Current Algebra: Quarks and What Else?

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Abstract

After receiving many requests for reprints of this article, describing the original ideas on the quark gluon gauge theory, which we later named QCD, we decided to place the article in the e–Print archive.

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I. Introduction

For more than a decade, we particle theorists have been squeezing predictions out of a mathematical field theory model of the hadrons that we don't fully believe – a model containing a triple of spin 1/2 fields coupled universally to a neutral spin 1 field, that of the "gluon". In recent years, the triplet is usually taken to be the quark triplet, and it is supposed that there is a transformation, presumably unitary, that effectively converts the current quarks of the relativistic model into the constituent quarks of the naive quark model of baryon and meson spectrum and couplings.

We abstract results that are true in the model to all orders of the gluon coupling and postulate that they are really true of the electromagnetic and weak currents of hadrons to all orders of the strong interaction. In this way we build up a system of algebraic relations, so–called current algebra, and this algebraic system gets larger and larger as we abstract more and more properties of the model.

In section III, we review briefly the various stages in the history of current algebra. The older abstractions are correct to each order of renormalized perturbation theory in the model[1)], while the more recent ones, those of light cone current algebra, are true to all orders only formally[3)]. We describe the results of current algebra[2)] in terms of commutators on or near a null plane, say $x_3 + x_0 = 0$.

In section IV, we attempt to describe, in a little more detail, using null plane language, the system of commutation relations valid in a free quark model that are known to remain unchanged (at least formally) when the coupling to a vector "gluon" is turned on. These equations give us a formidable body of information about the hadrons and their currents, supposedly exact as far as the strong interaction is concerned, for comparison with experiment. However, they by no means exhaust the degrees of freedom present in the model; they do not yield an algebraic system large enough to contain a complete description of the hadrons. In an Appendix, the equations of Section IV are related to form factor algebra.

In Section V, we discuss how further commutation of the physical quantities arising from light cone algebra leads, in the model field theory, to results dependent on the coupling constant, to formulae in which gluon field strength operators occur in bilocal current operators proliferate. Only when these relations are included do we finally get an algebraic system that contains nearly all the degrees of freedom of the model. We may well ask, however, whether it is the right algebraic system. We discuss briefly how the complete description of the hadrons involves the specification and slight enlargement of this algebraic system, the choice of representation of the algebra that corresponds to the complete set of hadron states, and the form of the mass or the energy operator, which must be expressible in terms of the algebra when it is complete. The choice of representation may be dictated by the algebra, and if so that would justify the use of a quark and gluon Fock space by some "parton" theorists.

Finally, in Section VI, it is suggested that perhaps there are alternatives to the vector gluon model as sources of information or as clues for the construction of the true hadron theory. Assuming we have described the quark part of the model correctly, can we replace the gluons by something else? The "string" or "rubber band" formulation, in ordinary coordinate space, of the zeroth approximation to the dual resonance model, is suggested as an interesting example.

Before embarking on our discussion of current algebra, we discuss in Section II the crucial point that quarks are probably not real particles and probably obey special statistics, along with related matters concerning the gluons of the field theory model.

II. FICTITIOUS QUARKS AND "GLUONS" AND THEIR STATISTICS

We assume here that quarks do not have real counterparts that are detectable in isolation in the laboratory – they are supposed to be permanently bound inside the mesons and baryons. In particular, we assume that they obey the special quark statistics, equivalent to "para–Fermi statistics of rank three" plus the requirement that mesons always be bosons and baryons fermions. The simplest description of quark statistics involves starting with three triplets of quarks, called red, white, and blue, distinguished only by the parameter referred to as color. These nine mathematical entities all obey Fermi–Dirac statistics, but real particles are required to be singlets with respect to the SU_3 of color, that is to say combinations acting like

$$\bar{q}_R q_R + \bar{q}_B q_B + \bar{q}_W q_W \quad \text{or} \quad q_R q_B q_W - q_B q_R q_W - q_R q_W q_B - q_W q_B q_R + q_W q_R q_B + q_B q_W q_R \,. \qquad (1)$$

The assumption of quark statistics has been common for many years, although not necessarily described in quite this way, and it has always had the following advantage: The constituent quarks as well as current quarks would obey quark statistics, since the transformation between them would not affect statistics, and the constituent quark model would then assign the lowest-lying baryon states (56 representation) to a symmetrical spatial configuration, as befits a very simple model.

Nowadays there is a further advantage. Using the algebraic relations abstracted formally from the quark–gluon model, one obtains a formula for the π^0 decay amplitude in the PCAC approximation, one that works beautifully for quark statistics but would fail by a factor 3 for a single Fermi–Dirac triplet[4)].

We have the option, no matter how far we go in abstracting results from a field theory model, of treating only color singlet operators. All the currents, as well as the stress–energy–momentum tensor $\Theta_{\mu\nu}$ that couples to gravity and defines the theory, are color singlets. We may, if we like, go further and abstract operators with three quark fields, or four quark fields and an antiquark field, and so forth, in order to connect the vacuum with baryon states, but we still need select only those that are color singlets in order to connect all physical hadron states with one another.

It might be a convenience to abstract quark operators themselves, or other non–singlets with respect to color, along with fictitious sectors of Hilbert space with triality non–zero, but it is not a necessity. It may not even be much of a convenience, since we would then, in describing the spatial and temporal variation of these fields, be discussing a fictitious spectrum for each fictitious sector of Hilbert space, and we probably don't want to load ourselves with so much spurious information.

We might eventually abstract from the quark–vector–gluon field theory model enough algebraic information about the color singlet operators in the model to describe all the degrees of freedom that are present.

For the real world of baryons and mesons, there must be a similar algebraic system, which may differ in some respects from that of the model, but which is in principle knowable. The operator $\Theta_{\mu\nu}$ could then be expressed in terms of this system, and the complete Hilbert space of baryons and mesons would be a representation of it. We would have a complete theory of the hadrons and their currents, and we need never mention any operators other than color singlets.

Now the interesting question has been raised lately whether we should regard the gluons as well as the quarks as being non–singlets with respect to color[5)]. For example, they could form a color octet of neutral vector fields obeying the Yang–Mills equations. (We must, of course, consider whether it is practical to add a common mass term for the gluon in that case – such a mass term would show up physically as a term in $\Theta_{\mu\nu}$ other than the quark bare mass term. In the past, we have referred to such an additional term that violates scale invariance, but does not violate $SU_3 \times SU_3$ as δ and its dimension as l_δ. Nowadays, ways of detecting expected values of δ are emerging.)[6)].

If the gluons of the model are to be turned into color octets, then an annoying asymmetry between quarks and gluons is removed, namely that there is no physical channel with quark quantum numbers, while gluons communicate freely with the channel containing the ω and ϕ mesons. (In fact, this communication of an elementary gluon potential with the real current of baryon number makes it very difficult to believe that all the formal relations of light cone current algebra could be true even in a "finite" version of singlet neutral vector gluon field theory.)

If the gluons become a color octet, then we do not have to deal with a gluon field strength standing alone, only with its square, summed over the octet, and with quantities like $\bar{q}\left(\partial_\mu - igo_A B_{A\mu}\right) q$, where the σ's are the eight 3×3 color matrices for the quark and the B's are the eight gluon potentials.

Now, suppose we look at such a model field theory, with colored quarks and colored gluons, including the stress–energy–momentum tensor. Basically the questions we are asking are the following:

1. Up to what point does the algebraic system of the color singlet operators for the real hadrons resemble that in the model? What is it in fact?

2. Up to what point does the representation of the algebraic system by the Hilbert space of physical hadron states resemble that in the model? What is it in fact?

3. Up to what point does $\Theta_{\mu\nu}$, expressed in term of the algebraic system, resemble that in the model? What is it in fact?

The measure of our ignorance is that for all we know, the algebra of color singlet operators, the representation, and even the form of $\Theta_{\mu\nu}$ could be exactly as in the model! We don't yet know how to extract enough consequences of the model to have a decisive confrontation with experiment, nor can we solve the formal equations for large g.

If we were solving the equations of a model, the first question we would ask is: Are the quarks really kept inside or do they escape to infinity? By restricting physical states and interesting operators to color singlets only, we have to some extent begged that question. But it re-emerges in the following form:

With a given algebraic system for the color singlet operators, can we find a locally causal $\Theta_{\mu\nu}$ that yields a spectrum corresponding to mesons and baryons and antibaryons and combinations thereof, or do we find a spectrum (in the color singlet states) that looks like combinations of free quarks and antiquarks and gluons?

In the next three Sections we shall usually treat the vector gluon, for convenience, as a color singlet.

III. REVIEW OF CURRENT ALGEBRA

In this section we sketch the gradual extension of algebraic results abstracted from free quark theory that remain true, either in renormalized perturbation theory or else only formally, when the coupling to a neutral vector gluon field is turned on.

The earlier abstractions were of equal–time commutation relations of current components. It was soon found that useful sum rules could best be derived from these by taking matrix elements between hadron states of equal P_3 as $P_3 \to \infty$, selecting the "gluon" components of the currents (those with matrix elements finite in this limit rather than tending to zero), and adding the postulate that, in the sum over intermediate states in the commutator, only states of finite mass need be considered. Thus formulae like the Adler–Weisberger and Cabibbo–Radicati sum rules were obtained and roughly verified by experiment.

Nowadays, the same procedure is usually accomplished in a slightly different way that is a bit cleaner – the hadron momenta are left finite instead of being boosted by a limit of Lorentz transformations, and the equal time surface is transformed by a corresponding limit of Lorentz transformations into a null plane, with $x_3 + x_0 =$ constant, say zero. The hypothesis of saturation by finite mass intermediate states is replaced by the hypothesis that the commutation rules of good components can be abstracted from the model not only on an equal time plane, but on a null plane as well[7,8].

In the last few years, the process of abstraction has been extended to a large class of algebraic relations (those of "light cone current algebra") that are true only formally in the model, but fail to each order of renormalized perturbation theory - they would be true to each order if the model were super–renormalizable. The motivation has been supplied by the compatibility of the deep inelastic electron scattering experiments performed at SLAC with the scaling predictions of Bjorken, which is the most basic feature of "light cone current algebra". The Bjorken scaling limit $q^2 \to \infty, 2p \cdot q \to \infty, \xi \equiv q^2/(-2p \cdot q)$ finite) corresponds in coordinate space to the singularity on the light cone $(x-y)^2 = 0$ of the current commutator $[j(x), j(y)]$, and the relations of light cone current algebra are obtained by abstracting the leading singularity on the light cone from the field theory model. The singular function of $x - y$ is multiplied by a bilocal current operator $\Theta(x, y)$ that reduces to a familiar local current as $x - y \to 0$. The Bjorken scaling functions $F(\xi)$ are Fourier transforms of the expected values of the bilocal operators. Numerous predictions emerge from the relations abstracted from the quark–gluon model for deep inelastic and neutrino cross–sections. For example, the spin 1/2 character for the quanta bearing the charge in the model is reflected in the prediction $\sigma_L/\sigma_T \to 0$, while the charges of the quarks are reflected in the inequalities $1/4sF^{\text{en}}(\xi)/F^{\text{ep}}(\xi) \leq 4$. So far there is no clear sign of my contradiction between the formulae and the experimental results.

We may go further and abstract from the model also the light–cone commutators of bilocal currents, in the limit in which all the intervals among the four points approach zero, that is to say, when all four points tend to lie on a light–like line. The same bilocal operators then recur as coefficients of the singularity, and the algebraic system closes.

The light cone results can be reformulated in terms of the null plane. We consider a commutator of local currents at two points x and y and allow the two points to approach the same null plane, say

$$x_+ \equiv x_3 + x_0 = 0, y_+ \equiv y_3 + y_0 = 0 \tag{2}$$

As mentioned above, when both current components are "good", we obtain a local commutation relation on the null plane, yielding another good component, or else zero. But when neither component is good, there is a singularity of the form

$$\delta(x_+ - y_+) \tag{3}$$

and the coefficient is a bilocal current on the null plane. It is this singularity, arising from the light–cone singularity, that gives the Bjorken scaling.

On the null plane, with $x_+ = 0$, the three coordinates are the transverse spacelike coordinates x_1 and x_2 (called $x_\perp$) and the lightlike coordinate $x_- \equiv x_3 - x_0$. Our bilocal currents $O(u, y)$ on the nullplane are functions of four coordinates: x_-, y_- and $x_\perp = y_\perp$, since the interval between x and y is lightlike.

We may now consider the commutator of two bilocal currents defined on neighboring null planes (in each case with a lightlike interval between the two arguments of the bilocal current). Again, when neither current component is good, there is a δ–function singularity of the spacing between the two null planes and the coefficient is a bilocal current defined on the common limiting null plane. In this language, as before in the light cone language, the system of bilocal currents closes.

We may commute two good components of bilocal currents on the same null plane, and, as for local currents, we obtain a good component on the right–hand side, without any δ–function singularity at coincidence of the two null planes. Thus the good components of the bilocal currents $O(u, y)$ form a Lie algebra on the null plane, a generalization of the old Lie algebra of local good components on the null plane (recovered by putting $x_- = y_-$).

Now, how far can we generalize this new Lie algebra on the null plane and still obtain exact formulae, formally true to all orders in the coupling constant, but independent of it, so that free quark formulae apply?

In the next section, we take up that question, but first we summarize the situation of current algebra on and near the null plane.

IV. SUMMARY OF LIGHT CONE AND NULL PLANE RESULTS

Let us now be a little more explicit. We are dealing with 144 bilocal quantities $\mathcal{F}_{j\alpha}, \mathcal{F}_{j\alpha}, S_j, P_j$ and $T_{j\alpha\beta}$ all functions of $x - y$ with $(x-y)^2 \to 0$. Let us select the 3–direction for our null planes. Then in the model we can set $B_+ \equiv B_3 + B_0 = 0$ for the gluon potential by a choice of gauge. The gauge–invariance factor exp ig $\int_y^x B \cdot dl$ for a straight line path on a null plane is just exp $\left[i\frac{g}{2}B_+ (x_- - y_-)\right] = 1$. Thus we have simple correspondences between our quantities and operators in the model:

$$\mathcal{F}_{j\alpha}(x, y) \sim \frac{i}{2}\bar{q}(x)\lambda_j\gamma_\alpha q(y), \text{ etc.}$$

and we have introduced the notation $\mathcal{D}\left(x, y, \frac{i}{2}\,\lambda_j\,\gamma_\alpha\right)$, etc., where

$$\mathcal{D}(x, y, G) \sim \bar{q}(x)Gq(y) \sim q^+(x)(\beta G)q(y)\,. \tag{4}$$

We are dealing with $\mathcal{D}(x, y, G)$ for every (12×12) matrix G, with

$$\mathcal{F}^5_{j\alpha}(x, y) \;=\; \mathcal{D}\left(x, y, \frac{i}{2}\lambda_j\gamma_\alpha, \gamma_5\right) S_j(x, y) = \mathcal{D}\left(x, y, \frac{1}{2}\,\lambda_j\right), \tag{5}$$

$$P_j(x, y) \;=\; \mathcal{D}\left(x, y, \frac{i}{2}\lambda_j\,\gamma_5\right), \text{ and } \mathcal{T}_{j\alpha\beta}(x, y) = \mathcal{D}\left(x, y, \frac{i}{2}\,\lambda_j\,\sigma_{\alpha\beta}\right). \tag{6}$$

The good components, in the old equal–time $P_3 \to \infty$ language, were those with finite matrix elements between states of finite mass and $P_3 \to \infty$. By contrast, bad components were those with matrix elements going like ${P_3}^{-1}$ and terrible components those with matrix elements going like ${P_3}^{-2}$.

In the null plane language, good components are those for which βG is proportional to $1+\alpha_3$; thus the 36 good components are $\mathcal{F}_{j+}, \mathcal{F}^5_{j+}, \mathcal{T}_{j1+}, \mathcal{T}_{j2+}$ for $j = 0 \ldots 8$. The terrible components are those for which βG is proportional to $1-\alpha_3$, hence $\mathcal{F}_{j-}, \mathcal{F}^5_{j-}, \mathcal{T}_{j1}$, and $\mathcal{T}_{j2-}$. The rest are bad; they have βG anticommuting with α_3 so that α_3 is -1 on the left and +1 on the right or vice versa.
Now the leading light cone singularity in the commutator of two bilocals is just given by the formula

$$[(\mathcal{D}\left(x,y,G\right), \mathcal{D}\left(u,v,G'\right))] \,\hat{=} \mathcal{D}\left(x,v,iG\gamma_\mu G'\right)\partial_\mu\Delta(y-u) - \mathcal{D}\left(u,y,iG'\gamma_\mu G\right)\partial_\mu\Delta(v-x), \tag{7}$$

with $\Delta(z) = (2\pi)^{-1}\,\varepsilon\left(z_0\right)\delta\left(z^2\right)$.

When we commute two operators with coordinates lying on neighboring null planes with separation Δx_+, a singularity of the type $\delta\left(\Delta x_+\right)$ appears (as we have mentioned in Section III) multiplied by a bilocal operator, with coordinates lying in the common null plane as $\Delta x_+ \to 0$, and it is this term that gives rise to Bjorken scaling. The term in question comes from the component $\frac{\partial}{\partial z_+}\Delta(z)$ in $\partial_\mu\Delta(z)$, and is thus multiplied by $\mathcal{D}\left(x,\nu,iG\gamma_+G'\right)$ and $\mathcal{D}\left(u,y,iG'\gamma_+G\right)$. Now $\beta\left(iG\gamma_+G'\right) = \left(\beta G\right)\left(1-\alpha_3\right)\left(\beta G'\right)$, so it is clear that the singular Bjorken scaling term vanishes for good–good and good–bad commutators. In the case of the other components, we have, schematically, [bad, bad] $\to$ good, [bad, terrible] $\to$ bad, and [terrible, terrible] $\to$ terrible for the Bjorken singularity.
The vector and axial vector local currents $\mathcal{F}_{j\alpha}(x,x)$ and $\mathcal{F}^5_{j\alpha}(x,x)$ occur, of course, in the electromagnetic and weak interactions. The local scalar and pseudoscalar currents occur in the divergences of the non–conserved vector and the axial vector currents, with coefficients that are linear combinations of the bare quark masses, m_u, m_d and m_s, treated as a diagonal matrix. (Here m_u would equal m_d if isotopic spin conservation were perfect, while the departure of m_s from the common value of m_u and m_d is what gives rise to SU_3 splitting; the non–vanishing of m is what breaks $SU_3 \times SU_3$).
We see that all the 144 bilocals are physically interesting, including the tensor currents, because they all occur in the commutators of these local V, A, S, and P densities as coefficients of the $\delta\left(\Delta x_+\right)$ singularity. Commuting a local scalar with itself or a local pseudoscalar with itself leads to the same bilocal as commuting a transverse component of a vector with itself, and thus the light cone commutator of current divergences is predicted to lead to Bjorken scaling functions that are proportional to those observed in the light cone commutation of currents, while the coefficients permit the experimental determination of the squares of the quark bare masses. Unfortunately, the relevant experiments are difficult. (The finiteness of the bare masses, as compared with the divergences encountered term in renormalized perturbation theory in a gluon model, presumably has the same origin as the scaling, which also fails term by term in renormalized perturbation theory.)

As we have outlined in Section III, we begin the construction of the algebraic system on the null plane by commuting the good bilocals with one another. The leading singularity on the light cone (Eq.(4.1)) gives rise to the simple closed algebra we have mentioned, but we need also the additional assumption that lower singularities on the light cone give no contribution to the good–good commutators on the null plane. This additional assumption can be squeezed out of the model in various ways. The simplest, however, is to use canonical quantization of the quark–gluon model on the null plane.
In the model, the quark field q is written as $q_+ + q_-$, where $q_\pm = \frac{1}{2}\left(1 \pm \alpha_3\right) q$. Then q_+ obeys the canonical rules $\{q_{+\alpha}(x), q_{+\beta}(y)\} = 0, \left\{q_{+\alpha}(x).q^+_{+\beta}(y)\right\} = \delta^{(3)}(x-y)\frac{1}{2}\left(l + a_3\right)_{\alpha\beta}$ on the null plane, where $\delta^{(3)}(x-y) = \delta\left(x_\perp - y_\perp\right)\delta\left(x_- - y_-\right)$. Thus for any good matrices βA_{++} and (βB_{++}), we have on the null plane

$$\left[\mathcal{D}\left(x, y, \beta A_{++}\right), \mathcal{D}\left(u, v, \beta B_{++}\right)\right] =$$

$$\mathcal{D}\left(x, v\beta A_{++}B_{++}\right)\delta^{(3)}(y-u) - \mathcal{D}\left(u, y, \beta B_{++}A_{++}\right)\delta^{(3)}(v-x),$$

which is just what we would get from (4.1) with no additional contribution from lower light cone singularities.
The good–good commutation relations (4.2) on the null plane, together with the equations (4.1) for the leading light cone singularity in the commutator of two bilocal currents, illustrate how far we can go with abstracting free quark formulae that are formally unchanged in the model when the gluon coupling is turned on.
One may go further in certain directions. For example, the formulae for the leading light cone singularity presumably apply to disconnected as well as connected parts of matrix elements, and thus the question of the vacuum expected value of a bilocal operator arises. In the model, the coefficient of the leading singularity as $(x-y)^2 \to 0$ of such an expected value is formally independent of the coupling constant, and we abstract that as well – the answer here is dependent on statistics, however, and we assume the validity of quark statistics. Thus we obtain predictions like the following:

$$\sigma\left(e^+ + e^- \to \text{ hadrons}\right)/\sigma\left(e^+ + e^- \to \mu^+ + \mu^-\right) \to 2 \tag{8}$$

at high energy to lowest order in the fine structure constant.

The leading light cone singularity of an operator product, or of a physical order (T^*) product, may also be abstracted from the model, except for certain subtraction terms (often calculable and / or unimportant) that behave like four–dimensional δ–functions in coordinate space. To go from a commutator formula to a physical ordered product formula, we simply perform the substitutions

$$(2\pi)^{-1}\varepsilon(z)\delta\left(z^2\right) \to \left(4\pi^2 i\right)^{-1}\left(z^2 - iz_0\varepsilon\right)^{-1} \to \left(4\pi^2 i\right)^{-1}\left(z^2 - i\varepsilon\right)^{-1}. \tag{9}$$

With the aid of the product formulae and the vacuum expected values, we obtain the PCAC value of the $\pi^0 \to 2\gamma$ decay amplitude.

Other exact abstractions from the vector gluon model that do not contain g are divergence and curl relations for local V and A currents:

$$\begin{aligned}\frac{\partial}{\partial x_\mu}\mathcal{D}\left(x,x,\frac{i}{2}\lambda_i\,\gamma_\mu\right) &= \mathcal{D}\left(x,x,\frac{i}{2}\left[m,\lambda_i\right]\right),\\ \frac{\partial}{\partial x_\mu}\mathcal{D}\left(x,x,\frac{i}{2}\lambda_i\gamma_\mu\gamma_5\right) &= \mathcal{D}\left(x,x,\frac{i}{2}\left\{m,\lambda_i\right\}\gamma_5\right),\end{aligned} \tag{10}$$

but we also have, as presented elsewhere[2)],

$$\begin{aligned}\frac{\partial}{\partial x_\nu}\mathcal{D}\left(x,x,\frac{1}{2}\lambda_i\,\sigma_{\mu\nu}\right) &= -\mathcal{D}\left(x,x,\frac{i}{2}\left\{m,\lambda_i\right\}\gamma_\nu\right)\\ &\quad+\left[\left(\frac{\partial}{\partial x_\nu}-\frac{\partial}{\partial y_\nu}\right)\mathcal{D}\left(x,y,\frac{i}{2}\lambda_i\right)\right]_{x=y}\end{aligned} \tag{11}$$

$$\begin{aligned}\frac{\partial}{\partial x_\nu}\mathcal{D}\left(x,x,\frac{1}{2}\lambda_i\sigma_{\mu\nu}\gamma_5\right) &= -\mathcal{D}\left(x,x,\frac{i}{2}\right)\left[m,\lambda_i\right]\gamma_\nu\gamma_5\\ &\quad+\left[\left(\frac{\partial}{\partial x_\nu}-\frac{\partial}{\partial y_\nu}\right)\mathcal{D}\left(x,y,\frac{i}{2}\lambda_i\gamma_5\right)\right]\end{aligned} \tag{12}$$

and a number of other formulae, including the following:

$$\left[\left(\frac{\partial}{\partial x_\nu}-\frac{\partial}{\partial y_\nu}\right)\mathcal{D}\left(x,y,\frac{i}{2}\lambda_i\gamma_\nu\right)\right]_{x=y} = \mathcal{D}\left(x,x,\frac{i}{2}\left\{\lambda_i,m\right\}\right) \tag{13}$$

In the last three formulae, it must be pointed out that for a general direction of $x-y$ we have the gauge–invariant correspondence

$$\mathcal{D}\left(x,y,G\right)\sim\bar{q}(x)Gq(y)\ \exp\ \mathrm{ig}\int_y^x B\cdot dl, \tag{14}$$

which is independent of the path from y to x when the coordinate difference and the path are taken as first order infinitesimals. The first internal derivative

$$\left[\left(\frac{\partial}{\partial x_\mu}-\frac{\partial}{\partial y_\mu}\right)\mathcal{D}(x,y,G)\right]_{x=y} \tag{15}$$

is physically interesting for all directions μ (and not just the – direction), as a result of Lorentz covariance.

In Eqs. (4.5–4.7), we have for the moment thrown off the restriction to a single null plane. In the next Section, we return to the consideration of the algebra on the null plane, and we see how further extensions give a much wider algebra, in which departures from free quark relations begin to appear.

V. THE FURTHER EXTENSION OF NULL PLANE ALGEBRA

We now look beyond the commutation relations of good bilocals on the null plane. In the model, then, we have to examine operators containing q_- or q_-^+ or both. The Dirac equation in the gauge we are using ($B_+ = 0$ on the null plane) tells us that we have

$$-2i\frac{\partial q_-}{\partial x_-} = \left(\alpha_\perp \cdot \left(-i \bigtriangledown_\perp - gB_\perp\right) + \beta m\right) q_+. \tag{16}$$

In terms of Eq. (5.1), we can review the various anticommutators on the null plane. We have already discussed the trivial one,

$$\left(q_+(x), q_+^+(y)\right) = \delta\left(x_- - y_-\right) . \frac{1}{2}\left(1+\alpha_3\right)\delta\left(x_\perp - y_\perp\right) . \tag{17}$$

Using (5.1), (5.2), the fact that $B_\perp$ commutes with q_+ on the null plane, and the equal–time anticommutator $\left\{q_-, q_+^+\right\} = 0$, we obtain well–known result

$$\left\{q_-(x), q_+^+(y)\right\} = \frac{i}{4}\varepsilon\left(x_- - y_-\right)\left[\alpha_\perp \cdot \left(i \bigtriangledown_\perp^{(y)} - gB_\perp(y)\right) + \beta m\right]\frac{1}{2}\left(1+\alpha_3\right)\delta\left(x_\perp - y_\perp\right) . \tag{18}$$

Using the same method a second time, one finds, for $y_- > x_-$,

$$\begin{aligned}\left\{q_-(x), q_-^+(y)\right\} = & -\frac{1}{8}\int_{x_-}^{y_-} dr_- \left[\alpha_\perp\left(-i \bigtriangledown_\perp^{(x)} - gB_\perp\left(x_\perp, r_-\right)\right) + \beta m\right]^2 \left(\frac{1-\alpha_3}{2}\right)\delta\left(x_\perp - y_\perp\right) \\ & + i\frac{g^2}{32}\int_{x_-}^{y_-} dy'_- \int_{x_-}^{y'_-} dx'_- \left[\alpha_\perp q_+\left(x_\perp, x'_-\right); q_+\left(y_\perp, y'_-\right)\alpha_\perp\right]\delta\left(x_\perp - y_\perp\right) \\ & + \delta\left(x_+ - y_+\right)\left(\frac{1-\alpha_3}{2}\right)\delta\left(x_\perp - y_\perp\right),\end{aligned} \tag{19}$$

where the singularity at the coincidence of the two null planes appears as an unpleasant integration constant. This singularity is, of course, responsible in the model for the Bjorken singularity in the commutator of two bad or terrible operators.

Because of the singularity, it is clumsy to construct the wider algebra by commuting all

our bilocals with one another. Instead, we adopt the following procedure. Whenever a bilocal operator corresponds to one in the model containing $q_-^+(x)$, we differentiate with respect to x_-; whenever it corresponds to one in the model containing $q_{(y)}$, we differentiate with respect to y_-. Thus we "promote" all our bilocals to good operators. We construct the wider algebra by starting with the original good bilocals and these promoted bad and terrible bilocals. We commute all of these, commute their commutators, and so forth, until the algebra closes. Then, later on, if we want to commute an unpromoted operator, we use the information contained in equations of the model like (5.1) - (5.3) to integrate over x_- or y_- or both and undo the promotion. (A similar situation obtains for operators corresponding to those in the model containing the longitudinal gluon potential B_-.)
Now let us classify the matrices βG into four categories:
the good ones, $\beta G = A_{++}$, with $\alpha_3 = 1$ on both sides;
the bad ones $\beta G = A_{+-}$ that have $\alpha_3 = 1$ on the left and -1 on the right;
the bad ones $\beta G = A_{-+}$ that have $\alpha_3 = -1$ on the left and $+1$ on the right;
and the terrible ones $\beta G = A_{--}$, with $\alpha_3 = -1$ on both sides.

Then, wherever q_- or q_-^+ appears, we promote the operator by differentiating q_- or q_-^+ with respect to its argument in the – direction. We obtain, then:

$$\mathcal{D}\left(x, y, \beta A_{++}\right),$$

the good operators, unchanged;

$\frac{\partial}{\partial x_-}\mathcal{D}\left(x, y, \beta A_{-+}\right)$ and $\frac{\partial}{\partial y_-}\left(x, y, \beta, A_{+-}\right)$ promoted bad operators:

and

$\frac{\partial}{\partial x_-}\frac{\partial}{\partial y_-}\mathcal{D}\left(x, y, \beta A_{--}\right)$, promoted terrible operators.

All 144 of these operators now are given, in the model, in terms of q_+ and q_+^+, but the promoted bad and terrible operators involve the expressions $\left(\bigtriangledown_\perp - igB_\perp\right) q_+$ and $\left(\bigtriangledown_\perp + igB_\perp\right) q_+^+$. In fact, substituting the Dirac equation for $\frac{\partial q_-}{\partial x_-}$ into the definitions of the promoted bad and terrible operators, we see that we obtain good operators (with coefficients depending on bare quark masses) and also good matrices sandwiched between $\left(\bigtriangledown_\perp + igB_\perp\right) q_+^+$ and q_+ or between q_+^+ and $\left(\bigtriangledown_\perp - igB_\perp\right) q_+$ or between $\left(\bigtriangledown_\perp + igB_\perp\right) q_+^+$ and $\left(\bigtriangledown_\perp - igB_\perp\right) q_+$.
The null plane commutators of all these operators with one another are finite, well–defined, and physically meaningful, but they lead to an enormous Lie algebra that is not identical with the one for free quarks, but instead contains nearly all the degrees of freedom of the model.
Let us first ignore any lack of commutation of the B's with one another. We keep commuting the operators in question with one another. When $\bigtriangledown_\perp \pm igB_\perp$ appears acting on a $\delta^{(3)}$

function, we can always perform an integration and fold it over onto an operator. Thus the number of applications of $\bigtriangledown_\perp \pm igB_\perp$ grows without limit. Since these gauge derivatives do not commute with one another, but give field strengths as commutators, it can easily be seen that we end up with all possible operators corresponding to $\bar{q}_+(x)Gq_+(y)$ acted on by any gauge invariant combination of transverse gradients and potentials. We have to put it differently, the operators corresponding to $\bar{q}_+(x)Gq_+(y)$ exp ig $\int_P B \cdot dl$ for any pair of points x and y on the null plane connected by any path P lying in the null plane. We could think of these as operators $\mathcal{D}(x, y, G, P)$ depending on the path P, with $\beta G = A_{++}$.

In fact the B's do not commute with another in the model, and so we get an even more complicated result. We have

$$[B_{\perp i}(x), B_{\perp j}(y)] \sim \varepsilon\,(x_- - y_-)\,\delta\,(x_\perp - y_\perp)\,\delta_{ij} \tag{20}$$

on the null plane, and the commutation of promoted bad and terrible bilocals with one another leads to operators corresponding to $\bar{q}_+(x)Gq_+(y)\bar{q}_+(a)G'q_+(b)$. Further commutation then introduces an unlimited number of sideways gradients, gluon field strengths, and additional quark pairs, until we end up with all possible operators of the model that can be constructed from equal numbers of $\bar{q}_+$'s and q_+'s at any points on the null plane and from exponentials of $ig \int B \cdot dl$ for any paths connecting these points.

If we keep track of color, we note that only color singlets are generated. If the gluons are a color octet Yang–Mills field, we must make suitable changes in the formalism but again we find that only color singlets are generated. The coupling constant g that occurs is, of course, the bare coupling constant. If may not be intrinsic to the algebraic system (equivalent to that of quarks and gluons) on the null plane, but it certainly enters importantly into the way we reach the system starting from well–known operators.

A troublesome feature of the extended null plane algebra is the apparent absence of operators corresponding to those in the model that contain only gluon field strengths and no quark operators; for a color singlet gluon, the field strength itself would be such an operator, while for a color octet gluon we could begin with bilinear forms in the field strength in order to obtain color singlet operators. Can we obtain these quark–free operators by investigating discontinuities at the coincidence of coordinates characterizing quark and antiquark fields in the model? At any rate, we certainly want these quarkfree operators included in the extended algebra.

Now when our algebra has been extended to include the analogs of all relevant operators of the model on the null plane that are color singlets and have baryon number $A = 0$, then the Hilbert space of all physical hadron states with $A = 0$ is an irreducible representation of the algebra.

If we wish, we might as well extend the algebra further by including the analogs of color singlet operators of the model (on the null plane) that would change the number of baryons. In that case, the entire Hilbert space of all hadron states is an irreducible representation of the complete algebra. From now on, let us suppose that we are always dealing with the complete color singlet algebra (whether the one abstracted from the quark–gluon model or some other) and with the complete Hilbert space, which is an irreducible representation of it.

The representation may be determined by the algebra and the uniqueness of the physical vacuum. We note that we are dealing with arbitrarily multilocal operators, functions of any number of points on the null plane. We can Fourier transform with respect to all these variables and obtain Fourier variables $(k_+, k_\perp)$ in place of the space coordinates. Since $B_+ = 0$, there is no formal obstacle to thinking of each k_+ as being like the contribution of the individual quark, antiquark or gluon to the total $P_+ = \sum k_+$. Now $P_+ = 0$ for the vacuum, and for any other state we can get $P_+ = 0$ only by taking $P_z \to -\infty$. The same kind of smoothness assumption that allows scaling can allow us to forget about matrix elements to such infinite momentum states. In that case, we have the unique vacuum state of hadrons as the only state of $P_+ = 0$, while all others have $P_+ > 0$. All Fourier components of multilocal operators for which $\sum k_+ < 0$ annihilate the physical vacuum. (Note in the null plane formalism we do not have to deal with a fictitious "free vacuum" as in the equal–time formalism.) The Fourier components of multilocal operator with $\sum k_+ > 0$ act on the vacuum to create physical states, and the orthogonality properties of these states and the matrix elements of our operators sandwiched between them are determined largely or wholly by the algebra. The details have to be studied further to see to what extent the representation is really determined. (The vacuum expected values contain one adjustable parameter in the case of free quarks, namely the number of colors.)

Once we have the representation of the complete color singlet algebra on the null plane, as well as the algebra itself, then the physical states of hadrons can all be written as linear combinations of the normalized basis states of the representation. These coefficients represent a normalized set of Fock space wave functions for each physical hadron state, with orthogonality relations for orthogonal physical states. Since the matrix elements of all null plane operators between basis states are known, the matrix elements between physical states of bilocal currents or other operators of interest are all calculable in terms of the Fock space wave functions[9)].

This situation is evidently the one contemplated by "parton" theorists such as Feynman and Bjorken; they suppose that we know the complete algebra, that it comes out to be a quark–gluon algebra, and that the representation is the familiar one, so that there is a simple Fock space of quark, antiquark, and gluon coordinates. In the Fourier transform, negative values of each k_+ correspond to destruction and positive values to creation.

Now the listing of hadron states by quark and gluon momenta is a long way from listing by meson and baryon moments. However, as long as we stick to color singlets, there is not necessarily any obstacle to getting one from the other by taking linear combinations. The operator $M^2 = -P^2 - P_+P_-$ has to be such that its eigenvalues correspond to meson and baryon configurations, and not to a continuum of quarks, antiquarks and gluons.

The important physical questions are whether we have the correct complete algebra and

representation, and what the correct form of $\Theta_{\mu\nu}$ or P_μ or M^2 is, expressed in terms of that algebra.
In the quark–gluon model we have $\Theta_{\mu\nu} = \Theta^{\text{quark}}_{\mu\nu} + \Theta^{\text{glue}}_{\mu\nu}$, where

$$\begin{aligned}\Theta^{\text{quark}}_{\mu\nu} &= \frac{1}{4}\bar{q}\gamma_\mu\left(\partial_\nu - igB_\nu\right)q + \ldots q + \frac{1}{4}\bar{q}\gamma_\nu\left(\partial_\mu - igB_\mu\right)q \\ &\quad -\frac{1}{4}\left(\partial_\mu + igB_\mu\right)\bar{q}\gamma_\nu q - \frac{1}{4}\left(\partial_\nu + igB_\nu\right)\bar{q}\gamma_\mu q\,, \end{aligned} \tag{21}$$

and $\Theta^{\text{glue}}_{\mu\nu}$ does not involve the quark variables at all. The term $\Theta^{\text{quark}}_{\mu\nu}$, by itself, has the wrong commutation rules to be a true $\Theta_{\mu\nu}$ (unless $g = 0$). For example, $\left(P^{\text{quark}}_1, P^{\text{quark}}_2\right) \neq 0$. The correct commutation rules are restored when we add the contribution from $\Theta^{\text{glue}}_{\mu\nu}$. We can abstract from the quark–gluon model some or all the properties of $\Theta_{\mu\nu}$, in terms of the null plane algebra. We see that in the model we have

$$\Theta^{\text{quark}}_{++} = \left[\left(\frac{\partial}{\partial y_-} - \frac{\partial}{\partial x_-}\right)\mathcal{D}\left(x, y, \frac{1}{2}\gamma_+\right)\right]_{x=y} \tag{22}$$

and, as is well–known, the expected value of the right–hand side in the proton state can be measured by deep inelastic experiments with electrons and neutrinos. All indications are that it is not equal to the expected value of Θ_{++}, but rather around half of that, so that half is attributable to gluons, or whatever replaces them in the real theory.

In general, using the gauge–invariant definition of $\mathcal{D}$, we have in the model

$$\Theta^{\text{quark}}_{\mu\nu} = \left[\left(\frac{\partial}{\partial y_\nu} - \frac{\partial}{\partial x_\nu}\right)\mathcal{D}\left(x, y, \frac{1}{4}\gamma_\mu\right) + \left(\frac{\partial}{\partial y_\mu} - \frac{\partial}{\partial x_\mu}\right)\mathcal{D}\left(x, y, \frac{1}{4}\gamma_\nu\right)\right]_{x=y} \tag{23}$$

and Eq. (4.7) then gives us the obvious result

$$-\,\Theta^{\text{quark}}_{\mu\nu} = \mathcal{D}\left(x, x, m\right)\,. \tag{24}$$

Whereas in (5.5) we are dealing with an operator that belongs to the null plane algebra generated by good, promoted bad, and promoted terrible bilocal currents, other components of $\Theta^{\text{quark}}_{\mu\nu}$ are not directly contained in the algebra, neither are the bad and terrible local currents, nor their internal derivatives in directions other than –. In order to obtain the commutation properties of all these operators with those actually in the algebra, we must, as we mentioned above, undo the promotions by abstracting the sort of information contained in (5.3) and (5.4). Thus we are really dealing with a wider mathematical system than the closed Lie algebra abstracted from that of operators in the model containing q^+_+, q_+ and $B_\perp$ only.

We shall assume that the true algebraic system of hadrons resembles that of the quark–gluon model at least to the following extent:

1) The null plane algebra of good components (4.2) and the leading light cone singularities (4.1) are unchanged.

2) The system acts as if the quarks had vectorial coupling in the sense that the divergence equation (4.3) and (4.4) are unchanged.

3) There is a gauge derivative of some kind, with path–dependent bilocals that for an infinitesimal interval become path–independent. Eqs. (4.5) - (4.7) are then defined and we assume they also are unchanged.

4) The expression (5.6) for $\Theta_{\mu\nu}^{\mathrm{quark}}$ is also defined and we assume it, too, is unchanged, along with its corollary (5.7).

About the details of the form of the path–dependent null plane algebra arising from the successive application of gauge derivatives, we are much less confident, and correspondingly we are also less confident of the nature of the gluons, even assuming that we can decide whether to use a color singlet or a color octet. What we do assert is that there is some algebraic structure analogous to that in quark–gluon theory and that it is in principle knowable.

One fascinating problem, of course, is to understand the conditions under which we can have an algebra resembling that for quarks and gluons and yet escape having real quarks and gluons. Under what conditions do the bilocals act as if they were the products of local operators without, in fact, being seen. We seek answers to this and other questions by asking "Are there models other than the quark–gluon field theory from which we can abstract results? Can we replace $\Theta_{\mu\nu}^{\mathrm{glue}}$ by something different and the gauge–derivative by a different gauge–derivative?"

VI. ARE THERE ALTERNATIVE MODELS?

In the search for alternatives to gluons, one case worth investigating is that of the simple dual resonance model. It can be considered in three stages: first, the theory of a huge infinity of free mesons of all spins; next, tree diagrams involving the interaction of these mesons; and finally loop diagrams. The theory is always treated as though referring to real mesons, and an S–matrix formulation is employed in which each meson is always on the mass shell.

Now the free stage of the model can easily be reformulated as a field theory in ordinary coordinate space, based on a field operator Φ that is a function not of one point in space, but of a whole path – it is infinitely multilocal. The free approximation to the dual resonance model is then essentially the quantum theory of a relativistic string or linear rubber band in ordinary space.

The coupling that leads, on the mass shell, to the tree diagrams of the dual resonance model has not so far been successfully reformulated as a field theory coupling but we shall assume that this can be done. Then the whole model theory, including the loops, would be a theory of a large infinity of local meson fields, all described simultaneously by a grand infinitely multilocal field Φ, couples to themselves and one another. The mesons, in the free approximation, lie on straight parallel Regge trajectories with a universal slope α'.

In the simplest form of such a theory, the grand field Φ (path) can be resolved into local fields $\phi(R), \Phi_{n\mu}(R), \Phi_{n\mu,n'\mu'}(R), \ldots$. There is a single scalar, a single infinity of vectors, a double infinity of tensors and scalars, and so forth. The matrices $a_{n\mu}$ and $a^{+}_{n\mu}$ of the dual theory connect these components of Φ with one another.

Perhaps the model theory of a gluon field can be replaced by a field theory version of a dual resonance model; the properties of operators, including $\Theta_{\mu\nu}$, would be abstracted from the new model instead of the old one. With $\alpha' \neq 0$, a term δ would naturally appear that violates scale invariance and is not related to the bare quark masses. (Probably $l_\delta = 0$ here rather than -2 as in the case of a gluon mass.) The gauge derivative in the other portion of $\Theta_{\mu\nu}$, referring to the quarks, would then involve a special linear combination of the $\Phi_{n\nu}(R)$ instead of the gluon potential $B_\mu(R)$.

An amusing point is that in the limit of a dual resonance theory as $\alpha' \to 0$ (so that the trajectories become flat), with attention concentrated on the value $\alpha = 1$, if the mathematics of a Lie group is built into the model, then the mass shell predictions become those of the corresponding massless Yang–Mills theory[10)]. That suggests that one might even try a dual resonance model as a replacement of a color octet Yang–Mills gluon model, with abstraction of the properties of color singlet operators.

We are not at all sure that what we are discussing here is a practical scheme, and if it is, we do not know how the resulting algebraic system differs from that of gluons. We put it forward merely in order to stimulate thinking about whether or not here are candidates for the algebra, the representation, and the form of $\Theta_{\mu\nu}$ other than those suggested by the gluon model.

Our attempt to use the dual model to construct a field theory has no bearing on whether the mass–shell dual model can lead to a complete S–matrix theory of hadrons; our suggestion resembles the use of limits of dual theories to obtain unified theories of weak and electromagnetic interactions or the theory of gravity.

One interesting speculation that is independent of what model we use for the stuff to which quarks are coupled is that perhaps when we perform the mathematical transformation from current quarks to constituent quarks and obtain the crude naive quark model of meson and baryon spectra and couplings, the gluons or whatever they are will also be approximately transformed into fictitious constituents, so that meson states would appear that act as if they were made of gluons rather than $q\bar{q}$ pairs. If there are indeed ten low–lying scalar mesons rather than nine, then we might interpret the tenth one (roughly speaking, the ε° meson) as the beginning of such a sequence of extra Su_3 singlet meson states. (A related question, much debated by specialists in the usual, mass–shell dual models, is whether the infinite sequence of meson and baryon Regge trajectories, all rising indefinitely and straight and parallel in zeroth approximation, should be extended to exotic channels, i. e., those with quantum numbers characteristic of $qqqq\bar{q}, q\bar{q}q\bar{q}$ etc.).

Let us end by emphasizing our main point, that it may well be possible to construct an explicit theory of hadrons, based on quarks and some kind of glue, treated as fictitious, but with enough physical properties abstracted and applied to real hadrons to constitute a complete theory. Since the entities we start with are fictitious, there is no need for any conflict with the bootstrap or conventional dual model point of view.

APPENDIX – BILOCAL FORM FACTOR ALGEBRA

We have described in Section III and IV a Lie algebra of good components of bilocal operators on a null plane. The generators are 36 functions of x_-, y_- and $x_\perp = y_\perp$, namely $\mathcal{F}_{j+}, \mathcal{F}^5_{j+}, \mathcal{T}_{jl+}$, and $\mathcal{T}_{j2+}$. We define $R \equiv 1/2(x+y)$ and $z \equiv x-y$; then we have functions of $R_\perp, R_-$, and z_-.
With z_- set equal to zero, we have just the usual good local operators on the null plane, related to the corresponding good local operators at equal times with $P_3 \to \infty$. We recall that in the early work using $P_3 \to \infty$ the most useful applications (fixed virtual mass sum rules) involved matrix elements with no change of longitudinal momentum, i. e., transverse Fourier components of the operators. Dashen and Gell–Mann[11)] studied these operators and found that between finite mass states their matrix elements do not depend separately on the transverse momenta of the initial and final states, but only on the difference, which is the Fourier variable $k_\perp$. Thus they obtained a "form factor algebra" generated by operators $F_i(k_\perp)$ and $F^5_i(k_\perp)$, to which, of course, one may adjoin $T_{il}(k_\perp)$ and $T_{i2}(k_\perp)$.
We may consider the analogous quantities using the null plane method and generating to bilocals:

$$F_i(k_\perp, z_-) \equiv$$
$$\int d^4R\,\delta(R_+)\,\mathcal{F}_{i+}(R, z_-) \ \ \exp\ ik_1\left[R_1 + P_+^{-1}(\Lambda_1 + J_2)\right] \ \ \exp\ ik_2\left[R_2 + P_+^{-1}(\Lambda_2 - J_1)\right] \tag{25}$$

and so forth. Here the integration over R_- assures us that $P_+ \equiv P_0 + P_3$ is conserved by the operator. (We note that Minkowski[12)] and others have studied the interesting problem of extracting useful sum rules from operators unintegrated over R_-, but we do not discuss that here.) The quantities $P_+^{-1}(\Lambda_1 + J_2)$ and $P_+^{-1}(\Lambda_2 - J_1)$ act like negatives of center–of–mass coordinates, $-\bar{R}_1$ and $-\bar{R}_2$, since on the null plane $x_+ = 0$ we have $\Lambda_1 + J_2 = -\int R_1 \Theta_{++} d^4R\,\delta(R_+)$ and $\Lambda_1 + J_1 = -\int R_2\Theta_{++} d^4R\,\delta(R_+)$, while $P_+ = \int \Theta_{++} d^4R\,(R_+)$.
Our bilocal form factor algebra has the commutation rules

$$\left[F_i(k_\perp, z_-), F_j\left(k'_\perp, z'_-\right)\right] = i\, f_{ijk} F_k\left(k_\perp + k'_\perp, z_- + z'_-\right), \tag{26}$$

etc., where the structure constants in general are those of $[U_6]_w$. Putting $z_- = z_-{}' = 0$, we obtain exactly the form factor algebra of Dashen and Gell–Mann. If we specialize further to $k_\perp = k'_\perp = 0$, we obtain the algebra $[U_6]_{w,\infty,\ \mathrm{currents}}$, of vector, axial vector, and tensor charges. It is not, of course, identical to the approximate symmetry algebra $[U_6]_{w,\infty\ \mathrm{strong}}$, for baryon and meson spectra and vertices, but is related to it by a transformation, probably unitary. That is the transformation which we have described crudely as connecting current quarks and constituent quarks.
The behavior of the operators $F_i(k_\perp)$, etc., with respect to angular momentum in the

s–channel is complicated and spectrum–dependent; it was described by Dashen and Gell–Mann in their angular condition[10)]. There is a similar angular condition for the bilocal generalizations $F_i\left(k_\perp, z_-\right)$, etc.

The behavior of $F_i\left(k_\perp, z_-\right)$ and the other bilocals with respect to angular momentum in the cross–channel is, in contrast, extremely simple. If we expand $F_i\left(k_\perp, z_-\right)$ or $F_i^5\left(k_\perp, z_-\right)$ in powers of z_-, each power z_-^n corresponds to a single angular momentum, namely $J = n + 1$.

As we expand $F_i\left(k_\perp, z_-\right)$, etc., in power series in z_-, we note that each term, in $z_-{}^{J-1}$, has a pole in $k_\perp^2$ at $k_\perp^2 + M^2 = 0$, where M is the mass of any meson of spin J. By an extension of the Regge procedure, we can keep $k_\perp^2$ fixed and let the angular momentum vary by looking at the asymptotic behavior of matrix elements of $F_i\left(k_\perp, z_-\right)$, etc., at large z_-. A Regge pole in the cross channel gives a contribution $z_-^{\alpha\left(-k_\perp^2\right)} \beta\left(k_\perp^2\right) \left[\sin\ \pi\alpha\left(-k_\perp^2\right)\right]^{-1}$ and a cut gives a corresponding integral over α. Thus the bilocal form factor $F_i\left(k_\perp, z_-\right)$ couples to each Reggeon in the non–exotic meson system in the same way that $\mathcal{F}_i\left(k_\perp\right)$ couples to each vector meson. The contribution of each Regge pole to the asymptotic matrix element of $F_i\left(k_\perp, z_-\right)$ between hadron states A and B is given by the coupling of $\mathcal{F}_i\left(k_\perp, z_-\right)$ to that Reggeon multiplied by the strong coupling constant of the Reggeon to A and B.

It would be nice to substitute the Regge asymptotic behavior of $F_i\left(k_\perp, x_-\right)$ etc., into the commutation rules and obtain algebraic relations among the Regge residues. Unfortunately, the asymptotic limit is not approached uniformly in the different matrix elements, and the asymptotic Regge formulae cannot, therefore, be used for the operators everywhere in the equations (A.2); only partial results can be extracted.

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