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NONLOCAL SYNCHRONIZATION IN NEAREST NEIGHBOUR COUPLED OSCILLATORS

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Abstract

We investigate a system of nearest neighbour coupled oscillators. We show that the nonlocal frequency synchronization, that might appear in such a system, occurs as a consequence of the nearest neighbour coupling. The power spectra of nonadjacent oscillators shows that there is no complete coincidence between all frequency peaks of the oscillators in the nonlocal cluster, while the peaks for neighbouring oscillators approximately coincide even if they are not yet in a cluster. It is shown that nonadjacent oscillators closer in frequencies, share slow modes with their adjacent oscillators which are neighbours in space. It is also shown that when a direct coupling between non-neighbours oscillators is introduced explicitly, the peaks of the spectra of the frequencies of those non-neighbours coincide.

I. INTRODUCTION

The study of the dynamics of systems consisting of a large number of mutually interacting units is attracting a lot of interest. Such a study is relevant in many fields, ranging from Physics to Chemistry to Biology. One of the features of a population of interacting oscillators is that the oscillators can synchronize to a common frequency [Kuramoto, 1984]. This self synchronization can be found in laser arrays, Josephson junction arrays as well as in chemical and biological systems [Winfree, 1990; Perez *et. al.*, 1992; Dominguez *et. al.*, 1993; Pikovsky *et. al.*, 1997]. Many investigations of the dynamical behavior of oscillators have been made in the context of coupled chaotic systems [Wiessenfeld *et. al.*, 1996; Lorenzo *et. al.*, 1996; Otsuka, 2000]. There are still many interesting features that remain unexplained.

Recently Zheng *et. al.* [1998, 2000] studied a ring of N coupled oscillators with nearest neighbour interaction. They found that the oscillators are led to complete synchronization through a sequence of partially synchronized elements (synchronization tree) as a function of the coupling strength, k. That means elements construct clusters with the same long time-average frequency which join for a larger value of k, to form larger clusters, until all elements synchronize to a common frequency (a cluster means two or more oscillators of common time-averaged frequency). One of the observed branches is formed by nonadjacent oscillators which have close values of frequencies. These nonadjacent oscillators synchronize while the oscillators spatially located in between them have considerably different frequencies; i.e, a nonlocal cluster can be formed. The nonlocal cluster brings the oscillators that lie spatially in between its members to a common synchronization state and thus form a larger cluster [Zheng *et. al.*, 2000; Zhan *et. al.*, 2000]. This kind of transitions seems to be unusual and contrary to the general intuitive thought that the oscillators should synchronize with their nearest neighbours, while the system is looking for the path to clustering. Thus, increasing the coupling between oscillators, it is expected that the formation of clusters of oscillators should occur according to the rule "closer in frequency and closer in space meet first". We believe that nonlocal synchronization is a very interesting effect that needs a deeper study. Is it a coincidence due to the chaotic motion of the oscillators that gives rise to some temporary nonlocal interaction or can it be explained as a consequence of the nearest-neighbour interaction? In this paper we investigate the features that lead to synchronization in a system of nearest neighbour coupled oscillators and try to understand whether such kind of nonlocal synchronization (or clusters) are robust or they happen accidentally. We shall investigate the possibility of having a direct coupling between non-neighbouring oscillators, which, in particular, should produce the effect of nonlocal synchronization.

This paper is organized as follows: in section II we present the model with nearest neighbour coupling and investigate the temporal behavior of the frequencies $\dot{\theta}_i$ for a given value of the coupling strength k in order to explore the occurrence of the synchronization due to nearest neighbour interactions. In section III, we study in detail the case of a nonlocal coupling between two oscillators among 15 oscillators specifying explicitly a direct interaction between nonadjacent elements. In section IV our conclusions are presented.

II. THE MODEL: NEAREST NEIGHBOUR COUPLING

Consider a set of N oscillators interacting via nearest neighbour coupling and with natural frequencies ω_i taken from a Gaussian distribution, such that:

$$\dot{\theta}_i = \omega_i + \frac{k}{3}[sin(\theta_{i+1} - \theta_i) + sin(\theta_{i-1} - \theta_i)] \tag{1}$$

where $i = 1, 2, .., N$, k is the coupling strength, θ_i is the instantaneous phase and $\dot{\theta}_i$ is the velocity. We use periodic boundary conditions $\theta_{i+N}(t) = \theta_i(t)$, and the constraint on ω_i, such that

$$\frac{1}{N}\sum_{i=1}^{N} \omega_i = 0. \tag{2}$$

System (1) exhibits synchronization and there exists a critical coupling k_c such that, for $k > k_c$ all oscillators are synchronized to one another and $\dot{\theta} = 0$ for all oscillators. When $k < k_c$, $\dot{\theta}_i(t)$ is nonzero and time dependent. It is found that synchronization occurs in the sense that the average frequency of the i^{th} oscillator, which is defined as

$$\overline{\omega_i} = \lim_{T\to\infty} \frac{1}{T}\int_0^T \dot{\theta}_i(t)dt, \tag{3}$$

is equal to $\overline{\omega}_j$ for $i \neq j$. Equation (2) in this case limits the synchronization to a zero frequency. However, when the previous constraint is not used, the general feature of the system will not be changed and the synchronization occurs to a common frequency value

$$\Omega = \frac{1}{N}\sum_{i=1}^{N} \omega_i. \tag{4}$$

Figure 1 shows the plot of the time-averaged frequency $< \dot{\theta}_i >$ for all oscillators versus the coupling strength k and for $N = 15$ oscillators. Since the interaction is between nearest neighbours (numbers of the oscillators are indicated in Fig. 1) it is expected that nearest neighbours will synchronize first. However, there are exceptions (see Fig. 1) and non-neighbouring oscillators, such as 5 and 8, form a cluster, which is later joined by oscillators 2 and 3, i.e.; nonlocal clusters are formed. It seems that these nonlocal clusters are responsible for the synchronization of the oscillators that lie in the space between them and form a large cluster of the same time-averaged frequency. The inset of Fig. 1 shows the case mentioned previously and the values of neighbouring frequencies are stated. In order to investigate the nonlocal behavior, Zheng *et. al.* [1998, 2000] explored the phase relation between the oscillators that form the nonlocal cluster in order to explain the nonlocal synchronization. They found a phase locking between the oscillators that form the nonlocal cluster. This effect of phase locking between non neighbouring oscillators has been termed 'nonlocal synchronization' [Zheng *et. al.*, 2000]. It has been assumed that the nonlocal cluster attracts the oscillators that lie between its constituents to synchronize with the elements in the cluster; i.e, 4, 6 and 7 join the cluster of 2,3,5 and 8 as shown in Fig 1. Since, the trajectory of $< \dot{\theta}_i >$ of the i^{th} oscillator for all values of k is controlled by the phase coupling between such oscillator and its nearest neighbours in addition to the value of the initial frequency (see (1)), we still believe that the explanation of the synchronization between nonadjacent oscillators has to be a consequence of the nearest neighbours interactions. The argument of phase locking may not be enough to explain the presence of the nonlocal synchronization. Therefore, in order to understand the behavior of the $< \dot{\theta}_i >$ of the i^{th} oscillator and explain how its path along k is determined, there should be detailed investigations for the $\dot{\theta}_i(t)$ of such oscillator at a particular value of k, where the nonlocal behavior seems to appear.

According to system (1), the frequency of each oscillator $\dot{\theta}_i$ depends on the value of the natural frequency ω_i and a phase coupling composed of two subterms, (C_{+i} and C_{-i}), of a periodic function of the phase difference between θ_i and, $\theta_{(i+1)}$ and $\theta_{(i-1)}$, respectively, and linearly dependent on the coupling strength. Increasing the coupling strength we observe (see Fig. 1) that oscillators synchronize and finally they reach a state of complete synchronization. In order to reach a frequency synchronized state of any two oscillators, we notice that if ω_i of a given oscillator is larger than ω_j, on the time average the coupling term reduces the natural frequency of the i^{th} oscillator and increases that of the j^{th} oscillator until $\overline{\omega}_i$ matches $\overline{\omega}_j$. For any two oscillators, the average frequencies are written as:

$$\overline{\omega_i} = \omega_i + \frac{k}{3}[\overline{sin(\theta_{i+1} - \theta_i)} + \overline{sin(\theta_{i-1} - \theta_i)}] \tag{5}$$

and

$$\overline{\omega_j} = \omega_j + \frac{k}{3}[\overline{sin(\theta_{j+1} - \theta_j)} + \overline{sin(\theta_{j-1} - \theta_j)}]. \tag{6}$$

Intuition tells us that, in general, oscillators closer in space and/or in frequency, should synchronize. However, this expectation may be violated, see for instance the example in Fig. 1. We take one example from these unusual cases, $\overline{\omega_5}$ and $\overline{\omega_8}$, where the oscillators are not nearest neighbours neither are close in frequency, and look at their behaviour for values of k for which they are synchronized, in order to see if nonlocal synchronization is accidental or it is a robust consequence of nearest neighbour coupling. In order to resolve the situation and reach a decision we do a detailed study of the spectra of each oscillator that show this effect and investigate also the behavior of their neighbours. Then we can decide whether our intuition holds true or not.

We select a value of k, at which it seems that $\overline{\omega_5}$ and $\overline{\omega_8}$ are equal, and we plot the temporal behavior of the coupling terms of the mentioned oscillators, this is shown in Figs. 2 and 3 at $k = 1$.

The value of $\overline{\omega_5}$ is determined by the addition of $\overline{C_{+5}}$ to $\overline{C_{-5}}$ and then adding both to ω_5. The same is done for $\overline{\omega_8}$, which receives contribution from $\overline{C_{+8}}$ and $\overline{C_{-8}}$ plus ω_8. We observe from Fig. 2(a) and 3(b), that C_{+5} and C_{-8} have the same features as a function of time but reverse in sign, with equal magnitude. These two terms come from the coupling of oscillator 5 to 6 and 8 to 7, respectively. With the aid of figures 1, 2 and 3, and equations (5) and (6), it is seen that oscillator 5 couples to 4 and 6, and also that 6 is closer in frequency to 5 than to 4, which is further away and in the same direction of 6 (Fig. 1). Oscillator 8 couples to 7 and 9 and it is closer in frequency to 7 than to 9 (Fig. 1). At the same time oscillators 5 and 8 are embedded in frequency between 6 and 7, which are neighbours (Fig. 1). The situation is that 5 is pushed towards 6 since it is weakly affected by 4, 6 moves towards 5 and 7, 7 towards 6 and 8, and 8 couples to 7 where it is weakly affected by 9, so that the paths of the average frequencies of 5 and 8 intersect. This can be attributed to 6 and 7 that are strongly coupled to each other (neighbours in space and relatively close in frequency). The coupling terms C_{+5} and C_{-8}, which result from the coupling of 5 to 6 and the coupling of 7 to 8, are the most effective terms and exist the strongest influence on the frequencies of 5 and 8 and have the same magnitude, while being opposite in sign. The effects of C_{-5} and C_{+8} on the frequencies of 5 and 8 are negligible due to the weak coupling of these two oscillators to 4 and 9, respectively. Therefore, the effect of the terms $\overline{C_{+5}}$ and $\overline{C_{-8}}$ on both time-averaged frequencies of 5 and 8 is dominant. As the coupling strength increases, the above statements hold true and the tracks of the average frequencies of 5 and 8 are determined by the coupling to the nearest neighbours.

In order to be sure of the explanation given above, and that the coincidence of the average frequencies of 5 and 8 may appear accidentally or results due to nearest neighbour coupling, we change the value of ω_4 among all the oscillators, from 1.44 to 0.55, that is, it comes closer to the four oscillators under consideration but it remains larger in frequency than oscillator 6. This can be seen in Fig. 4(a-b). We notice that this affects the range of k where oscillators 5 and 8 remain clustered, to the extent that they just cross each other when the frequency of oscillator 4 is very near that of oscillator 6. It is also clear from the figure that the oscillators are very sensitive to the coupling to the nearest neighbours. This argument now supports the appearance of nonlocal synchronization as a consequence of nearest neighbours interaction.

A clear justification can be reached from the spectra of each frequency $\dot{\theta}_i$, $i = 5, 6, 7, 8$. Figure 5 shows the Fourier transform of the quantity $\dot{\theta}_i(t)$ for the above mentioned four oscillators. It is clearly seen that the oscillators which are nearest neighbours show approximately complete coincidence of peaks which reflects the direct coupling of neighbours. If we look at any two non-neighbours, there is no complete coincidence of peaks, especially if we look at the spectra of 5 and 8 where no complete coincidence between peaks appears. However, it seems that the four oscillators share the same slow modes. These peaks of slow modes are similar in magnitude and positions also for 6 and 7. We attribute the nonlocal synchronization between 5 and 8 in Fig. 1 to this low modes which appear from the coupling to nearest neighbours. Even before oscillators 6 and 7 cluster with oscillators 5 and 8, there is a single slow mode that is shared between all of them, which may be equivalent to the effect mentioned by Zhan *et. al.* [2000] for Rössler oscillators, although the case here is different, since even the meaning of the phase is different and thus difficult to compare. Therefore, the situation now is that oscillator 5 shifts towards 6 and 7 is pushed by both 6 and 8 while 5 and 8 are located in between 6 and 7 and the matching between 5 and 8 comes due to nearest neighbours interaction while each oscillator is looking for a path to cluster as k varies.

The same investigation is repeated for the nonlocal synchronization of clusters (2,3) and (5,8). It should be noted that cluster (2,3) and cluster (5,8) have a common neighbour oscillator between them, which is 4. Therefore, 4 is attracted towards both 3 and 5 (see Fig. 1). If the initial frequency value of 4 is changed, then the path of the average frequency of oscillators 2 and 3 will be affected due to the coupling to 4. Fig. 1 shows that 4 goes to join the cluster (2,3) and the cluster (5,6) at the same value of k and a large cluster (2-6) is formed. However, when oscillator 4 has a value closer to its neighbours (see Fig. 4(a-b)), no intersection between the branches (2,3) and (5,8) occurs. The formation of clusters as a function of k is (2,3), (5,6), then (2-6) and finally (2-8) (see Fig. 4). It can also be seen from Fig. (4) that the final state of synchronization occurs at a frequency equal to Ω as indicated from (4), because the constraint (2) is not satisfied in this case. From the above we conclude that nonlocal frequency synchronization is not an unusual effect, but it is a rather common and robust consequence of nearest neighbour interaction, while the system is looking for the path to cluster according to the a rule *"closer in space and closer in frequency meet first"*.

III. NONLOCAL COUPLING

In this section we discuss the direct insertion of an extra nonlocal coupling between two among the 15 oscillators. We would like to investigate the changes in the nonlocal synchronization when a direct interaction between non-neighbouring oscillators is specified. Then the path of the average frequency, as the coupling strength increases, will depend both on the coupling to the nearest neighbours and on the nonlocal coupling. This path will not be affected if we change the values of the initial frequencies of the neighbouring elements; i.e the elements that form a nonlocal cluster due to nonlocal coupling should not be sensitive to the change of initial frequencies of their neighbours, but those which are in a nonlocal cluster will be sensitive to the change of the values of any of the nearest neighbour frequencies. We choose oscillators 5 and 8 and couple them such that their equations of motion become:

$$\dot{\theta}_5 = \omega_5 + \frac{k}{4}[sin(\theta_6 - \theta_5) + sin(\theta_4 - \theta_5) + sin(\theta_8 - \theta_5)] \quad (7)$$

and

$$\dot{\theta}_8 = \omega_8 + \frac{k}{4}[sin(\theta_9 - \theta_8) + sin(\theta_7 - \theta_8) + sin(\theta_5 - \theta_8)]. \quad (8)$$

The last subterms of the coupling terms of the above equations act on both oscillators with the same magnitude and opposite in signs. This is in addition to the interaction of each oscillator with its nearest neighbours. Figure 6 shows the simulation for the case described above. It is clearly seen that the synchronization takes place between 5 and 8 for a smaller value of k in comparison to Fig. 1. Now we are sure that a nonlocal cluster is formed. However, it is also seen that oscillators 2 and 3 constitute a cluster and synchronize to a common time-averaged frequency and follow the same path of 5 and 8 where it seems that a nonlocal cluster occurs. In order to get the above argument into considerations, we repeat the same simulation but with the initial frequency of oscillator 4 equal to 0.55. It is seen from Fig. 6 that oscillators 5 and 8 remain clustered (compare this figure with Fig. 4) while oscillators 2 and 3 are separated from 5 and 8 due to an effect which comes from oscillator 4 acting on 3. As revealed from Fig. 6(b) oscillator 4 joins the cluster of (2,3) and (5-8) at the same value of k. Thus, the nonlocality appears from direct interactions between nonadjacent elements while the appearance of what seems to be nonlocal is due to nearest neighbour coupling. Figure 7 shows the Fourier transform of the quantities $\dot{\theta}_i(t)$ for oscillators 5, 6, 7 and 8 at $k = 1$. It is clearly seen from this figure that all oscillators share the locations of the peaks whatever their magnitudes. This figure reveals the fact that, because of the nonlocal coupling between 5 and 8, there is a coincidence in the frequency spectra of these two oscillators and then the nonlocal synchronization occurs.

IV. CONCLUSION

In conclusion, we have studied the synchronization of nearest neighbour coupled oscillators as a function of the coupling strength. In particular we focused on the nonlocal synchronization that is observed through the investigations of the synchronization tree towards complete synchronization. It is shown that the nonlocal clustering is a clear consequence of the interaction between neighbouring oscillators: the nonlocal clustering does not happen by chance. The adjacent oscillators closer in frequency and closer in space are those that should synchronize first. On the other hand if there are neighbours to these adjacent oscillators which are closer in frequency and share slow modes with these adjacents, as k increases, these oscillators become closer to each other and are the first to synchronize. We also show that the nonlocal cluster in this case is sensitive to the change in the initial frequencies values of the other neighbours which are not closer in frequencies. In order to be sure of the above explanation, we introduced explicitly a direct interaction between nonadjacent oscillators. We investigated the effect of this nonlocal interaction and see that the nonlocal cluster is formed for smaller value of the coupling constant k and at the same time the nonadjacent oscillators share all frequency modes. In this case the nonlocal behaviour is clear and comes directly from the interaction between nonadjacent and it is not due to the consequence of the nearest neighbour interactions. At the same time, the nonlocal cluster is not sensitive to the change in the initial frequencies values of its neighbours.

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FIGURE CAPTIONS

Figure 1:

The average frequency versus k for 15 oscillators. The inset shows in a magnified view the cluster of elements $2, 3, 5, 6, 7$ and 8.

Figure 2:

Temporal behavior of the coupling terms and frequency of oscillator 5 at $k = 1$ where $\omega_5 = -0.47$. The average values are indicated on each graph by the dashed lines.

Figure 3:

Temporal behavior of the coupling terms and frequency of oscillator 8 at $k = 1$ where $\omega_8 = -0.03$. The average values are indicated on each graph by the dashed lines.

Figure 4:

Average frequency versus k for 15 oscillators when the value of the initial frequency of oscillator 4 is shifted to be relatively close to its neighbours.

Figure 5:

The spectra of the four oscillators $5, 6, 7$ and 8 at $k = 1$ starting from the slow mode in the left to the fast mode when goes to the right. The coupling is defined according to equation (1). The ranges of x and y coordinates are the same for all oscillators in order to compare.

Figure 6:

a) Average frequency versus k for 15 oscillators when nonlocal coupling occurs between oscillators 5 and 8.

b)Average frequency versus k when the initial frequency of 4 is shifted closer to its neighbours.

Figure 7:

The spectra of the four oscillators $5, 6, 7$ and 8 at $k = 1$ starting from the slow mode in the left to the fast mode when goes to the right. The coupling is defined according to equations (7) and (8). The ranges of x and y coordinates are the same for all oscillators in order to compare.