

$\mathcal{N} = 1$ Special Geometry, Mixed Hodge Variations and Toric Geometry

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Abstract

We study the superpotential of a certain class of $\mathcal{N} = 1$ supersymmetric type II compactifications with fluxes and D -branes. We show that it has an important two-dimensional meaning in terms of a chiral ring of the topologically twisted theory on the world-sheet. In the open-closed string B-model, this chiral ring is isomorphic to a certain relative cohomology group V , which is the appropriate mathematical concept to deal with both the open and closed string sectors. The family of mixed Hodge structures on V then implies for the superpotential to have a certain geometric structure. This structure represents a holomorphic, $\mathcal{N} = 1$ supersymmetric generalization of the well-known $\mathcal{N} = 2$ special geometry. It defines an integrable connection on the topological family of open-closed B-models, and a set of special coordinates on the space \mathcal{M} of vev's in $\mathcal{N} = 1$ chiral multiplets. We show that it can be given a very concrete and simple realization for linear sigma models, which leads to a powerful and systematic method for computing the exact non-perturbative $\mathcal{N} = 1$ superpotentials for a broad class of toric D -brane geometries.

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1. Introduction

Mirror symmetry [1] has proven to be a most valuable tool for doing exact computations in $\mathcal{N} = 2$ supersymmetric string theories. The geometric methods provide insights into the non-perturbative dynamics of such theories and, at a more conceptual level, they provide a glimpse of what one may call “stringy quantum geometry.” The underlying concept is that of the 2d topological field theory (TFT) on the string world-sheet, which relates non-perturbative space-time effects to computable geometric data of the compactification manifold.

While much of the work done in the past has been concerned with Calabi–Yau compactifications of the type II string with $\mathcal{N} = 2$ supersymmetry, methods have recently been developed for dealing with $\mathcal{N} = 1$ supersymmetric¹ compactifications as well. They apply in particular to open-closed type II string compactifications on Calabi–Yau manifolds X , having extra fluxes on cycles in X , and/or extra branes wrapped on cycles in X and filling space-time. Most notably, generalizations of mirror symmetry have been used to compute exact 4d $\mathcal{N} = 1$ superpotentials, including all corrections from world-sheet instantons of sphere [2][5] and disc [6]-[18] topologies. Instanton corrections to the superpotential and to other 4d $\mathcal{N} = 1$ amplitudes such as the gauge kinetic F -terms, have been also computed by localization techniques [19]-[22] and Chern-Simons theory [23,24].

The purpose of this note, as well as of the earlier companion paper [18], is to study the geometry of the superpotential on the space $\mathcal{M}_{\mathcal{N}=1}$ of scalar vev’s in $\mathcal{N} = 1$ chiral multiplets. These F-terms carry an interesting geometric structure inherited from the underlying 2d topological field theory on the world-sheet. This “holomorphic $\mathcal{N} = 1$ special geometry” is a close relative of the well-known $\mathcal{N} = 2$ special geometry [25,26] of the moduli space of type II compactifications without fluxes and branes. However, the geometry of the $\mathcal{N} = 1$ F-terms inherited from the TFT is *not* a consequence of space-time supersymmetry; rather it should be interpreted as a distinguished feature of the string effective supergravity, as opposed to a generic supergravity theory.

The study of the holomorphic² $\mathcal{N} = 1$ special geometry has been started in [11,15] in the context of a certain open-closed string duality, which relates the $\mathcal{N} = 1$ special geometry to the geometry of the moduli space of Calabi–Yau 4-folds. In [18] we have

¹ What is meant here is that the 4d theory can be described by an effective $\mathcal{N} = 1$ supergravity action. Supersymmetry may, and in many cases will, be broken by the presence of a superpotential.

² There is also a non-holomorphic part given by the D-terms, most notably the Kähler potential for the chiral multiplets. While this would be interesting to study as well, perhaps in the framework of the tt^* equations of [27,26], this is beyond the scope of the paper.

outlined how the $\mathcal{N} = 1$ special geometry arises as the consequence of systematically incorporating fluxes and branes into the familiar ideas and methods of mirror symmetry for Calabi–Yau 3-folds. In the present paper we complete the discussion of [18] in two respects.

First, we will develop a geometric representation of the chiral ring and the integrable connection ∇ on the (B-twisted) family of TFT’s over the $\mathcal{N} = 1$ deformation space $\mathcal{M}_{\mathcal{N}=1}$. Specifically, we show that the TFT concepts are equivalent to the mixed Hodge variation defined on a certain relative cohomology group associated with the flux and brane geometry. The fundamental rôle of the variation of the Hodge structure in the construction of $\mathcal{N} = 2$ mirror symmetry is therefore taken over by the *mixed* Hodge structure on the relative cohomology group. In particular, a system of differential equations associated with the Hodge structure gives a powerful and systematic technique for computing *i*) a set of special, flat coordinates t_A on $\mathcal{M}_{\mathcal{N}=1}$, and *ii*) exact expressions for the holomorphic potentials \mathcal{W}_K of $\mathcal{N} = 1$ special geometry. These potentials \mathcal{W}_K represent the basic topological data of the theory, and are in a quite precise sense the $\mathcal{N} = 1$ analogs of the familiar $\mathcal{N} = 2$ prepotential \mathcal{F} . Moreover, the holomorphic potentials \mathcal{W}_K have an important physical meaning, in that they describe the space-time instanton corrected superpotential of the effective $\mathcal{N} = 1$ supergravity in four dimensions.

Second, we will develop an explicit description of the relative cohomology and its mixed Hodge structure in terms of linear sigma models, or, equivalently, of toric geometry. Just like for the closed string, the toric formulation is highly effective also for the open-closed string compactification, and leads to a straightforward calculation of the special coordinates t_A (by the $\mathcal{N} = 1$ mirror map) and of the potentials \mathcal{W}_K . More specifically, the problem to compute the space-time instanton expansion to the four-dimensional superpotential is essentially reduced to finding an appropriate generalized hypergeometric series that solves a certain system of linear differential equations associated with the toric D -brane geometry.

The organization of this note is as follows. Sect. 2 contains an overview over the main ideas and results that were outlined in [18] and will be further developed in the present paper. In sect. 3, we discuss the 2d chiral ring of the open-closed B-model, which is an extension of the bulk chiral ring by boundary operators. We identify the chiral ring elements with sections of a certain relative cohomology group, $H^3(X, Y)$. In sect. 4 we consider continuous families of topological B-models over the space $\mathcal{M}_{\mathcal{N}=1}$ of the open-closed string deformations z_A , which represent scalars vev’s of 4d chiral $\mathcal{N} = 1$ multiplets. The central objects are the variation of the mixed Hodge structure on the relative cohomology group and an integrable connection ∇ which is constructed as the Gauss-Manin connection on the relative cohomology bundle.

In sect. 5 we elaborate on the powerful methods of toric geometry to study the relative cohomology and mixed Hodge structure for open-closed compactifications described by linear sigma models. The system of differential equations associated with ∇ takes a simple, generalized hypergeometric form. Accordingly, its solutions, which represent the $\mathcal{N} = 1$ mirror map and the holomorphic potentials \mathcal{W}_K , are generalizations of hypergeometric series. Moreover, a certain ambiguity for non-compact geometries, the so-called framing ambiguity discovered in [10], is identified with the degree of a certain hypersurface Y that defines the relative cohomology $H^3(X, Y)$. Finally, in sect. 6 we illustrate the toric methods at the hand of a detailed sample computation.

2. Overview

Recall that the vector multiplet moduli space $\mathcal{M}_{\mathcal{N}=2}$ of the type II string compactified on a Calabi–Yau manifold X to four dimensions is a restricted Kähler manifold with so-called $\mathcal{N} = 2$ special geometry [25,26]. The $\mathcal{N} = 2$ supersymmetric effective action for the vector multiplets is defined by a holomorphic prepotential $\mathcal{F}(z_a)$, where z_a are generic coordinates on the space $\mathcal{M}_{\mathcal{N}=2}$ of scalar vev’s. The special geometry of $\mathcal{M}_{\mathcal{N}=2}$ can be traced back to properties of the underlying topological field theory on the string world-sheet, which leads to beautiful connections between physics and geometry. Indeed the key concepts of the geometry can be directly identified with properties of the internal twisted TFT [26]. A central object of this TFT is the ring of two-dimensional, primary chiral superfields [28]. Its moduli dependent structure constants, or operator product coefficients, defined by³

$$\mathcal{R}_{cl} : \phi_a \cdot \phi_b = C_{ab}{}^c(t.) \phi_c, \tag{2.1}$$

are related to three-point functions which are given by derivatives of the prepotential:

$$C_{ab}{}^c(t.) = \mathcal{F}_{abc}(t.) = \partial_a \partial_b \partial_c \mathcal{F}(t.). \tag{2.2}$$

The prepotential, and thus the effective action for the vector multiplets, can be recovered from the chiral ring coefficients $C_{ab}{}^c(t.)$ through integration.

³ Here $t_a = t_a(z_b)$ denote the special, topological coordinates that will be discussed in detail later. Moreover $\partial_a = \partial/\partial t_a$.

In a certain variant of the 2d TFT, the so-called B-model [29], there are no quantum corrections to the chiral ring, and the $C_{ab}{}^c(t.)$ can be computed via classical geometry. They appear as particular components of the period matrix for the elements of $H^3(X, \mathbf{C})$:

$$\Pi_i^\alpha = \langle \Gamma^\alpha, \Phi_i \rangle = \int_{\Gamma^\alpha} \Phi_i = \begin{pmatrix} 1 & t_a & \mathcal{F}_a & 2\mathcal{F} - t_a \mathcal{F}_a \\ 0 & \delta_{ab} & \mathcal{F}_{ab} & \mathcal{F}_b - t_a \mathcal{F}_{ab} \\ 0 & 0 & \mathcal{F}_{abc} & -t_a \mathcal{F}_{abc} \\ 0 & 0 & \dots & \dots \end{pmatrix}. \quad (2.3)$$

Here the forms $\Phi_i \in H^3(X, \mathbf{C})$ are, up to normalization, geometric representatives for the elements of the chiral ring, and the $\{\Gamma^\alpha\}$ constitute a fixed basis for the homology 3-cycles, $H_3(X, \mathbf{Z})$.

The underlying geometric structure of the period integrals is the variation of Hodge structure on a family of Calabi–Yau manifolds, whose complex structures are parametrized by points in $\mathcal{M}_{\mathcal{N}=2}$. The integrable Gauss–Manin connection ∇_a on the Hodge bundle with fibers $H^3(X)$ leads to a system of differential equations in the moduli that enables one to compute the period matrix Π_i^α from the solutions. Thus, solving these differential equations provides a simple way of computing the flat coordinates t_a and the holomorphic prepotential \mathcal{F} for a given threefold X .

Our purpose here is to derive similar results for $\mathcal{N} = 1$ supersymmetric theories obtained from compactifying open-closed type II strings on a Calabi–Yau manifold X . In these theories, supersymmetry is reduced by the presence of fluxes of the closed string gauge potentials, and/or by the presence of an open string sector from background (D-)branes that wrap cycles in X and fill space-time. Specifically, we will mainly consider D5-branes of the type IIB string that wrap certain 2-cycles in X . The latter are mapped by mirror symmetry to type IIA D6-branes partially wrapped on 3-cycles.

A type IIB compactification with such branes and fluxes may be described by an effective $\mathcal{N} = 1$ supergravity with a non-trivial superpotential on the space $\mathcal{M}_{\mathcal{N}=1}$ of vev’s in $\mathcal{N} = 1$ chiral multiplets.⁴ This space comprises the complex structure moduli z_a from the closed, and brane moduli \hat{z}_α from the open string sector. As discussed in [18], the

⁴ We will sometimes loosely refer to this space as the “ $\mathcal{N} = 1$ moduli space”, despite of the fact that it is generically obstructed by the superpotential $W(\phi)$ that lifts flat directions and may break supersymmetry spontaneously. This notion is well motivated if the superpotential is completely non-perturbative, as is the case for the moduli of branes wrapping supersymmetric cycles in X .

general superpotential on $\mathcal{M}_{\mathcal{N}=1}$, depending on both the closed and open string moduli, can be written as

$$W_{\mathcal{N}=1} = W_{cl}(z_a) + W_{op}(z_a, \hat{z}_\alpha) = \sum_{\Sigma} N_{\Sigma} \Pi^{\Sigma}(z_A), \quad (2.4)$$

where

$$z_A \equiv (z_a, \hat{z}_\alpha), \quad A = 1, \dots, M \equiv \dim \mathcal{M}_{\mathcal{N}=1}, \quad (2.5)$$

and where Π^{Σ} is the *relative period vector*

$$\Pi^{\Sigma}(z_A) = \int_{\Gamma^{\Sigma}} \Omega, \quad \Gamma^{\Sigma} \in H_3(X, B; \mathbf{Z}). \quad (2.6)$$

Here $\Gamma^{\Sigma} \in H_3(X, B; \mathbf{Z})$ is a fixed basis of relative homology 3-cycles in X with boundaries on the union of 2-cycles $B = \cup_{\nu} B_{\nu}$, where the B_{ν} denote the 2-cycles wrapped by 5-branes. The elements of $H_3(X, B; \mathbf{Z})$ are *i*) the familiar 3-cycles in X without boundaries, whose volumes specify the flux superpotential [2,11,30], and *ii*) the 3-chains with boundaries on the 2-cycles B_{ν} , whose 3-volumes govern the brane superpotential [31,9,8].⁵ The vector Π^{Σ} thus uniformly combines the period integrals of Ω over 3-cycles Γ^{α} , with integrals over the 3-chains $\hat{\Gamma}^{\nu}$ whose boundaries are wrapped by D-branes.⁶ The coefficients N_{Σ} in (2.4) label the various flux and brane sectors. Specifically, $N_{\Sigma} = n_{\Sigma} + \tau m_{\Sigma}$, where n_{Σ} measures either the integral RR flux on Γ^{Σ} if Σ labels a 3-cycle, or the number of D5-branes if Σ labels a 3-chain. Similarly, the integers m_{Σ} specify the NS fluxes and branes.

In fact, there is an alternative interpretation of the 4d superpotential, in terms of the tension of a BPS domain wall obtained by wrapping a 5-brane on a relative 3-cycle in X [30,33][2]. The possible wrappings are determined by the relative homology group $H_3(X, Y; \mathbf{Z})$. We are thus led to identify the integral relative cohomology group with the lattice of 4d BPS charges (with two copies for the NS and RR sector, respectively)

$$\Gamma_{BPS} = H^3(X, Y; \mathbf{Z}) \otimes (1 \oplus \tau),$$

where the charge $Q \in \Gamma_{BPS}$ is specified by the integers n_{Σ} and m_{Σ} . The holomorphic $\mathcal{N} = 1$ special geometry of the string effective supergravity may thus also be interpreted as the BPS geometry of 4d domain walls with charges $Q \in \Gamma_{BPS}$, very much as the $\mathcal{N} = 2$ special geometry describes the BPS geometry of four-dimensional particles.

⁵ See also [32] for a further discussion of branes and fluxes in the effective supergravity.

⁶ More precisely, the pairing between homology and cohomology is defined in relative cohomology, as discussed in sect. 4.2

Let us now briefly summarize our results on the underlying topological structure of the superpotential, partly recapitulating statements made in [18], and partly previewing the discussion in the later sections of the present paper.

A central object in the 2d TFT on the string world-sheet is the extended chiral ring of the open-closed type II theory:

$$\mathcal{R}_{oc} : \quad \phi_I \cdot \phi_J = C_{IJ}^K(z_A) \phi_K. \quad (2.7)$$

This open-closed chiral ring is an extension of the bulk chiral ring (2.1) by boundary operators. Its structure constants depend on both the open and closed string moduli z_A parametrizing $\mathcal{M}_{\mathcal{N}=1}$. The 2d superfields ϕ_I form a basis for the local BRST cohomology of the 2d TFT.

As will be discussed, in the topological B-model the ring elements ϕ_I can be identified with elements of a certain relative cohomology group, $V = H^3(X, Y)$; here Y is a union of hypersurfaces in X that is determined by the D -brane geometry. The space V may be thought of as the appropriate concept for systematically combining differential forms on the Calabi–Yau X (describing the closed string sector) with forms on the boundary Y (describing the open string sector).

The relative cohomology group V comes with an important mathematical package, namely the mixed Hodge structure, which defines an integral grading, $V = \bigoplus_q V^{(q)}$, on it. Moreover, the Gauss-Manin connection ∇ on the relative cohomology bundle over $\mathcal{M}_{\mathcal{N}=1}$ (with fibers V) represents an integrable connection on the space of topological B-models. The integrability implies the existence of special, topological coordinates t_A on $\mathcal{M}_{\mathcal{N}=1}$, in which the Gauss-Manin derivatives reduce to ordinary derivatives.

Moreover, the geometric representation of the open-closed chiral ring allows to describe its moduli dependence by the “relative period matrix” Π_I^Σ . It consists of the period integrals of the elements of V , with respect to a basis of topological cycles with boundaries on Y . In the special coordinates t_A , the relative period matrix takes the form:

$$\Pi_I^\Sigma = \begin{pmatrix} 1 & t_A & \mathcal{W}_K & \dots \\ 0 & \delta_{AB} & \partial_B \mathcal{W}_K & \dots \\ 0 & 0 & C_{AB}^K & \dots \\ \vdots & \vdots & \vdots & \dots \end{pmatrix}. \quad (2.8)$$

The first row, Π_0^Σ , coincides with the relative period vector Π^Σ which enters the definition of the superpotential W in (2.4). The entries $t_A(z.)$ of the relative period matrix define the $\mathcal{N} = 1$ *mirror map*, in terms of the ratios of certain period and chain integrals on X .

This map determines the special coordinates t_A in terms of arbitrary coordinates z_A on $\mathcal{M}_{\mathcal{N}=1}$.

The entries \mathcal{W}_K of the relative period matrix represent the *holomorphic potentials of $\mathcal{N} = 1$ special geometry*. They are, in a quite precise sense, the $\mathcal{N} = 1$ counterparts of the holomorphic prepotential \mathcal{F} of $\mathcal{N} = 2$ special geometry.⁷ In particular, their derivatives determine the structure constants of the 2d chiral ring (2.7):

$$C_{AB}^K(t.) = \partial_A \partial_B \mathcal{W}_K(t.). \quad (2.9)$$

Moreover, the potentials \mathcal{W}_K have also an important physical meaning as they encode the space-time instanton expansion of the $\mathcal{N} = 1$ superpotential in the string effective space-time theory. This expansion has been conjectured to have remarkable integrality properties [23]. Specifically, when expressed in terms of the special coordinates t_A , the potentials have the following expansions:

$$\mathcal{W}_K(t.) = \sum_{\{n_C\}} N_{n_1 \dots n_M}^{(K)} \sum_k \frac{1}{k^2} \left(\prod_C e^{2\pi i k n_C t_C} \right), \quad (2.10)$$

where the coefficients $N_{n_1 \dots n_M}^{(K)}$ are supposedly integral. In writing the above expression, we have assumed that the physically motivated definition of the special coordinates given in [23,9], namely in terms of the tension of certain domain walls, is equivalent to the above 2d definition in terms of the vanishing of the Gauss-Manin connection. This is a quite non-trivial and important relation. The results in the following sections give substantial evidence for the equivalence of these two definitions.

Note that the coefficients $N_{n_1 \dots n_M}^{(K)}$ have, besides other interpretations, a well-known interpretation in the language of the A-model mirror as corrections from world-sheet instantons of disc [23,9] and sphere [3,4] topologies. However, since the scalar vev's t_A determine the coupling constants of the RR sector in the four-dimensional string effective supergravity, the expression (2.10) can often be directly interpreted, alternatively, also as corrections to the $\mathcal{N} = 1$ superpotential from space-time instantons.

In the following, we will work out the details of the geometrical structure outlined above; the next two sections provide a general discussion of the topological field theoretic and geometric aspects, while in sects. 5 and 6 we will present some additional structure that is specific to toric geometries, and some explicit computations in such geometries.

⁷ Note that a subset of the \mathcal{W}_K may coincide with the derivatives, \mathcal{F}_a , of the bulk prepotential. However, in general the \mathcal{W}_K cannot be integrated to a single prepotential; the fact that one needs more than one holomorphic function to define the $\mathcal{N} = 1$ special geometry, reflects the fact that it is less restricted than the special geometry associated with $\mathcal{N} = 2$ space-time supersymmetry.

3. Observables in the open-closed B-model and relative cohomology

As is well-known, the elements of the chiral ring in the closed string B-model on the Calabi–Yau X have an geometric representation as elements of the cohomology group $H^3(X)$ [29]. Moreover the gradation by $U(1)$ charge of the chiral ring corresponds to the Hodge decomposition $H^3(X) = \oplus_q H^{3-q,q}(X)$. We will now describe a similar representation of the chiral ring \mathcal{R}_{oc} of the open-closed B-model in terms of a certain relative cohomology group.

The open-closed chiral ring is an extension of the closed string chiral ring, and combines operators from both the bulk and boundary sectors. Geometrically, the new structure from the open string sector is the submanifold $B \subset X$ wrapped by the D-branes.⁸ Since the bulk sector of the closed string is represented by $H^3(X)$, the open-closed chiral ring should correspond to an extension of this group by new elements originating in the open string sector on B . It is natural to expect that this extension is simply the dual $H^3(X, B)$ of the space $H_3(X, B)$ that underlies the flux and brane induced superpotential (2.4).

Under a certain assumption discussed in [18], one may in fact replace the group $H^3(X, B)$ by a simpler relative cohomology group, $H^3(X, Y)$. Here Y denotes a union of hypersurfaces that pass through the 2-cycles in B wrapped by the 5-branes. Let us here recall briefly the geometry of the hypersurface Y and the assumption which underlies it. As in [18], we will restrict the discussion to a single 2-cycle B with a single modulus. In fact the superpotential for a collection of non-intersecting branes wrapped on a set of 2-cycles $\{B_\nu\}$ will be the sum of the individual superpotentials and may be treated similarly.

One way to solve the minimal volume condition for a special Lagrangian 3-cycle Γ , is to slice it into a family of 2-cycles along a path I . This can be achieved by intersecting Γ with a suitable one-parameter family of holomorphic hypersurfaces $Y(z)$, with z a complex parameter. The intersection of the hypersurface $Y(z)$ with Γ is a family of 2-cycles $\Gamma_2(z)$ of minimal volume $V(z)$ and the integral (2.6) can be written as

$$W_\Gamma = \int_\Gamma \Omega = \int_{z_0}^{z_1} V(z) dz. \quad (3.1)$$

Here the contour I in the z -plane is determined by the minimal volume condition for Γ .

For a 3-chain $\hat{\Gamma}$, the interval in the z -plane ends at a specific value, say $z = \hat{z}$, for which the hypersurface $Y(\hat{z})$ passes through the boundary 2-cycle $B_\nu = \partial\hat{\Gamma}^\nu \subset B$ (around

⁸ We will restrict our discussion to two-cycles and trivial line bundles on them.

which the 5-brane is wrapped). Varying the position of the 5-brane leads to the following variation of the brane superpotential:

$$\delta W_{\hat{\Gamma}} \sim V(\hat{z}) = \int_{\Gamma_2(\hat{z})} \omega,$$

where ω is an appropriate holomorphic form on $Y(z)$. As will be discussed below, ω is the $(2,0)$ form on $Y(z)$ obtained from a Poincaré residue of Ω . Note that δW_{Γ} vanishes if Γ_2 is holomorphic [31,9,8].

The moral of the foregoing discussion is that the variations of the relative period vector (2.6), related to the boundary of a chain Γ , are detected by the periods of a holomorphic $(2,0)$ form ω on the hypersurface $Y(z)$.⁹ In other words, to describe the variations of the volumes of the special Lagrangian chains, we may replace the relative cohomology group $H^3(X, B)$ by the group $H^3(X, Y)$. This connection will be made explicit in the next section, by associating the topological observables of the open-closed B-model with elements of the relative cohomology group $H^3(X, Y)$.

3.1. Space of RR ground states and relative cohomology

The closed string B-model for the Calabi–Yau 3-fold X is defined by a twisted, two-dimensional $N = (2, 2)$ superconformal theory on the string world-sheet Σ [29]. The map $\varphi : \Sigma \rightarrow X$ from the world-sheet to the target space X is defined by three complex scalar fields φ^i on Σ . The fermionic superpartners are three complex fermions ψ_{\pm}^i which are sections of the holomorphic tangent bundle $T = T^{1,0}(X)$. The BRST cohomology is generated by the linear combinations

$$\begin{aligned} \eta^{\bar{i}} &= \psi_+^{\bar{i}} + \psi_-^{\bar{i}}, \\ \theta_i &= g_{i\bar{j}}(\psi_+^{\bar{j}} - \psi_-^{\bar{j}}), \end{aligned} \tag{3.2}$$

which satisfy $\delta_Q \eta = \delta_Q \theta = 0$; here Q is the BRST operator which corresponds to $\bar{\partial}$. The local BRST observables have the form

$$\phi_{\bar{i}_1, \dots, \bar{i}_p}^{j_1, \dots, j_q} \eta^{\bar{i}_1} \dots \eta^{\bar{i}_p} \theta_{j_1} \dots \theta_{j_q}. \tag{3.3}$$

The local BRST cohomology is thus isomorphic to the space of sections of $H^p(X, \wedge^q T)$.

⁹ See also [34] for a related discussion.

In the open string sector, the B-type boundary conditions set to zero a certain linear combination of the fermionic fields, in the simplest form one has [35,36]:

$$\begin{aligned}\eta^{\bar{i}} &= 0 \quad (D), \\ \theta_i &= 0 \quad (N),\end{aligned}\tag{3.4}$$

where D and N denote a Dirichlet and Neumann boundary condition in the i -th direction, respectively. The cohomology of the boundary BRST operator can also be represented by elements of the form (3.3), however with the interpretation as sections of $H^p(Y, \wedge^q N_Y)$. Here N_Y denotes the normal bundle to the submanifold Y in X on which the Dirichlet boundary conditions are imposed. More generally, one may couple the fermions to a gauge bundle E on the D-brane, leading to a generalized BRST operator $\bar{\partial}_E$ acting on a Hilbert space isomorphic to the space of sections $H^p(Y, \wedge^q N_Y \otimes E)$ [6].

We interpret a section $\theta \in H^0(Y, N_Y)$ as the restriction $\theta = \Theta|_Y$ of a C^∞ section Θ of some bundle E on X . The zero locus of the section Θ defines a submanifold in X that can be wrapped by a D-brane. Deformations of the section Θ , corresponding to a variation of the location of the D-brane, represent the open string moduli \hat{t}_α . These deformations correspond to physical fields, $\hat{\phi}_\alpha$, of $U(1)$ charge one¹⁰, associated to the section θ_α :

$$\hat{\phi}_\alpha \longleftrightarrow \theta_\alpha \in H^0(Y, N_Y).\tag{3.5}$$

On the world-sheet the deformations correspond to adding boundary terms $\delta S = \sum_\alpha \hat{t}_\alpha \int_{\partial\Sigma} dz d\theta \hat{\Phi}_\alpha$. See [37] for a discussion of ordering effects involving such terms.

There are two basic multiplications induced by wedge products with sections of $H^0(Y, N_Y)$:

$$\begin{aligned}\hat{\phi}_\alpha \cdot \hat{\phi}_\beta &= C_{\alpha\beta}^\mu \hat{\phi}_\mu \longleftrightarrow H^0(Y, N_Y) \times H^0(Y, N_Y) \rightarrow H^0(Y, \wedge^2 N_Y), \\ \phi_a \cdot \hat{\phi}_\alpha &= C_{a\alpha}^\mu \hat{\phi}_\mu \longleftrightarrow H^1(X, T) \times H^0(Y, N_Y) \rightarrow H^1(Y, T|_Y \wedge N_Y).\end{aligned}\tag{3.6}$$

Together with the closed string chiral ring operator product (2.1), these products generate the extended, open-closed chiral ring \mathcal{R}_{oc} . Here it is understood that the open and closed string operators are inserted on a world-sheet with boundary, and the restriction of the closed string operator to the boundary is defined by the geometric restrictions $T \rightarrow T_{X|Y}$ and $T^* \rightarrow T_{X|Y}^*$.

¹⁰ Quantities in the open string sector will be denoted with a hat and greek indices; letters from the beginning of the alphabet refer to fields with $U(1)$ charge one.

More judiciously one should replace the bulk operators ϕ_a in (3.6) by a set of boundary operators $\hat{\phi}_a$ defined by the collision of the bulk operators ϕ_a with the boundary. At the level of the topological observables considered in this paper, this step is already built into the geometric representation $H^3(X, \mathbf{C})$ of the “closed” string Hilbert space by the following simple observation. The integral dual homology cycles in $H_3(X, \mathbf{Z})$ define a basis of D-branes wrapped on the 3-cycles of X . The transition from bulk to boundary observables is the obvious pairing $H_3(X) \times H^3(X) \rightarrow \mathbf{C}$ [38,36]. The bulk-boundary product is thus implicit in the following discussion when passing to a flat basis dual to $H_3(X, \mathbf{Z})$.

Similarly as was done in [26] for the closed B-model, we would like to describe the space spanned by the elements of the chiral ring by a geometric cohomology group V . Recall that the chiral fields ϕ_I in the NSNS are in one-to-one correspondence with the supersymmetric ground states in the RR sector. In particular a ground state $|I\rangle_{RR}$ can be obtained from the canonical vacuum $|0\rangle_{RR}$ by inserting the operator $\phi_I \in \{\phi_a, \hat{\phi}_\mu\}$ in the twisted 2d path integral on a world-sheet with boundary, $|I\rangle_{RR} = \phi_I |0\rangle_{RR}$ [26]. The cohomology group V may thus be alternatively identified with the space of 2d RR ground states.

As is well-known, the elements of the bulk chiral ring arising from sections of $H^p(\wedge^p T)$ can be identified with sections of $H^{3-p,p}(X)$ by the isomorphisms

$$\rho_\Omega : H^p(\wedge^p T) \xrightarrow{\Omega} H^{3-p,p}(X, \mathbf{C}), \quad (3.7)$$

provided by the unique holomorphic $(3,0)$ form Ω . Importantly, the holomorphic $(3,0)$ form defines also an isomorphism on the elements in the open string sector associated with the boundary Y ,

$$\rho_\Omega : H^0(Y, N_Y) \xrightarrow{\Omega} H^{2,0}(Y), \quad H^1(Y, T|_Y \wedge N_Y) \xrightarrow{\Omega} H^{1,1}(Y) \quad (3.8)$$

This isomorphism can be interpreted as a Poincaré residue for Ω , which may be written as [39]

$$\tilde{\phi}(s_1, \dots, s_p; \theta_\alpha) = \phi|_Y(s_1, \dots, s_p, \Theta_\alpha), \quad s_k \in T_Y \quad (3.9)$$

where $\tilde{\phi} \in H^{p,0}(N_Y^*)$ and ϕ is a holomorphic $(p+1)$ -form on X .¹¹

¹¹ This definition, which is directly related to (3.8), is equivalent to the more familiar definition used e.g. in [40].

For closed strings, the isomorphism ρ_Ω has the simple interpretation of a choice of normalization for the canonical ground state $|0\rangle_{RR} \sim H^{3,0}$ [26]. In the next section we define a grading of the space V of open string RR ground states that again identifies $H^{3,0}$ with the canonical vacuum. Thus Ω has precisely the same meaning also in the open string context, with the space $H^{3,0}$ identified as the RR vacuum $|0\rangle_{RR}$ on world-sheets with boundaries, on which the operators ϕ_i and $\hat{\phi}_\mu$ act to generate the full space V of RR ground states.

We are now ready to argue that the total space of RR vacua can be described by the relative cohomology group $H^3(X, Y)$. The latter is defined as follows. The inclusion $i : Y \rightarrow X$ induces the morphism of the complexes of sheaves $i^* : \Omega_X^* \rightarrow \Omega_Y^*$. The mapping cone of i^* is defined as the complex $\Omega_X^* \oplus \Omega_Y^{*-1}$ with differential $d = (d_X + i^*, -d_Y)$. The relative cohomology is the hyper cohomology of this complex. From the exact sequence:

$$0 \longrightarrow \Omega^n(X, Y) \longrightarrow \Omega^n(X) \longrightarrow \Omega^n(Y) \longrightarrow 0$$

it follows that the n -th hyper cohomology is the space

$$\text{Ker}[H^n(X) \rightarrow H^n(Y)] \oplus \text{Coker}[H^{n-1}(X) \rightarrow H^{n-1}(Y)].$$

In the present context, with $V = H^3(X, Y)$, the first factor will be all of $H^3(X)$ for the D-brane geometries that we will consider. This statement will be easy to check in practice, as $H^3(Y)$ is trivial for these geometries. The second factor is the variable cohomology¹² $H_{var}^2(Y)$ of Y . The image of the Poincaré residue precisely maps into this space. In summary we thus have¹³

$$\text{Space of RR groundstates } V \sim \text{Im}(\rho_\Omega) \subseteq H^3(X, Y) . \quad (3.10)$$

¹² The variable cohomology is the part of the cohomology which varies with the embedding $i : Y \rightarrow X$. E.g. the cohomology of the quintic Y in \mathbf{P}^4 splits over the rationals into the even-dimensional cohomology $H^{2n}(Y) = i^* H^{2n}(\mathbf{P}^4)$, which is the “fixed” cohomology inherited from the ambient \mathbf{P}^4 , and the odd-dimensional “variable” cohomology $H^3(Y)$ which does not descend from the ambient space.

¹³ In many cases the following inequality will be an equality; however this statement may sometimes fail for a chosen algebraic realization of the geometry.

4. Variations of mixed Hodge structures and topological flat connection

The next step is to extend the previous discussion to families of the open-closed B-model over the space $\mathcal{M}_{\mathcal{N}=1}$ of open-closed string deformations. The main concepts here will be the variation of mixed Hodge structures [41] defined on the relative cohomology group $H^3(X, Y)$, and a topological flat connection ∇ on the vacuum bundle \mathcal{V} with fibers $V \sim H^3(X, Y)$. The connection ∇ is the Gauss-Manin connection on the relative cohomology bundle with fiber $H^3(X, Y)$.

4.1. Infinitesimal deformations and variations of mixed Hodge structure

Our first aim is to identify geometric cohomology classes that are related to infinitesimal variations in the moduli and generate the space $V = H^3(X, Y)$ of RR ground states from a “canonical vacuum”. This leads to a natural gradation

$$V = \bigoplus_q V^{(q)}, \quad (4.1)$$

which can be identified with a $U(1)$ charge in the TFT. In the present case it will also be necessary to distinguish between infinitesimal variations in the bulk and boundary sectors, respectively. The resulting filtration is in fact well-known in mathematics and defines a mixed Hodge structure on the relative cohomology group V .

In relative cohomology¹⁴, a RR ground state may be represented by a pair of forms

$$\vec{\Theta} = (\Theta_X, \theta_Y) \in H^3(X, Y), \quad (4.2)$$

where Θ_X is a closed 3-form on X and θ_Y a 2-form on Y such that $i^*\Theta_X - d\theta_Y = 0$. The equivalence relation is

$$\vec{\Theta} \sim \vec{\Theta} + (d\omega, i^*\omega - d\phi), \quad (4.3)$$

with ω (ϕ) a 2-form on X (1-form on Y).

We will start from the ansatz

$$(\Omega, 0) \quad (4.4)$$

as a representative for the canonical ground state $|0\rangle$. To identify the geometric classes that generate the total space V from $|0\rangle$, consider the infinitesimal deformations for a family of B-models defined by the geometry (X, Y) . The moduli space $\mathcal{M}_{\mathcal{N}=1}$ of open-closed string deformations of the geometry (X, Y) consists of complex deformations of X and deformations of the sub-manifold $Y \subset X$.

¹⁴ See e.g., [42] for an introduction.

The general geometric argument which identifies infinitesimal variations in the moduli with geometric classes goes as follows. Consider an analytic family $\mathcal{A} = \{A_b\}_{b \in \Delta}$, where Δ is a disc that represents a local patch in the moduli space for an algebraic variety A_b . The infinitesimal displacement mapping of Kodaira is

$$T_b(\Delta) \longrightarrow H^0(A_b, N_{A_b}), \quad (4.5)$$

where N_{A_b} is the normal bundle of A_b within \mathcal{A} . On the other hand, the co-boundary of the exact sequence

$$0 \longrightarrow T_{A_b} \longrightarrow T_{\mathcal{A}|A_b} \longrightarrow \pi^* T_{\Delta}|_{A_b} \longrightarrow 0, \quad (4.6)$$

where $\pi : \mathcal{A} \rightarrow \Delta$, gives the Kodaira-Spencer map:

$$\kappa : T_b(\Delta) \longrightarrow H^1(T_{A_b}). \quad (4.7)$$

Applying the above maps to the present geometric situation, one obtains altogether four classes, which define the action of the closed and open string deformations, respectively, on Θ_X and θ_Y in (4.2).

Explicitly, this works as follows. Consider first the family of Calabi–Yau manifolds $\mathcal{X} \rightarrow \mathcal{M}_{CS}$ with fiber $X_{\{t_a\}}$. The Kodaira-Spencer map for this family gives a class in $H^1(X, T_X)$ that defines a map on the first entry of $\vec{\Theta}$ by taking the wedge product and contracting the form with the vector field. This map represents the geometric counter-part of the well-known TFT argument that identifies an infinitesimal variation δ_a in the closed string moduli with multiplication by a charge one operator $\phi_a \in H^1(X, T_X)$. Restricting the sequence (4.6) to Y one obtains a class in $H^1(Y, T_X|_Y)$ that acts on the second entry of $\vec{\Theta}$ in a similar way.

As for an open string variation $\hat{\delta}_\alpha$, one may likewise consider the Kodaira-Spencer map for the family $X \rightarrow \Delta_z$ with fiber $Y(z)$ to obtain a class in $H^1(Y, T_Y)$ that defines a map on the second entry of $\vec{\Theta}$. On the other hand, the action of $\hat{\delta}_\alpha$ on the first entry of $\vec{\Theta}$, which is a form on X , cannot be obtained from the Kodaira-Spencer map; one really needs a class on the total space X of the family. Such a class may be defined by the Kodaira map (4.5), and it acts on the first entry by the Poincaré residue (3.9).

The common property of the maps defined by these four classes associated with the open-closed string deformations, is that they all lower the holomorphic degree of a form by one. To complete the discussion of how these classes generate the space V and define a natural gradation on it, let us pass right away to the mathematical definition of a mixed Hodge structure [41] on V which is the appropriate language for the present problem.

A mixed Hodge structure is defined by *i*) a \mathbf{Z} module $V_{\mathbf{Z}}$ of finite rank; *ii*) a finite decreasing Hodge filtration F^p on $V = V_{\mathbf{Z}} \otimes \mathbf{C}$; *iii*) a finite increasing weight filtration W_p on $V_{\mathbf{Q}} = V_{\mathbf{Z}} \otimes \mathbf{Q}$ such that F^p defines a pure Hodge structure of weight p on the quotient $Gr_p^W = W_p/W_{p-1}$. In the present case we have $V = H^3(X, Y)$ and the filtrations¹⁵ are

$$\begin{aligned}
F^3 &: H_X^{3,0}, & W_3 &: \oplus_{p+q=3} H_X^{p,q} \\
F^2 &: F^3 + H_X^{2,1} + H_Y^{2,0} & W_4 &: W_3 + \oplus_{p+q=2} H_Y^{p,q} \\
F^1 &: F^2 + H_X^{1,2} + H_Y^{1,1}, \\
F^0 &: F^1 + H_X^{0,3} + H_Y^{0,2},
\end{aligned} \tag{4.8}$$

The quotients Gr_p^W separate the observables into bulk and boundary operators on which separate Hodge filtrations may be defined. On the other hand, the quotients $Gr_p^F = F^p/F^{p+1}$ define the spaces (4.1) of pure (left) $U(1)$ charge in the TFT. The motivation to prefer the filtered spaces F^p over the spaces of pure grades defined by the quotient is the same as in the closed string case: the bundles over $\mathcal{M}_{\mathcal{N}=1}$ with fibers F^p are holomorphic bundles, whereas the bundles with the quotient space as the fibers are not.

In terms of the above basis, the action of the infinitesimal variations on V can be written more explicitly as

$$\begin{aligned}
\delta_a &\in \text{Hom}(H^{p,q}(X), H^{p-1,q+1}(X)) \oplus \text{Hom}(H^{p,q-1}(Y), H^{p-1,q}(Y)) \\
\hat{\delta}_\alpha &\in \text{Hom}(H^{p,q}(X), H^{p-1,q}(Y)) \oplus \text{Hom}(H^{p,q-1}(Y), H^{p-1,q}(Y)).
\end{aligned} \tag{4.9}$$

Schematically, the open-closed string deformations act as

$$\begin{array}{ccccccc}
(3,0)_X & \xrightarrow{\delta_a} & (2,1)_X & \xrightarrow{\delta_a} & (1,2)_X & \xrightarrow{\delta_a} & (0,3)_X \\
& \searrow^{\hat{\delta}_\alpha} & & \searrow^{\hat{\delta}_\alpha} & & \searrow^{\hat{\delta}_\alpha} & \searrow^{\delta_a, \hat{\delta}_\alpha} \\
& & (2,0)_Y & \xrightarrow{\delta_a, \hat{\delta}_\alpha} & (1,1)_Y & \xrightarrow{\delta_a, \hat{\delta}_\alpha} & (0,2)_Y \xrightarrow{\delta_a, \hat{\delta}_\alpha} 0
\end{array} \tag{4.10}$$

From the maps defined in the above diagram one may obtain an explicit basis for the extended ring \mathcal{R}_{oc} at each point in the moduli space $\mathcal{M}_{\mathcal{N}=1}$, given the classes δ_a and $\hat{\delta}_\alpha$ at that point.¹⁶ The appropriate framework to address the latter question is to construct the topological connection ∇ on the family of B-models reached by finite deformations, which will be discussed now.

¹⁵ In the following equation we have used the Poincaré residue to write a basis of the relative cohomology for (X, Y) in terms of elements of the cohomologies on X and Y . This will be also used below. More generally, the filtrations are defined on the hypercohomology of the complexes $\Omega_X^*(\log Y)$.

¹⁶ For intersecting brane configurations, there may be additional steps in the vertical direction in (4.10), describing cohomology at the intersections of higher codimension.

4.2. Moduli dependent open-closed chiral ring and topological flat connection

The spaces $V = H^3(X, Y)$ for a family of open-closed B-models parametrized by some coordinates $z_A \in \mathcal{M}_{\mathcal{N}=1}$ are all isomorphic. The bundle \mathcal{V} over $\mathcal{M}_{\mathcal{N}=1}$, with fiber $V(z_A, \bar{z}_{\bar{A}})$, is thus locally constant and admits a flat connection ∇ . This is the Gauss-Manin connection on the relative cohomology bundle with fiber $H^3(X, Y; \mathbf{C})$. The existence of a flat connection on the family of twisted, two-dimensional $\mathcal{N} = 2$ SCFT's on the world-sheet has been predicted in full generality in [27] as a consequence of the so-called tt^* equations. Their arguments also apply to world-sheets with boundaries, although the content of the tt^* equations for the open-closed B-model has not been worked out so far¹⁷. We proceed with the geometric approach provided by the open-closed B-model; clearly it would be interesting to study and extend the results in the full framework of tt^* geometry.

To complete the geometric representation of the open-closed chiral ring, we are looking for: *i*) A graded basis of cohomology classes $\vec{\Phi}_I^{(q)}(z_A)$ for \mathcal{V} , that represents the elements of the chiral ring \mathcal{R}_{oc} ; *ii*) Flat topological coordinates $t_A(z_B)$ on the moduli space $\mathcal{M}_{\mathcal{N}=1}$ that correspond to the coupling constants in the 2d world-sheet Lagrangian. These are, on very general grounds [26], the good coordinates for the instanton expansion of the correlation functions. The topological basis $\{\vec{\Phi}_I^{(q)}(t_A)\}$ is singled out by the property

$$\frac{\partial}{\partial t_A} \vec{\Phi}_I^{(q)}(t_B) = C_{AI}^K(t_B) \vec{\Phi}_K^{(q+1)}(t_B). \quad (4.11)$$

This equation expresses that an infinitesimal variation in the t_A direction is equivalent to multiplication by the grade one field $\vec{\Phi}_A^{(1)}(t_B)$ in the chiral ring. Moreover the basis $\{\vec{\Phi}_I^{(q)}(t_A)\}$ is generated from the canonical vacuum by repeated application of (4.11).

To guarantee holomorphicity of the bundle, of which the $\vec{\Phi}_I^{(q)}(z_A)$ are sections¹⁸, one must define the classes $\vec{\Phi}_I^{(q)}(z_A)$ as an element of the Hodge filtration F^{3-q} rather than of pure grade q . In particular the bundles $\mathcal{F}^q = F^q \mathcal{V}$ are holomorphic sub-bundles of \mathcal{V} , whereas the bundles with fibers $Gr_F^q \mathcal{V}$ are not holomorphic. In this way one may preserve holomorphicity of the basis for the vacuum bundle; one may always project the $\vec{\Phi}_I^{(q)}(z_A)$ to the pieces of highest grade $3 - q$ at the end to obtain eventually a basis of pure Hodge type (varying non-holomorphically with the moduli z_A).

¹⁷ For a discussion of some aspects of the open string case, see [26].

¹⁸ We continue to denote arbitrary coordinates on $\mathcal{M}_{\mathcal{N}=1}$ by z_a and the flat topological coordinates by t_A .

To construct the basis $\{\vec{\Phi}_I^{(q)}\}$, we start from the unique element $\vec{\Phi}^{(0)}(z_A) = (\Omega(z_A), 0) \in \mathcal{F}^3$. The multi-derivatives $\partial/\partial z_{A_1} \dots \partial/\partial z_{A_k} \Omega$ for $k \leq 3$ generate a basis $\{\vec{\Psi}_I^{(q)}\}$ of V , where $\vec{\Psi}_I^{(q)}(z_A) \in \mathcal{F}^{3-q}$. This is a consequence of Griffiths' transversality

$$\nabla : F^p \mathcal{V} \longrightarrow F^{p-1} \mathcal{V} \otimes \Omega_{\mathcal{M}_{\mathcal{N}=1}},$$

which says that the action of the Gauss-Manin derivative increases the grade of the Hodge filtration by one. Note that the derivatives $\partial/\partial z_A$ differ from the covariant derivatives ∇_A by terms which do not increase the grade and thus are irrelevant at this step. Note also that in general not all of the multi-derivatives $\partial/\partial z_{A_1} \dots \partial/\partial z_{A_l} \Omega$ for $l \leq k$ will be linearly independent for given k , but only $\dim(F^{3-k}V)$ of them.

One may describe the moduli dependence of the elements $\vec{\Psi}_I^{(q)}(z_A)$ by expressing them in terms of a flat, moduli independent basis $\{\vec{\Gamma}_\Sigma\}$ for \mathcal{V} :

$$\vec{\Psi}_I^{(q)}(z_A) = \Pi_I^\Sigma(z_A) \vec{\Gamma}_\Sigma. \quad (4.12)$$

The basis $\{\vec{\Gamma}_\Sigma\}$ may be defined as the basis dual to a constant basis $\{\Gamma^\Sigma\}$ of moduli independent cycles in $H_3(X, Y)$. The transition matrix

$$\Pi_I^\Sigma(t) = \langle \Gamma^\Sigma, \vec{\Psi}_I \rangle, \quad (4.13)$$

is the *relative period matrix*. The pairing $\langle \Gamma^\Sigma, \vec{\Theta} \rangle$ in relative co-homology is defined by

$$\langle \Gamma^\Sigma, \vec{\Theta} \rangle = \int_{\Gamma^\Sigma} \Theta - \int_{\partial \Gamma^\Sigma} \theta, \quad (4.14)$$

for $\vec{\Theta} = (\Theta, \theta) \in H^3(X, Y)$ and $\Gamma^\Sigma \in H_3(X, Y)$. It is the appropriate pairing in relative co-homology, $H^3(X, Y) \times H_3(X, Y) \rightarrow \mathbf{C}$, invariant under the equivalence relation (4.3).

Let us order the basis $\{\vec{\Psi}_I^{(q)}\}$ by increasing grade q . The searched for basis $\{\vec{\Phi}_I^{(q)}\}$, representing the chiral ring, may be obtained from the basis $\{\vec{\Psi}_I^{(q)}\}$ by a linear transformation with holomorphic coefficient functions that preserves the Hodge filtration. By such a transformation one may put the relative period matrix Π_I^Σ into upper triangular form, implying the relations

$$\nabla_A \vec{\Psi}_I^{(q)}(z_B) = C_{AI}^K(z_B) \vec{\Psi}_K^{(q+1)}(z_B), \quad (4.15)$$

where we use again $\vec{\Psi}_I^{(q)}(z_A)$ to denote the elements of the transformed basis. It follows that the piece of pure type $d - q$ in $\vec{\Psi}_I^{(q+1)}$ represents the chiral ring, up to a moduli dependent normalization.

One may further find the *flat* topological coordinates $t_A(z_B)$ by requiring the connection pieces in ∇_A to vanish. This is achieved by a moduli dependent, holomorphic change of normalization of the classes $\vec{\Psi}_I^{(q)}$ that makes the block-diagonal terms in the transition matrix Π_I^Σ constant in this basis¹⁹. The classes so obtained provide the searched for basis $\{\vec{\Phi}_I^{(q)}\}$ that represents the open-closed chiral ring \mathcal{R}_{oc} .

Note that the flat coordinates t_A , as defined by the above procedure, are the solutions of

$$\nabla_A \Pi^\Sigma = \partial_A \Pi^\Sigma = \delta_{A'}^\Sigma, \quad \Sigma' = 0, \dots, M, \quad A = 1, \dots, M.$$

Here $\Pi^\Sigma \equiv \Pi_0^\Sigma$ is the relative period vector representing the first row of the period matrix and $M = \dim \mathcal{M}_{\mathcal{N}=1}$. Thus the flat coordinates t_A are given by ratios of relative period integrals, and this is what determines the $\mathcal{N} = 1$ mirror map $t_A(z_B)$ [11]:

$$t_A(z_B) = \frac{\langle \Gamma_A, \vec{\Phi}^{(0)}(z_B) \rangle}{\langle \Gamma_0, \vec{\Phi}^{(0)}(z_B) \rangle}, \quad A = 1, \dots, M. \quad (4.16)$$

For the choice of vacuum $\vec{\Phi}^{(0)} = (\Omega, 0)$, this expression is identical in form to the well-known mirror map for the flat coordinates for the closed string with $\mathcal{N} = 2$ supersymmetry [40,43]. However, in the present context, it describes the flat coordinates on the space of chiral $\mathcal{N} = 1$ multiplets $\mathcal{M}_{\mathcal{N}=1}$, with the pairing in (4.16) defined in relative co-homology. Another definition of flat coordinates had been previously given in [23,9,10], in terms of the tension of D-brane domain walls; it leads to the same functional dependence $t_A(z_A)$, at least for the class of open-closed compactifications studied so far.

Finally note that the equations (4.15) determine the chiral ring coefficients for grade $q > 1$ in terms of the derivatives of the relative period matrix. In fact, it follows from eq.(4.15) that the relative period matrix $\Pi_I^\Sigma(z_A)$ satisfies the system of linear differential equations

$$(\nabla_A - C_A) \Pi_I^\Sigma(z_A) = 0. \quad (4.17)$$

Eliminating the lower rows of Π_I^Σ in favor of the first row, one may thus obtain all chiral ring coefficients C_{AI}^K from derivatives of the relative period vector Π^Σ . In particular, at $q = 2$ one obtains the promised relations

$$C_{AI}^K = \partial_A \partial_B \mathcal{W}_K, \quad \mathcal{W}_K = \langle \Gamma^\Sigma, \vec{\Phi}_K^{(2)} \rangle, \quad (4.18)$$

¹⁹ In particular the diagonal block at grade $q = 1$ can be chosen to be the unit matrix $\mathbf{1}_M$.

which express the holomorphic potentials \mathcal{W}_K of $\mathcal{N} = 1$ special geometry as certain entries of the relative period matrix. In the flat coordinates t_A , the first rows and columns of Π_I^Σ take the form

$$\Pi_I^\Sigma = \langle \Gamma_\Sigma, \vec{\Phi}_I \rangle = \begin{pmatrix} 1 & t_A & \mathcal{W}_K & \dots \\ 0 & \delta_{AB} & \partial_B \mathcal{W}_K & \dots \end{pmatrix}. \quad (4.19)$$

By iterative elimination of the lower rows in (4.17) one obtains a system of homogeneous, linear differential equations for the period vector Π^Σ of higher order. This is a Picard-Fuchs system for the relative cohomology group $H^3(X, Y)$. One may determine the exact expressions for the mirror map (4.16) and the holomorphic potentials \mathcal{W}_K from the solutions to the Picard-Fuchs system with appropriate boundary conditions by standard methods (see [11,15] for some explicit examples).

As it turns out, this approach is particularly powerful for D -branes on Calabi–Yau manifolds that can be represented as hypersurfaces in toric varieties; this class of geometries will be discussed in the next section.

5. Relative cohomology in toric geometry, and GKZ type differential equations

In this section we illustrate the previous ideas in detail for a class of open-closed string Calabi–Yau backgrounds that may be realized by a linear sigma model (LSM) [44]. The aim is to describe how toric geometry gives an efficient and systematic framework for generating a basis for the relative cohomology group $H^3(X, Y)$ and a system of differential equations associated to it.

5.1. Relative cohomology in toric Calabi–Yau X

We consider a toric Calabi–Yau d -fold defined by a 2d LSM with N matter fields y_n , whose mirror X can be described by a superpotential of the form²⁰

$$\widetilde{W} = \sum_n a_n y_n, \quad \prod_n (y_n)^{l_n^{(a)}} = 1, \quad a = 1, \dots, h^{d-1,1}(X). \quad (5.1)$$

Here the $l^{(a)}$ are $N - d = h^{d-1,1}(X)$ integral vectors that define relations between the fields y_n . Moreover the coefficients a_n are N complex parameters that specify (with some redundancy) the complex structure of X .

²⁰ We use a tilde to distinguish the 2d superpotential \widetilde{W} from the superpotential in 4d space-time. For background material on the following definitions, see [45,36].

In view of the example that we will present further below, we will here discuss non-compact Calabi-Yau's X , for which the y_n are variables in \mathbf{C}^* and for which we do *not* impose $\widetilde{W} = 0$ [45]. We denote the true \mathbf{C} variables by $u_n = \ln(y_n)$. The following discussion can be also adapted to compact Calabi-Yau manifolds, where one imposes $\widetilde{W} = 0$ and where the y_n are variables with values in \mathbf{C} .

The relations in (5.1) can be solved in terms of d of the N variables y_n . In the following we order the variables y_n such that these d variables are $y_i, i = 0, \dots, d-1$, whereas variables $y_{\hat{k}}$ with hatted indices, $\hat{k} \geq d$, denote the variables eliminated in this step. The resulting superpotential in the d variables y_i can be written as

$$\widetilde{W}(y_i) = \sum_{n=0}^{N-1} a_n \left(\prod_{i=0}^{d-1} y_i^{v_n^i} \right),$$

where v_n^i are d null vectors of the matrix l defined by the choice of coordinates y_i . They satisfy $\sum_n l_n^{(a)} v_n^i = 0$ for all a and i .

In the patch parametrized by the variables $u_i = \ln(y_i)$, the holomorphic d -form can be written as [46,45]

$$\Omega = \left(\prod_{i=1}^d du_i \right) e^{-\widetilde{W}}. \quad (5.2)$$

The set of N derivatives:

$$\frac{\partial}{\partial a_n} \Omega = -y_n \Omega, \quad n = 0, \dots, N-1, \quad (5.3)$$

can be reduced to a basis of $h^{d-1,1}$ preferred derivatives in the following way. A natural choice for this basis is to keep the derivatives $\partial/\partial a_{\hat{k}}$ with respect to the parameters $a_{\hat{k}}, \hat{k} \geq d$ (which are the coefficients of the variables that have been eliminated in the first step).

The derivatives

$$\widetilde{W}_i = \frac{d}{du_i} \widetilde{W} = \sum_{n=0}^{N-1} a_n v_n^i y_n,$$

provide d equations to eliminate the d variables y_i for $i < d$ in the expression (5.3) for $\partial/\partial a_i$ in favor of the derivatives \widetilde{W}_i . An expression involving \widetilde{W}_i can then be further simplified by noting that the form

$$\left(\prod du_j \right) \widetilde{W}_i e^{-\widetilde{W}} = d(\omega_i), \quad \omega_i = (-)^i \left(\prod_{j \neq i} du_j \right) e^{-\widetilde{W}}$$

is exact in the absolute cohomology on X .²¹ The d derivatives $\frac{\partial}{\partial a_i} \Omega$ for $i < d$ can thus be expressed in terms of the $h^{d-1,1}$ independent derivatives $\Omega_n = \frac{\partial}{\partial a_n} \Omega$ for $n \geq d$ as follows:

$$a_i \frac{\partial}{\partial a_i} \Omega = d(\omega_i) - \sum_{n \neq i} v_n^i a_n \Omega_n. \quad (5.4)$$

Note that (5.4) can be used to express any product of derivatives in the a_n in terms of multiple derivatives with respect to the $h^{d-1,1}$ chosen derivatives $\frac{\partial}{\partial a_k}$, plus exact pieces that are derivatives of the exact piece in (5.4). Thus the problem of rewriting an arbitrary form obtained by acting with derivatives on Ω , in terms of elements of a chosen basis for $H^d(X)$ plus exact pieces, is already completely solved at this step.

This should be contrasted with Griffiths' reduction method [47,48], which applies to the general, non-toric case but is technically by far more complicated. The key point is that the toric parameters a_n linearize the moduli problem, a fact that leads to a well-known simplification in the derivation of the Picard-Fuchs equations for the absolute cohomology $H^d(X)$. The equations (5.4) describe a similar simplification for the treatment of exact pieces, which are relevant for the derivation of the differential equations for the relative cohomology group $H^d(X, Y)$. Specifically these exact pieces become important in open string backgrounds, where the extended chiral ring \mathcal{R}_{oc} lives in the relative cohomology $H^d(X, Y)$ and where one has to keep track of boundary terms.

From now on we specialize to the case $d = 3$, i.e., a Calabi–Yau 3-fold, and to a single D -brane on top of it. In order to define the relative cohomology group $H^3(X, Y)$, we need to specify an appropriate family of hypersurfaces $Y(\hat{z})$ in the toric Calabi–Yau X that slice the relevant special Lagrangian 3-chains. We will use the following ansatz, whose form will be derived in sect. 5.3,

$$Y(\hat{z}) : y_0 = \hat{z} y_1 \left(\frac{y_1}{y_2}\right)^\nu, \quad \hat{z} = \frac{a_0}{a_1} \left(\frac{a_2}{a_1}\right)^\nu, \quad (5.5)$$

where y_i , $i = 0, 1, 2$, denote the coordinates on the 3-fold X . Moreover \hat{z} is the open string modulus that specifies the position of the hypersurface $Y(\hat{z})$ that passes through the boundary 2-cycle wrapped by the D -brane. As argued later, with the above ansatz the parameter \hat{z} is one of the good algebraic coordinates on the moduli space $\mathcal{M}_{\mathcal{N}=1}$. That is, $\hat{z} \rightarrow 0$ defines a classical limit in which the associated flat coordinate has the leading behavior $\hat{t} \sim \frac{1}{2\pi i} \ln(\hat{z})$.

²¹ Although our presentation illustrates the underlying idea of the computation, this interpretation is really oversimplified. As we are using here the parameters a_n , some of which are redundant by \mathbf{C}^* rescalings of the coordinates, the equations for Ω written in this section should be interpreted in the \mathbf{C}^* equivariant cohomology on X .

5.2. *A system of GKZ equations for the relative cohomology*

From the above considerations, one may obtain a complete system of differential equations for the relative cohomology as follows. The holomorphic 3-form Ω on the Calabi–Yau manifold X satisfies a system of differential equations, a so-called GKZ [49] system

$$\mathcal{L}_a \Omega = \left(\prod_{l_i^{(a)} > 0} \partial_{a_i}^{l_i^{(a)}} - \prod_{l_i^{(a)} < 0} \partial_{a_i}^{-l_i^{(a)}} \right) \Omega = 0, \quad a = 1, \dots, h^{2,1}(X). \quad (5.6)$$

These equations follow straightforwardly from the explicit expression (5.2) for Ω . An important point is that the differential equations (5.6) continue to hold in the relative cohomology $H^3(X, Y)$, if the canonical vacuum is defined by $\vec{\Omega} = (\Omega, 0)$ as in (4.4). However the content of these equations is different; first they now describe differential equations that involve both, the open and closed string moduli $z_A = (z_a, \hat{z})$, defined respectively by (5.5) and

$$z_a = \prod_n (a_n)^{l_n^{(a)}}, \quad a = 1, \dots, h^{2,1}(X). \quad (5.7)$$

The z_a are good local parameters for the complex structure, and are invariant under those \mathbf{C}^* rescalings of the y_i that preserve the form of the LG potential for X .

Second, derivatives of $(\Omega, 0)$ produce terms that, despite being exact in the absolute cohomology $H^3(X)$, generate non-trivial elements in the relative cohomology $H^3(X, Y)$ on the boundary Y , as discussed in sect. 5.1. Thus, the differential operators \mathcal{L}_a now represent linear constraints on the space spanned by the linearly independent elements in $H^3(X, Y)$ (which represents the open-closed chiral ring \mathcal{R}_{oc}).

There are additional differential equations needed to augment the GKZ system (5.6) to a complete system of differential equations for the periods of the relative cohomology $H^3(X, Y)$. The origin of these equations is obvious: They are the differential equations satisfied by the holomorphic 2-form

$$\omega = du_1 du_2 e^{-\widetilde{W}_Y}, \quad \widetilde{W}_Y(y_1, y_2) = \widetilde{W}|_Y, \quad (5.8)$$

on the hypersurface Y defined by the LG potential \widetilde{W}_Y . It is straightforward to see from these expressions that ω satisfies the following differential equations:

$$\mathcal{L}_a \omega = 0, \quad \widehat{\mathcal{L}} \omega = 0, \quad (5.9)$$

where $\widehat{\mathcal{L}}$ is a differential operator of the form (5.6) defined by the vector

$$\hat{l} = (1, -\nu - 1, \nu, 0, \dots, 0). \quad (5.10)$$

Moreover, eqs.(5.5) and (5.7) imply that

$$\hat{\theta} \Omega = \hat{z} \frac{d}{d\hat{z}} \Omega = \text{const. } \omega .$$

It follows that the following differential operators provide a complete system that determines the periods of the relative cohomology:

$$\begin{aligned} \mathcal{L}_a \Omega &= 0, & a = 1, \dots, h^{2,1}(X), \\ \widehat{\mathcal{L}} \hat{\theta} \Omega &= 0. \end{aligned} \tag{5.11}$$

In passing we note that the above discussion extends the open-closed string dualities found in [11], from the special case $\nu = 0$ to all values of ν . Specifically, there exists a “dual” closed string compactification on a Calabi–Yau 4-fold M without branes for any of the open-closed string geometries considered above. In particular, the holomorphic potentials \mathcal{W}_K of the open-closed string are reproduced by the genus zero partition function \mathcal{F}_0 of the closed string on M . This implies a quite amazing coincidence of the “numbers” of sphere and disc instantons counted by the A-model for the D-brane geometry on the mirror of X , and the “numbers” of sphere instantons computed by the A-model on the mirror of the 4-fold M . As the necessary modifications of the 4-fold geometries for $\nu \neq 0$ are straightforward when starting from the 4-folds described in [11], we will not go into the details, except for a brief comment in sect. 6.

5.3. Degree ν hypersurfaces Y , framing ambiguity, winding sectors and open-closed string moduli

It remains to justify the ansatz (5.5) for the family of hypersurfaces Y in the toric Calabi–Yau X that slice the relevant special Lagrangian 3-chains.

In fact, the definition of the family of hypersurfaces $Y \subset X$ is parallel to the definition of the Calabi–Yau X itself. To start with, note that the relations defined by the $h^{2,1}$ charge vectors $l^{(a)}$ define homogeneous “hypersurfaces”²² H_a and X is defined on the intersection $\cap_a H_a$ of these $h^{2,1}$ hypersurfaces. One may associate moduli to these hypersurfaces as in (5.7), by writing

$$H_a : \prod_n (y_n)^{l_n^{(a)}} = z_a, \quad z_a = \prod_n (a_n)^{l_n^{(a)}}. \tag{5.12}$$

A modulus z_a is, by definition, invariant under the \mathbf{C}^* action that preserves the hypersurface H_a , but on the other hand parametrizes a \mathbf{C}^* action that moves the hypersurface H_a

²² This terminology is not quite precise, as will be discussed momentarily.

in the ambient space. Specifically, the (closed string) moduli z_a , $a = 1, \dots, h^{1,2}$, provide coordinates on the moduli space of complex structures on X that are invariant under those \mathbf{C}^* rescalings of the coordinates y_i that preserve the LG potential.

The ansatz (5.5) for the hypersurface Y

$$Y(\hat{z}) : y_0 = \hat{z} y_1 \left(\frac{y_1}{y_2}\right)^\nu, \quad \hat{z} = \frac{a_0}{a_1} \left(\frac{a_2}{a_1}\right)^\nu,$$

is of the same general form as (5.12). It is subject to the following two constraints: *i*) it is homogeneous in the coordinates y_i ; *ii*) in the limit $\hat{z} \rightarrow 0$, $Y(\hat{z})$ approaches the hypersurface $y_0 = 0$ with an asymptotic behavior linear in \hat{z} .²³

Let us now explain the origin of these conditions. Although it seems natural, the homogeneity of the ansatz for $Y(\hat{z})$ is not really obvious. In fact this constraint is mirror to that derived in [9] on a *supersymmetric* D-brane in the A-model. This constraint imposes the vanishing of the sum over the components of \hat{l} in (5.10). Note that in the B-model that we are considering, it implies the homogeneity of the equation and is indeed nothing but the condition that the intersection $Y \cap X$ is a (possibly singular) Calabi–Yau manifold of two dimensions²⁴.

The second constraint, item *ii*), needs a little more explanation. It is related to a fundamentally new geometric structure in the open-closed string compactification, which is related to the logarithmic multi-valuedness of the good coordinates $u_i = \ln(y_i)$.²⁵ First note that an equation like (5.12) does not really define a hypersurface, but a lattice of hypersurfaces in the true coordinates u_i , one for each of the sheets of the logarithm. Specifying a Dirichlet boundary condition for an open string in X includes therefore the selection of a particular sheet of the logarithm $u_i = \ln(y_i)$. This implies that there will also be extra winding states in the open string sector that interpolate between the hypersurfaces Y on different log sheets [10]. In contrast, the observables in the closed string B-model are related to absolute homology cycles in X and thus live on a single, connected sheet.

²³ Of course the choice of singling out y_0 is a convention, taken for simplicity of notation. There are similar phases of D-branes located in other patches of X , described by hypersurfaces Y whose classical limit is $y_n = 0$ for some other n .

²⁴ For $\nu = 0$, the intersection $Y \cap X$ is easily recognized as the standard form for the mirror of an ALE space, as illustrated further in sect. 6.2.

²⁵ The following discussion is very similar to the argument for the identification of the open string modulus with a D-brane tension in ref. [10]. It is more concrete in the sense that it identifies the “framing ambiguity” discussed there, with the degree of the hypersurface Y .

The lattice generated by the log sheets for the open string sector should be seen as the way the Calabi–Yau manages to meet the expectations from T-duality. As is well-known, the Wilson line on a D-brane wrapped on S^1 is mapped by T-duality to a position on the dual S^1 . The phase of the open string modulus for the A-type geometry which is mirror to our B-model, is precisely a Wilson line on the type IIA D-brane [6,7]. This phase needs to be mapped to a position on a circle in the Calabi–Yau X for the B-model, but there are no S^1 's in the Calabi–Yau manifold.²⁶ This S^1 structure, which is supposedly seen in the open string sector but not in the closed string sector, is realized by the lattice of log sheets which appear in the very definition of the manifold X . T-duality suggests that a similar logarithmic structure could be also relevant for compact Calabi–Yau manifolds with open string sectors.

The constraint $y_0 \sim \hat{z}$ near $\hat{z} = 0$ in (5.5) is explained by the above considerations as follows. Near a classical limit, where the instanton corrections to the superpotential are suppressed, the good local open string modulus $\hat{z} = 0$ will satisfy $\hat{z} \sim e^{2\pi i \hat{t}}$. In this limit, the classical mirror A-model geometry is valid and we can identify the real part of \hat{t} with the Wilson line on the A-type brane. An integral shift $\hat{t} \rightarrow \hat{t} + 2\pi$ of the Wilson line is mirror to a full rotation on the T-dual circle, and it must map the hypersurface Y to “the same” hypersurface, up to a change of the sheet of the logarithm induced by $y_0 \rightarrow e^{2\pi i} y_0$.

As this is a classical argument which is valid in the limit $\hat{z} \rightarrow 0$, it fixes the asymptotic behavior $y_i \sim \hat{z}$, for a hypersurface Y that is classically defined by $y_i = 0$; it does not fix the degree of the hypersurface Y in (5.5), which is given by the integer ν .

The integer ν in the definition (5.5) of Y is in fact an geometric invariant of our B-model which represents the “framing ambiguity” discovered in [10]. It is related to the choice of a framing for a knot in the Chern-Simons theory on the A-type mirror brane.

The geometric identification of the framing number ν in the D-brane geometry of the A-model has been given in [22], as the number of times the boundary of an world-sheet instanton ending on the A-type brane wraps around the D-brane in transverse space. This wrapping number can be specified by choosing a $U(1)$ action on the three coordinates of the A-model, relative to the $U(1)$ defined by the boundary of the world-sheet instanton.

From the equation (5.5) we see that the interpretation of the framing number as the degree of the hypersurface Y is precisely the mirror of this statement. Rewriting the constraint as $\hat{z} = (y_0/y_1)(y_2/y_1)^\nu$, one observes that the phase of \hat{z} (identified with the $U(1)$ acting on the S^1 boundary of an world-sheet instanton) defines a particular $U(1)$

²⁶ Note that the crux of the T-duality is that the gauge degree of freedom on the D-brane becomes geometric in the B-model.

action T on the phases of the coordinates y_i . The integer ν specifies the orientation of the $U(1)$ action T w.r.t to the classical definition of the $U(1)$ on the D-brane, which is the phase of y_0/y_1 .

6. A case study: D5-branes on the mirror of the non-compact Calabi–Yau $X = \mathcal{O}(-3)_{\mathbf{P}^2}$

To illustrate the somewhat abstract ideas presented above, we will now study a particular D-brane geometry in some detail. Specifically, we consider the non-compact Calabi–Yau X that is mirror to the canonical bundle on \mathbf{P}^2 . The superpotential for a D5 brane in this geometry has been computed already in [10,11,15].

The single integral vector for the LSM describing this non-compact Calabi–Yau manifold is given by $l^{(1)} = (-3, 1, 1, 1)$. Accordingly, the mirror geometry is captured by the following Landau-Ginzburg (LG) superpotential:

$$\widetilde{W} = a_0 y_0 + a_1 y_1 + a_2 y_2 + a_3 y_3, \quad \text{where } y_3 = y_0^3 / y_1 y_2. \quad (6.1)$$

The D -brane geometry is specified by the “hypersurface” Y , defined as in (5.5). The classical limit $y_0 = 0$ for Y corresponds to a D -brane in the “outer phase” in the nomenclature of ref. [15]. In order to describe geometries with general ν , it is convenient to change variables $y_0 \rightarrow y_0' (y_1/y_2)^\nu$. The surface Y can then be represented by setting $y_0' = y_1$ in the superpotential

$$\widetilde{W}(\nu) = a_0 y_0' y_1^\nu y_2^{-\nu} + a_1 y_1 + a_2 y_2 + a_3 y_0'^3 y_1^{3\nu-1} y_2^{-3\nu-1}.$$

The derivatives of the holomorphic (3,0) form Ω defined as in (5.2) obey the relations

$$\begin{aligned} a_0 \partial_{a_0} \Omega &= d\omega_{0'} - 3a_3 \partial_{a_3} \Omega \\ a_1 \partial_{a_1} \Omega &= d\omega_1 - \nu d\omega_{0'} + a_3 \partial_{a_3} \Omega \\ a_2 \partial_{a_2} \Omega &= d\omega_2 + \nu d\omega_{0'} + a_3 \partial_{a_3} \Omega, \end{aligned} \quad (6.2)$$

where $\omega_{0'} = d(du_1 du_2 e^{-\widetilde{W}(\nu)})$, and similarly for the other ω_i . The forms $d\omega_i$ are exact in $H^3(X)$. However, as emphasized before, they contribute in chain integrals with non-zero boundary components, and represent non-trivial elements in the relative cohomology. Specifically, under “partial integration” one has $\omega_2|_Y = 0$, $\omega_0|_Y = -\omega_1|_Y = \omega$, where ω is the holomorphic (2,0)-form on Y :

$$\omega = du_1 du_2 e^{-\widetilde{W}_Y(\nu)}. \quad (6.3)$$

By iteratively applying (6.2) and reducing the exact pieces to forms on Y , one may express any derivative of Ω in terms of a_3 -derivatives of Ω and ω .

In eq. (6.3), the expression $\widetilde{W}_Y(\nu)$ denotes the restriction of the superpotential \widetilde{W} to Y :

$$\widetilde{W}_Y(\nu) = a_0 y_1^{\nu+1} y_2^{-\nu} + a_1 y_1 + a_2 y_2 + a_3 y_1^{3\nu+2} y_2^{-3\nu-1}, \quad (6.4)$$

Note that for a hypersurface Y of degree 1, that is, for the hyperplanes defined with $\nu = 0, -1$, this turns into the standard LG superpotential for the mirror of an ALE space with A_1 singularity, however with an unusual parametrization of the moduli. E.g., for $\nu = 0$ one finds

$$\widetilde{W}_Y(0) = (a_0 + a_1) y_1 + a_2 y_2 + a_3 y_3, \quad \text{where } y_3 = y_1^2 / y_2. \quad (6.5)$$

Note also that after a rescaling $y_i \rightarrow a_i^{-1} y_i$, the potential $\widetilde{W}(\nu)$ and the constraint for Y depend only on the algebraic coordinates $z_A = (z_1; z_2)$, where

$$z_1 = \frac{a_1 a_2 a_3}{a_0^3}, \quad z_2 = \hat{z} = \frac{a_0}{a_1} \left(\frac{a_2}{a_1} \right)^\nu, \quad (6.6)$$

represent the bulk modulus and the brane modulus, respectively.

In the following two sections, we will discuss separately the cases where Y is either a hyperplane, or a hypersurface of higher degree.

6.1. The hyperplane case, $\nu = 0$

We should mention here that the motivation for this sub-section is *not* the derivation of the differential equations for the flat coordinates and the superpotential; these equations have been written down in full generality in sect. 5.2. Rather, the following computations demonstrate how the various 3-forms on the Calabi–Yau X , and the 2-forms on the ALE space defined by the “hyperplane” Y , nicely fit together to a basis for the relative cohomology group $H^3(X, Y)$. This will allow to give an explicit representation of the open-closed chiral ring, and to write down its structure constants. From these we will then determine, via the matrix differential equations (4.17), the relative period matrix.

The starting point is to express arbitrary derivatives w.r.t. the moduli of the holomorphic $(3, 0)$ form, in terms of a suitable basis of differential forms spanning $H^3(X, Y)$. By iteratively applying (6.2) (with $\nu = 0$), we can express any derivative of Ω in terms of a_3 -derivatives of Ω and ω . We have collected some sample expressions in the following table:

	$a_3^3\Omega'''$	$a_3^2\Omega''$	$a_3\Omega'$	$a_0^2h^2\omega''$	$a_0a_1h^2\omega''$	$a_1^2h^2\omega''$	$a_0h\omega'$	$a_1h\omega'$	ω
θ_1			1						
θ_2									1
θ_1^2		1	1						
$\theta_1\theta_2$							1	1	
θ_2^2							1	3	
θ_1^3	1	3	1						
$\theta_1^2\theta_2$				1	2	1	1	1	
$\theta_1\theta_2^2$				1	4	3	1	3	
θ_2^3				1	6	9	$\frac{a_0}{a_0+a_1}$	$\frac{9a_1+4a_0}{a_0+a_1}$	

(6.7)

Here a prime denotes $\frac{\partial}{\partial a_3}$, $h = \frac{a_3}{a_0+a_1}$, and $\theta_A = z_A \partial_{z_A}$ are the logarithmic derivatives with respect to the good local coordinates (6.6) (for $\nu = 0$) of the combined open-closed string moduli space.

The list of the forms displayed on the top of the table can be further reduced, by making use of relations between the derivatives $\partial/\partial a_n$ that originate from the specific form of \widetilde{W} and \widetilde{W}_Y for $\nu = 0$. The first such relation is the GKZ system (5.6) for $X = \mathcal{O}(-3)_{\mathbf{P}^2}$:

$$\mathcal{L}_1\Omega = (\partial_{a_1}\partial_{a_2}\partial_{a_3} - \partial_{a_0}^3)\Omega = 0.$$

It allows one to eliminate Ω''' in favor of lower-order derivatives.

As indicated earlier, there are additional differential equations (5.9) satisfied by the $(2,0)$ form ω (6.3) on the A_1 ALE space, Y . The differential operator associated with \hat{l} is

$$\widehat{\mathcal{L}}_1\omega = (\partial_{a_1} - \partial_{a_0})\omega = 0, \quad (6.8)$$

It expresses the redundancy of the parametrization of the coefficients in \widetilde{W}_Y . Using the trivial identities

$$\partial_{a_0} = \frac{1}{a_0}(\theta_2 - 3\theta_1), \quad \partial_{a_1} = \frac{1}{a_1}(\theta_1 - \theta_2),$$

it gives rise to the second PF operator in eq.(6.11) below.

Next we consider a certain linear combination of the differential relations in (5.9), corresponding to the following vector of relations: $l^{(1)} - \hat{l}$; this gives an alternative viewpoint of the geometric meaning of the differential operators. In fact, this second differential

operator annihilating ω describes the ordinary GKZ equation for the ALE space with A_1 singularity. The latter is defined by the charge vector $l = (-2, 1, 1)$, leading to the relation

$$\widehat{\mathcal{L}}_2 \omega = (\partial_{\tilde{a}_2} \partial_{\tilde{a}_3} - \partial_{\tilde{a}_1}^2) \omega = 0 . \quad (6.9)$$

From (6.5), the parameters \tilde{a}_n for the A_1 singularity are related to the complex structure parameters for X by $\tilde{a}_1 = a_1 + a_0$, $\tilde{a}_2 = a_2$, $\tilde{a}_3 = a_3$. The differential equation (6.9) can be used to eliminate ω'' in the table in favor of ω' :

$$a_3 \omega'' = \left(\frac{6\tilde{z} - 1}{1 - 4\tilde{z}} \right) \omega' ,$$

where $\tilde{z} = \tilde{a}_2 \tilde{a}_3 / \tilde{a}_1^2$.

The three differential equations thus allow one to reduce the forms in the table to a minimal basis given by $\{\Omega, \Omega', \Omega'', \omega, \omega'\}$.

From now on it will be more convenient to switch to the good local variables z_A defined in (6.6) (with $\nu = 0$). In terms of these variables we obtain, after making some convenient linear combinations, the following system of differential equations:

$$\begin{aligned} \mathcal{L}_1 \Omega &= 0, \\ \widehat{\mathcal{L}}_{1,2} \omega &= \widehat{\mathcal{L}}_{1,2} \theta_2 \Omega = 0 , \end{aligned} \quad (6.10)$$

with

$$\begin{aligned} \mathcal{L}_1 &= \theta_1^2 (\theta_1 - \theta_2) + z_1 (3\theta_1 - \theta_2) (1 + 3\theta_1 - \theta_2) (2 + 3\theta_1 - \theta_2) , \\ \widehat{\mathcal{L}}_1 &= (3\theta_1 - \theta_2) - z_2 (\theta_1 - \theta_2) , \\ \widehat{\mathcal{L}}_2 &= (z_2 - 3) ((1 - z_2)^2 + 4z_2^3 z_1) \theta_2^2 + 2z_2 (1 - z_2 - 9z_2^2 z_1 + z_2^3 z_1) \theta_2 . \end{aligned} \quad (6.11)$$

As discussed in sect. 5.2, one of the operators, i.e., $\widehat{\mathcal{L}}_2$, is in fact redundant. A complete system of differential operators for Ω , namely $\{\mathcal{L}_1, \widehat{\mathcal{L}}_1 \theta_2\}$, had already been obtained in [11,15]²⁷.

Note that due to the simple form of the second and the third operators, one can easily write down the general solution for the system $\widehat{\mathcal{L}}_{1,2} \omega = 0$, which yields the periods of the $(2, 0)$ form ω . It is given by the logarithmic derivative of the space-time superpotential:

$$\theta_2 \mathcal{W} \equiv \text{const.} \log \left[\frac{1 - z_2 + \sqrt{(z_2 - 1)^2 + 4z_1 z_2^3}}{\sqrt{4z_1 z_2^3}} \right] + \text{const.} \quad (6.12)$$

²⁷ See also [12] for a closely related discussion.

The operators (6.11) enable us to express any derivative of Ω in terms of the following minimal basis for the relative cohomology $H^3(X, Y)$:

$$\vec{\pi}(z) = \{(\Omega, 0), (\theta_1\Omega, 0), (0, \omega), (\theta_1^2\Omega, 0), (0, \theta_1\omega)\}^t. \quad (6.13)$$

The action of single logarithmic derivatives θ_A on $\vec{\pi}$ can be expressed in terms of the linear system (4.15), i.e.,

$$D_A \cdot \vec{\pi}(z) = 0, \quad A = 1, 2, \quad (6.14)$$

where

$$D_1 = \mathbf{1}\theta_1 - \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -\frac{6z_1}{\Delta_1} & \frac{2z_1}{\Delta_1} & -\frac{27z_1}{\Delta_1} & \frac{z_1 B}{\Delta_1 \Delta_2 \Delta_3} \\ 0 & 0 & 0 & 0 & -\frac{2z_2^3 z_1}{\Delta_2} \end{pmatrix} \quad (6.15)$$

$$D_2 = \mathbf{1}\theta_2 - \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{z_2-3}{\Delta_3} \\ 0 & 0 & 0 & 0 & -\frac{2z_2^3 z_1}{\Delta_2} \\ 0 & 0 & 0 & 0 & \frac{2(3-z_2)z_2^3 z_1}{\Delta_3 \Delta_2} \end{pmatrix}.$$

Here $B = -9 + 31z_2 - 37z_2^2 + z_2^3(17 - 18z_1) + 2z_2^4(11z_1 - 1)$, and the Δ_i are discriminant factors associated with the PF system:²⁸

$$\begin{aligned} \Delta_1 &= 1 + 27z_1, \\ \Delta_2 &= (1 - z_2)^2 + 4z_2^3 z_1, \\ \Delta_3 &= z_2 - 1. \end{aligned} \quad (6.16)$$

The matrices (6.15) give an explicit representation of the maps in the following diagram of the variations of mixed Hodge structures:

$$\begin{array}{ccccc} (\Omega, 0) & \xrightarrow{\theta_1^{+1}} & (\theta_1\Omega, 0) & \xrightarrow{\theta_1^{+1}} & (\theta_1^2\Omega, 0) \\ & \searrow \theta_2^{+1} & & \searrow \theta_2^{+1} & \searrow \theta_1^{+1}, \theta_2^{+1} \\ & & (0, \omega) & \xrightarrow{\theta_1^{+1}, \theta_2^{+1}} & (0, \theta_1\omega) \xrightarrow{\theta_1^{+1}, \theta_2^{+1}} 0 \end{array} \quad (6.17)$$

²⁸ Note that $\Delta_2 = 0$ is reminiscent of the splitting of classical singularities in $N = 2$ SYM theory. It would be interesting to investigate this from the view point of non-perturbative brane physics; for example, in terms of domain walls becoming tensionless.

Here the superscript on θ_a^{+1} denotes the grade one piece in the derivative. Note that the diagram truncates at grade two because of the non-compactness of both of the 3-fold $X = \mathcal{O}(-3)_{\mathbf{P}^2}$ and the embedded ALE space Y .

A consistency check on the matrices (6.15) is that they satisfy

$$[D_A, D_B] = 0, \quad (6.18)$$

and this explicitly demonstrates the flatness and integrability of the combined closed and open string moduli space. As discussed in sect. 4.2, we know that in general: $D_A = \nabla_A - C_A = \theta_A - \mathcal{A}_A - C_A$. Moreover, we know that the flatness property (6.18) allows us to introduce flat coordinates, t_A , and go to a gauge in which the Gauss-Manin connection \mathcal{A}_A vanishes (the ring structure constants C_A are distinguished by the property of being strictly upper triangular, *i.e.*, having zeroes on the diagonal).

This is easy to explicitly verify in the present example, by making use of the solutions of the differential system (6.11). As discussed before, a classical limit is defined by $z_A = 0$ and the leading behavior of the flat coordinates is $t_A = \ln(z_A) + \mathcal{O}(z_A)$. A power series in the z_A then represents an instanton expansion in the exponentials $q_A = \exp(2\pi i t_A)$, which is invariant under the shifts of the real parts, $t_A \rightarrow t_A + 2\pi$. The exact solutions to the differential operators (6.11) with the requisite leading behavior are known and given by

$$t_1(z.) = \log(z_1) - 3S(z_1), \quad t_2(z.) = \log(z_2) + S(z_1), \quad (6.19)$$

where

$$S(z_1) = - \sum_{n_1 > 0} \frac{(-)^{n_1} (3n_1 - 1)!}{n_1!^3} z_1^{n_1}. \quad (6.20)$$

One may check that after transforming to the coordinates t_A , the lower-triangular parts of D_A indeed vanish. There are still some degree zero terms in the connection due to the fact that we have not properly fixed the normalization of the forms in (6.13). Moreover there is the further ambiguity in that forms of the same degree can mix, so the complete flattening of the connection, including all of its block-diagonal parts, will require rescalings of the basis of differential forms along with a careful choice of linear combinations. Ultimately we are led to the following flat basis for the relative cohomology:

$$\vec{\pi}(t) = \{(\Omega, 0), (\partial_{t_1} \Omega, 0), (0, \eta), (f \partial_{t_1}^2 \Omega, -f h \partial_{t_1} \eta), (0, g \partial_{t_1} \eta)\}^t \in H^3(X, Y), \quad (6.21)$$

where we have defined $d\eta \equiv \partial_{t_2} \Omega$, and the rescalings are given by $f = 1/\partial_{t_1}^3 \mathcal{F}$, $g = 1/\partial_{t_2} \partial_{t_1} \mathcal{W}$, and $h = g \partial_{t_1}^2 \mathcal{W}$; the functions \mathcal{F} and \mathcal{W} will be defined below. This period vector is a solution of the completely flattened Gauss-Manin system:

$$\left(\frac{\partial}{\partial t_A} - C_A(t) \right) \cdot \vec{\pi}(t) = 0, \quad A = 1, 2. \quad (6.22)$$

where the open-closed chiral ring structure constants have been obtained from (6.15) and the mirror map (6.19) as follows:

$$\begin{aligned}
C_1(t) &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \partial_{t_1}^3 \mathcal{F} & \partial_{t_1}^2 \mathcal{W} \\ 0 & 0 & 0 & 0 & \partial_{t_2} \partial_{t_1} \mathcal{W} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
C_2(t) &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial_{t_2} \partial_{t_1} \mathcal{W} \\ 0 & 0 & 0 & 0 & \partial_{t_2}^2 \mathcal{W} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned} \tag{6.23}$$

As expected, $\mathcal{F} = \mathcal{F}(t_1)$ as computed in this way turns out to precisely coincide with the known bulk $\mathcal{N} = 2$ prepotential associated with $X = \mathcal{O}(-3)_{\mathbf{P}^2}$, which is of the form²⁹ $\mathcal{F}(t_1) = -\frac{1}{18}t_1^3 + \sum_{n_1} N_{n_1}^{(1)} Li_3(e^{2\pi i n_1 t_1})$; the precise values of the sphere instanton coefficients $N_{n_1}^{(1)}$ are known but are not important here.

Moreover, $\mathcal{W} = \mathcal{W}(t_A)$ coincides with the known superpotential on the world-volume of the D-brane, for which the instanton expansion is of the form:

$$\begin{aligned}
\mathcal{W}(t_A) &= \sum_{n_1 \geq 0, n_2 > n_1} \frac{(-)^{n_1} (n_2 - n_1 - 1)!}{(n_2 - 3n_1)! n_1!^2 n_2} z_1^{n_1} z_2^{n_2} \\
&= \sum_{n_1, n_2} N_{n_1, n_2}^{(2)} Li_2(e^{2\pi i (n_1 t_1 + n_2 t_2)}).
\end{aligned} \tag{6.24}$$

For explicit values for some of the disk instanton coefficients $N_{n_1, n_2}^{(2)}$, see e.g. [10,11].

In the above basis for the chiral ring, adapted to the special coordinates t_A on $\mathcal{M}_{\mathcal{N}=1}$, the relative period matrix $\Pi_I^\Sigma(t_A)$ takes the form

$$\Pi_I^\Sigma(t_A) = \langle \Gamma^\Sigma, \pi_I(t_A) \rangle = \begin{pmatrix} 1 & t_1 & t_2 & \partial_{t_1} \mathcal{F} & \mathcal{W} \\ 0 & 1 & 0 & \partial_{t_1}^2 \mathcal{F} & \partial_{t_1} \mathcal{W} \\ 0 & 0 & 1 & 0 & \partial_{t_2} \mathcal{W} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{6.25}$$

where Γ^Σ are 3-cycles in the flux sector, $\Sigma \in \{1, 2, 4\}$, and 3-chains in the brane sector, $\Sigma \in \{3, 5\}$. This matches the form (2.8) upon setting $\mathcal{W}_1 = \partial_{t_1} \mathcal{F}$, $\mathcal{W}_2 = \mathcal{W}$; the diagonal terms can be restored by undoing the rescalings by f, g .

²⁹ The k^{th} polylogarithm is defined by $Li_k(q) = \sum_{n>0} q^n / n^k$ for $k \geq 1$.

The relative period matrix (6.25) has been previously computed in [11], as the *ordinary* period matrix on the absolute cohomology $H^4(M)$ of a certain Calabi–Yau 4-fold M that is related, via an open-closed string duality, to the D-brane geometry on $\mathcal{O}(-3)_{\mathbf{P}^2}$. In particular, the holomorphic $\mathcal{N} = 1$ special geometry of the open-closed string type II string is in this case identical to the special geometry of the moduli space of the Calabi–Yau 4-fold M .

To conclude this section, we note that it should be possible to interpret the open-closed chiral ring structure constants (6.23) in terms of correlation functions of a boundary TFT. However, this goal is hampered in the present example for practical reasons, namely by the non-compactness of both the target space and the D-brane world-volumes. At any rate, the fact that $(C_i(t)C_j(t))_1^{\rho_{op-cl}} = \partial_{t_i}\partial_{t_j}\mathcal{W}(t)$ is highly reminiscent of the well-known relation $(C_i(t)C_j(t)C_k(t))_1^{\rho_{cl}} = \partial_{t_i}\partial_{t_j}\partial_{t_k}\mathcal{F}(t)$ in the bulk TFT (here ρ_{\dots} refers to the top elements of the open-closed and closed chiral rings, respectively).

6.2. Families of hypersurfaces of degree > 1

As discussed in sect. 5, the instanton expansion of the superpotential can be obtained readily from the system of differential equations given in eqs.(5.11), for all integers ν in (5.5). Below we sketch briefly the effectiveness of this type of computations for some hypersurfaces of degree larger than one.

Defining

$$l^{(1)} = (-3, 1, 1, 1, 0, 0), \quad l^{(2)} = (1, -\nu - 1, \nu, 0, 1, -1), \quad (6.26)$$

the moduli z_A and the GKZ system for the relative cohomology on the hypersurface Y follow from the standard toric definitions (5.6) and (5.7) for all ν , upon setting $a_4 = a_5 = 1$.³⁰ Specifically, the GKZ system (5.11) associated with (6.26),

$$\begin{aligned} \mathcal{L}_1(\nu) &= \theta_1(\theta_1 - (\nu+1)\theta_2)(\theta_1 + \nu\theta_2) + z_1(3\theta_1 - \theta_2)(1 + 3\theta_1 - \theta_2)(2 + 3\theta_1 - \theta_2), \\ \widehat{\mathcal{L}}_1(\nu) &= (3\theta_1 - \theta_2) \prod_{\ell=0}^{\nu-1} (\theta_1 + \nu\theta_2 - \ell) - z_2 \prod_{\ell=0}^{\nu} (\theta_1 - (\nu+1)\theta_2 - \ell), \end{aligned} \quad (6.27)$$

leads to a quick determination of the instanton corrected superpotential in terms of hypergeometric series (for simplicity we consider only the case $\nu \geq 0$; $\nu < 0$ leads to very similar expressions). The solutions with single logarithmic leading behavior in the z_A are

³⁰ The vectors (6.26) can also be associated to the ordinary Hodge variation on the middle cohomology $H^4(M)$ on a Calabi–Yau 4-fold M ; see [11] for details.

still given by (6.19) and thus the $\mathcal{N} = 1$ mirror map does not depend on the integer ν . This is a special property of the present example which is not true in general.

The instanton expansion of the superpotential is obtained, as in [11,15], from the solution to (6.27) with double logarithmic leading behavior. The leading logarithmic piece proportional to $(\ln z_1 + 3 \ln z_2)^2$ is fixed by the solution for $\nu = 0$ (c.f., (6.12)), and its presence reflects the fact that we considered the cohomology with compact support. Thus our computation of the superpotential differs by boundary terms at infinity from the computation for the supersymmetric, non-compact branes performed in [9]. The leading logarithms (as well as more generally terms depending only on the closed string moduli) can be subtracted by adding boundary terms to obtain the superpotential for a supersymmetric brane, which is entirely given by the instanton part. Alternatively, one may add inhomogeneous pieces to the differential operators (6.27) that reflect the boundary terms at infinity. This is the same type of subtraction as that made in [9] and it leads to a solution with the same instanton expansion but without logarithmic terms. At any rate, the instanton contribution to the superpotential is given for general ν by the following generalized hypergeometric series

$$\mathcal{W}(\nu) = \sum_{n_1, n_2} \frac{(-)^{n_1+n_2\nu} \Gamma(-n_1 + (\nu + 1) n_2) \Gamma(n_2)}{\Gamma(-3n_1 + n_2 + 1) \Gamma(n_1 + \nu n_2 + 1) \Gamma(n_1 + 1) \Gamma(n_2 + 1)} z_1^{n_1} z_2^{n_2}. \quad (6.28)$$

Inserting the inverted mirror map into $\mathcal{W}(\nu)$ then leads to the instanton expansion(2.10) of the superpotential in the topological flat coordinates t_A . We have collected some coefficients N_{k_1, k_2} for a few choices for ν in the following tables. The coefficients are all integral, as predicted in [23], and moreover agree with the results of [22] obtained by a localization computation in the A-model.

										$\nu=1$
1	0	0	0	0	0	0	0	0	0	0
-2	3	-7	19	-56	174	-561	1859	-6292	21658	-75582
5	-10	30	-100	350	-1274	4741	-17844	67704	-258556	992256
-32	63	-182	620	-2296	8907	-35486	143572	-586110	2404978	-9895632
286	-545	1530	-5116	18816	-73242	295625	-1222080	5131594	-21766158	92908536
-3038	5655	-15600	51249	-185510	713610	-2859758	11793782	-49646532	212102095	-915808632

$\nu=2$									
-1	0	0	0	0	0	0	0	0	0
2	5	19	85	416	2156	11628	64581	366850	2121405
-5	-18	-84	-460	-2665	-16036	-98583	-615436	-3883803	-24712454
32	105	510	2900	17836	114591	755310	5056812	34187916	232578905
-286	-905	-4284	-23980	-147056	-952252	-6384435	-43787296	-304865506	-2143839945
3038	9425	43680	240295	1452830	9317700	62170632	426440826	2983278342	21168794600

$\nu=13$									
1	0	0	0	0	0	0	0	0	0
-2	27	-547	13131	-346256	9692892	-282809528	8506210779		
5	-94	2460	-72180	2249525	-72674420	2403644343	-80816694516		
-32	567	-14942	459180	-15221596	526243689	-18654989678	671776623828		
286	-4895	125460	-3801204	125909616	-4398712716	158946297875	-5871829419360		
-3038	50895	-1279200	38097081	-1245280910	43136402580	-1553377686384	57471237302598		

$\nu=1712$									
-1	0	0	0	0	0	0	0	0	0
2	3425	8797969	26784928225	89588261367041	318131499524284985				
-5	-11988	-39590859	-147317100100	-582323680556665	-2385986201234773711				
32	71925	240477810	937472436500	3941883280200586	17285143983849650460				
-286	-620780	-2019133809	-7760932508800	-32610125084743556	-144511224912780917884				
3038	6456125	20587246680	77783427082075	322535637493687955	1417275737631184228875				

Table: Low degree invariants for various choices of framing ν . The degrees $k_1 \geq 0$ in the closed string variable and $k_2 \geq 1$ in the open string variable are listed in the vertical and horizontal directions, respectively. Sign conventions are as in [22].

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