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**A GENERALIZED JAYNES-CUMMINGS MODEL:  
NONLINEAR DYNAMICAL SUPERALGEBRA  $u(1/1)$   
AND SUPERCOHERENT STATES**

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**Abstract**

The generalization of the Jaynes-Cummings (GJC) Model is proposed. In this model, the electromagnetic radiation is described by a Hamiltonian generalizing the harmonic oscillator to take into account some nonlinear effects which can occur in the experimental situations. The dynamical superalgebra and supercoherent states of the related model are explicitly constructed. Relevant quantities (total number of particles, energy and atomic inversion) are computed.

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# 1 Introduction

The Jaynes-Cummings (JC) model [1] which is extensively used in quantum optics describes, in its simplest version, the interaction of a cavity mode with two-level system

$$H_{JC} = \omega(a^+ a^- + \frac{1}{2})\sigma_0 + \frac{\omega_0}{2}\sigma_3 + \kappa(a^+ \sigma_- + a^- \sigma_+), \quad (1)$$

where  $a^+$  and  $a^-$  are the photon creation and annihilation operators,  $\sigma_{\pm} = \frac{1}{2}(\sigma_1 \pm \sigma_2)$ , with  $\sigma_1$ ,  $\sigma_2$  and  $\sigma_3$  are the Pauli matrices and  $\sigma_0$  is the identity matrix. Moreover,  $\kappa$  is a coupling constant,  $\omega$  is the radiatif field [1] mode frequency and  $\omega_0$  the atomic frequency. The interest of this model, its solvability and its applications, has long been discussed [1, 2, 3]. Over the last two decades, there has been intensive study [4 and references quoted therein] on the solvable Jaynes-Cummings model and its various extensions, such as intensity depending coupling constants, two photons or multiphoton transitions and two or three cavity modes for three-level atoms. These models have found their applications in laser trapping and cooling atoms [5] and quantum non-demolition measurements [6]. Furthermore, the Jaynes-Cummings model constitutes now the basis for a vast array of the current experiments on foundations of quantum mechanics involving entangled states [7, and references therein]. On the other hand, the supergroup theoretical approach to Jaynes-Cummings model has opened the way to relate the exact solvability of this model and representation theory of superalgebras. Indeed the Hamiltonian  $H_{JC}$  is an element of the  $u(1/1)$  superalgebra [8]. In the absence of coupling ( $\kappa = 0$ ) and for exact resonance ( $\omega = \omega_0$ ), the  $u(1/1)$  dynamical superalgebra reduces to a superalgebra  $sl(1/1)$  and the JC model coincides with the supersymmetric harmonic oscillator. More recently, the investigations of a class of shape-invariant bound state problem, which represents two-level system, leads to the generalized Jaynes-Cummings model [9, 10, 11]. In the case of the simplest shape-invariant system, namely the harmonic oscillator, the generalized Jaynes-Cummings model reduces to standard one.

In this paper we shall address the generalization and quantum characteristics of the Jaynes-Cummings model. Besides the eigenvalues and eigenvectors, we give the supercoherent states of the related model. It is found that the generalized Jaynes-Cummings model is governed by a nonlinear superalgebra  $u(1/1)$  which reduces to the well-known  $u(1/1)$  occurring in the standard (JC) model [12, 13]. We compute the total number of photons and the energy. We find that the atomic inversion exhibits Rabi oscillations.

The paper is organized as follows: In Section 2, we introduce the generalized supersymmetric quantum oscillator and we construct the corresponding supercoherent states. Exact spectrum of the generalized Jaynes-Cummings model is given in section 3. Section 4 is devoted to nonlinear dynamical superalgebra  $u(1/1)$  of the (GJC) model which is useful to construct the supercoherent states adapted in our model (Section 5). Using the latter set of super-states, we compute in

section 6 some relevant physical quantities. The last section concerns the conclusion of this work.

## 2 Generalized Supersymmetric Quantum Oscillators

We begin by introducing the generalized supersymmetric quantum oscillators. Let us consider a Hamiltonian  $H$  with a discrete spectrum which is bound below and has been adjusted so that  $H \geq 0$ . We assume that the eigenstates of  $H$  are non-degenerate. The eigenstates  $|\Psi_n\rangle$  of  $H$  are orthonormal vectors and they satisfy

$$H|\Psi_n\rangle = e_n|\Psi_n\rangle. \quad (2)$$

In a general setting, we also assume that the energies  $e_0, e_1, e_2, \dots$  are positive and verify  $e_{n+1} > e_n$ . The ground state energy is  $e_0 = 0$ . We define the creation and the annihilation operators  $A^+$  and  $A^-$ , respectively, such that the Hamiltonian can be factorized as

$$H = A^+ A^-. \quad (3)$$

The action of the operators  $A^+$  and  $A^-$  on the states  $|\Psi_n\rangle$  are given by

$$\begin{aligned} A^+|\Psi_n\rangle &= (e_{n+1})^{\frac{1}{2}}|\Psi_{n+1}\rangle \\ A^-|\Psi_n\rangle &= (e_n)^{\frac{1}{2}}|\Psi_{n-1}\rangle \end{aligned} \quad (4)$$

implemented by the action of  $A^-$  on the ground state  $|\Psi_0\rangle$

$$A^-|\Psi_0\rangle = 0. \quad (5)$$

The commutator of  $A^+$  and  $A^-$  is defined by

$$[A^-, A^+] = G(N), \quad (6)$$

where the operator  $G(N)$  is defined through this action on  $|\Psi_n\rangle$

$$G(N)|\Psi_n\rangle = (e_{n+1} - e_n)|\Psi_n\rangle. \quad (7)$$

We define the number operator  $N$  as

$$N|\Psi_n\rangle = n|\Psi_n\rangle, \quad (8)$$

$N$  is in general different from the product  $A^+ A^- (=H)$ . We can see that the number operator satisfies

$$\begin{aligned} [A^+, N] &= A^+ \\ [A^-, N] &= -A^-. \end{aligned} \quad (9)$$

Here, we consider two generalized oscillators systems which have been extensively studied in the literature. The first one concerns the so-called generalized deformed oscillator [14] and the second one is the  $x^4$ -anharmonic oscillator [15]. The physical interests of these two systems

have been extensively enumerated [see the references quoted in [14, 15]]. Here, we recall their eigenstates and eigenvalues to construct the supersymmetric generalized quantum oscillator and the corresponding supercoherent states.

To introduce the generalized deformed oscillator, the procedure of [14] requires the existence of a map from the usual harmonic oscillator algebra generated by annihilation and creation operators  $a^-$  and  $a^+$  satisfying the standard canonical commutation relations, to the new one generated by  $A^-$  and  $A^+$

$$A^- = a^- f(N) \quad A^+ = f(N) a^+ \quad (10)$$

$N$  being the number operator  $N = a^+ a^-$  and the function  $f$  is given by

$$f(N) = N + m \quad (11)$$

The Hamiltonian of the obtained generalized harmonic oscillator is then given by

$$H = A^+ A^- = N(N + m) \quad (12)$$

with eigenvalues

$$e_n = n(n + m) \quad (13)$$

The Fock states  $|\Psi_n\rangle \equiv |n, m\rangle$  are labelled by the integers  $m$  and  $n = 0, 1, 2, \dots$

It is clear that the operator  $G(N)$  (7), in this case is given by

$$G(N) = 2N + m + 1. \quad (14)$$

Then, the operators  $A^+$ ,  $A^-$  and  $G(N)$  satisfy the relations (6) and (9). Note that other choices of the function  $f$  are possible. We remark also that when  $f(N) = 1$ , we have the ordinary harmonic oscillator.

The other nonlinear oscillator that we consider is the  $x^4$ -anharmonic oscillator. The Hamiltonian, describing this system, is

$$H = a^+ a^- + \frac{\epsilon}{4} (a^- + a^+)^4 - \delta \quad (15)$$

where  $a^+$  and  $a^-$  are the creation and annihilation operators for the harmonic oscillator. The quantity  $\delta$  is given by

$$\delta = \frac{3}{4}\epsilon - \frac{21}{8}\epsilon^2 \quad (16)$$

which vanishes when  $\epsilon = 0$  and  $H$  reduces in this limit to the standard harmonic oscillator Hamiltonian. The Hamiltonian  $H$  can be factorized in the following form [15]

$$H = A^+ A^- \quad (17)$$

in terms of  $A^+$  and  $A^-$  which are expressed as some functions of  $a^+$  and  $a^-$  (for the expressions of these functions see[15]). The energy levels are given by

$$e_n = n + \frac{3}{2}\epsilon(n^2 + n), \quad n = 0, 1, 2, \dots \quad (18)$$

The positive parameter  $\epsilon$  can be seen as one taking into account some non linearity of the radiatif field arising from some perturbative effects occurring in experimental situations. Note that we keep terms only up to  $\epsilon$  which is the standard first-order perturbation result. The Hilbert space of this system is easily constructed in the same way as the standard harmonic oscillator. It is spanned by the states  $|n, \epsilon\rangle$ ,  $n = 0, 1, 2, \dots$  which is generated by the action of  $A^+$  on the ground state  $|0, \epsilon\rangle$ . The operator  $G(N)$  is

$$G(N) = 1 + 3\epsilon(n + 1) \quad (19)$$

Here, again one can verify that the relations (6) and (9) are satisfied by the creation and annihilation operators corresponding to the  $x^4$ -anharmonic system.

In supersymmetric quantum mechanics, one can consider the so-called supersymmetric Hamiltonian which is defined by

$$H_{susy} = \begin{pmatrix} A^- A^+ & 0 \\ 0 & A^+ A^- \end{pmatrix} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}, \quad (20)$$

where  $H_+ = A^- A^+$  and  $H_- = A^+ A^- = H$  are the so-called supersymmetric partner Hamiltonians.

Working in the Hilbert space

$$h = h_b \otimes h_f = \left\{ |\Psi_n, -\rangle = \begin{pmatrix} 0 \\ |\Psi_n\rangle \end{pmatrix}, |\Psi_n, +\rangle = \begin{pmatrix} |\Psi_n\rangle \\ 0 \end{pmatrix}; n = 0, 1, 2, \dots \right\}, \quad (21)$$

the eigenstates are

$$\begin{aligned} |\phi_0\rangle &= \begin{pmatrix} 0 \\ |\Psi_0\rangle \end{pmatrix}, \\ |\phi_{n>0}\rangle &= c_n^+ \begin{pmatrix} |\Psi_{n-1}\rangle \\ 0 \end{pmatrix} + c_n^- \begin{pmatrix} 0 \\ |\Psi_n\rangle \end{pmatrix}, \quad |c_n^+|^2 + |c_n^-|^2 = 1, \end{aligned} \quad (22)$$

with the energies  $E_0 = 0$  and  $E_{n>0} = e_n$ . Because, we are interested by generalized quantum oscillator, the states  $|\Psi_n\rangle$  are  $|n, m\rangle$  for the generalized deformed oscillator and  $|\Psi_n\rangle$  are  $|n, \epsilon\rangle$  for the  $x^4$ -anharmonic oscillator. As we have mentioned previously these two quantum systems will be used to extend the JC model and we will compute some relevant physical quantities, like the mean values of the total number operators, the energy and the atomic inversion, over the coherent states of the (GJC) model. The latter will be obtained from the supercoherent states corresponding the generalized supersymmetric oscillator. So for  $H_{susy}$ , we consider the supercoherent states (linear combination of the fundamental coherent states)

$$|z, \beta\rangle = \cos \frac{\theta}{2} \begin{pmatrix} 0 \\ |z\rangle \end{pmatrix} + \sin \frac{\theta}{2} e^{i\Phi} \begin{pmatrix} |z\rangle \\ 0 \end{pmatrix}, \quad (23)$$

with  $\beta = \frac{\theta}{2} e^{i\Phi}$  and  $|z\rangle$  are the coherent states, corresponding to  $H_-$ , defined by [16, 17, 18, 19]

$$|z\rangle = \aleph(|z|) \sum_{n=0}^{\infty} \frac{z^n}{(e(n))^{\frac{1}{2}}} |\Psi_n\rangle, \quad (24)$$

where  $e(n) = e_1 \dots e_n$  for  $n = 1, 2, \dots$  and we set  $e(0) = 1$ . The normalization constant  $\aleph(|z|)$  is calculated from the normalization condition  $\langle z|z \rangle = 1$  and it is given by

$$(\aleph(|z|))^{-2} = \sum_{n=0}^{\infty} \frac{|z|^{2n}}{e(n)}. \quad (25)$$

Let us mention that the coherent states for  $x^4$ -anharmonic oscillator has been studied in [19]. They are given by

$$|z\rangle = a(|z|) \sum_{n=0}^{\infty} \left( \frac{2^n}{(3\epsilon)^n \Gamma(n+1) \Gamma(n+2 + \frac{2}{3\epsilon})} \right)^{\frac{1}{2}} z^n |n, \epsilon\rangle, \quad (26)$$

with

$$a(|z|) = \frac{(\Gamma(2 + \frac{2}{3\epsilon}))^{\frac{1}{2}}}{({}_0F_1(2 + \frac{2}{3\epsilon}, \frac{2}{3\epsilon}|z|^2))^{\frac{1}{2}}}. \quad (27)$$

For the deformed generalized harmonic oscillator, we construct the coherent states in the same way that one Barut-Girardello coherent states of the  $su(1, 1)$  algebra was given [20]. Note that the algebra generated by  $\{A^+, A^-, G(N) = 2N + m + 1\}$  is isomorphic to  $su(1, 1) \sim sl(2, (\mathbf{R})) \sim so(2, 1)$ . Indeed, the creation  $A^+$  and annihilation  $A^-$  operators satisfy the following commutation relations

$$[A^-, A^+] = G(N), \quad [A^{\pm}, G(N)] = \mp A^{\pm}. \quad (28)$$

A more familiar basis for  $su(1, 1)$  algebra is given by

$$J_- = \frac{1}{\sqrt{2}}A^-, \quad J_+ = \frac{1}{\sqrt{2}}A^+, \quad J_{12} = \frac{1}{2}G(N), \quad (29)$$

with the following commutation relations

$$[J_-, J_+] = J_{12}, \quad [J_{\pm}, J_{12}] = \mp J_{\pm}. \quad (30)$$

Barut and Girardello introduced the  $su(1, 1)$  coherent states as eigenvectors of  $J_-$  [20]

$$\begin{aligned} J_-|z\rangle &= z|z\rangle \\ |z\rangle &= b(|z|) \sum_{n=0}^{\infty} \frac{(\sqrt{2}z)^n}{(n!\Gamma(n+m+1))^{\frac{1}{2}}} |n, m\rangle. \end{aligned} \quad (31)$$

The normalization constant is given by

$$b(|z|) = \frac{(\Gamma(m+1))^{\frac{1}{2}}}{({}_0F_1(m+1, 2|z|^2))^{\frac{1}{2}}} \quad (32)$$

where the  ${}_0F_1(m+1, 2|z|^2)$  is the hypergeometric function. The coherent states (26) (for the  $x^4$ -anharmonic oscillator) and (31) (for the generalized deformed oscillator) can be obtained simply from equation (24) by replacing in the expressions of  $e(n)$  the energies by their corresponding values for each considered system. Note that the coherent states (26) and (31) will be useful to build up ones of (GJC) model (see section 5). Remark also that the resolution to identity such states has not been discussed here. However, the measures with respect of the previous sets of coherent states can be computed in a very easy way following the approach developed in [13, 17].

### 3 Eigenstates and Eigenvalues

To generalize the JC model, let us consider the Hamiltonian

$$H_{GJC} = \frac{1}{2}\omega\{A^-, A^+\} + \frac{1}{2}\omega_0[A^-, A^+]\sigma_3 + \kappa(A^+\sigma_- + A^-\sigma_+). \quad (33)$$

This last expression generalizes the ordinary JC Hamiltonian. In fact, when the creation and the annihilation operators are those associated with the harmonic oscillator, the above Hamiltonian reduces to the well-known JC model (Eq.(1)). We also note that the Hamiltonian  $H_{GJC}$  is supersymmetric when  $\omega = \omega_0$  (exact resonance) and  $\kappa = 0$  (absence of coupling)

$$H_{GJC}(\kappa = 0, \omega = \omega_0) = \{Q^-, Q^+\}, \quad (34)$$

where the supercharge operators are defined by

$$Q^- = i\sqrt{\omega}a^+\sigma_-, \quad Q^+ = -i\sqrt{\omega}a^-\sigma_+, \quad (35)$$

and satisfy the relations

$$(Q^-)^2 = 0, \quad (Q^+)^2 = 0, \quad [Q^\pm, H] = 0. \quad (36)$$

We note that the generalized Hamiltonian of JC model (33) is different from the ones considered in the references [21, 22, 23]. The main purpose of this section is to show that the generalized JC model can be solved analytically.

The diagonalization of the Hamiltonian  $H_{GJC}$  is easily carried out (in the same way as the standard JC model) and leads to the eigenstates

$$\begin{aligned} |E_0^-\rangle &= |\Psi_0, -\rangle \\ |E_n^+\rangle &= \frac{1}{P(n+1)}[S(n+1)|\Psi_n, +\rangle - Q(n+1)|\Psi_{n+1}, -\rangle] \\ |E_{n+1}^-\rangle &= \frac{1}{P(n+1)}[Q(n+1)|\Psi_n, +\rangle + S(n+1)|\Psi_{n+1}, -\rangle], \end{aligned} \quad (37)$$

where

$$\begin{aligned} Q(n+1) &= \kappa(e_{n+1})^{\frac{1}{2}} \\ S(n+1) &= \kappa(e_{n+1} + \frac{\Delta_-}{2\kappa^2}(\frac{e_{n+2}-e_n}{2})^2)^{\frac{1}{2}} + \frac{\Delta_-}{2} \frac{e_{n+2}-e_n}{2}, \\ P(n) &= ((S(n))^2 + (Q(n))^2)^{\frac{1}{2}} \end{aligned} \quad (38)$$

with  $\Delta_- = \omega - \omega_0$ .

The corresponding eigenvalues are given by

$$\begin{aligned} E_n^+ &= \frac{\Delta_+}{2}e_{n+1} + \frac{\Delta_-}{4}(e_{n+2} + e_n) - (\kappa^2 e_{n+1} + \frac{\Delta_-^2}{4}(\frac{e_{n+2}-e_n}{2})^2)^{\frac{1}{2}} \\ E_{n+1}^- &= \frac{\Delta_+}{2}e_{n+1} + \frac{\Delta_-}{4}(e_{n+2} + e_n) + (\kappa^2 e_{n+1} + \frac{\Delta_-^2}{4}(\frac{e_{n+2}-e_n}{2})^2)^{\frac{1}{2}}, \end{aligned} \quad (39)$$

where  $\Delta_+ = \omega + \omega_0$ .

Interesting particular cases arise from the former results. We start with the exact resonance ( $\Delta_- = 0$ ). In this case, the Hamiltonian  $H_{GJC}$  takes the form

$$H_{GJC}(\Delta_- = 0) = \frac{1}{2}\omega\{A^-, A^+\} + \frac{1}{2}\omega[A^-, A^+]\sigma_3 + \kappa(A^+\sigma_- + A^-\sigma_+). \quad (40)$$

The eigenstates of the resonant generalized JC model are then

$$\begin{aligned} |E_n^+\rangle &= \frac{1}{\sqrt{2}}[|\Psi_n, +\rangle - |\Psi_{n+1}, -\rangle] \\ |E_{n+1}^-\rangle &= \frac{1}{\sqrt{2}}[|\Psi_n, +\rangle + |\Psi_{n+1}, -\rangle]. \end{aligned} \quad (41)$$

and the corresponding eigenvalues are

$$\begin{aligned} E_n^+ &= \omega e_{n+1} - \kappa\sqrt{e_{n+1}} \\ E_{n+1}^- &= \omega e_{n+1} + \kappa\sqrt{e_{n+1}}. \end{aligned} \quad (42)$$

The resonant generalized JC model is reduced to supersymmetric Hamiltonian when the coupling constant  $\kappa$  vanishes

$$H_{susy} = H_{GJC}(\Delta_- = 0 = \kappa) = \omega \begin{pmatrix} A^-A^+ & 0 \\ 0 & A^+A^- \end{pmatrix}. \quad (43)$$

Finally, we remark that when  $e_n = n$  (spectrum of the quantized radiatif field), the eigenstates and eigenvalues (37) and (39) coincide with ones corresponding to the standard JC model [12] (see also [13]).

## 4 Dynamical Superalgebra

Like the standard JC model [12], the generalized JC model can be written as a linear combination of two even operators  $N_1$  and  $N_2$  and two odd operators  $Q^-$  and  $Q^+$ ;

$$H_{GJC} = \frac{\Delta_+}{2}N_1 - \frac{\Delta_-}{2}N_2 + \frac{i\kappa}{\sqrt{\omega}}(Q^+ - Q^-), \quad (44)$$

where

$$N_1 = \begin{pmatrix} A^-A^+ & 0 \\ 0 & A^+A^- \end{pmatrix}, \quad N_2 = \begin{pmatrix} -A^+A^- & 0 \\ 0 & -A^-A^+ \end{pmatrix} \quad (45)$$

$$Q^- = i\sqrt{\omega}A^+\sigma_-, \quad Q^+ = -i\sqrt{\omega}A^-\sigma_+.$$

These operators satisfy the following relations

$$\begin{aligned} [N_1, N_2] &= 0, & [N_1, Q^-] &= [N_1, Q^+] = 0, \\ [N_2, Q^+] &= (Q^+g_N + g_NQ^+), & [N_2, Q^-] &= -(Q^-g_N + g_NQ^-), \\ \{Q^-, Q^+\} &= \omega N_1, & (Q^-)^2 &= (Q^+)^2 = 0, \end{aligned} \quad (46)$$

where the even operator  $g_N$  is defined by

$$g_N = G(N)\sigma_0. \quad (47)$$

The algebra generated by  $\{N_1, N_2, Q^-, Q^+\}$  can be seen as a nonlinear (or deformed) version of the superalgebra  $u(1/1)$ . Indeed, when  $G(N) = 1$  (a situation which occurs in the ordinary JC



model), we have the  $u(1/1)$  superalgebra. In the presence of an exact resonance  $w = w_0$ , the generalized JC model is written as

$$H_{GJC} = \omega N_1 + \frac{i\kappa}{\sqrt{\omega}}(Q^+ - Q^-), \quad (48)$$

in terms of the operators  $N_1$ ,  $Q^+$  and  $Q^-$  which satisfy the structural relations of the superalgebra  $sl(1/1)$ . It is also easy to see that for  $\omega = \omega_0$  and  $\kappa = 0$ , we have

$$\begin{aligned} H_{GJC} &= \omega N_1 = \{Q^+, Q^-\}, \\ [Q^\pm, H_{GJC}] &= 0, \quad (Q^-)^2 = (Q^+)^2 = 0. \end{aligned} \quad (49)$$

## 5 Supercoherent states for $H_{GJC}$

To construct the supercoherent states for  $H_{GJC}$ , we start by looking for an unitary operator  $U$  which connects the Hamiltonians  $H_D$  (diagonal in the fundamental space of states  $\{|\Psi_n, +\rangle, |\Psi_n, -\rangle\}$ ) and  $H_{GJC}$

$$H_D = U^+ H_{GJC} U = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}, \quad (50)$$

where

$$\begin{aligned} H_+ &= \frac{\Delta_+}{2}g(N+1) + \frac{\Delta_-}{4}(g(N+2) + g(N)) - (\kappa^2 g(N+1) + \frac{\Delta_-}{4}(\frac{g(N+2)-g(N)}{2})^2)^{\frac{1}{2}} \\ H_- &= \frac{\Delta_+}{2}g(N) + \frac{\Delta_-}{4}(g(N+1) + g(N-1)) + (\kappa^2 g(N) + \frac{\Delta_-}{4}(\frac{g(N+2)-g(N)}{2})^2)^{\frac{1}{2}}. \end{aligned} \quad (51)$$

The operators  $g(N+k)$  ( $k = -1, 0, 1, 2$ ) are defined by

$$g(N+k)|\Psi_n\rangle = e_{n+k}|\Psi_n\rangle. \quad (52)$$

The operator  $U$  takes the following form

$$U = \begin{pmatrix} \frac{1}{P(N+1)}S(N+1) & \frac{\kappa}{P(N+1)}A^- \\ -A^+ \frac{\kappa}{P(N+1)} & \frac{1}{P(N)}S(N) \end{pmatrix}, \quad (53)$$

where

$$S(N) = \frac{\Delta_-}{2} \frac{g(N+1) - g(N-1)}{2} + \kappa(g(N) + \frac{\Delta_-}{2\kappa}(\frac{g(N+1) - g(N-1)}{2}))^{\frac{1}{2}}, \quad (54)$$

and

$$P(N) = (S(N)^2 + \kappa^2 g(N))^{\frac{1}{2}}. \quad (55)$$

The operator  $U$  can be written as follows

$$U = e^{-Z}, \quad (56)$$

with

$$Z = A^+ h(N+1)\sigma_- - h(N+1)A^-\sigma_+. \quad (57)$$

Developing  $e^{-Z}$  and using the following identities of the Hermitian skew operator  $Z$ ,

$$Z^{2p} = (-1)^p (h(N+1))^{2p} (g(N+1))^p \sigma_+ \sigma_- + (-1)^p (h(N))^{2p} (g(N))^p \sigma_- \sigma_+ \quad (58)$$

and

$$Z^{2p+1} = (-1)^{p+1} (h(N+1))^{2p+1} (g(N+1))^p A^- \sigma_+ + (-1)^p A^+ (h(N+1))^{2p+1} (g(N+1))^p \sigma_-. \quad (59)$$

It becomes that  $U$  is given

$$U = \begin{pmatrix} \cos(h(N+1)(g(N+1))^{\frac{1}{2}}) & \frac{-1}{(g(N+1))^{\frac{1}{2}}} \sin(h(N+1) \frac{1}{(g(N+1))^{\frac{1}{2}}}) A^- \\ A^+ \frac{1}{(g(N+1))^{\frac{1}{2}}} \sin(h(N+1)(g(N+1))^{\frac{1}{2}}) & \cos(h(N)(g(N))^{\frac{1}{2}}) \end{pmatrix}, \quad (60)$$

where  $h(N) = \frac{-1}{\sqrt{g(N)}} \arctan(\kappa \frac{\sqrt{g(N)}}{S(N)})$ . Of course, for  $g(N) = N$  (standard JC model), we recover the results obtained in [12].

Using the unitary operator  $U$ , we introduce the coherent states for generalized JC Hamiltonian as follows

$$\begin{aligned} |z, \beta\rangle_{GJC} &= U|z, \beta\rangle \\ &= \aleph(|z|) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{e(n)}} (\sin \frac{\theta}{2} e^{i\Phi} |E_n^+\rangle + \cos \frac{\theta}{2} |E_n^-\rangle). \end{aligned} \quad (61)$$

To compute some relevant physical quantities of the generalized JC model, we consider the time evolution of states (61)

$$|z, \beta, t\rangle_{GJC} = e^{-itH_{GJC}} |z, \beta\rangle. \quad (62)$$

We get

$$|z, \beta, t\rangle_{GJC} = \aleph(|z|) \sum_{n=0}^{\infty} \frac{z^n}{(e(n))^{\frac{1}{2}}} (\sin \frac{\theta}{2} e^{i\Phi} e^{-itE_n^+} |E_n^+\rangle + \cos \frac{\theta}{2} e^{-itE_n^-} |E_n^-\rangle). \quad (63)$$

Using the latter states, we will compute the average value of the total value of photons, energy and atomic inversion.

## 6 Total number of photons, energy and atomic inversion

### 6.1 Total number of particles

The mean values of the operator  $N$  over the states (63) is given by

$$\langle N \rangle = \aleph(|z|)^2 \sum_{n=0}^{\infty} \frac{|z|^{2n}}{e(n)} (\sin^2 \frac{\theta}{2} \langle E_n^+ | \hat{N} | E_n^+ \rangle + \cos^2 \frac{\theta}{2} \langle E_n^- | \hat{N} | E_n^- \rangle). \quad (64)$$

A direct computation of matrix elements occurring in the last expression gives

$$\langle N \rangle = \aleph(|z|)^2 \sum_{n=0}^{\infty} \frac{|z|^{2n}}{e(n)} (\sin^2 \frac{\theta}{2} e_{n+1} + \cos^2 \frac{\theta}{2} e_n). \quad (65)$$

where the energies  $e_n$  are given (13) (resp.(18)) for the generalized deformed oscillator (resp. for the  $x^4$ -anharmonic system).

## 6.2 Energy

The energy in coherent states (63) is, simply, given by

$$\langle H_{GJC} \rangle = \aleph(|z|)^2 \sum_{n=0}^{\infty} \frac{|z|^{2n}}{e(n)} \left( \sin^2 \frac{\theta}{2} E_n^+ + \cos^2 \frac{\theta}{2} E_n^- \right). \quad (66)$$

The  $E_n^+$  and  $E_n^-$  are given by Eqs (39).

## 6.3 Atomic inversion

First, if we start with supercoherent states (61), we see that the average value of the third component of the spin is time independent. However, as it is well known, if the radiatif field is prepared in a coherent state, the atomic inversion consists of Rabi oscillations. The temporal dependence of  $\langle \sigma_3 \rangle$  appears when we use the generalized supercoherent states (63). Indeed, we get

$$\langle \sigma_3 \rangle = \langle \sigma_3 \rangle_{++} + \langle \sigma_3 \rangle_{--} + \langle \sigma_3 \rangle_{+-} + \langle \sigma_3 \rangle_{-+}, \quad (67)$$

with

$$\begin{aligned} \langle \sigma_3 \rangle_{++} &= \frac{1}{2}(1 - \cos \theta) \aleph(|z|)^2 \sum_{n=0}^{\infty} \frac{|z|^{2n}}{e(n)} q(n+1) \\ \langle \sigma_3 \rangle_{--} &= -\frac{1}{2}(1 + \cos \theta) \aleph(|z|)^2 \sum_{n=0}^{\infty} \frac{|z|^{2n}}{e(n)} q(n) \\ \langle \sigma_3 \rangle_{+-} &= \frac{1}{2} \sin \theta \aleph(|z|)^2 \sum_{n=0}^{\infty} \frac{|z|^{2n+1}}{e(n)} e^{it(2\kappa s(n+1) + \Phi - \alpha)} \frac{1}{s(n+1)} \\ \langle \sigma_3 \rangle_{-+} &= \frac{1}{2} \sin \theta \aleph(|z|)^2 \sum_{n=0}^{\infty} \frac{|z|^{2n+1}}{e(n)} e^{-it(2\kappa s(n+1) + \Phi - \alpha)} \frac{1}{s(n+1)}, \end{aligned} \quad (68)$$

where

$$\begin{aligned} q(n+1) &= \frac{\frac{\Delta_-}{4\kappa}(e_{n+2} - e_n)}{\sqrt{e_{n+1} + (\frac{\Delta_-}{4\kappa})^2(e_{n+2} - e_n)^2}}, & q(0) &= 1 \\ s(n+1) &= \frac{1}{\sqrt{e_{n+1} + (\frac{\Delta_-}{4\kappa})^2(e_{n+2} - e_n)^2}}, & s(0) &= 1 \\ \alpha &= \arctan z, & & \end{aligned} \quad (69)$$

The result of the computation of  $\langle \sigma_3 \rangle$  shows that the time dependence comes from the value  $\langle \sigma_3 \rangle_{+-} + \langle \sigma_3 \rangle_{-+}$  and we obtain

$$\langle \sigma_3 \rangle_{+-} + \langle \sigma_3 \rangle_{-+} = \aleph(|z|)^2 \sum_{n=0}^{\infty} \frac{|z|^{2n+1}}{e(n)} \left( \frac{\sin \theta \cos(t(2\kappa s(n+1) + \Phi - \alpha))}{s(n+1)} \right). \quad (70)$$

This expression has an oscillating behaviour that characterizes the atomic inversion in the generalized JC model.

## 7 Conclusion

In this work, we developed the generalization of the Jaynes-Cummings model where the radiatif field is replaced by generalized harmonic oscillators. We showed the exact solvability of the model. The role of the nonlinear dynamical superalgebra  $u(1/1)$ , governing the evolution of the related model, is important in the construction of the supercoherent states over which we

compute the energy, average number of photons and the atomic inversion. These states have been constructed using the unitary transformations expressed in terms of the generators of the  $u(1/1)$  superalgebra. Furthermore, it is shown that the time dependent supercoherent states have the advantage in obtaining the Rabi oscillations.

Finally, we note that the importance of the generalized Jaynes-Cummings model is due not only to its exact solvability but also arises from its quantum effects such as revival of atomic inversion. In our opinion the generalization and results presented here can be used to give a realistic description of the nonlinear process of the interaction of an atom and radiation field. Clearly, only a confrontation with experimental measures can validate the results of this work.

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