# Four–loop logarithms in 3d gauge + Higgs theory*

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We discuss the logarithmic contributions to the vacuum energy density of the three-dimensional SU(3) + adjoint Higgs theory in its symmetric phase, and relate them to numerical Monte Carlo simulations. We also comment on the implications of these results for perturbative and non-perturbative determinations of the pressure of finite-temperature QCD.

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## 1. INTRODUCTION

Strongly interacting quantum field theories, such as QCD, require extensive numerical simulations, to obtain a non-perturbative understanding from first principles. Some regions in parameter space might however be amenable to analytic methods, which can then be used to obtain a clearer physical picture as well as an independent check on the Monte Carlo (MC) simulations. Furthermore, since in practice there are upper limits on computing power, one might combine numerical and analytic methods, to supplement each other and provide for a sufficient tool in cases where either method alone would fail.

As a concrete example of this interplay, let us study the free energy density $f$ (which, in the thermodynamic limit, equals the negative pressure $p$) of QCD, at finite temperature $T$ and vanishing baryon chemical potential,

$$e^{-f(T)\frac{V}{T}} = \int \mathcal{D}\left[A\bar{\psi}\psi\right] e^{-\int_0^{1/T} d\tau \int d^3x \mathcal{L}_E\left[A\bar{\psi}\psi\right]},$$

where $\mathcal{L}_E$ is the standard QCD Lagrangian. The free energy can be expected to be a good candidate to witness the change of the properties of QCD matter around a critical temperature $T_c \sim 200$ MeV. While the low-temperature phase is governed by bound states, such as mesons, the high-temperature phase should, due to asymptotic freedom, look more like a gas of free quarks and gluons.

A direct lattice measurement of $f$ can be and has been performed, see e.g. [1]. The results show the pressure rising sharply around $T_c$, to level off at a few times $T_c$. At higher temperatures, the direct numerical approach gets increasingly harder, since ensuring clean continuum as well as thermodynamic limits one is facing a multiscale problem, $a \ll \frac{1}{T} \ll \frac{1}{T_c} \approx \frac{1}{\Lambda_{\overline{\rm MS}}} \approx 1$ fm $\ll Na$, where $a$ and $N$ are the lattice spacing and the number of lattice sites, respectively.

On the other hand, the temperature being the only scale in the problem, perturbative methods are guaranteed to work well at high $T$, due to asymptotic freedom. In fact, at vanishing coupling one reaches the ideal-gas limit,

$$p_{\rm ideal}(T) = \frac{\pi^2 T^4}{45}\left(N_c^2 - 1 + \frac{7}{4}\,N_c N_f\right), \qquad (1)$$

where $N_c$ and $N_f$ denote the number of colours and flavours, respectively. With decreasing temperature, the value of the effective coupling con-

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stant $g(T)$ however increases, rendering a perturbative series [2] meaningless below some point.

## 2. METHOD

Progress can be made by exploiting the scale hierarchy $\pi T > gT > g^2T$ at high temperatures, enabling one to use the powerful analytic method of effectice theories, allowing to reduce numerical simulations needed for the QCD pressure to a much less demanding three-dimensional (3d) bosonic theory [3]. The partition function factorizes, and hence the free energy decomposes into $f_{\rm QCD} = f_{\rm hard} + f_{\rm soft}$ [4]. The effective theory for the *soft*, $\mathcal{O}(gT)$ modes turns out to be dimensionally reduced,

$$e^{-Vf_{\rm soft}} = \int \mathcal{D}[A_iA_0]e^{-\int d^3x \mathcal{L}_{\rm 3d}[A_iA_0]} \,, \qquad (2)$$

where $\mathcal{L}_{\rm 3d}$ is a 3d SU(3) + adjoint Higgs theory,

$$\mathcal{L}_{\rm 3d} = \tfrac{1}{4} F_{ij}^2 + \tfrac{1}{2}[D_i, A_0]^2 + \tfrac{1}{2} m_3^2 A_0^2 + \tfrac{1}{4}\lambda_3 A_0^4. \quad (3)$$

Its coefficients ($g_3^2$,$m_3^2$,$\lambda_3$) are functions of $T$ via perturbative matching [5], e.g.,

$$y \equiv \frac{m_3^2}{g_3^4} \approx \frac{11}{8\pi^2} \ln \frac{8.086T}{\Lambda_{\overline{\rm MS}}} \ . \qquad (4)$$

Let us note that for simplicity, this relation refers to the case of pure 4d SU(3), while the inclusion of fermions, as well as (small) chemical potentials, is also possible.

For a more detailed account of the setup, we refer to [3]. In the following we wish to highlight the specific role of logarithmic terms in $f_{\rm soft}$.

## 3. 3d LATTICE MEASUREMENTS

Related to the fact that $\mathcal{L}_{\rm 3d}$ defines a confining theory, it turns out that $f_{\rm soft}$ is perturbatively computable only up to 3-loop level, while all higher loop orders contribute at the next level [6]. The parametric form of $f_{\rm soft}$ can however still be written down for large $y$,

$$\begin{aligned} f_{\rm soft}(y) &= f_{\rm soft,pert}(y) \\ &+ \frac{g_3^6}{(4\pi)^4}\Big(c_1 \ln y + c_2 + \mathcal{O}(\frac{1}{y^{1/2}})\Big). \end{aligned} \quad (5)$$

The coefficient $c_1$ here can now be accessed with lattice methods. Indeed, $\partial_y f_{\rm soft}$ is related

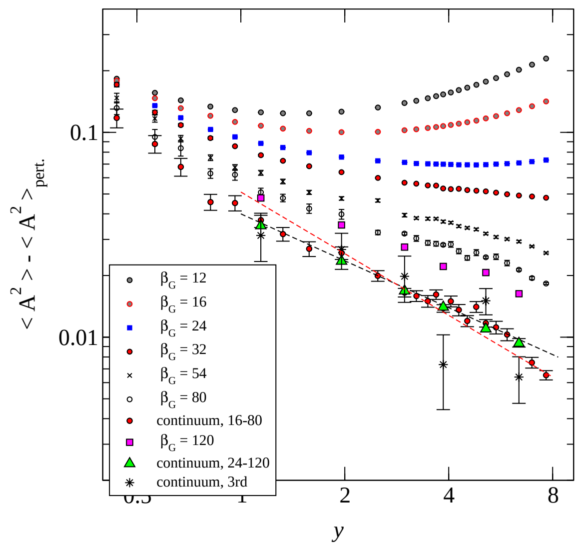


Figure 1. Lattice results for $\langle {\rm Tr}(A_0/g_3)^2\rangle_{\overline{\rm MS}} - \langle {\rm Tr}(A_0/g_3)^2\rangle_{\overline{\rm MS}}^{\rm pert}$ as a function of $y$, $\beta_{\rm G}$. Various continuum extrapolated values are also shown.

to a gauge-invariant condensate,

$$\partial_y f_{\rm soft}(y) = g_3^4 \left\langle {\rm Tr} A_0^2 \right\rangle . \qquad (6)$$

If we also subtract the known perturbative part, we see that

$$\begin{aligned} &\left\langle {\rm Tr}(A_0/g_3)^2 \right\rangle - \left\langle {\rm Tr}(A_0/g_3)^2 \right\rangle_{\rm pert} \\ &\quad = \frac{1}{(4\pi)^4}\Big(c_1 \frac{1}{y} + \mathcal{O}(\frac{1}{y^{3/2}})\Big). \end{aligned} \qquad (7)$$

An additional issue which has to be addressed is the renormalisation of the condensate. However, due to the super-renormalisability of $\mathcal{L}_{\rm 3d}$, this problem can be taken care of, with a perturbative 2-loop computation [7,8]. Denoting

$$\beta_{\rm G} = \frac{6}{ag_3^2} \,, \qquad (8)$$

the result is, schematically, that a lattice measurement can be converted to a continuum regularisation (such as $\overline{\rm MS}$) through a relation

$$\left\langle {\rm Tr} A_0^2 \right\rangle_{\overline{\rm MS}} \sim \lim_{\beta_{\rm G}\to\infty} \left\{ \left\langle {\rm Tr} A_0^2 \right\rangle_L + \beta_{\rm G} + \ln \beta_{\rm G} + 1 \right\} . \quad (9)$$

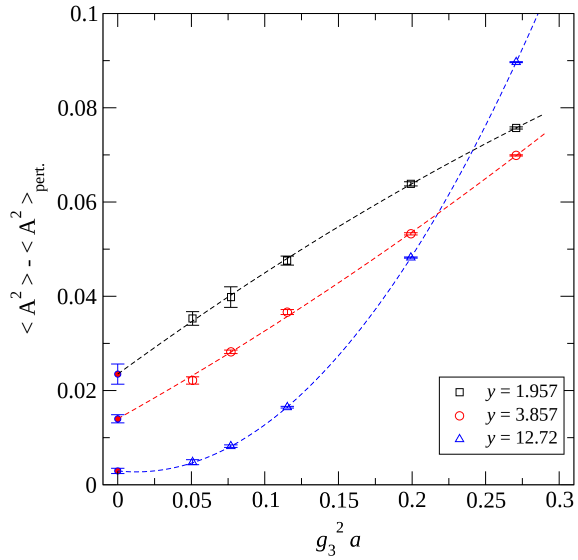


Figure 2. Examples of continuum extrapolations for $\langle \mathrm{Tr}(A_0/g_3)^2 \rangle_{\overline{\mathrm{MS}}} - \langle \mathrm{Tr}(A_0/g_3)^2 \rangle^{\mathrm{pert}}_{\overline{\mathrm{MS}}}$, at a few selected $y$. The fits are polynomial.

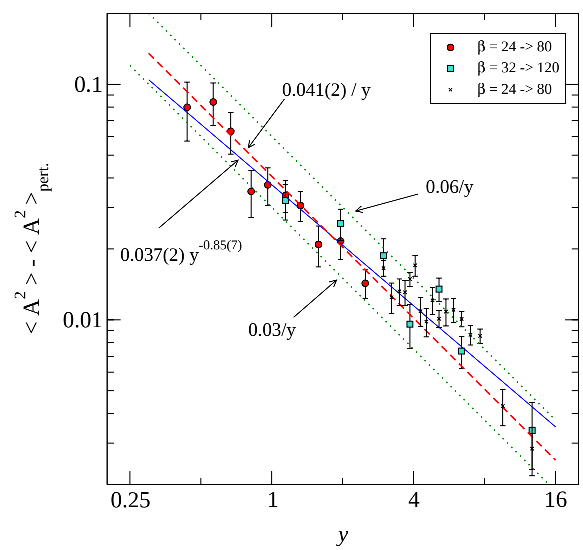


Figure 3. The continuum extrapolated values of $\langle \mathrm{Tr}(A_0/g_3)^2 \rangle_{\overline{\mathrm{MS}}} - \langle \mathrm{Tr}(A_0/g_3)^2 \rangle^{\mathrm{pert}}_{\overline{\mathrm{MS}}}$ as a function of $y$, together with various fits. A fit linear in $y^{-1}$ describes the data well in a wide range of $y$.

In Figs. 1, 2 we show measurements of the condensate with various finite $\beta_{\mathrm{G}}$, as well as continuum extrapolations. The (very preliminary) final result, after the subtraction of the 3-loop perturbative part, is shown in Fig. 3.

We find that the data can indeed be well described by the functional form in Eq. (7), with what appears to be a definite coefficient $c_1$. This clearly calls for an analytic evaluation of $c_1$.

Previously [3], we have discussed how the non-perturbative measurement of $\langle \mathrm{Tr} A_0^2 \rangle$ allows to estimate $f_{\mathrm{soft}}(y)$, and correspondingly $f_{\mathrm{QCD}}(T)$, down to temperatures of a few times $T_c$. Once $c_1$ is reliably extracted, it will be interesting to see how well these results can be reproduced by keeping in the expression only this single logarithm.

Concluding, we have discussed a method that in principle allows to determine the free energy of full QCD from the known analytic limit at high $T$, down to a few times $T_c$. A small set of perturbative constants remains to be determined, but these can already be partly constrained with numerical 3d MC data.

## ACKNOWLEDGEMENTS

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