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Holomorphic $N = 1$ Special Geometry of Open–Closed Type II Strings

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Abstract

We outline a general geometric structure that underlies the $\mathcal{N} = 1$ superpotentials of a certain class of flux and brane configurations in type II string compactifications on Calabi-Yau threefolds. This "holomorphic $\mathcal{N} = 1$ special geometry" is in many respects comparable to, and in a sense an extension of, the familiar special geometry in $\mathcal{N} = 2$ supersymmetric type II string compactifications. It puts the computation of the instanton-corrected superpotential W of the four-dimensional $\mathcal{N} = 1$ string effective action on a very similar footing as the familiar computation of the $\mathcal{N} = 2$ prepotential $\mathcal{F}$ via mirror symmetry. In this note we present some of the main ideas and results, while more details as well as some explicit computations will appear in a companion paper.

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1. Introduction

As is well-known, the manifold $\mathcal{M}_{\mathcal{N}=2}$ of scalar vev's z_a of the vector multiplets of a four-dimensional $\mathcal{N}=2$ supergravity is characterized by a "special geometry". In the context of an effective supergravity theory obtained from the compactification of a type II string on a Calabi–Yau manifold X, the $\mathcal{N}=2$ special geometry can be understood from (at least) three different points of view:

i) as a consequence of local $\mathcal{N}=2$ space-time supersymmetry [1];

ii) from the structure of the underlying 2d topological field theory (TFT) on the string world-sheet, whose correlation functions are summarized by the $\mathcal{N}=2$ effective supergravity [2];

iii) for type IIB strings, in terms of the Hodge structure on the middle cohomology of X which varies with the complex structure moduli z_a [3].

The last formulation is particularly important, as it allows to determine the *exact* non-perturbative effective action (up to two derivatives) for the vector multiplets from classical geometric data; that is, from the periods of the holomorphic (3,0) form, integrated over an integral basis for the middle homology $H_3(X,\mathbf{Z})$. This fact has been exploited with large success in the context of closed string mirror symmetry [4].

The purpose of this letter is to report similar results on a special geometry of the holomorphic F-terms of certain $\mathcal{N}=1$ supersymmetric string compactifications. Specifically, these are type II compactifications on Calabi–Yau manifolds with extra fluxes and (D-)branes. As is evident, this holomorphic $\mathcal{N}=1$ special geometry will not be a general consequence of $\mathcal{N}=1$ space-time supersymmetry, but really a property of the string effective theory. However, as will be discussed, the counterparts of the above items *ii*) and *iii*) still exist in this phenomenologically relevant class of superstring theories. Some aspects of the special geometry have been studied already in [5,6], based on a class of D-brane geometries defined and studied in the important work [7].

In the following, we describe the $\mathcal{N}=1$ special geometry as the consequence of systematically incorporating fluxes and branes into the familiar ideas and methods of mirror symmetry for Calabi–Yau 3-folds. Starting with the $\mathcal{N}=1$ counterpart of item ii) above, the TFT description involves the chiral ring $\mathcal{R}_{oc}$ of the open-closed B-model, and an integrable, topological connection on the space of 2d RR ground states generated by the elements of $\mathcal{R}_{oc}$. These concepts are, as in the case of $\mathcal{N}=2$ special geometry, particular aspects of the general tt^* geometry [8,2] of the 2d SCFT on the string world-sheet.

As for item iii), what is needed conceptually is to pass from the middle cohomology $H^3(X)$, which enters the usual form of mirror symmetry, to a certain relative cohomology group $H^3(X,Y)$ defined by a submanifold $Y \subset X$ associated to the background branes.

Accordingly, the rôle of the variation of Hodge structure in the construction of the usual mirror map is taken over by the variation of the mixed Hodge structure on this relative cohomology group. Together this leads to a nice correspondence between the concepts of the fundamental 2d world-sheet theory and of the target space geometry, such as a geometric representation for the open-closed chiral ring.

These concepts in TFT and geometry manifest themselves in the structure of the $\mathcal{N} = 1$ superpotential of the effective four-dimensional space-time theory. In particular, we will show how the superpotential W and its derivatives specify the moduli dependent chiral ring $\mathcal{R}_{oc}$, or, equivalently, the mixed Hodge structure on the relative cohomology group.

The basic topological data are a set of holomorphic potentials $\mathcal{W}_K(z_A)$ that are, in a quite precise sense, the $\mathcal{N} = 1$ counterparts of the holomorphic prepotential $\mathcal{F}(z_a)$ of $\mathcal{N} = 2$ special geometry. They are among the building blocks of the superpotential W in the various flux and D-brane sectors labeled by the index K. Above, the z_A are the scalar vev's of the chiral multiplets, both from the closed and open string sectors, which parametrize the $\mathcal{N} = 1$ moduli space $\mathcal{M}_{\mathcal{N}=1}$.[1] As will be discussed, on this space there exist flat, topological coordinates t_A such that the derivatives

$$\frac{\partial}{\partial t_A}\frac{\partial}{\partial t_B}\,\mathcal{W}_K(t.) = C_{AB}^{\ K}(t.)$$

represent the structure constants $C_{AB}^{\ K}$ of the open-closed chiral ring $\mathcal{R}_{oc}$. The special coordinates t_A are defined by an integrable connection ∇, that defines a system of differential equations of Picard-Fuchs type for the superpotential. The instanton corrected superpotential is a solution of this system and can be determined by standard methods.

Since the mathematical details are a little involved, we will present in this note some of the basic ideas and results, and defer more thorough arguments and computations to a companion paper [9]. In Appendix A we include a brief summary of the basic concepts in 2d TFT, such as the chiral ring, and their connection to ordinary special geometry. This section will be useful for reference and provides some background material for our discussion of $\mathcal{N} = 1$ special geometry.

[1] We will loosely refer to the coupling space $\mathcal{M}_{\mathcal{N}=1}$ as a "moduli space", although in general there will be a superpotential that lifts the flat directions. In particular this makes sense for a scalar with a perturbatively flat potential, as is the case for certain brane moduli.

2. $\mathcal{N}=1$ Superpotentials in type II Calabi–Yau compactifications

We begin the discussion of the holomorphic $\mathcal{N}=1$ special geometry with a physical characterization of an object that will be central to much of the following: this is the relative period vector Π^Σ which encodes the instanton corrected $\mathcal{N}=1$ superpotential of the open-closed string theory.

We consider a compactification of the closed type IIB string on a Calabi–Yau 3-fold X with Hodge numbers $h^{p,q} = \dim H^{p,q}$. The effective theory at low energies is an $\mathcal{N}=2$ supergravity with a generic Abelian gauge group $U(1)^{h^{1,2}+1}$. There are two closely related modifications of this $\mathcal{N}=2$ supersymmetric type IIB background that break supersymmetry in a way that may be described by an effective $\mathcal{N}=1$ supergravity action.

The first modification is a deformation in the closed string theory, obtained by adding background fluxes $H = H^{RR} + \tau H^{NS}$ of the 2-form gauge fields on X. This leads to an $\mathcal{N}=1$ superpotential of the form [10,11,12]:

$$W_{cl}(z_a) = \int \Omega \wedge H \; = \; \sum_\alpha N_\alpha \cdot \Pi^\alpha(z_a). \tag{2.1}$$

This superpotential depends on the vev's of certain scalars z_a in $\mathcal{N}=1$ chiral multiplets, and these vev's represent the complex structure deformations from the closed string sector. The moduli dependence is encoded in the period vector of the holomorphic $(3,0)$-form Ω on X:

$$\Pi^\alpha = \int_{\Gamma^\alpha} \Omega(z_a), \qquad \Gamma^\alpha \in H_3(X, \mathbf{Z}).$$

The parameters N_α in (2.1) specify the integer 3-form fluxes on X. That is, $N_\alpha = n_\alpha + \tau m_\alpha$, where the n_α, m_α are integers and τ is the type IIB string dilaton [11].

The second supersymmetry-breaking modification is to introduce an open string sector by adding background (D-)branes that wrap supersymmetric cycles $B_\nu \in H_{2n}(X)$ and simultaneously fill space-time. The $\mathcal{N}=1$ superpotential for these branes is computed by the holomorphic Chern-Simons functional [13]. Specifically, we will consider 5-branes wrapped on a set of 2-cycles $\{B_\nu\}$, for which the superpotential is given by [7,14,15]:

$$W_{op}(z_a, \hat{z}_k) \; = \; N_\nu \cdot \int_{\hat{\Gamma}^\nu} \Omega(z_a) \; = \; \sum_\nu N_\nu \cdot \Pi^\nu(z_a, \hat{z}_\alpha)\,. \tag{2.2}$$

Here $\hat{\Gamma}^\nu$ denotes a special Lagrangian 3-chain with boundary $\partial\hat{\Gamma}^\nu \supset B_\nu$ and the $\hat{z}_\alpha$ are the brane moduli from the open string sector. Moreover $N_\nu = n_\nu + \tau m_\nu$, where n_ν (m_ν)

are the numbers of D5 (NS5) branes. The moduli $\hat{z}_\alpha$ parametrize the position of the D-branes in X in a special way, namely by measuring the volumes of the 3-chains $\hat{\Gamma}^\nu$ whose boundaries are wrapped by the D-branes. The precise definition of the good open string moduli will be one of the outcomes of the following discussion.

It is natural to consider a combination of the two types of supersymmetry breaking backgrounds, and to study the general superpotential on the full deformation space $\mathcal{M}_{\mathcal{N}=1}$ parametrized by the closed and open string moduli, z_a and $\hat{z}_\alpha$, respectively. In the low energy effective action, the two contributions combine into the section of a single line bundle $\mathcal{L}$ over $\mathcal{M}_{\mathcal{N}=1}$, and are really on the same level. Note that the line bundle $\mathcal{L}$ is identified with the bundle of holomorphic $(3,0)$ forms on X. Also, in string theory, the distinction between RR fluxes and background D-branes is somewhat ambiguous: In the large N transition of [16] a type IIB background with N D-branes is replaced by a type IIB background on a different manifold X' without branes but N units of flux.

We thus consider the most general superpotential from background fluxes and branes of the form

$$W_{\mathcal{N}=1} \;=\; W_{cl}(z_a) + W_{op}(z_a, \hat{z}_\alpha) \;=\; \sum_\Sigma N_\Sigma \cdot \Pi^\Sigma(z_A), \tag{2.3}$$

where Π^Σ is the *relative period vector*[2]

$$\Pi^\Sigma(z_A) = \int_{\Gamma^\Sigma} \Omega\,, \qquad \Gamma^\Sigma \in H_3(X, B, \mathbf{Z}), \qquad z_A \equiv \{z_a, \hat{z}_\alpha\}. \tag{2.4}$$

Here $H_3(X,B)$ is the group of relative homology cycles on X with boundaries on $B = \cup_\nu B_\nu$, where $B_\nu \subset \partial\hat{\Gamma}^\nu$. Its elements are *i*) the familiar 3-cycles without boundaries, whose volumes specify the flux superpotential and *ii*) the 3-chains with boundaries on the 2-cycles B_ν. The vector Π^Σ thus uniformly combines the period integrals of Ω over 3-cycles Γ^α, with integrals over 3-chains $\hat{\Gamma}^\nu$ whose boundaries are wrapped by D-branes.

In the following we will show that, for appropriate normalization of Ω, the relative period vector is of the form

$$\Pi^\Sigma = (1, t_A, \mathcal{W}_K, \ldots)\;, \tag{2.5}$$

where t_A are flat topological coordinates on $\mathcal{M}_{\mathcal{N}=1}$ and $\mathcal{W}_K$ the *holomorphic potentials of $\mathcal{N}=1$ special geometry*. The latter determine the ring structure constants $C_{AB}^{\ \ K}$ of the extended open-closed chiral ring $\mathcal{R}_{oc}$

$$\phi_A^{(1)} \cdot \phi_B^{(1)} = C_{AB}^{\ \ K}\ \phi_K^{(2)}.$$

[2] More precisely, the pairing is defined in relative (co-)homology, as discussed in sect. 5.

Here the $\phi_I^{(q)}$ denote a basis of 2d superfields that span the local BRST cohomology of the topological sector of the 2d world-sheet theory for the compactification on X.[3] The ring structure constants are given in terms of the derivatives

$$C_{AB}^{\ \ K}(t.) = \frac{\partial}{\partial t_A}\frac{\partial}{\partial t_B}\mathcal{W}_K(t.). \tag{2.6}$$

This formula is the $\mathcal{N}=1$ counterpart of the well-known equation (A.3) that describes the chiral ring constants of $\mathcal{N}=2$ special geometry in terms of the prepotential $\mathcal{F}$.

In the next two sections we will outline the relation between the relative period vector Π^Σ that defines the holomorphic $\mathcal{N}=1$ superpotential (2.3), and the chiral ring of the underlying TFT on the string world-sheet. Concretely, the elements ϕ_A of the chiral ring $\mathcal{R}_{oc}$ have a geometric representation in the B-model as elements of a certain relative cohomology group $H^3(X,Y)$.[4] On the bundle of open-closed B-models over $\mathcal{M}_{\mathcal{N}=1}$, there exists a topological *flat* connection ∇, which is the Gauss-Manin connection on the relative cohomology bundle. The flatness of ∇ predicts the existence of special topological coordinates t_A, for which the covariant derivatives ∇_A reduce to the ordinary derivatives $\partial_A = \frac{\partial}{\partial t_A}$; these are precisely the special coordinates on the $\mathcal{N}=1$ moduli space $\mathcal{M}_{\mathcal{N}=1}$ that appear in (2.5). Moreover, the geometric representation of the chiral ring $\mathcal{R}_{oc}$ leads to an exact expression for the holomorphic potentials $\mathcal{W}_K$ in terms of the "relative period matrix" for $H^3(X,Y)$. This matrix describes the projection of the moduli dependent chiral ring onto a fixed, topological field basis and represents the counterpart of the familiar period matrix (A.4) in $\mathcal{N}=1$ special geometry.

Before we turn to this discussion, it is worth to comment on the physical meaning of this extra structure in the effective four-dimensional supergravity theory. This is important, as it distinguishes the string effective theory from a generic $\mathcal{N}=1$ supergravity.

Recall that the type IIB string has 5-branes in the NS and the RR sectors, which may be wrapped on supersymmetric 3-cycles in $H_3(X,B)$. These states are interpreted as domain walls in four dimensions with non-zero BPS charge [14,17]. We are thus led to identify the integral homology lattice appearing in (2.4), with the four-dimensional lattice of BPS charges

$$\Gamma^{4d}_{BPS} = H_3(X,B;\mathbf{Z}). \tag{2.7}$$

[3] See sect. 4 and App. A for a further discussion and references.

[4] Here Y denotes a union of hypersurfaces in X passing through the cycles B_ν, as discussed below.

The BPS tension of a domain wall with charge $Q \in \Gamma_{BPS}$ is then determined by the volume of the wrapped 3-manifold, as measured by the relative period vector Π^Σ. In fact the tension of the domain wall is proportional to a change in the $\mathcal{N}=1$ superpotential on the two sides of the domain wall [17,11]. Thus the space-time interpretation of the holomorphic $\mathcal{N}=1$ special geometry of the string effective supergravity is in terms of the BPS geometry of the 4d domain walls with charges $Q \in \Gamma_{BPS}$, very much as the $\mathcal{N}=2$ special geometry describes the more familiar BPS geometry of four-dimensional particles.

One may also interpret the $\mathcal{N}=1$ special geometry in terms of masses of BPS particles, by compactifying on a further T^2 to a two-dimensional theory with $\mathcal{N}=2$ supersymmetry, with the 5-branes wrapped on the extra T^2. The same type of theories also arises in a Calabi–Yau 4-fold compactification of the type IIB string, where the BPS particles are represented by 5-branes wrapped on special Lagrangian 4-cycles in the 4-fold. In some cases such a closed string 4-fold compactification is dual to the open-closed type II string compactification on the 3-fold X times the extra T^2 [5]. This then provides an identification of the presently discussed $\mathcal{N}=1$ BPS geometry of 4d domain walls on the 3-fold with D-branes, with the 2d $\mathcal{N}=2$ geometry of BPS *states* in the 4-fold compactification.

Another interesting aspect is that the ring structure constants enjoy an integral instanton expansion [18]:

$$C_{AB}^{(inst)\ K} = \sum_{\{n_C\}} \sum_k n_A n_B N^{(K)}_{n_1 \ldots n_M} \prod_C e^{2\pi i k n_C t_C}, \tag{2.8}$$

where $M = \dim \mathcal{M}_{\mathcal{N}=1}$ and the coefficients $N^{(K)}_{\cdot}$ are integers that count the number of certain BPS multiplets of the theory compactified to two dimensions, in the topological sector labelled by K. Alternatively, these numbers may be thought of as counting the (appropriately defined) number of world-sheet instantons of sphere or disc topologies.[5]

Most importantly, the instanton expansion (2.8) can in many cases, if not in all, be directly identified with genuine *space-time* instanton corrections to the superpotential in the RR sector. The reason is the familiar fact that the four-dimensional coupling constants in the RR sector are not determined by the four-dimensional string coupling, but rather by the geometric moduli t_A. The statement that the space-time instanton expansion has integrality properties, attributable to the counting of BPS states, is quite remarkable and distinguishes this class of effective supergravities from generic ones.

[5] In the case of sphere topologies, the above multi-covering formula is again defined in a two-dimensional compactification; see [5] for a discussion.

3. Open-closed chiral ring and relative cohomology

As reviewed in App. A, the elements of the chiral ring of the closed string B-model on X have a geometric representation as the cohomology elements in $H^3(X)$. Moreover the gradation by $U(1)$ charge of the chiral ring corresponds to the Hodge decomposition $H^3(X) = \oplus_p H^{p,3-p}(X)$. We will now describe a similar geometric representation of the chiral ring of the open-closed B-model, in terms of a relative cohomology group $H^3(X,Y)$.[6]

The open-closed chiral ring is an extension of the closed string chiral ring, and combines operators from both the bulk and boundary sectors. Geometrically, the new structure from the open string sector is the submanifold $B \subset X$ wrapped by the D-branes.[7] Since the bulk sector of the closed string is represented by $H^3(X)$, the open-closed chiral ring should correspond to an extension of this group by new elements originating in the open string sector on B.[8] It is natural to expect that this extension is simply the dual of the space $H_3(X,B)$ which is underlying the superpotential (2.3) from the fluxes and branes. The dual space is the relative cohomology group $H^3(X,B)$, and we will see that it gives indeed the right answer.

Let us discuss at this point more generally a relative cohomology group $H^3(X,Y)$, where $Y = B$ for the moment; the motivation for this is that we will eventually give an argument that allows to replace the boundary B by a simpler object.

The relative cohomology group $H^3(X,Y)$ fits into a long exact sequence

$$... \to H^2(Y) \to H^3(X,Y) \to H^3(X) \to ... \tag{3.1}$$

To simplify the discussion, let us assume that the maps on the left and right hand side are trivial. This assumption can be easily checked in practice and is justified for a large class of D-brane geometries [9]. In this case the group $H^3(X,Y)$ is essentially an extension of a space which combines the 3-forms on X and the 2-forms on $Y \subset X$. The elements of $H^3(X)$ have an obvious interpretation as describing the (topological) closed string states in the open-closed string compactification[9]. On the other hand the group $H^2(Y)$ will describe the new degrees of freedom originating in the open string sector.

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[6] For an introduction to relative cohomology, see e.g. [19].

[7] We will restrict our discussion to trivial line bundles on the two-cycle B_ν.

[8] More precisely, the ring multiplication for the bulk operators could be a priori different on closed and open string world-sheets.

[9] In 2d CFT language, these are bulk operators inserted in the interior of a world-sheet with boundary.

An element $\vec{\Theta} \in H^3(X,Y)$ may be specified by a pair of differential forms

$$\vec{\Theta} = (\Theta_X, \theta_Y),$$

where Θ_X is a 3-form on X and θ_Y a 2-form on Y. The differential is

$$d\vec{\Theta} = (d\Theta_X, i^*\Theta_X - d\theta_Z),$$

where i^* is the map on forms deriving from the embedding $i: \; Z \to X$. Thus $d\vec{\Theta} = 0$ implies that Θ_X is closed on X and restricts to an exact form on Z. Moreover the equivalence relation is

$$\vec{\Theta} \sim \vec{\Theta} + (d\omega, i^*\omega - d\phi), \tag{3.2}$$

where ω (ϕ) is a 2-form on X (1-form on Z). Note that this equation says that the exact form $-d\omega$ on X is not necessarily trivial in $H^3(X,Y)$, but equivalent to the 2-form $i^*\omega$ on Y. Thus a form that represents a trivial operator in the closed string theory, such as an exact piece of the holomorphic $(3,0)$ form, may lead to non-trivial elements in the open string extension of the chiral ring – specifically from "boundary terms" on the submanifold $Y \subset X$.

Our aim is to connect the moduli dependence of the chiral ring $\mathcal{R}_{oc}$ for a family of geometries parametrized by the couplings z_A, to the superpotential W and its derivatives. Geometrically, the coordinates $z_A = (z_a, \hat{z}_\alpha)$ on the moduli space $\mathcal{M}_{\mathcal{N}=1}$ describe the complex structure of the manifold X (closed string sector) and the "location" of the D-branes specified by the map $Y \hookrightarrow X$ (open string sector), respectively. The relative cohomology groups $H^3(X,Y)$ fit together to a bundle $\mathcal{V}$ over $\mathcal{M}_{\mathcal{N}=1}$ that may be identified with the bundle of RR groundstates $|I\rangle_{RR}(z_A, \bar{z}_{\bar{A}})$ for the B-model on the family of D-brane geometries parametrized by the couplings z_A. Here we are using the correspondence between the chiral ring elements ϕ_I and groundstates $|I\rangle_{RR}$ in the RR sector; specifically the ground state $|I\rangle_{RR}$ may be obtained from the canonical vacuum $|0\rangle_{RR}$ by inserting the operator ϕ_I in the twisted path integral on a world-sheet with boundary [2].

In the following we discuss the case of a single brane wrapped on a 2-cycle B; the superpotential for a compactification with several non-intersecting branes wrapped on a collection of cycles $\{B_\nu\}$ is the sum of the individual superpotentials and can be treated similarly. As is well-known, a supersymmetric configuration of a D-brane wrapped on a 2-cycle $B \in X$ requires B to be holomorphic [20]. On general grounds, the condition for supersymmetry in field theory is of the form $W'(z_A) = 0$, where the prime denotes an arbitrary derivative in the moduli. A non-trivial superpotential in the moduli z_A is thus defined on a family $\mathcal{B}$ of 2-cycles, whose members are in general non-holomorphic, except at those values of z_A where W has critical points.

A study of the relative cohomology group $H^3(X,B)$ for $B \in \mathcal{B}$ would thus involve in general non-holomorphic maps $B \hookrightarrow X$. To avoid such a complication, we will now give an argument that allows to replace the group $H^3(X,B)$ by another relative cohomology group $H^3(X,Y)$, where Y is a member of a family of holomorphic hypersurfaces in X that pass through the boundary cycles in $\mathcal{B}$.[10]

The superpotential for the brane wrapped on B is proportional to the volume (2.2) of a minimal volume 3-chain Γ with boundary $\partial\Gamma \supset B$. One way to solve the minimal volume condition is to slice Γ into 2-cycles along a path I. This can be achieved by intersecting Γ with a family[11] of holomorphic hyperplanes $Y(z)$, where z is a complex parameter. The intersection of the hyperplane $Y(z)$ with Γ is a family of 2-cycles $\Gamma_2(z)$ of minimal volume $V(z)$, and the integral (2.4) can be written as

$$W_\Gamma = \int_\Gamma \Omega = \int_{z_0}^{z_1} V(z) dz. \tag{3.3}$$

Here the path I in the z-plane is determined by the minimal volume condition for Γ. The existence of the appropriate family of hypersurfaces is the assumption used in the following discussion.[12]

The interval in the z-plane ends at a specific value, say $z = \hat{z}$, for which the hyperplane $Y \equiv Y(\hat{z})$ passes through the boundary 2-cycle B which is wrapped by the D5-brane. Varying the position of the D-brane leads to the variation of the brane superpotential:

$$\delta W_{\hat{\Gamma}} \;\sim\; V(\hat{z}) \;=\; \int_{\Gamma_2(\hat{z})} \omega, \tag{3.4}$$

[10] Simply speaking, the possibility to replace the group $H^3(X,B)$ by $H^3(X,Y)$ is due the fact that the topological B-model depends only on the complex structure type of moduli, not on the Kähler type. Thus, in general there may be a whole family of deformations of the 2-cycle Γ_2, whose members all have the same superpotential. In the situation discussed below, the group $H^3(X,Y)$, as defined by an appropriate holomorphic hypersurface Y passing through this family, is a good object for capturing the superpotential. See also [21] for a related discussion.

[11] For simplicity, we will consider here only one-parameter families, corresponding to brane wrappings with a single modulus.

[12] This is a relatively mild assumption that is in particular satisfied for a large class of D-brane geometries [9]. The above idea was used in [22] in the context of (mirrors of) ALE fibrations, which give a natural slicing of a chain Γ into a 1-parameter family of 2-cycles in the ALE fiber parametrized by a path on the base.

where ω is a 2-form on Y. As discussed in more detail in [9], the form ω is a holomorphic $(2,0)$ form on Z obtained from a Poincaré residue of Ω. Note that δW_Γ vanishes if Γ_2 is holomorphic [14,7,15], as expected.

By the above construction, the open string extension of the chiral ring is represented by the holomorphic 2-form ω on the hypersurface Y. As shown in [9], products of chiral fields in the bulk with ω generate additional elements in $\mathcal{R}_{oc}$ which can be identified with 2-forms of lower holomorphic degree in $H^2(Z)$. These elements in $H^2(Z)$, together with the elements in $H^3(X)$ from the closed string sector, combine into the relative cohomology group $H^3(X,Y)$, as described by the sequence (3.1).

4. Exact 4d $\mathcal{N}=1$ superpotential from the integrable connection ∇

The relative cohomology bundle $\mathcal{V}$ over $\mathcal{M}_{\mathcal{N}=1}$ with fibers $H^3(X,Y)$ comes with a structure that comprises all the ingredients needed to define the generalized special geometry of the open-closed B-model. This is the family of mixed Hodge structures [23] on the bundle $\mathcal{V}$ over $\mathcal{M}_{\mathcal{N}=1}$, and a flat Gauss-Manin connection ∇ defined on it. We will be brief in the following, leaving more detailed explanations and computations to [9]. Here we sketch how the integrable connection ∇ on $\mathcal{V}$ determines a set of flat topological coordinates on the $\mathcal{N}=1$ moduli space $\mathcal{M}_{\mathcal{N}=1}$ and the exact instanton corrected $\mathcal{N}=1$ superpotential of the effective 4d theory. The following discussion will be in many respects be similar to the one in the Appendix, where we review how the flat connection for the moduli dependent closed string chiral ring leads to the exact formula for the $\mathcal{N}=2$ prepotential $\mathcal{F}$.

Specifically we want to relate the holomorphic potentials $\mathcal{W}_K$ to the moduli dependence of the structure constants of the open-closed chiral ring, $\mathcal{R}_{oc}$. For this purpose, we would like to know at each point in the coupling space $z_A \in \mathcal{M}_{\mathcal{N}=1}$, the basis $\{\vec{\Phi}_I\}$ for $H^3(X,Y)$ that represents the elements ϕ_I of the chiral ring, or equivalently, the RR ground states $|I\rangle_{RR}$ created by the operators ϕ_I from a canonical vacuum $|0\rangle_{RR}$.

An important structure in the TFT is the existence of a preserved $U(1)$ charge. This gives an integral grading to the space V of RR ground states

$$V = \oplus_q V^{(q)}.$$

Special geometry essentially describes the position of the subspaces $V^{(q)}$ in V, defined relative to a constant basis for V.

At grade zero, there is a unique element of the chiral ring corresponding to the unit operator $\phi^{(0)}$ that flows to the canonical vacuum in the RR sector, $|0\rangle_{RR}$.[13]

The most interesting sector is that of the fields $\phi^{(1)}_A$ of grade one, which represent the deformations of the 2d TFT parametrized by the couplings $z_A \in \mathcal{M}_{\mathcal{N}=1}$. In the present context, these fields generate the full chiral ring. For example, grade two fields can be generated by the OPE:

$$\phi^{(1)}_A(z.) \cdot \phi^{(1)}_B(z.) = C_{AB}^{\ \ K}(z.)\, \phi^{(2)}_K(z.). \tag{4.1}$$

The moduli dependence of a chiral field $\phi^{(q)}_I(z.)$ can be described by projecting to a fixed, constant basis for V. Such a basis $\{\vec{\Gamma}_\Sigma\}$ may be defined as the dual of a basis Γ^Σ of topological cycles for the relative homology group $H_3(X,Y)$. The transition to the constant basis is

$$\vec{\Phi}^{(q)}_I(z.) = \Pi_I^\Sigma(z.)\ \cdot\ \vec{\Gamma}_\Sigma\,.$$

Here $\Pi_I^\Sigma(z.)$ is the *relative period matrix* for the relative cohomology group $H^3(X,Y)$:

$$\Pi_I^\Sigma = \langle \vec{\Phi}_I, \Gamma^\Sigma \rangle = \int_{\Gamma^\Sigma} \Phi_i - \int_{\partial\Gamma^\Sigma} \phi_I, \tag{4.2}$$

for $\vec{\Phi}_I = (\Phi_I, \phi_I) \in H^3(X,Y)$. The linear combination of integrals on the r.h.s. of the above equation defines the dual pairing in relative (co-)homology, invariant under the equivalence relation (3.2).

The grade one elements $\vec{\Phi}^{(1)}_A$ can be identified by the fundamental property

$$(\nabla_A - C_A)\,|I\rangle_{RR} = 0,$$

which expresses the fact that an insertion of the grade one operator $\phi^{(1)}_A$ (represented by the matrix C_A) in the path integral is equivalent to taking a derivative in the A direction. The connection terms in ∇_A depend on the choice of coordinates z_A on $\mathcal{M}_{\mathcal{N}=1}$ and are not determined by this argument. However, ∇_A is a *flat* connection on the bundle $\mathcal{V}$ over $\mathcal{M}_{\mathcal{N}=1}$ with fibers V, namely the Gauss-Manin connection on the relative cohomology bundle $\mathcal{V} \sim H^3(X,Y) \otimes \mathcal{O}_{\mathcal{M}_{\mathcal{N}=1}}$.

In particular, the flatness of the connection ∇ implies the existence of flat coordinates $t_A(z.)$ for which the covariant derivatives reduce to ordinary ones, $\nabla_A \ \to\ \partial_A = \partial/\partial t_A$.

[13] The $U(1)$ charge of $|0\rangle_{RR}$ is shifted by $-\hat{c}/2$ under the spectral flow from the NSNS to the RR sector; we will refer to integral grades without this shift also in the RR sector.

In these coordinates the relative period matrix satisfies the following system of linear differential equations:

$$\Big(\frac{\partial}{\partial t_A} - C_A(t.)\Big)\,\Pi_I^\Sigma(t.) = 0. \tag{4.3}$$

If we order the basis $\{\vec{\Phi}_I^{(q)}\}$ by increasing grade q, these equations imply that Π_I^Σ can be put into upper block triangular form with constant entries on the block diagonals. In fact one may chose $\vec{\Phi}^{(0)} = (\Omega, 0)$ as a basis element for this space [9], such that the first row

$$\Pi^\Sigma \equiv \Pi_0^\Sigma = \int_{\Gamma^\Sigma} \Omega$$

describes the periods of the holomorphic $(3,0)$ form Ω on the basis $\{\Gamma_\Sigma\}$ of topological 3-cycles and 3-chains for $H_3(X,Y)$.

At grade one, the content of the equations (4.3) is the definition of the flat coordinates $t_A(z.)$:

$$t_A(z) = \frac{\Pi_0^A}{\Pi_0^0} = \frac{\int_{\Gamma^A}\Omega}{\int_{\Gamma^0}\Omega}, \tag{4.4}$$

for $A = 1, ..., \dim(\mathcal{M}_{\mathcal{N}=1})$. This is the $\mathcal{N}=1$ *mirror map* for the $\mathcal{N}=1$ chiral multiplets, which defines the flat coordinates on the deformation space $\mathcal{M}_{\mathcal{N}=1}$ as the ratio of certain period and chain integrals on the manifold X [5]. A different definition of the open string moduli, namely in terms of the BPS tension of 4d domain walls, had been first given in [18,7]; it agrees with the above definition of flat topological coordinates, at least for the cases studied so far.

In the flat coordinates t_A, the relative period matrix has the general form

$$\left(\begin{array}{c|c|c|c||c} 1 & t_A & \mathcal{W}_\mathcal{K} & ... & (q=0) \\ \hline 0 & \delta_B^A & \partial_B \mathcal{W}_\mathcal{K} & ... & (q=1) \\ \hline 0 & 0 & \eta_L^K & ... & (q=2) \end{array}\right) \tag{4.5}$$

where η_L^K is some constant matrix. At grade two we then get from (4.3) and (4.5):

$$\frac{\partial}{\partial t_A}\frac{\partial}{\partial t_B}\mathcal{W}_K(z.) = C_{AB}^{\ \ K}(z.)$$

This is the promised relation that expresses the ring structure constants in terms of the holomorphic potentials

$$\mathcal{W}_K(z.) = \int_{\Gamma^K} \Omega. \tag{4.6}$$

Inverting the mirror map (4.4), and inserting the result into the potentials (4.6), one then finally obtains the instanton expansion (2.8) of the superpotentials.

The linear system (4.3) terminates at grade $q \leq 3$. Eliminating the lower rows in this system in favor of the first row, namely the relative period vector Π^Σ, one obtains a system of coupled, higher order differential equations

$$\mathcal{L}_A \, \Pi^\Sigma(z.) = 0. \tag{4.7}$$

These equations comprise a Picard-Fuchs system for the of the relative cohomology group $H^3(X, Y)$. Solutions to the above differential operators, most notably expansions around the classical point $z_A = 0$, can then easily be obtained by standard methods, as has been exemplified in [5,6].

This then gives a very effective way to determine the holomorphic potentials $\mathcal{W}_K(t.)$, very similar to how $\mathcal{N} = 2$ prepotentials were computed in the past. Via (2.3), these expansions represent non-perturbatively exact space-time instanton contributions to the $\mathcal{N} = 1$ superpotential $W(t.)$ of the four-dimensional string effective supergravity theory.

Although many of the above equations will have looked familiar to the reader who is acquainted with the connection between 2d TFT and $\mathcal{N} = 2$ special geometry, the validity of these equations in the present, $\mathcal{N} = 1$ supersymmetric context is very non-trivial. As alluded to before, the underlying mathematical structure is the mixed Hodge structure on the relative cohomology bundle $\mathcal{V}$, and the Gauss-Manin connection defined on it. The precise arguments why these mathematical concepts can be identified with the TFT concepts discussed here, leading to the equations presented in this section, will be given in ref. [9].

5. Conclusion and outlook

Summarizing, what we have outlined in the present letter is a general geometric structure that underlies the $\mathcal{N} = 1$ superpotentials of certain flux and brane configurations in type II string compactifications on Calabi-Yau threefolds. This holomorphic $\mathcal{N} = 1$ special geometry is in many respects similar to (and in a sense an extension of) the special geometry of $\mathcal{N} = 2$ supersymmetric type II string compactifications. It puts the computation of the instanton-corrected superpotential W of the four-dimensional $\mathcal{N} = 1$ string effective action on a very similar footing as the familiar computation of the $\mathcal{N} = 2$ prepotential $\mathcal{F}$ via mirror symmetry.

As already mentioned, the purpose of this letter is to give an overview of these of our main ideas and results, which will be further developed and explained in more detail in the companion paper [9]. We will in particular show there how these concepts can be made very explicit in the framework of linear sigma models, or toric geometry. For this broad class of toric geometries, $\mathcal{N} = 1$ special geometry provides an efficient and systematic method to compute the instanton corrected superpotentials W.

Specifically, we will derive a simple toric representation of the open-closed chiral ring in terms of the relative cohomology, and of a complete system of GKZ type differential equations. In passing we will also show that for non-compact threefolds, the degree of the slicing hypersurface Y is a free parameter and gives a natural geometric interpretation to the so-called "framing ambiguity", a discrete quantum number in the open string sector discovered in ref.[24].

Finally we like to recall that, as discussed in sect. 2, the space-time instanton expansion (2.8) of W in the string effective theory has certain integrality properties, leading to a highly distinguished class of $\mathcal{N} = 1$ supergravities. It is interesting to ask whether the integrality of the expansion can be directly linked to the holomorphic special geometry discussed in this paper. Indeed both structures point to some non-perturbative "duality group" acting on the coupling space $\mathcal{M}_{\mathcal{N}=1}$ of the string compactification, which is likely to originate from the monodromy group of the solutions of the differential equations. The superpotential W must then transform properly under S transformations. For appropriate groups S, these "automorphic" functions often have miraculous integrality properties, like for example the famous j-function for the group $SL(2, \mathbf{Z})$. The idea that the effective theory might be largely constrained by a non-perturbative duality group has been discussed already in the string phenomenology of the twentieth century; there the strategy was to assume a – simple enough – duality group, and then to attempt to determine the superpotential by identifying the appropriate automorphic functions (see e.g. [25]).

We are here in a somewhat opposite situation, where we can compute the automorphic function, and in principle derive also the duality group, from a system of differential equations. If an open-closed string compactification has an interestingly enough duality group, this may lead to far reaching constraints on other quantities in the effective action, such as the Kähler potential for the moduli of the B-model. Clearly it would be interesting to study this further, perhaps in the context of the tt^* equations of refs.[8,2].

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Appendix A. $\mathcal{N}=2$ special geometry and 2d TFT

Here we give a concise summary of the relation between the $\mathcal{N}=2$ special geometry and the 2d TFT on the string world-sheet; for a thorough discussion of this connection, we refer to ref.[2].

The basic topological datum of the $\mathcal{N}=2$ supergravity theory is the holomorphic prepotential $\mathcal{F}(z_a)$, where z_a are some coordinates on the vector moduli space $\mathcal{M}_{\mathcal{N}=2}$. The 2d interpretation of the prepotential $\mathcal{F}(z_a)$ in the TFT is as a generating function for the structure constants of the chiral ring $\mathcal{R}$ [26]:

$$\phi_i^{(q)} \cdot \phi_j^{(q')} = C_{ij}^{\ \ k} \phi_k^{(q+q')}. \qquad \text{(A.1)}$$

Here the $\phi_i^{(q)}$ denote a basis of 2d superfields that span the local BRST cohomology of the TFT. The superscript denotes the charge $q=(q_L,q_R)$ of a field under the left- and right-moving $U(1)$'s of the super-conformal algebra on the world-sheet. The fields ϕ_a of $U(1)$ charges $(1,-1)$, labeled by subscripts from the beginning of the alphabet, represent the marginal deformations of the 2d field theory and are in one-to-one correspondence with the moduli z_a.

By spectral flow, the operators ϕ_i are in one-to-one correspondence with the 2d ground states $|i\rangle_{RR}$ in the RR sector. The topological RR ground states can be generated from a canonical vacuum $|0\rangle_{RR}$ by inserting fields ϕ_i in the twisted path integral, which leads to a representation of the form: $|i\rangle_{RR} = \phi_i|0\rangle_{RR}$. An important property of the TFT is the existence of a *flat* topological connection ∇ on the bundle $\mathcal{V}$ of RR ground states $|i(z_a,\bar{z}_{\bar{a}})\rangle_{RR}$ over $\mathcal{M}_{\mathcal{N}=2}$. By a familiar argument, which identifies a derivative with respect to the deformation parameter with an insertion of the operator in the path integral, one obtains the relation:

$$D_a|i\rangle_{RR} = (\nabla_a - C_a)\,|i\rangle_{RR} = 0. \qquad \text{(A.2)}$$

Here C_a is interpreted as a matrix representing the multiplication with the field ϕ_a. The flatness of the connection ∇ and the precise form of the connection terms in (A.2) have been studied in [2]; in particular the object D_a is the tt^* connection of ref.[8].

The basic relation connecting the chiral ring coefficients C_a with the $\mathcal{N}=2$ prepotential $\mathcal{F}$ in the supergravity is

$$C_{ab}^{\ \ c} = \partial_a\partial_b\partial_c\mathcal{F}(t.), \qquad \partial_a = \frac{\partial}{\partial t_a}. \qquad \text{(A.3)}$$

Here we have introduced the topological flat coordinates t_a, which are the local deformation parameters in the 2d world-sheet action such that the connection terms vanish, i.e., $\nabla_a \to \partial_a$.

The TFT concepts we were just discussing have an explicit geometric realization in the topological B-model, in terms of the family of Hodge structures on the middle cohomology $H^3(X,\mathbf{C})$. The elements $\phi_i^{(q)}$ of the chiral ring are identified with elements of the Hodge spaces $H^{3-q,q}(X)$. The unique element $\phi^{(0)}$ of zero $U(1)$ charge is identified with the unique holomorphic $(3,0)$ form Ω on X. Moreover, deformations arising from the charge one fields are identified with the deformations of the complex structures on X. The spaces $H^3(X,\mathbf{C})$ fit together to a locally trivial bundle $\mathcal{V}$ over $\mathcal{M}_{\mathcal{N}=2}$ which admits a flat connection ∇, the so-called Gauss-Manin connection. As the complex structure varies with the moduli z_a, the definition of a (p,q) form changes and therefore the bundles with fiber $H^{3-q,q}(X)$ are non-trivial. The moduli dependence of a representative $\Phi_i^{(q)}(z.) \in H^{3-q,q}(X)$ for the field $\phi_i^{(q)}$ of definite $U(1)$ charge, can be specified by its projection onto a fixed, moduli independent basis $\{\Gamma_\alpha\} \in H^3(X)$. The transition functions are summarized in the period matrix:

$$\Pi_i^\alpha(z.) = \ \langle i|\alpha\rangle_{RR} = \int_{\Gamma^\alpha} \Phi_i^{(q)}(z.), \qquad \Gamma^\alpha \in H_3(X,\mathbf{Z})\,. \tag{A.4}$$

Here $\Gamma^\alpha \in H_3(X,\mathbf{Z})$ is a basis dual to the constant basis Γ_α. From (A.2) we see that the period matrix satisfies the differential equation

$$D_a\Pi_i^\alpha(z.) = (\nabla_a - C_a(z.))\,\Pi_i^\alpha(z.) = 0. \tag{A.5}$$

Let us order the basis $\{\Phi_i^{(q)}(z.)\}$ by increasing grade q. By iterative elimination of the lower rows, corresponding to the fields with non-zero $U(1)$ charge, one obtains from (A.5) a system of linear differential equation of higher order for the first row, $\Pi^\alpha \equiv \Pi_0^\alpha$. These are the well-known Picard-Fuchs equations for the period integrals $\Pi^\alpha = \int_{\Gamma^\alpha}\Omega$ over the holomorphic $(3,0)$ form on X. Imposing the appropriate boundary conditions, the solutions to these equations determine the period vector Π^α from which all chiral ring coefficients C_i may be obtained by differentiation. The period vector also determines the flat coordinates and eventually, one obtains the holomorphic prepotential $\mathcal{F}$ by integration of (A.3). This provides an extremely effective means to obtain the exact holomorphic data of the topological theory.

Given a topological flat metric on the space V of RR ground states, the period vector Π^α determines also the non-holomorphic Kähler potential $K(z_a,\bar{z}_{\bar{a}})$ on $\mathcal{M}_{CS}$, and thus the metric on moduli space [3]. The curvature of this metric is of a restricted form that is compatible with the general properties of local $\mathcal{N}=2$ supersymmetry in four dimensions [1].

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