

Vortex free energies in SO(3) and SU(2) lattice gauge theory *

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Lattice gauge theories with gauge groups SO(3) and SU(2) are compared. The free energy of electric twist, an order parameter for the confinement-deconfinement transition which does not rely on centre-symmetry breaking, is measured in both theories. The results are used to calibrate the scale in SO(3).

1. What makes SO(3) interesting

The motivation for a comparison of SO(3) and SU(2) lattice gauge theory is to clarify an apparent paradox. On one hand, both groups have the same local structure, $SO(3) = SU(2)/\mathbb{Z}_2$, so the naive continuum limit of both theories is the same. Universality of the continuum fixed point suggests that this should be true also non-perturbatively. On the other hand, the deconfinement transition is usually associated with the breakdown of the centre symmetry, which does not exist in SO(3). If SO(3) has a deconfinement transition, its characterisation must be different; and there should also be another order parameter.

Further doubt about universality comes from the observation [1] that the weak coupling phase of SO(3) LGT features 2 metastable states: in addition to the state with positive adjoint Polyakov loop expected in a deconfined phase, there is a state with negative adjoint Polyakov loop. This state has not found a satisfactory explanation so far.

2. Centre vortices

Even though the centre of SO(3) is trivial, centre vortices exist both in SU(2) and SO(3). A centre vortex is a 2-dimensional configuration which tends to a pure gauge at infinity. In SU(2), the corresponding gauge function g changes its sign as one goes around the vortex. In SO(3), the sign

of the group elements is discarded, so $g(\varphi)$ becomes single-valued, but now it is a non-trivial element of $\pi_1[SO(3)]$. For general gauge groups G , the relevant quantity is $\pi_1[G/\text{Centre}(G)]$.

A convenient way to study centre vortices in SU(2) are twisted boundary conditions on a torus. These are imposed by introducing a mismatch in the transition functions along 2 directions, $\Omega_\mu\Omega_\nu = -\Omega_\nu\Omega_\mu$. These boundary conditions change the number of centre vortices through the plane with twist from even to odd. Accordingly, one can define the free energy of an (additional) centre vortex as the ratio of partition functions with twisted and periodic boundary conditions

$$e^{-\beta F_{CV}} = Z^{\text{tbc}}/Z^{\text{pbc}}. \quad (1)$$

For electric twist, this ratio is an order parameter of the deconfinement transition: for large volumes, it tends to 1 in the confined phase while it is exponentially suppressed in the deconfined phase [2].

3. Twist in SO(3)

In SO(3), twisted boundary conditions are void because the sign of the transition functions is discarded; twist becomes an ordinary topological quantum number. In order to understand this in more detail, we invoke the formulation of Wilson

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SU(2) in terms of SO(3) and \mathbb{Z}_2 variables [3],

$$Z_{\text{SU}(2)}^{\text{Wilson}} = \sum_{\alpha_p = \pm 1} \int_{\text{SU}(2)} DU e^{\frac{1}{2}\beta \sum_p \alpha_p \text{Tr}_F U_p} \times \prod_{c \in \text{cubes}} \delta\left(\prod_{p \in \partial c} \alpha_p - 1\right). \quad (2)$$

The constraint forbids monopoles in the plaquette field α . Without the constraint, one obtains the Villain partition function of SO(3) LGT,

$$Z_{\text{SO}(3)}^{\text{Villain}} = \sum_{\alpha_p = \pm 1} \int_{\text{SU}(2)} DU e^{\frac{1}{2}\beta \sum_p \alpha_p \text{Tr}_F U_p}. \quad (3)$$

The sum over α_p makes the action independent of the sign of each link matrix U , so (3) is really a redundant formulation of an SO(3) theory.

So far the argument holds for an infinite volume. On a 4-torus, 6 additional global constraints are required [4], namely

$$\prod_{p \in \mu\nu\text{-plane}} \alpha_p \stackrel{!}{=} +1 \quad (\text{periodic SU}(2)). \quad (4)$$

Here, the product is over all plaquettes in a fixed 2-dimensional section in $\mu\nu$ -direction of the torus. Equation (4) ensures periodic boundary conditions in SU(2). If the product is negative for some μ, ν , one obtains twisted boundary conditions in that direction. For then one U_p in each $\mu\nu$ -plane enters the action with a negative sign.

The above relation suggests the following definition of a “twist” observable in the SO(3) theory (with monopoles):

$$z_{\mu\nu} \equiv \frac{1}{L_\rho L_\sigma} \sum_{\mu\nu\text{-planes}} \prod_{\text{plane}} \text{sgn Tr}_F U_p. \quad (5)$$

At weak coupling, $\text{sgn Tr}_F U_p$ and α_p are strongly correlated; $z_{\mu\nu}$ is a genuine SO(3) observable because each link appears twice in the product, so its sign drops out. The average over parallel planes has been introduced because, in the presence of monopoles, the twist can vary between planes. So z in general takes fractional values. It turns out that it is always close to ± 1 above the bulk phase transition at $\beta \approx 4.45$ below which monopoles condense. This is illustrated in Fig. 1 by a Monte Carlo history obtained at $\beta = 4.5$.

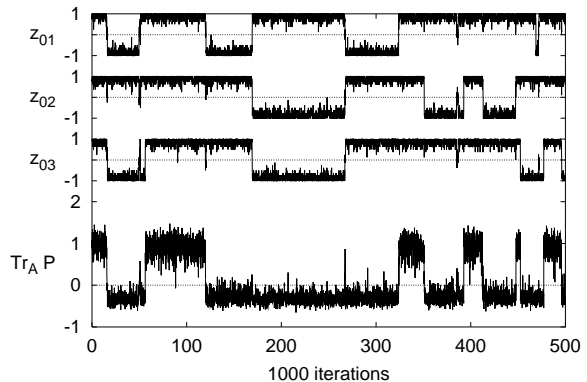


Figure 1. Monte Carlo history of the 3 electric twist variables (top) and the adjoint Polyakov loop (bottom) (4^4 lattice, $\beta = 4.5$).

We also find a simple explanation for the mysterious phase with negative adjoint Polyakov loop as a state with electric twist. A classical ground state in the sector with $z_{01} = -1$, say, can be obtained by setting all links to 1, except $U_1(1, y, z, t) = i\sigma_1$ and $U_4(x, y, z, 1) = i\sigma_2$ [5]. The Polyakov loop of this configuration is $\text{Tr}_F P = 0$ in the fundamental representation and $\text{Tr}_A P \equiv (\text{Tr}_F P)^2 - 1 = -1$ in the adjoint representation. So we expect a negative $\text{Tr}_A P$ in sectors with electric twist. Figure 1 shows that this is indeed the case.

4. Vortex free energies

We use the observable (5) to measure the free energy of electric twists in the Villain SO(3) theory. The technical difficulty here is that there are high barriers between the different twist sectors, so it is very difficult to maintain ergodicity. The density of states as a function of the 3 twist variables z_{0i} extends over 12 orders of magnitude [6]. The remedy is a multicanonical algorithm where the barriers are removed by a bias, which is corrected by reweighting the observables. The bias depends on 3 variables (z_{0i}) and is represented by a 3-dimensional table determined iteratively.

In Fig. 2, the free energies of 1, 2 and 3 electric twists (an additional vortex winding around 1, 2 or 3 directions) obtained at $\beta_{\text{SO}(3)} = 4.5$ on $4^3, 6^3$

and $8^3 \times 4$ lattices are compared with results for $SU(2)$ at various $\beta_{SU(2)}$. The latter were obtained with the method of [7]. The $SO(3)$ data can be reproduced by $SU(2)$ with $\beta_{SU(2)} = 4.12(3)$. The good quality of the joint fit ($\chi^2/\text{dof} = 1.35$) lends support to the hypothesis of universality of the continuum limit.

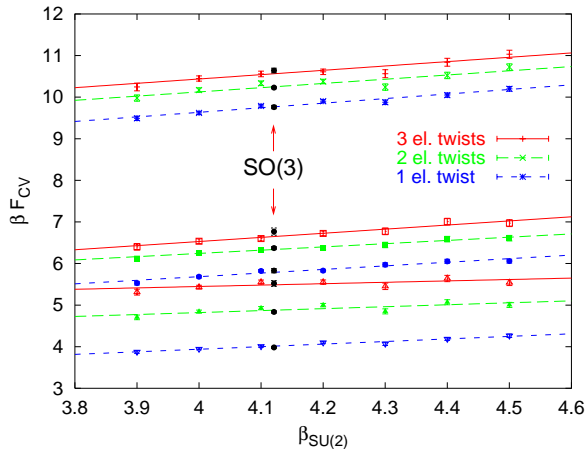


Figure 2. Comparison of the vortex (electric twist) free energies between $SO(3)$ at $\beta = 4.5$ and $SU(2)$ at various β . Lattice sizes are 8^3 , 6^3 and $4^3 \times 4$ from the top (3 lines each).

Asymptotic scaling suggests that the inverse lattice spacing at this coupling is about 200 GeV. As the bulk transition prohibits coarser lattices, we conclude that lattices larger than about 700^4 are needed to simulate confined (Villain) $SO(3)$. We would like to emphasise that the scale of 200 GeV is in no way related with continuum physics. It is just due to the bulk phase transition of the $SO(3)$ lattice gauge theory beyond which lattice artifacts dominate, and gives a lower bound on the possible cutoffs one can use. This value can be shifted by suppressing (or enhancing) lattice artifacts [8].

The fact that the couplings in $SU(2)$ and $SO(3)$ are not very different is not a coincidence: since the actions of the two theories differ only by terms exponentially small in the coupling, the perturbative scale parameters Λ_L are the same for both.

The difference is thus of purely non-perturbative origin.

5. Conclusions

To conclude, we have recalled that $SO(3)$ features twist sectors like $SU(2)$; but unlike $SU(2)$ they are summed over within periodic boundary conditions rather than imposed by the boundary conditions. The vortex free energy can serve as an order parameter in both theories, so the centre symmetry – which does not exist in $SO(3)$ – is not needed. The free energies of spatial centre vortices are very similar in $SO(3)$ and $SU(2)$ provided the bare couplings are adjusted. This supports universality of the continuum limit. However, one has to keep in mind that the systems studied here are very small. Because of a bulk phase transition, one needs lattices larger than 700^4 to simulate the confined phase.

The vortex free energy can be defined whenever $\pi_1[G/\text{Centre}(G)] \neq 1$, independent of whether there is a centre or not, i.e. for all simple Lie groups except G_2 , F_4 and E_8 . The absence of a vortex order parameter in the latter prompts speculations about the nature of the confinement/deconfinement phase transition. The study of G_2 proposed in [9] will be interesting in this connection.

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