 7)}{25R^2(\lambda + 1)^2(C\sqrt{1 - a^2/R^2} + D)}. \tag{21}$$

The anisotropy vanishes at the core boundary and the envelope is thus represented by perfect fluid distribution, which we discuss in the next section.

2.2. *ENVELOPE*

We now determine the equation of state of the envelope which is described by the radial limit $a \le r \le b$. In this case the isotropic fluid distribution in the envelope leads to a very simple relation from the condition of pressure isotropy :

$$\nu'' + \nu'^2 - \mu'\nu' - \frac{\nu'}{r} - \frac{\mu'}{r} - \frac{1}{r^2}\left(1 - e^{-2\mu}\right) = 0. \tag{22}$$

If we write

$$\psi = e^{\nu},$$

$$x^2 = 1 - \frac{r^2}{R^2},$$
$$z = \sqrt{\frac{\lambda}{\lambda+1}}\, x \tag{23}$$

the pressure isotropy condition now gives rise to a second order differential equation

$$(1 - z^2)\, \psi''(z) + z\, \psi'(z) + (\lambda + 1)\, \psi(z) = 0. \tag{24}$$

This equation admits a general solution [11]

$$\psi = A \left[\frac{\cos[(n+1)\zeta + \gamma]}{n+1} - \frac{\cos[(n-1)\zeta + \gamma]}{n-1} \right], \tag{25}$$

where $\zeta = \cos^{-1} z$ and A and γ are constants to be determined by matching the solution with the exterior Schwarzschild solution

$$ds^2 = -\left(1 - \frac{2M(r)}{r}\right) dt^2 + \left(1 - \frac{2M(r)}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\varphi^2). \tag{26}$$

At the boundary $r = b$ one gets

$$e^{2\nu(b)} = 1 - \frac{2M}{b}$$
$$e^{-2\mu(b)} = 1 - \frac{2M}{b} \tag{27}$$

In this model the energy density and the pressure are given by

$$\rho = \frac{1}{R^2(1-z^2)} \left[1 + \frac{2}{(\lambda+1)(1-z^2)} \right], \tag{28}$$

$$p = -\frac{1}{R^2(1-z^2)} \left[1 + \frac{2z\psi'}{(\lambda+1)\psi} \right]. \tag{29}$$

We note that ρ is obviously positive for $\lambda > -1$. Thus inside the core of a compact star energy density is always positive as we require $\lambda > 2$. The mass contained inside a radius r is given by

$$M(r) = \frac{1}{2} \int_0^r r'^2 \rho(r') dr' \tag{30}$$

which on integration for $r = b$ yields

$$\frac{M(b)}{b} = \frac{(1+\lambda) b^2/R^2}{2(1 + \lambda b^2/R^2)}. \tag{31}$$

Now one can determine the radius of a compact star from the condition that the pressure should vanish at the boundary $r = b$. Thus for a given mass, the reduced radius $\frac{b}{R}$ is determined from the condition $P(b) = 0$ which leads to

$$\frac{\psi'(z_b)}{\psi(z_b)} = -\frac{\lambda+1}{2z_b} \tag{32}$$

with $z_b = \sqrt{\frac{\lambda}{\lambda+1}\left(1 - \frac{b^2}{R^2}\right)}$, for a given λ. Thus λ determines the equation of state for the CORE as well as for the envelope too. There are four unknowns λ, R, A and γ in the CORE region and we have another four unknowns λ, R, C and D in the envelope region. If the values of mass and radius are given , we have two free parameters, one of which is utilized to match the exterior Schwarzschild metric. However, for a given mass of a star one determines the size of the star and *vice versa* for a given λ. Thus it now leads to determination of only two unknowns C and D as λ, R, A and γ are determined from the boundary matching condition of the envelope.

Our model is valid for $\lambda > 2$. Thus we find that to describe a core model with anisotropic fluid found here, the class of general solutions with perfect fluid obtained for $\lambda \leq 2$ are not acceptable. The surface condition that the pressure is zero determines γ and the constant A is determined from the equation(27). Knowing A and γ one can determines the two other unknowns C and D from the matching conditions at the core-envelope boundary (i.e., $r = a$). Thus one obtains

$$\left(1+\lambda\frac{a^2}{R^2}\right)^{1/4}\left[C\sqrt{1-a^2/R^2}+D\right] = A\left[\frac{cos((n+1)\zeta+\gamma)}{n+1} - \frac{cos((n-1)\zeta+\gamma)}{n-1}\right] \tag{33}$$

where $\zeta = cos^{-1}z_a$, $z_a = \sqrt{\frac{\lambda}{\lambda+1}\left[1 - \frac{a^2}{R^2}\right]}$ and $n = \sqrt{\lambda^2+2}$. The other condition is

$$\frac{C\sqrt{1-\frac{a^2}{R^2}}\left(3+\lambda(\lambda+4)\frac{a^2}{R^2}\right) + D\left(1+\lambda(\lambda+2)\frac{a^2}{R^2}\right)}{R^2\left(1+\lambda\frac{a^2}{R^2}\right)\left(C\sqrt{1-\frac{a^2}{R^2}}+D\right)}$$

$$= -\frac{1}{R^2(1-z_a^2)}\left[1+\frac{2z_a}{\lambda+1}\left(\frac{\psi_{z_a}}{\psi}\right)_{r=a}\right] \tag{34}$$

3. DISCUSSIONS

To conclude we present a core-envelope model of compact stars which follows from the exact general solutions of the Einstein equations for a superdense star in hydrostatic equilibrium satisfying all physical constraints for the CORE and the ENVELOPE. The matter content for the core is described by an anisotropic fliud distribution and that of the envelope is described by perfect fluid assumptions. The equation of state for the core is determined and is found that it requires $\lambda > 2$ for consistency. This lower bound on λ is different from that if one considers a perfect fluid which is $\lambda > \frac{3}{17}$ [11]. Thus the the geometrical parameter decides the equation of state inside the compact star with spheroidal geometry. Inside the core of star we introduce density variation parameter $\frac{\rho(a)}{\rho(0)} = Q$ (where $\rho(a)$ and $\rho(0)$ represent the density of the core and centre respespectively) to know the density profile in the model and we get

$$\frac{a^2}{R^2} = \frac{1-6Q\pm\sqrt{24Q+1}}{6Q\lambda} \leq 1.$$

Consequently, one obtains a restriction on Q given by

$$Q \leq \frac{3+\lambda}{3(1+\lambda)}$$

which is determined by λ once again. It is evident that large values of λ ($>> 3$) leads to $Q \leq \frac{1}{3}$ whereas for lower values, say $\lambda = 3$ ($\lambda > 2$) one gets $Q \leq \frac{1}{2}$. Thus the anisotropic core shows a high degree of density variation as one moves from the centre to the core boundary. In the envelope region we denote the density variation parameter by

$$\bar{Q} = \frac{\rho(b)}{\rho(a)} = \frac{(3+\lambda\frac{b^2}{R^2})(1+\lambda\frac{a^2}{R^2})^2}{(3+\lambda\frac{a^2}{R^2})(1+\lambda\frac{b^2}{R^2})^2}$$

(where $\rho(b)$ represents the density of the star) which may not admit high density variation as in our model we have both $\frac{a}{R}$ and $\frac{b}{R}$ are possesing values less than unity.

Table 1.

	b in Km.	R in Km.	$M(b)$ in Km.	a in Km.	Q	$\bar{Q}$	$\rho(a)$
	18.635	34.022	6.708	0.0027	0.84	0.21	9.4×10^{14}
	17.931	40.095	5.379	0.0030	0.84	0.29	6.8×10^{14}
$\lambda = 5$	15.814	50.008	3.163	0.0033	0.84	0.46	4.4×10^{14}
	6.669	66.689	0.191	0.0039	0.84	0.81	2.5×10^{14}
	18.209	33.245	7.511	0.0024	0.8	0.12	1.7×10^{15}
	18.096	40.465	6.635	0.0026	0.8	0.18	1.1×10^{15}
$\lambda = 10$	17.168	54.289	4.721	0.0030	0.8	0.32	0.6×10^{14}
	8.690	86.896	0.435	0.0039	0.8	0.82	0.2×10^{15}
	16.404	29.950	8.183	0.00027	0.7	0.001	1.6×10^{17}
	16.417	36.701	8.176	0.00030	0.7	0.002	1.1×10^{17}
$\lambda = 1000$	16.456	52.039	8.155	0.00036	0.7	0.004	5.6×10^{16}
	16.975	169.751	7.724	0.00065	0.7	0.038	5.2×10^{15}

Table 1. The variation of parameter R, mass of a star $M(b)$, corresponding core size (a), density profile inside the core (Q), density profile inside the envelope ($\bar{Q}$) and core density with different size of the star (b) are shown for a given λ and star density $\rho(b) = 2 \times 10^{14}$ gm/cc.

It is evident from the table that a dense core of a star is obtained once one takes larger values of λ, also in that case the size of the core diminishes.

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