

HISTORY OF SPIN AND STATISTICS

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Abstract

These lectures were given in the framework of the “Dixième séminaire rhodanien de physique” entitled “Le spin en physique”, given at Villa Gualino, Turin, March 2002. We have shown how the difficulties of interpretation of atomic spectra led to the Pauli exclusion principle and to the notion of spin, and then described the following steps: the Pauli spin with 2×2 matrices after the birth of “new” quantum mechanics, the Dirac equation and the magnetic moment of the electron, the spins and magnetic moments of other particles, proton, neutron and hyperons. Finally, we show the crucial role of statistics in the stability of the world.

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1 Introduction

Before speaking of spin, I have to speak of angular momentum, and specifically classical angular momentum. We know since a long time that angular momentum is a conserved quantity. Specifically one of Kepler's laws of the motion of planets around the sun, the "law of areas" is nothing but the conservation of angular momentum:

If a planet moves around the sun, the time taken by the planet to go from 1 to 2 is equal to the time taken to go from 3 to 4 if the shaded areas, delimited by the trajectory and rays coming from the sun, are equal.

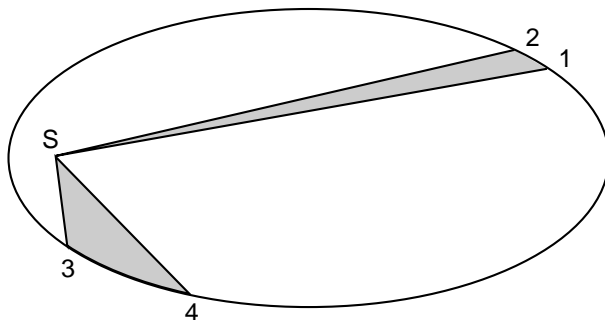


Figure 1

The angular momentum of the planet is

$$\vec{x}\Lambda\vec{p} = m\vec{x}\Lambda\frac{d\vec{x}}{dt} , \quad (1)$$

its derivative with respect to time is

$$m \frac{d}{dt} \left[\vec{x}\Lambda\frac{d\vec{x}}{dt} \right] = m \frac{d\vec{x}}{dt}\Lambda\frac{d\vec{x}}{dt} + \vec{x}\Lambda\frac{d^2\vec{x}}{dt^2} m . \quad (2)$$

The first term is obviously zero. From Newton's law for a central force

$$m \frac{d^2\vec{x}}{dt^2} = \frac{\vec{x}}{r}F(r) , \text{ where } r = |\vec{x}| , \quad (3)$$

and hence the second term in (2) is also zero. The angular momentum is a constant of motion, and this does not depend on the force behaving like r^{-2} . However, $\vec{x}\Lambda d\vec{x}$ represents precisely twice the area spanned by the rays going from the sun to the planet during an interval dt . Hence the "area law" holds at the infinitesimal level.

Since Bohr's model of the atom (1913) angular momentum was quantized

$$|\vec{L}| = \ell\hbar , \quad (4)$$

and, in addition, the projection of the angular momentum along the z axis, M was also quantized:

$$M = m\hbar . \tag{5}$$

The energy levels of atoms, especially Alcaline atoms (the first row of Mendeleieff's classification) were characterized by n (principal quantum number) and ℓ (orbital angular momentum, called k by the pioneers – but I prefer to use the modern notation).

However, take the yellow light of Sodium, that light which illuminates tunnels and makes you look like a corpse. With a rather ordinary spectrometer that first year students were using at the Ecole Normale Supérieure when I was there, you discover that this $3P \rightarrow 2S$ line ($n = 3, \ell = 1$ to $n = 2, \ell = 0$) is in fact a doublet, the relative spacing between the two lines being about 1/1000. This means that there is something wrong with the classification of energy levels and spectral lines.

2 The Pre-Spin, Pre-Quantum Mechanics Periods and the Discovery of Spin

In first approximation, in the Bohr model, the energy levels of Hydrogen depended only one quantum number the “Principal quantum number”, n , as indicated on Fig. 2

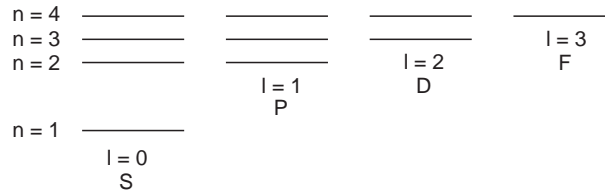


Figure 2

The energy levels were degenerate in ℓ , the orbital angular momentum.

In Alcaline atoms, with a single valence electron, the energy levels also depend on ℓ , in such a way

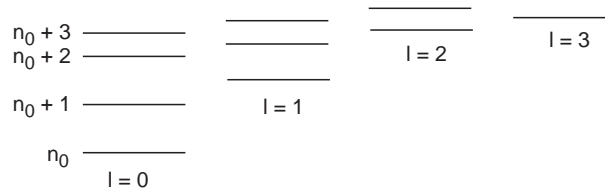


Figure 3

that for a fixed principal quantum number, the energy increases as ℓ increases. This is outside the subject but let me say that the best, most rational explanation of this fact was only given relatively recently by Baumgartner, Grosse and myself [1]. It is due to the fact that the valence electron is submitted to the field of the nucleus which is pointlike at this scale and of the electron cloud which has a negative charge and hence produces a potential with negative laplacian.

The observed transitions between levels satisfy the rule $|\Delta\ell| = 1$. However, in fact, as we already mentioned it, the levels of the Alkaline atoms are doublets. Similarly, the levels of the Alkaline earths, with Magnesium as a prototype, with two valence electrons are either singlets or triplets. This is illustrated on Fig. 4.

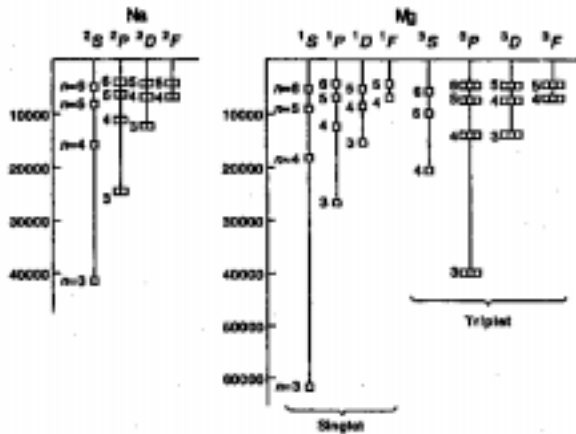


Figure 4

Another important piece of the puzzle is the Zeeman effect: the splitting of the energy levels inside a uniform magnetic field. In a weak magnetic field, one observes, in the case of Alkaline atoms, that except for the $\ell = 0$ state which is a singlet, all other doublets, for $\ell \neq 0$, split in such a way that one member gives 2ℓ levels and the other $2\ell + 2$ levels. This could not be explained by what was known at the time.

What could be explained was the so-called “NORMAL” Zeeman effect, according to which a level with orbital angular momentum component m along the z axis would be shifted by a magnetic field H in the z direction by

$$\Delta E(n, m, \ell) = \frac{e\hbar}{2mc} H m , \quad (6)$$

$\frac{e\hbar}{2mc}$ is called the Bohr magneton.

This is because an electron circulating around the atom is equivalent to a current producing a magnetic moment.

Similarly, in strong magnetic fields (the so-called Paschen-Back effect), what is seen is also very different. We also see more levels than we should. Both situations are illustrated on Fig. 5.

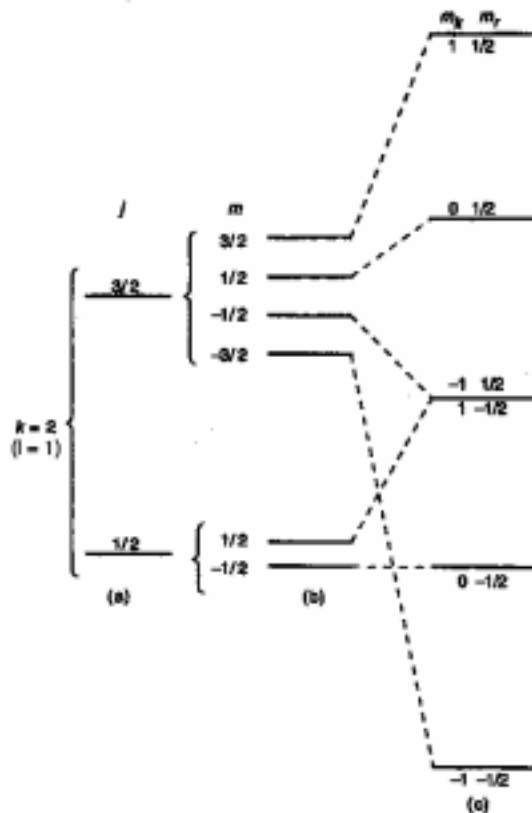


Figure 5

It is of course difficult to describe the difficulties of the pioneers seeing this, because we are somehow like in a TV episode where we know already who is guilty, but inspector Columbo does not know yet!

Among the main protagonists were Landé, Pauli and Sommerfeld. All of them realized that an extra quantum number beside n, ℓ and m was needed. Their models fitted the data, but they knew they were only models. They were called “ERSATZ” models. Those of you who have lived in occupied Europe during the war know what an ersatz is. It is a bad substitute for something much better. However, in this case, the ERSATZ was replacing something which did not exist yet.

Sommerfeld and Landé were putting the fourth quantum number in the “core” of the atom without a very precise definition of the core (nucleus? or nucleus plus inner electrons?). The core was carrying angular momentum, $1/2$ in the case of the alkaline atoms, had a magnetic moment, twice what you get in the case orbital angular momentum: $2 \times 1/2 \times$ electron Bohr magneton. That was in 1923.

Pauli did not like that, for many reasons. For instance, if you remove an electron from Mg , you get Mg^+ which behaves like an Alkaline atom, except for the fact that it is not neutral. By miracle, the “core” which had angular momentum 0 or 1 suddenly gets angular momentum

1/2.

Pauli, in early 1925, decided that an extra quantum number was a “classically undescrivable two-valuedness” which belonged to the electron, and formulated his famous exclusive principle, according to which two electrons cannot be in the same state. With this principle you “explain” the filling of the successive shells. For instance, sodium is

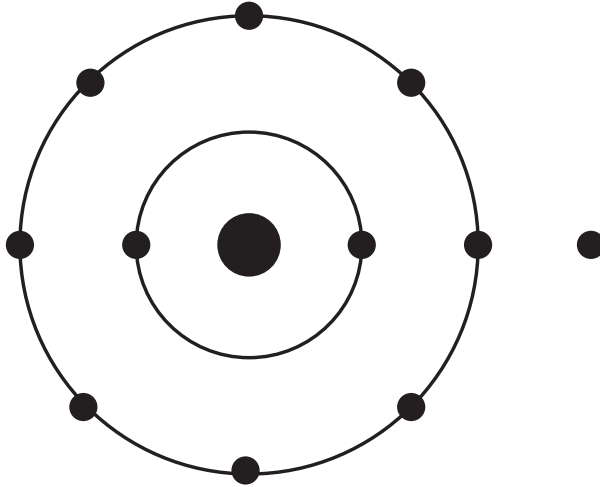


Figure 6

You explain also the counting of levels in weak and strong magnetic fields as it is seen in Fig. 5.

Pauli was not, however, willing to make the big jump that this is due to an intrinsic angular momentum $1/2 \hbar$ of the electron. It was R.L. Kronig, arriving in Germany from the U.S. who proposed this idea first. However, this idea was not well received by Pauli, as well as in Copenhagen where Kronig went visiting. There was also a problem about the spacing of the levels which gave doubts to Kronig himself. Then in the fall of 1925, Uhlenbeck and Goudsmit, in Leiden, proposed the same idea which they sent for publication to *Naturwissenschaften*. After discussions with Lorentz, they tried to withdraw their paper, but it was too late (fortunately) and it was published!

Naturally the gyromagnetic factor of the electron was taken to be 2 to fit the experiment, so that the magnetic moment of the electron was $2 \times 1/2 \times 1$ Bohr magneton.

There was a problem, however, which is that a naïve calculation of the energy interval in a doublet, due to the coupling of the orbital motion with the spin of the electron, gave a result twice too big as stressed by a letter of Kronig against the Uhlenbeck–Goudsmit hypothesis. It was G.H. Thomas who realized that relativistic effects in the motion of the electron should be treated more carefully. It amounts to replacing g , the gyromagnetic factor of the electron, which was empirically 2, by $g - 1$, i.e. 1. This is treated in the lectures of E. Leader and so I shall not attempt to describe the reasoning of Thomas.

Before going to the true quantum mechanical situation, I would like to show to you how, in the time of “old” quantum mechanics, people calculated the “Landé factor” giving the magnetic splittings in weak fields. The magnetic interaction is given by

$$W = -\frac{e\hbar}{2mc} \langle H_{\mu''} \rangle \quad (7)$$

where μ'' is the projection of the total magnetic moment on the total angular momentum $\vec{L} + \vec{S} = \vec{J}$;

$$\mu'' = \left(\frac{\vec{J} \cdot \vec{\mu}}{J^2} \right) J \quad (8)$$

and $\vec{\mu} = \vec{L} + g_0 \vec{S}$ with $g_0 = 2$. So

$$\begin{aligned} \langle H_{\mu''} \rangle &= - \left[1 + (g_0 - 1) \frac{\vec{J} \cdot \vec{S}}{J^2} \right] (\vec{H} \cdot \vec{J}) \\ \langle H_{\mu''} \rangle &= -g(H \cdot J), \end{aligned} \quad (9)$$

where g is Landé factor.

$$g = 1 + (g_0 - 1) \frac{J^2 + S^2 - L^2}{2J^2}. \quad (10)$$

If you put the actual values of $j, S (= 1/2)$, and ℓ , it does not work! You should use the substitution rule

$$J^2 \rightarrow j(j+1), \quad S^2 \rightarrow s(s+1), \quad L^2 \rightarrow \ell(\ell+1), \quad (11)$$

that I learnt from my old master Alfred Kastler who, in spite of the fact that he was a great physicist (he got the Nobel prize for “optical pumping” a method to polarize atoms by sending circularly polarized light on them), never took the pain to learn modern quantum mechanics.

The fact is that this substitution rule gives the correct result:

$$g = \frac{3}{2} + \frac{s(s+1) - \ell(\ell+1)}{2j(j+1)} \quad (12)$$

To me this was not obvious, and I checked it by hand!

Finally, I would like to speak of something which historically took place before the critical years of the discovery of spin and should have accelerated things, namely the Stern–Gerlach experiment (1921), which consists

in sending a beam of silver atom through an inhomogeneous magnetic field. The atoms behave like tiny magnetic dipoles and depending on their orientation are deviated upward or downward. At the end one observes two dots of silver atoms and nothing else (Fig. 7). Remembering that Ag is monovalent, this fits with the spin picture. However, this experiment seems to have had little influence on the mind of the great theoreticians in 1921. This experiment is fundamental from another point of view. It shows that spin is absolutely not classical. You expect that Ag atoms, in the source, have their spins oriented at RANDOM. Yet there are two and only two

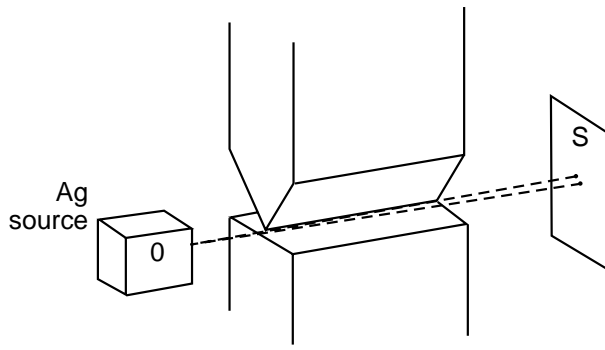


Figure 7

dots at the other end. This experiment, besides proving that the silver atom carry magnetic moment with two possible orientations, is also absolutely fundamental because it leads to a complete revision of our ideas on what things are observables and how they can be observed. I shall come back on this later.

3 Spin after the birth of “modern” quantum mechanics

When new quantum theory was born? In 1925 with Heisenberg’s matrix mechanics or in fact before, with the famous equation of Louis de Broglie (1924) giving the wave length associated to a particle:

$$\lambda = \frac{h}{p} = \frac{h}{mv} \quad (\text{in the non - relativistic case}) . \quad (13)$$

Anyway, shortly after the work of Heisenberg, Pauli at the end of 1925, succeeded in getting the energy levels of Hydrogen, by purely algebraic methods, using matrix mechanics.

However, the news of De Broglie’s relation (which Langevin had agreed to present at the French Academy of Sciences after consulting Einstein) was carried to Zürich by a chemist named Victor Henry (information from David Speiser, son in law of Hermann Weyl). Debye then told Schrödinger: “you should find an equation for these waves”. He did, in 1926! Schrödinger and others proved the equivalence of the Schrödinger approach and of Heisenberg’s approach. Schrödinger too found the energy levels of hydrogen using properties of “special functions”.

The Schrödinger equation (with units such that $\hbar=1$) looks like:

$$-\frac{1}{2m}\Delta\psi + (V - E)\psi = 0 ,$$

which, for a central potential in polar co-ordinates gives

$$-\frac{1}{2m} \frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} \psi$$

$$\begin{aligned}
& + \frac{1}{2mr^2} \times (-1) \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] \psi \\
& + (V - E)\psi = 0 .
\end{aligned} \tag{14}$$

It is tedious but trivial to check that the second line in Eq. (14) is nothing but the square of the angular momentum operator acting on ψ :

$$L^2 = L_x^2 + L_y^2 + L_z^2 , \tag{15}$$

$$\vec{L} = \vec{x} \wedge \vec{p} . \tag{16}$$

On the one hand p admits the representation

$$p_x = -i \frac{\partial}{\partial x} , \quad p_y = -i \frac{\partial}{\partial y} , \quad p_z = -i \frac{\partial}{\partial z} . \tag{17}$$

On the other hand, it has the commutation relation

$$\left. \begin{aligned}
[p_x, x] &= -i \\
[p_x, y] &= 0 \\
&etc. \\
&(\times \hbar \text{ in general units})
\end{aligned} \right\} \tag{18}$$

At that point, one can take two attitudes.

- i) Use the fact that the Schrödinger equation is separable for a central potential and look for the eigenfunctions and eigenvalues of L^2 , represented by the differential operator on the second line of Eq. (14). To do this it suffices to open a good book on special functions (for instance, the “Bateman” Project). One finds that the eigenvalue of L^2 are $\ell(\ell + 1)$ where ℓ is an integer and the eigenfunctions are the spherical harmonics $Y_{\ell m}(\theta, \phi)$

$$Y_{\ell, m}(\theta, \varphi) = C(\sin \theta)^m \left(\frac{d}{d \cos \theta} \right)^m P_{\ell}(\cos \theta) e^{im\phi} , \tag{19}$$

the P_{ℓ} 's being Legendre polynomials.

- ii) The second attitude consists in using the algebraic properties of the operators L_x, L_y, L_z and in particular their commutation relations:

$$\left. \begin{aligned}
L_x L_y - L_y L_x &= i L_z \\
L_y L_z - L_z L_y &= i L_x \\
L_z L_x - L_x L_z &= i L_y
\end{aligned} \right\} \tag{20}$$

from which it follows that

$$[L_z, L^2] = 0, \text{ etc...} \tag{21}$$

You start from an eigenstate of L^2 and L_z . Say that

$$L^2 \psi = \ell(\ell + 1) \psi . \tag{22}$$

ℓ , so far, is anything real (L^2 is a Hermitean operator!)

$$L_z \psi = m \psi . \tag{23}$$

From positivity of L_x^2, L_y^2, L_z^2 , it is already clear that

$$|m| \leq \sqrt{\ell(\ell + 1)} .$$

Now it is easy to see that the operators

$$\left. \begin{aligned} L_+ &= L_x + iL_y \\ \text{and} \\ L_- &= L_x - iL_y \end{aligned} \right\} , \quad (24)$$

applied to ψ raise or increase L_z by one unit without changing L^2 . It is also easy to see that these operators must terminate both ways and that $m_{\text{maximum}} = \ell, m_{\text{minimum}} = -\ell$, therefore $2\ell + 1$ is integer, and indeed it is acceptable to have integer orbital momentum as in the Schrödinger equation. However, there is the possibility, if we use only the algebraic structure of angular momentum, to have half integer angular momentum.

The group theoretical interpretation of the conservation of angular momentum is that physics, in the case of a central potential, is invariant under rotations around the origin. L_x, L_y, L_z are the generators of the rotation group around respectively the x axis, the y axis, the z axis. A rotation around L_x of angle θ_x is

$$\exp i\theta_x L_x .$$

A general eigenstate of L^2 , with eigenvalue $\ell(\ell + 1)$, can be represented as a vector column with components corresponding to $L_z = -\ell, -\ell + 1, \dots, +\ell$. Acting with $\exp i\theta_x L_x$ on this column will produce a new state again with $L^2 = \ell(\ell + 1)$. Clearly, the action of $\exp i\theta_x L_x$ will be represented by a matrix with $(2\ell + 1) \times (2\ell + 1)$ components, i.e., we generate in this way a representation of the rotation group of dimension $2\ell + 1$.

Now, take the simple case of $\exp i\theta_z L_z$, acting on a column made of components which are eigenstates of L_z . If $\theta_z = 2\pi$, we come back to the initial state, i.e., we should find the same vector we started from. A component with $L_z = m$ will be multiplied by $\exp 2i\pi m$, which will be unity if ℓ is integer. So, to represent the true rotation group, we must use only integer ℓ and m .

However, there is another group, which is the covering group of the rotation group, and which is in fact SU_2 , the group of 2×2 unitary transformations, which has the same Lie algebra, i.e., the same commutation relations between L_x, L_y and L_z the generators of the group, but which admits both integer values of ℓ and half integer values of ℓ . In the case of an half integer ℓ , a column vector will be multiplied by -1 in a complete turn. The half integer angular momenta will be also acceptable if this -1 factor is invisible in physical results, which is the case.

In what follows, ℓ will be restricted to designate the orbital angular momentum, always integer. From what we have said, an angular momentum 1/2 for the electron is acceptable, and this is precisely what Pauli exploited. He understood that it was pointless to try to represent the spin wave function in x space, and it should be in fact represented as a two-component

column, $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ with $\alpha^2 + \beta^2 = 1$, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ being a state with total angular momentum 1/2 and projection on the z axis +1/2, while $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ has a projection on the z which is -1/2.

The spin operators satisfy the same algebra as the components of L :

$$\begin{aligned} S_x S_y - S_y S_x &= i S_z \\ S_y S_z - S_z S_y &= i S_x \\ S_z S_x - S_x S_z &= i S_y \end{aligned} \quad (25)$$

and $\vec{S}^2 = 3/4$.

They can be represented very simply by the ‘‘Pauli’’ hermitian matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (26)$$

with

$$S_x = \frac{\sigma_x}{2}, \quad S_y = \frac{\sigma_y}{2}, \quad S_z = \frac{\sigma_z}{2} \quad (27)$$

the σ 's have two remarkable properties:

$$\left. \begin{aligned} \sigma_\mu^2 &= 1 \\ \sigma_\mu \sigma_\nu &= -\sigma_\nu \sigma_\mu \quad \text{if } \nu \neq \mu \end{aligned} \right\} \quad (28)$$

Now if we take into account:

- The postulate that the gyromagnetic factor of the electron is 2, based on experiment
- The Thomas precession,

we get the Pauli Hamiltonian in a magnetic field H associated to a potential vector A :

$$H_p = \frac{(\vec{p} - e\vec{A})^2}{2m} + V(r) + \frac{ie}{2m} \vec{\sigma} \cdot \vec{H} + \frac{1}{4m^2} \frac{\vec{\sigma} \cdot \vec{L}}{2} \left\langle \frac{1}{r} \frac{dV}{dr} \right\rangle \quad (29)$$

The double quotes indicates that this Hamiltonian is not really a Hamiltonian if $\frac{1}{r} \frac{dV}{dr}$ is more singular than $\frac{1}{r^2}$ near the origin. For a purely Coulombic potential $\frac{1}{r} \frac{dV}{dr}$ behaves like r^{-3} . Then strictly speaking, H_p is not lower bounded, which means that you can find a trial function which gives an expectation value of H_p arbitrarily negative. There was a period when this difficulty was just disregarded because the spin-orbit term was just regarded as a perturbation. Only relatively recently [2] as we shall see later, it was understood that the spin orbit term must be indeed treated as a perturbation, at least for a one-electron system. Hence, the recipe is:

- Solve the Schrödinger equation without spin and without magnetic field for a given orbital angular momentum;
- Take the expectation value of $\frac{1}{r} \frac{dV}{dr}$, which we denote as $\langle \frac{1}{r} \frac{dV}{dr} \rangle$;
- Then diagonalize the Hamiltonian which, for a constant magnetic field in the Z direction and neglecting higher orders in H is

$$H = \frac{p^2}{2m} + V(r) + \frac{ie}{2m} (L_Z + \sigma_z) H + \frac{1}{4m^2} \frac{\vec{L} \cdot \vec{\sigma}}{2} \left\langle \frac{1}{r} \frac{dV}{dr} \right\rangle \quad (30)$$

L^2 commutes with H , and $J_Z = L_Z + \frac{\sigma_Z}{2}$ also commutes with H since $\vec{L} \cdot \vec{\sigma} = J^2 - L^2 - \left(\frac{\sigma}{2}\right)^2$.

So, for any magnetic field, all we have to do is to diagonalize a 2×2 matrix. For instance, in the basis where L_Z is diagonal we take as basis vectors

$$\begin{aligned} & |\ell, L_Z = J_Z + 1/2 \rangle \quad |S_Z = -1/2 \rangle \\ \text{and} \quad & |\ell, L_Z = J_Z - 1/2 \rangle \quad |S_Z = +1/2 \rangle . \end{aligned} \quad (31)$$

In the extreme case where

$$\frac{eH}{2m} \ll \frac{1}{4m^2} < \frac{1}{r} \frac{dV}{dr} >$$

we get the Zeeman effect where J^2 is a good quantum number and in the case

$$\frac{eH}{2m} \gg \frac{1}{4m^2} < \frac{1}{r} \frac{dV}{dr} > ,$$

we get the Paschen-Back effect (see Fig. 5).

Here I would like to introduce a digression about the spin-orbit interaction and the one-electron model. Of course, the one-electron model is good – almost perfect – for hydrogen. For alkaline atoms, with B. Baumgartner, H. Grosse and J. Stubbe [1],[3] we have obtained a lot of very stringent inequalities on the energy levels link to the fact that the potential $V(r)$, due to the nucleus and the electron cloud satisfies $r^2 \Delta V(r) < 0$ as long as spin is disregarded (i.e., if one takes averages over spin multiplets), these inequalities are very well satisfied. We have already mentioned

$$E(n, \ell) < E(n, \ell + 1) , \quad (32)$$

which allows to understand why the third electron of lithium is in an s state and not a p state. There are others linking three successive “angular excitations”, i.e., levels with $n = \ell + 1$, which work beautifully [3].

However, if one tries to go further and gets inequalities on the fine splittings [4] themselves, things do not always work. For levels such that $\ell = n - 1$, define:

$$E(J = \ell + 1/2) - E(J = \ell - 1/2) = \frac{2\ell + 1}{4m^2} \left\langle \frac{1}{r} \frac{dV}{dr} \right\rangle = \delta(\ell) \quad (33)$$

the first “theorem” is that it is positive for $\Delta V < 0$. Furthermore, denoting $E(\ell + 1, \ell)$ as $E(\ell)$

$$\delta(\ell) > \frac{1}{m\ell} \frac{(\ell + 2)^4}{(2\ell + 3)^2} \left(E(\ell + 1) - E(\ell) \right)^2 = \Delta(\ell) \quad (34)$$

and

$$\frac{\delta(2)}{\delta(1)} < \frac{8}{81} \quad (35)$$

These inequalities are almost satisfied by the LiI isoelectronic sequence of ions, and really satisfied starting from Carbon IV.

Table 1
 Fine splittings for the Li I isoelectronic sequence in $(\text{cm})^{-1}$.
 In the one-electron model, we should have
 $\delta(1) > \Delta(1), \delta(2) > \Delta(2), \delta(2)/\delta(1) < 8/81$

		$\delta(1)$	$\delta(2)$	$\Delta(1)$	$\Delta(2)$	$\delta(2)/\delta(1)$
Li	I	0.3372	0.037	0.4218	0.0362	0.1097
Be	II	6.58	0.55	6.8748	0.480	0.0836
B	III	34.1	3.1	34.287	2.540	0.0909
C	IV	107.1	10.5	106.6	9.278	0.098
N	V	258.7	22.0	256.54	22.64	0.085
O	VI	532.5	51.1	525.7	46.9	0.096
F	VII	975.8	90.0	926.66	86.97	0.092
Ne	VIII	1649.3	150	1638.4	148.3	0.091
Na	IX	2650	224	2600		0.085
Mg	X	3978	470	3938	362	0.118
Al	XI	5800	560	5713	530	0.096
Si	XII	8180	990	8112	752	0.121
P	XIII	11310	1000	11100		0.088
S	XIV	15130	1420	14900		0.094
Cl	XV	19770	1000	19560		0.051
Ar	XVI	25655	3600	253090		0.140
K	XVII	32538	3040			0.093
Ca	XVIII	40843	3810	40400		0.093
Sc	XIX	50654	4730	50200		0.094
Ti	XX	62143	5780	61500		0.093
V	XXI	75499	7020	74800		0.093
Cr	XXII	90910	8440	90000		

However, if one looks at the Sodium isoelectronic sequence, one finds strong violation of the inequalities, in particular $3D^{5/2} < 3D^{3/2}$, so that even the sign of the splitting is wrong. So the spin splittings are more sensitive than the levels themselves to the detailed structure of the many-body wave function.

Before going to the Dirac equation, I would like:

a) to return to the Stern-Gerlach experiment that I have mentioned in Section 2.

It allows not only to show that atoms, and in fact electrons, have a magnetic moment, which can be measured in this way, but the fact that at the end of the apparatus, the silver jet is split and produces two and only two dots aligned with the direction of the magnetic field is of fundamental importance. Suppose that silver atoms were just tiny classical dipole magnets. In the oven, via collisions, their orientation should be randomly distributed, and for instance a dipole oriented in the horizontal plane, perpendicular to the direction of the magnetic field would not be deflected vertically, neither up nor down. So, one would expect to see on the final plate a continuous segment. This, however, is not the case.

This means that the measuring apparatus projects the atoms on the eigenstates of σ_Z if the magnetic field is in the direction Z , and deflects then because of the inhomogeneity. Yet the initial assembly of atoms can be as well seen as a statistical assembly of eigenstates of σ_X or σ_Y . The measuring apparatus behaves like a projector of the states on the particular eigenstates it prefers, whether we like it or not. Many very refined tests of quantum mechanics, as opposed to classical mechanics, have been performed [5] but this very old one stands and defies any classical explanation.

b) to speak about the two-electron system, which is only the beginning of the many electron systems.

After the birth of new quantum mechanics, you could not anymore state the Pauli principle by saying that two electrons are in a different state, because, if you accept that one can write a many-particle Schrödinger equation, with

$$H = - \sum_i \frac{1}{2m_i} \Delta_i + V(x_i) + \sum_{i>j} W(r_{ij}) + \text{spin terms} \quad (36)$$

the wave function $\psi(x_1, s_1, x_2, s_2, \dots, x_n, s_n)$ cannot be written as a product. It seems that it is Heisenberg who understood that the “new” Pauli principle should be that the wave function should be completely antisymmetric, including the spin variables. If you switch off the interaction between electrons, you could have a product wave function, but according to the new rule, it should be replaced by what is called a “Slater determinant”. If the one-particle wave functions in the determinant are mutually orthogonal and if there is no interaction, the energy is the same as with the product wave function, but as soon as the interaction between electrons is on, the Slater determinant is infinitely superior and can be used as a trial wave function. In the special case of two electrons, the wave function is a product of the co-ordinate wave function and of the spin wave function. Then the Pauli principle requires that

- if the space wave function is symmetrical the spin wave function should be antisymmetrical;
- if the space wave function is antisymmetrical, the spin wave function should be symmetrical.

Things are extraordinarily simple because:

If the spin wave function is symmetrical the total spin is ONE. In the special case where $\sigma_{z1} = \sigma_{z2} = 1$ the wave function is just

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \times \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 \quad (37)$$

which has $s = 1, s_z = 1$ and which is symmetrical.

Applying to this wave function, the lowering operator

$$(\sigma_X - i\sigma_Y)_1 + (\sigma_X - i\sigma_Z)_2$$

does not change the symmetry. Hence all spin 1 wave functions are symmetrical.

The $S = 1, S_z = 0$ wave function is:

$$C \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 + \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 \right] \quad (38)$$

so the wave function

$$C \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 - \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 \right] \quad (39)$$

is orthogonal to it. It has necessarily spin 0, otherwise you would get $2(S = 1 \quad S_z = 0)$ wave functions which is impossible. Therefore the antisymmetrical wave function has necessarily spin zero. After the preliminaries, you can embark into atomic and molecular physics, and more generally quantum chemistry, magnetism, etc., the first steps being the molecular hydrogen ion, the Helium atom, the hydrogen molecule.

Many numerical results have been obtained, some very accurate, but rigorous results are still coming. It is only relatively recently that the stability of the hydrogen molecule has been rigorously established by J. Fröhlich, J.M. Richard, G.M. Graf and H. Seifert [6], without using the Born-Oppenheimer approximation which is a two-step process:

- a) solving the Schrödinger equation for fixed positions of the protons;
- b) using the electron energy plus the Coulomb repulsion between the protons as potential between the protons.

Another application of the structure of the spin wave function of two electrons concerns Cooper pairs. At least in normal superconductors it is known that one finds “Cooper pairs”, pairs of electrons bound by interaction with the lattice which are in an S state. Hence from the Pauli principle, their spin wave function is antisymmetric with spin zero. Suppose, like G.B. Lesovik, T. Martin and G. Blatter [7], that a Cooper pair crosses the border between a superconductor and a normal conductor. The pair will dissociate but it will “remember” that it has total spin zero, at least as long as depolarizing collisions are negligible. Then, if one measures the spins of the two electrons, one sees typical quantum effects.

Naïvely, one would expect that if the spin of electron Nr. 1 is up in the z direction, the spin of electron Nr. 2 is necessarily down in the z direction. This would be like to what happens

with Bertlmann's socks [8] (R. Bertlmann is a slightly excentric Viennese physicist, friend of John Bell): if he has a red sock on the left foot you can predict that he has a blue sock on the right foot.

But quantum physics is not like Bertlmann. If, for instance, the spin of electron 1 is found to be $+1/2$ in the z direction, the spin of electron 2 can be found to be $+1/2$ in the x direction. This is because the scalar product

$$\left\langle \left(\begin{matrix} 1 \\ 0 \end{matrix} \right)_1 \times \left(\begin{matrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{matrix} \right)_2 \middle| \frac{1}{\sqrt{2}} \left(\left(\begin{matrix} 1 \\ 0 \end{matrix} \right)_1 \quad \left(\begin{matrix} 0 \\ 1 \end{matrix} \right)_2 - \left(\begin{matrix} 1 \\ 0 \end{matrix} \right)_2 \quad \left(\begin{matrix} 0 \\ 1 \end{matrix} \right)_1 \right) \right\rangle$$

is not zero.

- c) to prepare the Dirac equation by pointing that already before Dirac it was known that if in zeroth approximation the energy levels of hydrogen depend only on n , in first approximation they depend on n and on j the total angular momentum, but not on ℓ .

It is only since the discovery of the Lamb Shift (1949) that we know that there is a small dependence on the orbital angular momentum: $E(2S_{1/2} - 2P_{1/2}) = 1058$ Megacycles.

We come now to the Dirac equation (1928). Paul Adrien Maurice Dirac was a British physicist whose father was coming from Switzerland, exactly Saint Maurice, with ancestors in Poitou who fled to Switzerland to escape forced recruitment in Napoleon's army (story that Dirac told me. He even said "there are more famous people than me coming from Poitou, like Mr. Cadillac").

Dirac was guided essentially by aesthetic reasons. He wanted to have a wave equation linear in p and not only in d/dt , because of the relativistic links between space and time, which, for $A = V = 0$ would reduce to the Klein-Gordon equation which gives the energy of a freely moving particle

$$(p^2 + m^2 - E^2) \psi = 0 \tag{40}$$

There was no possible solution using ordinary numbers, but Dirac was strongly influenced by Pauli's recent work using matrices. So he tried

$$H_0 = \sum_{i=1}^3 \alpha_i p_i + \beta m, \tag{41}$$

and found that, to reproduce the Klein-Gordon equation you need

$$\begin{aligned} \alpha_i^2 &= 1 & \alpha_i \alpha_j + \alpha_j \alpha_i &= 0 \text{ if } i \neq j \\ \beta^2 &= 1 & \alpha_i \beta + \beta \alpha_i &= 0 \end{aligned}$$

The solution consists in taking the α_i 's and β to be 4×4 matrices.

It was proved by Van Der Waerden [9] that the only other solutions are just repetitions of 4×4 matrices, modulo a change of basis.

The α matrices can be written in terms of 2×2 matrices. Then, using the notation of H. Grosse one has

$$H - mc^2 = \begin{vmatrix} V & \vdots & c\vec{\sigma} \cdot \vec{\pi} \\ \dots & \dots & \dots \\ c\vec{\sigma} \cdot \vec{\pi} & \vdots & V - 2mc^2 \end{vmatrix}$$

with $\vec{\pi} = -i\hbar\vec{\nabla} - e\vec{A}$, each subsquare being a 2×2 matrix. For instance,

$$V = \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix}$$

For $\vec{A} = 0$, $V = -\frac{Ze^2}{r}$, Dirac found the solution. The energy of the levels, including the rest mass is given by

$$E(n, j) = mc^2 \left[1 + \frac{(Z\alpha)^2}{(n - j - 1/2 + \sqrt{(j + 1/2)^2 - (Z\alpha)^2})^2} \right]^{-1/2} \quad (42)$$

$$E(n, j) \simeq mc^2 - \frac{m c^2 (Z\alpha)^2}{2n^2} \left[1 + \frac{(Z\alpha)^2}{n} \left(\frac{1}{j + 1/2} - \frac{3}{4n} \right) \right] \quad (43)$$

where j is the angular momentum. There is, however, an extra quantum number

$$k = \pm |j + 1/2|, \quad (44)$$

such that $k = +|j + 1/2|$ if $j = \ell - 1/2$ and $k = -|j + 1/2|$ if $j = \ell + 1/2$ where ℓ is the orbital angular momentum. However, for a pure Coulomb potential the two levels coincide, for instance

$$E(2P_{1/2}) = E(2S_{1/2}).$$

As we said already, the Lamb Shift, an effect of quantum electrodynamics, leads to a violation of this equality.

It has been shown by Grosse, Martin and Stubbe [10] that if the potential V is attractive and such that $\Delta V < 0$, the energy levels satisfy:

$$\left. \begin{array}{l} 2P_{3/2} > 2P_{1/2} > 2S_{1/2} \\ 3D_{5/2} > 3D_{3/2} > 3P_{3/2} > 3P_{1/2} > 3S_{1/2} \end{array} \right\} \text{ etc.} \quad (45)$$

Eq. (45) shows that even if the outer electron of sodium is treated relativistically, the problem of the ordering of the D states remains. So it is due to the one-electron approximation.

In the non-relativistic limit, Dirac finds:

- 1) that the electron has a gyromagnetic factor $g = 2$,
- 2) that the spin-orbit coupling is correct, including the Thomas precession.

This was a tremendous success.

The usual method to get the non-relativistic Hamiltonian is to carry a Foldy-Wouthuysen transformation. The problem, with this Hamiltonian is that there are terms like

$$(\vec{\sigma} \cdot \vec{L}) \frac{1}{r} \frac{dV}{dr}$$

which as we said already are not lower bounded in the case of a Coulomb potential. The trick of Gestezy, Grosse and Thaller [2] is to consider the Pauli Hamiltonian (without the spin-orbit term) as the limit for $c \rightarrow \infty$ of the Dirac Hamiltonian and to show that the resolvent of the Dirac Hamiltonian is holomorphic in $1/c$ around $c = \infty$:

$$(H_c - mc^2 - Z)^{-1} = \begin{pmatrix} (H_P - Z)^{-1} & 0 \\ 0 & 0 \end{pmatrix} + 0 \left(\frac{1}{c} \right) \quad (46)$$

with

$$H_P = \frac{(\vec{\sigma} \cdot \vec{\pi})^2}{2m} + V(x) \quad (47)$$

they show that

$$\begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} (H_c - mc^2 - Z)^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^{-1} \quad (48)$$

can be expanded in powers of $1/c^2$.

If E_0 is the eigenvalue of H_p , with eigenvalue f_0 , the first correction is

$$E_1 = \left(\frac{\vec{\sigma} \cdot \vec{\pi}}{2m} f_0, (V - E_0) \frac{\vec{\sigma} \cdot \vec{\pi}}{2m} f_0 \right) . \quad (49)$$

They calculate the next one, but it is too lengthy to be written here, and they could iterate their procedure. E_1 can be rewritten as

$$\begin{aligned} E_1 &= \frac{1}{4m^2} \langle f_0 | (V - E) (\vec{\sigma} \cdot \vec{\pi})^2 | f_0 \rangle \\ &+ \frac{1}{4m^2} \langle f_0 | \frac{dV}{dr} \frac{d}{dr} | f_0 \rangle \\ &+ \frac{1}{4m^2} \langle f_0 | \frac{1}{r} \frac{dV}{dr} (\vec{\sigma} \cdot x \wedge \vec{\pi}) | f_0 \rangle . \end{aligned} \quad (50)$$

The last term is exactly the spin orbit coupling (for a constant magnetic field). The second term is not affecting the fine structure. The first term, if one disregards higher-order terms in A is just

$$- \langle f_0 | \frac{p^4}{8m^3} | f_0 \rangle , \quad (51)$$

a relativistic correction to the kinetic energy which is exactly what you would get using the semi-relativistic kinetic energy:

$$E_{kin} = \sqrt{m^2 + p^2} - m = \frac{p^2}{2m} - \frac{p^4}{8m^3} + \dots \quad (52)$$

It is an elementary exercise to check that for zero magnetic field, E_1 preserves the degeneracy in ℓ for a given j , by a cancellation between the first and the last term.

All this is very nice, but what to do if you have many electrons and you want to use the Pauli hamiltonian? You have to decide to treat the spin-orbit terms as perturbations, as well as spin-spin terms which also exist, or to put a cut-off in the spin orbit interaction. For one electron one can show that this cut-off should be at r_0 such that $|V(r_0)| \simeq 2m$. One could decide to use the same rule for many electrons.

The other very remarkable property of the Dirac equation is the occurrence of negative energy states. This seemed to be a terrible handicap for the Dirac equation but turned out to lead to the fantastic discovery of antiparticles and antimatter.

The trick that Dirac invented was to assume that all negative energy states were filled, and then, because of the Pauli exclusion principle, they could not be occupied by other electrons, and became harmless. However, if one had enough energy to extract one of these electrons from the sea, one would see an electron moving normally and a hole which would behave like a positively charged particle. At the beginning, Dirac believed that this positively charged particle was the proton. However, Oppenheimer pointed out that in that case the proton would annihilate with the electron in hydrogen to produce photons (in fact, many years later, Martin Deutsch manufactured positronium, a system made of an electron and an antielectron which do annihilate rapidly into photons). Then Hermann Weyl pointed out that the field theory of electrons should be completely symmetrical between particles and antiparticles. As you know the antielectron, called positron, was discovered in 1932 by Anderson. Positrons are now very easily produced in large amounts and used in particular to study electron-positron collisions in machines, the last one being LEP in Geneva.

4 Spin of Particles Other than the Electron

You might think that the spin of the proton was obtained first from an experiment of the Stern-Gerlach type. Indeed, Stern did such an experiment, but only later. The first indication that the proton had spin 1/2 came from the observation of an anomaly in the specific heat of the molecular hydrogen around 1927.

If protons have spin 1/2, the two protons inside the hydrogen can have a spin 1 and hence a symmetric wave function – this is called orthohydrogen – or a spin 0, with an antisymmetric wave function, which is called parahydrogen. These two protons are bound by a potential which is produced by the electrons (in the framework of the Born-Oppenheimer approximation), and they have rotation levels (vibrations also exist but are much higher).

Orthohydrogen has only odd angular momentum rotation levels because of the Pauli principle for protons, while parahydrogen has only even rotation levels. Taking this into account in counting the degrees of freedom of hydrogen, with a ratio 3:1 of ortho parahydrogen at room temperature all anomalies in the specific heat are removed. I cannot reproduce the rather

intricate details of the argument which are given in Tomonaga's book.

Then Otto Stern and Extermann did a Stern-Gerlach experiment on the proton in 1933, confirming of course the spin 1/2 but they also measured the magnetic moment of the proton. There are two stories about this. Jensen (the German physicist who got the Nobel prize with Maria Goeppert-Mayer for the nuclear shell model) told Tomonaga that Pauli was visiting Stern's laboratory and, when Stern explained that he wanted to measure the magnetic moment of the proton, Pauli said it was useless because the gyromagnetic factor would be 2, like for the electron. The second story, Gian Carlo Wick (who incidentally, spent the last years of his life here, in Turin) told me. Otto Stern entered a seminar room in Leipzig, where all the elite of physics was present, Heisenberg, Oppenheimer, etc., told he had measured the magnetic momentum of the proton and asked the people present to indicate on a sheet of paper which value it had, with their signature. Most said it would be one proton Bohr magneton, i.e., $g = 2$. Then Stern announced that he had found 2.5. Bohr magnetons! Now we know that it is 2.79 Bohr magnetons.

This discrepancy is the only reason why one might doubt that the proton, in spite of its spin 1/2, is a good Dirac particle with an associated antiparticle, the antiproton, and design a special experiment in 1956 to prove the existence of the antiproton, and give the Nobel prize to Segré and Chamberlain. Yet, already then, it was known that the gyromagnetic factor of the electron was not exactly 2, but

$$g_e = 2 + \frac{\alpha}{\pi} + \dots, \quad \text{with } \alpha = \frac{1}{137}, \quad (53)$$

the first correction being due to J. Schwinger. Higher corrections have been calculated by our friends Kinoshita ..., Remiddi, etc ..., and very recently, De Rafael, Knecht, Nyffeler and Perrotet [11].

Now let me present an argument "explaining" the anomalous gyromagnetic ratio of the proton. I hope that my QCD friends, like E. De Rafael, H. Leutwyler, etc., will forgive me.

You know that the proton and the neutron are made of quarks:

$$\begin{aligned} \text{proton} &= u \ u \ d \\ \text{neutron} &= u \ d \ d \\ \text{where } u &\text{ has a charge } +2/3 \\ &d \text{ has a charge } -1/3 . \end{aligned}$$

In a very naïve view, the binding energy of the quark could be small. Then

$$m_n = m_d = \frac{m_p}{3} \simeq \frac{m_n}{3} \simeq 312 \text{ MeV} \quad (54)$$

(The QCD masses of the u and d quarks are a few MeV.)

Assume now that the gyromagnetic ratio of u and d is 2, like good Dirac particles. Then you get

$$\mu_p = 3 \text{ Bohr proton magnetons} \left(\begin{array}{c} \text{experiment} \\ 2.79 \end{array} \right)$$

$$\mu_n = -2 \text{ Bohr proton magnetons} \begin{pmatrix} \text{experiment} \\ -1.91 \end{pmatrix} \quad (55)$$

Since questions were asked about that, we give the proof:

$$\text{proton} = u u d$$

The colour wave function of the three quarks is a singlet completely antisymmetric. Hence in the ground state the $u u$ system has a space and spin wave functions which are symmetric, i.e., spin 1. We construct a proton with $s_Z = +1/2$ using Clebsch-Gordan coefficients:

$$\left| \frac{1}{2}, \frac{1}{2} \right\rangle = \sqrt{\frac{2}{3}} \left| 1 \ 1 \right\rangle_{uu} \left| \frac{1}{2} - \frac{1}{2} \right\rangle_d + \sqrt{\frac{1}{3}} \left| 1 \ 0 \right\rangle_{uu} \left| \frac{1}{2} + \frac{1}{2} \right\rangle_d \quad (56)$$

Hence the magnetic moment is

$$\left(\frac{2}{3} \left[2 \times \frac{2}{3} + \frac{1}{3} \right] + \frac{1}{3} \left[0 - \frac{1}{3} \right] \right) \frac{\hbar}{2m_u c} = 3 \left(\frac{e\hbar}{2m_p c} \right)$$

if $m_u = m_d = \frac{m_p}{3}$.

In the neutron case, it is the dd pair which has spin 1 and we get

$$\left(\frac{2}{3} \left[2 \times \left(\frac{-1}{3} \right) - \frac{2}{3} \right] + \frac{1}{3} \left[+ \frac{2}{3} \right] \right) 3 \frac{e\hbar}{2m_p c} = -2 \left(\frac{e\hbar}{2m_p c} \right) .$$

You can apply this to all baryons made of u, d, s quarks. The strange quark must be heavier than the $u - d$ quarks. From the masses of the Λ and Σ hyperons we see that the strange quark must be about 200 MeV heavier, which gives

$$\frac{m_{u,d}}{m_s} = 0.625 . \quad (57)$$

With this value, we get the following table, which is in my opinion remarkable (remember that just getting the right signs is not trivial). $m_u/m_s = 1$ is much less good. All the particles listed have a spin 1/2 except the Ω^- .

Notice that there is one prediction which is mass independent which is:

$$\mu_{\Omega^-}(\text{spin}3/2) = 3\mu_{\Lambda} \begin{pmatrix} \text{spin}1/2 \\ u \ d \ \text{singlet} \end{pmatrix} , \quad (58)$$

which is approximately satisfied by experiment.

Table 2

	exp	μ/μ_P Quark Model $m_u/m_s = 1$	Quark Model $m_u/m_s = 0.625$
N	-0.68	-0.67	-0.67
Λ	-0.22 ± 0.02	-0.333	0.207
Σ^+	0.85 ± 0.01	1	0.958
Σ^-	-0.50 ± 0.059	-0.333	-0.37
Ξ^-	-0.24 ± 0.01	-0.333	-0.17
Ξ^0	-0.45 ± 0.05	-0.666	-0.50
Ω^-	-0.72 ± 0.07	-1	-0.625

Finally, since the proton has a magnetic moment, it has a spin-spin interaction with the electron, which gives rise to an hyperfine splitting. The “rule” is that it is given by

$$W = \frac{a}{2} \left[F(F+1) - I(I+1) - J(J+1) \right] \quad (59)$$

F = total angular momentum

I = angular momentum of the nucleus

J = angular momentum of the electron

$$a = \frac{16\pi}{3} g_e g_{Nucl} \mu_{BN} \mu_{Be} |\psi(0)|^2 \quad (60)$$

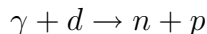
where $\psi(0)$ is the Schrödinger wave function at the origin (the Dirac wave function at the origin is infinite!). This is not trivial! (worse than the spin-orbit coupling!). For Hydrogen this gives the famous 21 cm line, produced by an excited state which has a lifetime of 10^7 years.

Of course the year 1932 is also the year of the discovery of the neutron by Frederic and Irène Joliot-Curie and James Chadwick. The Joliot's had seen a mysterious penetrating neutral particle. According to Gian-Carlo Wick, when Majorana read the “note aux Comptes Rendus” of the Joliot's, he exclaimed: “Stronzi! Non hanno capito ché è il neutrone!” “Idiots! They have not understood that it is the neutron!”

Chadwick really demonstrated that it is a neutron by a kind of billiard experiment: when a neutron hits a proton, the recoiling proton and the deflected neutron go at right angle because they have equal masses.

That the neutron has also spin 1/2 it is not a surprise. It was shown by J. Chadwick and M. Goldhaber (still alive!) in 1934 that the deuteron was made of a proton and a neutron by

dissociating it with a gamma ray:



They showed that the masses of the proton and the neutron were very close, the neutron being slightly heavier. The spin and statistics of the neutron were obtained from the spin and statistics of the deuteron from the band spectrum of the deuteron molecule. The deuteron has spin 1. The neutron has hence a half-integer spin, 1/2 or 3/2. 3/2 was very unlikely and not so long ago (since nuclear reactors producing neutrons exist!) Stern-Gerlach experiments have been performed and the spin found to be 1/2 with a magnetic moment $-1.93 \mu_{Bn}$. The magnetic moment was first deduced from the magnetic moment of the deuteron, using an extremely plausible theoretical model of the deuteron.

Previously, Rasetti, who died very old only a few months ago, here in Turin I believe, had proved that the spin of N_{14} which has charge 7 is zero from the band structure of the Nitrogen molecule. This demonstrated that it was made of an equal number of protons and neutrons.

5 Spin and Statistics and the Stability of the World

We have seen that electrons have spin 1/2 and satisfy the Pauli principle. They are usually said to satisfy Fermi (or Fermi-Dirac) statistics. Why Fermi? Probably because Fermi studied the free electron gas which is, to a good approximation, what you find in a conductor. The energy levels are filled up to a certain level, the Fermi level.

There is another type of statistics: Bose statistics. Bosons are integer spin particles with a wave function which is completely symmetrical. They were invented in 1927 by S.M. Bose, who sent a letter to Einstein, who immediately understood the importance of Bose's discovery and developed it so that now people speak of "Bose-Einstein" particles.

Concerning Dirac particles with spin 1/2 it is intuitively clear that if one does not want to have a catastrophe they must satisfy the Pauli principle. Otherwise Dirac's original argument with the filling of the energy levels breaks down. This remains true in the modern field theoretical picture for electrons and positrons. I have neither the time nor the ability to explain this further.

As for integer spin particles, spin 0 like the pion, or 1 like the intermediate vector bosons W^\pm and Z_0 , they must satisfy Bose statistics as shown very early by Pauli and Weisskopf (who just died!). Bosons, except for the motion due to uncertainty relations, can – crudely speaking – be all at the same place. This is realized in boson condensates, the observation of which has been rewarded by the 2001 Nobel prize.

In our world, with one time and three space dimensions there is only room for these two kinds of statistics. In two spatial dimensions there is room for other statistics. This is important for the quantum Hall effect but we shall not speak about that here.

If the world exists as it is, it is essentially because electrons have spin 1/2 and satisfy the

Pauli principle. The late Weisskopf told me that if he puts his hand on a table and the hand does not go through the table it is because the electrons of the hand cannot be at the same place as the electrons of the table. If we start with atoms satisfying the Schrödinger equation, heavy atoms are well described by the Thomas-Fermi model which gives [12]

$$E \sim -0.77 \dots Z^2 N^{1/3} \alpha^2 m ,$$

which has been shown to be asymptotically exact by Lieb [13] for $Z \rightarrow \infty$ (this limit is of course purely mathematical since the Schrödinger equation is no longer valid because of relativistic effects) and furthermore is presumably a lower limit of the binding energy of the atom according to Lieb and Thirring [14], modulo a certain unproved technical assumption on bound states in potentials. The two ingredients of the Thomas-Fermi model are:

- i) the electrostatic repulsion between electrons and the electric attraction of the nucleus;
- ii) a crude form of the Pauli principle: locally the electrons are considered as free in a potential well and their density is determined by the Fermi level.

Dyson and Lennard [15] have gone further than that and established what they call the “stability of matter” in a non-relativistic world where only electric forces are taken into account, the fact that the binding energy of an assembly of N atoms has an energy which behaves like $-CN$ and a minimum volume which behaves like $C'N$. Only the Fermi statistics of the electrons is essential. The spin of the nuclei can be either integer or half integer (for instance liquid Helium has a volume proportional to N). What is remarkable in the work of Lieb-Thirring [14] is that they succeed in getting a constant C which is not more than twice a realistic value.

Things change if either all particles satisfy Bose statistics or if instead of electric forces we have gravitational forces. Table 3 [16] gives a summary, for the non-relativistic situation (Schrödinger equation).

Table 3

Nature of the Forces	Statistics	Volume	Energy
electric, with <u>finite mass</u>	Fermi	N	$-N$
positive and negative particles	Bose	$N^{-3/5}$	$-N^{7/5}$
gravitational	Fermi	N^{-1}	$-N^{7/3}$
	Bose	N^{-3}	$-N^3$

Now, what happens if we put in relativistic effects? We already know that the Dirac equation with a point nucleus of charge Z and an electron breaks down if

$$Z\alpha = Z \frac{e^2}{\hbar c} = 1 , \quad \text{i.e.,} \quad Z \sim 137 .$$

the Klein-Gordon equation breaks down for $Z\alpha = 1/2$ and the “Herbst” or Salpeter equation,

$$\left(\sqrt{p^2 + m^2} - \frac{Z\alpha}{r} - E \right) \psi = 0$$

breaks down for $Z\alpha = 2/\pi$. This occurs also for many-body systems.

Take the case of an assembly of N particles with all equal masses (for simplicity) m [17]. Call $E(N, m)$ the non-relativistic energy of this system:

$$\left[\sum \frac{p_i^2}{2m} + V(x_1 \dots x_n) \right] \psi = E(N, m) \psi$$

In a semi-relativistic treatment, one replaces $m + p_i^2/m$ by $\sqrt{p_i^2 + m^2}$. To get the ground state energy one has to minimize the expectation value of

$$H_{Rel} = \sum \sqrt{p_i^2 + m^2} + V(n_1 \dots n_n)$$

but

$$\sqrt{p_i^2 + m^2} \geq \frac{1}{2} \left[M + \frac{p^2 + m^2}{M} \right]$$

So

$$\langle H_{Rel} \rangle \leq \inf_M \left[\frac{n}{2} \left(M + \frac{m^2}{M} \right) + E(N, M) \right]$$

So, if

$$E(N, m) \sim -CmN^\alpha \quad \text{for large } N$$

(it comes from homogeneity), we have to minimize

$$\frac{N}{2} \left(M + \frac{m^2}{M} \right) - C M N^\alpha .$$

We see immediately that if $\alpha > 1$ there is no minimum for sufficiently large N and the system collapses. This is the case of an electric system with Bose statistics or of any gravitational system. The above equation gives an upper bound of the Chandrasekhar limit for a neutron star which is not unreasonable and also a limit on the mass of a boson state [18].

In the case of fermions, or Fermion-Boson, the occurrence of the collapse will depend on the value of C , which is itself linked to the strength of the coupling (this was already the case for one or two particles).

The (temporary) existence of our world depends on a careful balance between masses and interactions of its constituents.

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