CONVERGENCE AND STABILITY OF THE RENORMALISATION GROUP [1]

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Within the exact renormalisation group approach, it is shown that stability properties of the flow are controlled by the choice for the regulator. Equally, the convergence of the flow is enhanced for specific optimised choices for the regularisation. As an illustration, we exemplify our reasoning for $3d$ scalar theories at criticality. Implications for other theories are discussed.

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1 Introduction

Renormalisation group techniques are important tools to describe how classical physics is modified by quantum fluctuations. Integrating-out all quantum fluctuations provides the link between the classical theory and the full quantum effective theory [1]. A useful method is given by the Exact Renormalisation Group (ERG) [2], which is based on the Wilsonian idea of integrating-out infinitesimal momentum shells. ERG flows have a simple one-loop structure. They admit non-perturbative truncations and are not bound to weak coupling.

An application of the ERG requires some approximations like the derivative expansion or expansions in powers of the fields. It has been known since long that approximations induce a spurious dependence on the regularisation [3, 4, 5, 6, 7]. This is somewhat similar to the scheme dependence within perturbative QCD, or within truncated solutions of Schwinger-Dyson equations. While this scheme dependence should vanish at sufficiently high order in the expansion, practical applications are always bound to a finite order, and hence to a non-vanishing scheme dependence. A partial understanding of the interplay of approximations and scheme dependence has been achieved previously. For scalar QED [8], the scheme dependence in the region of first order phase transition has been studied in [4, 5]. For $3d$ scalar theories, the interplay between the smoothness of the regulator and the resulting critical exponents has been addressed in [9] using a minimum sensitivity condition. The weak scheme dependence found in these cases suggests that higher order corrections remain small, thereby strengthening the results existing so far.

In this contribution, we review how the convergence and stability of ERG flows is optimised, thereby providing improved results already to low orders within a given approximation [10, 11, 12, 13, 14]. This involves a discussion on the origin of the spurious scheme dependence, and its link with convergence and stability properties of truncated ERG flows. We exemplify the basic reasoning for the universality class of $O(N)$ symmetric scalar theories in three dimensions. It is expected that insights gained from this investigation will also prove useful for applications to more complex scalar theories, gauge theories [15] or gravity [16], which are more difficult to handle.

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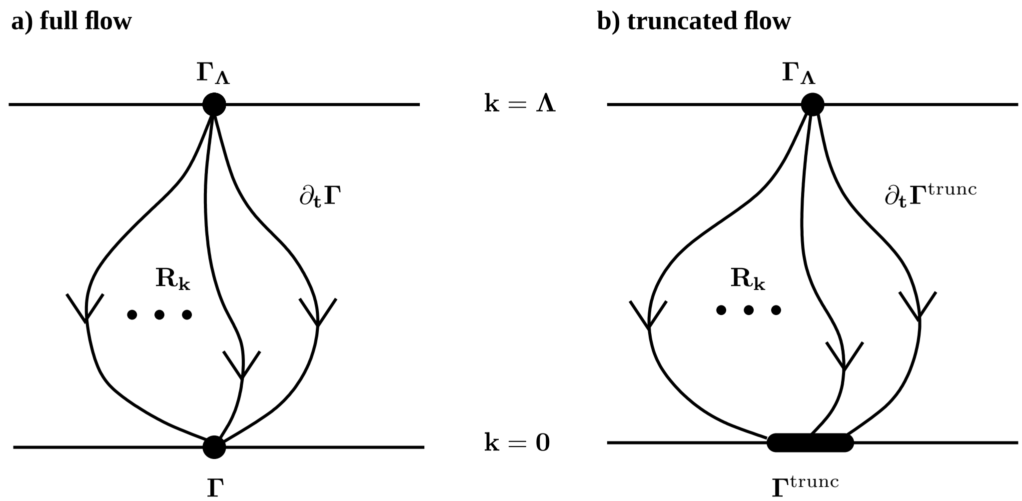

Fig. 1. Schematic diagram of a full (left panel) or truncated (right panel) renormalisation group flow connecting an initial effective action at $k = \Lambda$ with the full (truncated) quantum effective action at $k = 0$. The upper line corresponds to the space of all initial effective actions. The lower line corresponds to the space of effective actions. For $k \neq 0$, the flow trajectories depend on the regulator R_k. For the full flow, all trajectories join at $k = 0$. For a truncated flow, the endpoint depends, in general, on R_k.

2 Renormalisation group flows and truncations

ERG flows are based on the Wilsonian idea of integrating out momentum modes within a path integral representation of quantum field theory. In its modern form, the ERG flow for an effective action Γ_k for bosonic fields ϕ is given by the simple one-loop expression [2]

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} {\rm Tr} \left(\frac{\delta^2 \Gamma_k}{\delta\phi\delta\phi} + R_k \right)^{-1} \partial_t R_k \tag{1}$$

Here, $\Gamma_k^{(2)}$ denotes the second functional derivative of the effective action, $t \equiv \ln k$ is the logarithmic scale parameter, and $R_k(q^2)$ is an infrared (IR) regulator at the momentum scale k. The regulator R_k obeys a few restrictions, which have been discussed at length in the literature [2]. They ensure that the flow equation is well-defined, thereby interpolating between an initial action in the UV and the full quantum effective action in the IR. In order to solve (1), we have to specify an initial effective action Γ_Λ at some ultraviolet (UV) scale $k = \Lambda$, and a regulator R_k. Clearly, the flow trajectory of Eq. (1) in the space of all action functionals depends on the IR regulator function R_k. For the full flow, this is of no relevance. Starting from an initial effective action Γ_Λ, the integrated full flow approaches the full quantum effective action, independently on the choice for R_k along the flow. Schematically, this is depicted in Fig. 1a.

The situation changes once truncations have been made. Here, "truncations" mean that some vertex functions are neglected in the Ansatz for the functional form of $\Gamma_k^{\rm trunc}$ entering (1). Schematically, this scenario is depicted in Fig. 1b. Still, the flow trajectories in the space of all action functionals depend on R_k. However, it cannot be guaranteed that the endpoint of the integrated flow is independent on R_k. In general, it is not. The origin of this spurious scheme dependence is easily understood: while regulating the flow, the regulator R_k also modifies all vertex functions and their interactions at $k \neq 0$. Hence, the "missing" back-coupling of neglected vertex functions is responsible for a spurious scheme dependence. An immediate consequence of this observation is that varying the regulator influences the physical content of a given truncation. Hence, the scheme dependence within a given truncation, and convergence properties of ERG flows are entangled [10].

3 Optimisation and stability

Next we turn to the stability of the flow (1), and a simple optimisation condition [10, 11]. The two ingredients of (1) are the full regularised propagator $(\Gamma_k^{(2)} + R_k)^{-1}$ -which contains the physical information of the flow-, and the insertion $\partial_t R_k$. Typically, $\partial_t R_k$ is peaked around $q^2 \approx k^2$, and decays exponentially for large momenta. For small momenta, the flow (1) is regularised due to R_k in the full propagator. The regulator implies that the inverse propagator displays a gap,

$$\min_{q^2 \geq 0} \left(\left. \frac{\delta^2 \Gamma_k[\phi]}{\delta\phi\,\delta\phi} \right|_{\phi=\phi_0} + R_k \right) > C\,k^2 \tag{2}$$

with $C > 0$. The minimum is achieved for $q^2 \approx k^2$. In general, C depends on R_k and on ϕ_0. The flow (1) receives its dominant contributions from the region in momentum space where $\partial_t R_k$ is large and the inverse propagator is small. In consequence, the flow is more stable against small changes in Γ_k the larger the full inverse propagator. This observation leads to a simple criterion to optimise the stability of flows. To that end, let us consider a theory with a standard propagator and $\Gamma^{(2)}(\phi) = q^2 + U_k''(\phi)$. This corresponds to the leading order in a derivative expansion. Inserting this expression into (2), we require the gap to be maximal w.r.t. the regularisation scheme. Dropping irrelevant momentum-independent terms, the optimisation condition becomes

$$\max_{\text{(RS)}} \left[\min_{q^2 \geq 0} \left(q^2 + R_k(q^2) \right) \right] \Rightarrow R_{\text{opt}} \tag{3}$$

for any fixed k. Eq. (3) states that an optimised regulator maximises the gap (2) w.r.t. the regularisation scheme (RS) [10]. The condition is based only on properties of the flow (1), and not on the specific theory under investigation. To leading order in the derivative expansion, solutions to the condition (3) are independent on the specific theory. In general, solutions R_{opt} to (3) are not unique and depend on the class of regulators chosen for the optimisation. Still, it is worthwhile noticing that (3) is a rather mild condition: it fixes only one out of countable infinitely many parameters describing a regulator R_k [10].

We stress that the present considerations are based on the structure of ERG flows of the form (1). Similar considerations can be applied to other exact RG flows based upon momentum shell integrations, like Wilsonian flows, the Polchinski RG, Wegner-Houghton flows, Hamiltonian flows or generalised proper-time flows [17]. In contrast, an implementation is less transparent for RG flows based upon reparametrisation invariance.

As an example, we consider a scalar theory to leading order in the derivative expansion, using a standard kinetic term. Higher order corrections can be treated as well. Then, a simple solution to the optimisation condition (3) is given by [11]

$$R_{\rm opt}(q^2) = (k^2 - q^2)\theta(k^2 - q^2)\,. \tag{4}$$

For momenta $q^2 > k^2$, it leads to

$$\Gamma^{(2)}_k[\phi] + R_{\rm opt}(q^2) = q^2 + U''_k(\phi) \tag{5}$$

Eq. (5) states that the regularisation is absent for large momenta. For $q^2 < k^2$, we find

$$\Gamma^{(2)}_k[\phi] + R_{\rm opt}(q^2) = k^2 + U''_k(\phi)\,. \tag{6}$$

In this region, the inverse propagator (6) is “flat”, *i.e.* independent of momenta. Hence, all IR modes are treated equally. The regulator (4) has a number of interesting properties [11]. It leads to the fastest decoupling of heavy modes, it disentangles the contribution of quantum and thermal fluctuations along the flow, it leads to a factorisation of a homogeneous wave function renormalisation, it leads to a smooth approach to convexity for a theory in the phase with spontaneous symmetry breaking, and it improves the convergence of the derivative expansion [13]. The link to a minimum sensitivity condition has been established as well [12]. Finally, the choice (4) is also useful from a technical point of view, because it leads to a simple analytic flow.

More generally, (most of) these properties hold as well for other optimised flows, different from (4), as long as (6) holds approximately in the momentum region where the flow receives its dominant contributions. We restricted the discussion to bosonic fields. Extensions to fermions and gauge fields have been considered as well [11].

4 Stability and convergence

In the remaining part, we apply our reasoning for $O(N)$-symmetric real scalar theories at the Wilson-Fisher fixed point in $d = 3$ Euclidean dimensions. The universality class is characterised by the critical exponent $\nu_{\rm phys}$, given by the inverse of the negative eigenvalue of the stability matrix at criticality, and $\eta_{\rm phys}$, the anomalous dimension. It is known from experiment that $\eta_{\rm phys}$ is at most of the order of a few percent. Hence, it is believed that the derivative expansion is a good approximation for a reliable computation of universal critical exponents. Within the derivative expansion, the physical critical exponents at the scaling solution are computed as the series

$$\nu_{\rm phys} = \nu_0({\rm RS}) + \nu_1({\rm RS}) + \nu_2({\rm RS}) + \cdots \tag{7}$$

Here, the index corresponds to the order of the derivative expansion. The anomalous dimension η vanishes to leading order. Notice that every single order in the expansion — due to the approximations employed — depends on the regularisation scheme. The independence of physical observables on the regulator scheme (RS) can only be guaranteed in the limit where *all* operators of the effective action are retained during the flow. In turn, the physical value $\nu_{\rm phys}$ is independent of the precise form of the infrared regulator. Hence, the infinite sum on the right-hand side adds up in a way such that the physical values are scheme independent. The convergence of the expansion (7) is best if a regulator is found such that the main physical information is contained in a few leading order terms.

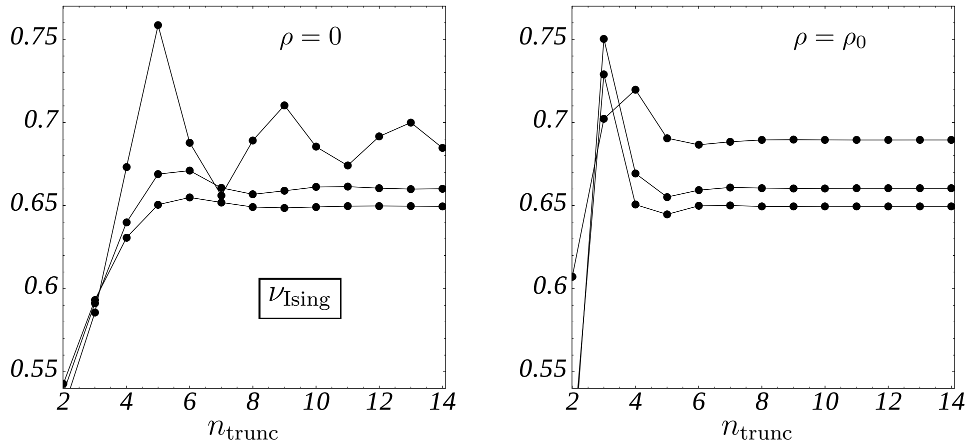

Fig. 2. The critical index ν for the Ising universality class. Results are given for an expansion about $\rho = 0$ (left panel) and $\rho = \rho_0$ (right panel), and for the sharp cutoff (upper curves) the quartic regulator (middle curves) and the optimised regulator (lower curves). The stability is largely improved by replacing the non-optimised sharp cutoff by optimised ones.

In Fig. 2, we have computed ν for the Ising universality class within a polynomial approximation for the scaling potential, using a sharp cutoff $R_{\rm sharp}$ (upper curves), the quartic regulator $R_{\rm quart} = k^4/q^2$ (middle curves), and $R_{\rm opt}$ (lower curves) [13, 14]. Both $R_{\rm quart}$ and $R_{\rm opt}$ are optimised regulators [solutions to (3)], while $R_{\rm sharp}$ is not. For the left panel, we have expanded the scaling potential in polynomials of $\rho \equiv \phi^2/k$ around vanishing field up to order $n_{\rm trunc}$. For the right panel, the expansion has been performed around the local minimum $\rho = \rho_0$. A few lessons can be learnt from Fig. 2:

First, it is seen that the convergence and stability of the sharp cutoff flow is poor. The expansion depends strongly on the expansion point. For an expansion about vanishing fields, it does not even converge beyond a certain accuracy [18, 19]. In contrast, the polynomial approximation converges rapidly for both $R_{\rm quart}$ and $R_{\rm opt}$. Also, the convergence depends only weakly on the expansion point. This picture holds true for any N. These findings confirm that optimised flows are more stable. In this light, the non-convergence of the sharp cut-off flow within an expansion about vanishing field is considered as a deficiency of the sharp cut-off regularisation, and not of the expansion.

Second, we notice that the numerical values for the critical exponent ν depends on the regulator. In particular, the values obtained from optimised flows are closer to the physical value. This holds true for all $N \geq 0$ [13, 14]. Based on an investigation of a large class of regulator functions, it has even been argued that the value $\nu_{\rm opt}$ as obtained from (4) corresponds to a minimum [14],

$$\nu_{{\rm large}-N} \geq \nu_{\rm ERG} \geq \nu_{\rm opt} > \nu_{\rm phys} \,. \tag{8}$$

Here, the upper bound denotes the large-N limit, for all N. The result (8) shows that the regulator (4) correponds to a solution of a minimum sensitivity condition [12].

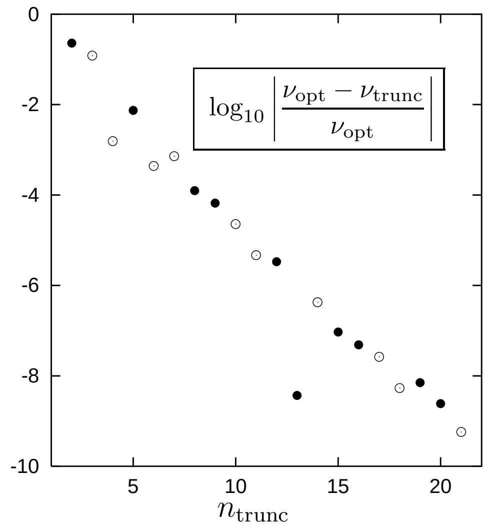

Fig. 3. Ising universality class. Convergence of $\nu_{\rm trunc}$ towards $\nu_{\rm opt}$ with increasing truncation. Points where $\nu_{\rm trunc}$ is larger (smaller) than $\nu_{\rm opt}$ are denoted by o (●).

Third, it is interesting to note that $\nu_{\rm opt}$ agrees to all published digits with the results obtained from the Polchinski RG [3, 20]. This is remarkable insofar as the two flows, ERG and Polchinski RG, are related by a Legendre transform and appropriate field rescalings. Hence, their derivative expansions are not equivalent. Also, to leading order, the result from a Polchinski flow is scheme independent, in marked contrast to what has been found within the ERG. The agreement between the Polchinski RG result and the ERG result based on (4) suggests that the optimisation has removed a redundant scheme dependence from the ERG flow.

Next, we emphasize that the numerical convergence of $\nu_{\rm trunc}$ from the optimised flow towards $\nu_{\rm opt}$ is very fast (Fig. 3): typically, increasing $n_{\rm trunc}$ by $2-2.5$ increases the numerical accuracy by one decimal point. Given that the accuracy of ν cannot be better than a few percent (contributions $\sim \eta$ are suppressed to leading order in the derivative expansion), it suffices to retain $\nu_{\rm trunc} = 4(6)$ independent couplings in the Ansatz for the effective potential, in order to achieve an accuracy for $\nu_{\rm trunc}$ below 1% (0.1%). This efficiency is remarkable.

Finally, we discuss in Fig. 4 the relative improvement due to an optimised regulator for all N [13]. For comparison, we again took the sharp and the quartic cutoff. It is interesting to note that both the large-N limit and the case $N=-2$ lead to universal leading-order results for ν [21, 22]. For intermediate values, the results deviate significantly from the best one, up to nearly $9-10\%$ for the sharp cutoff, and $2-3\%$ for the quartic one. An improvement by up to 10% is very important, given that the physical value lies a few percent below the values found for $\nu_{\rm opt}$ (for all N of physical relevance). Hence, for flows based on the sharp cutoff or similar regulators, one expects that a higher order in the derivative expansion is required to achieve the same accuracy in comparison to optimised flows. For a more detailed discussion of

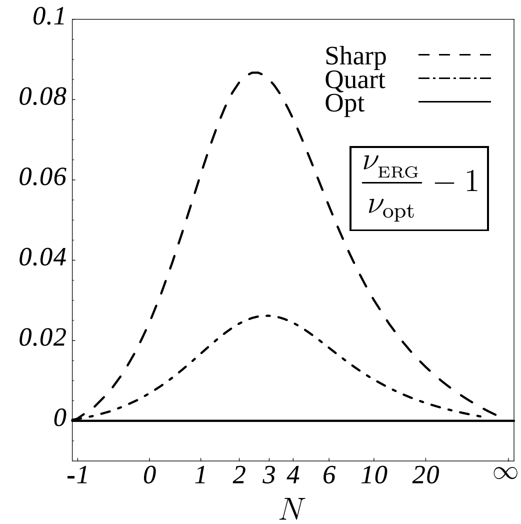

Fig. 4. The relative improvement of $\nu_{\rm ERG}$ in comparison with $\nu_{\rm opt}$ (various regulators).

the link between the convergence of the derivative expansion, and the optimisation, we refer to the discussion in [13].

5 Conclusions

Within the framework of the ERG, we have studied the link between stability and convergence properties of ERG flows, and their dependence on the regularisation. This understanding is a prerequisite for reliable applications of the formalism. In this context, the exactness of the flow (1) plays an important role. These considerations have lead to a simple optimisation condition for ERG flows. When applied to scalar theories at criticality in $3d$, we have shown explicitly that the optimisation leads to improved results already to leading order in the derivative expansion. This understanding of the spurious scheme dependence has reduced the ambiguity in ν to a small range about $\nu_{\rm opt}$.

Some of our results are based on the particularly simple choice (4) for the regulator. However, many more optimised regulators are available, and other choices may even be more appropriate depending on the order of the truncation, or on the physical problem under investigation. This can be seen already from the present results. To leading order in the derivative expansion, and as a function of the regularisation, the critical index ν is very flat [14]. Higher order corrections $\sim \eta$ are subleading. However, it is expected that the (nearly) flat region for ν is resolved to higher order in the derivative expansion, once $\eta \neq 0$.

Based on the understanding achieved so far, we expect that optimised flows should be useful for applications to higher in the derivative expansion, or for applications to quantum gravity [16], to more complex scalar theories [23], to fermionic ones [24], or to non-Abelian gauge theories [25, 26, 27]. In all these cases, an implementation of the ERG is technically much more demanding, and approximations are often bound to lower orders as compared to (simpler) scalar theories. Therefore, it may be most helpful to apply optimised flows and to achieve improved results already to lower orders in the truncation.

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