Particle Accelerators, 1994, Vol. 43(4), pp.231–233
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SOLUTION TO THE STATIONARY LONGITUDINAL ENVELOPE EQUATION

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(Received 22 June 1993; in final form 16 August 1993)

The longitudinal envelope equation[1] can be used to describe the evolution of a bunched beam under the influence of linear longitudinal forces in induction linacs, rf linacs, and circular accelerators. Unlike the radial envelope equation of Kapchinskij and Vladimirskij,[2] it does not yield a simple analytic solution for the stationary case. We present an approximate analytic solution that shows the scaling with respect to emittance, space charge and external focusing, and which is always within 3.4% of the exact solution.

KEY WORDS: Collective effects, Particle dynamics, Stationary longitudinal envelope equation

Neuffer[1] has derived an equation for the half-length of a bunched charged particle beam with a parabolic line charge density $\rho_L(z) = \rho_L(0)(1 - z^2/Z^2)$, where $Z = z_{max}$ is the bunch half-length and $z = s - s_o$ is the longitudinal distance from the bunch center. Here, s is the distance along the direction of propagation and s_o is the distance of the bunch centroid along the direction of propagation. It is assumed that the change in energy due to acceleration occurs adiabatically (i.e., on a time scale that is large compared to the longitudinal oscillation period). This equation is

$$Z'' + k_{z0}^2 Z - \frac{K_L}{Z^2} - \frac{\epsilon_z^2}{Z^3} = 0, \tag{1}$$

where Z'' is the second derivative of Z with respect to s_o, $\epsilon_z \pi = Z(z')_{max}\pi$ is the longitudinal emittance ($(z')_{max}$ is the maximum derivative of z with respect to s_o), k_{z0} is the longitudinal focusing constant, and K_L is the longitudinal perveance.

For longitudinal motion in a circular machine, $k_{z0} = ((2\pi q E_m|\eta \sin \varphi_s|)/(\lambda\beta^3\gamma mc^2))^{1/2}$ and $K_L = -3gNr_c\eta/(2\beta^2\gamma^3)$, where E_m is the peak rf field, φ_s is the synchronous phase, λ is the rf wavelength, m is the particle mass, q is the particle charge, g is the geometry factor,[3] N is the total number of particles in the bunch, r_c is the classical particle radius, $\beta = v/c$, the beam centroid velocity divided be the speed of light, and $\gamma = (1 - \beta^2)^{-1/2}$. $\eta = \gamma_t^{-2} - \gamma^{-2}$ is the slip factor, and $\gamma_t mc^2$ is the transition energy. In a linear accelerator $\gamma_t \to \infty$ and $\eta = -\gamma^{-2}$. For an induction linac one uses the electric field gradient E'_{za} instead of $-2\pi E_m|\sin\varphi_s|/(\lambda\beta)$, so $k_{z0} = \left(qE'_{za}/(mc^2\beta^2\gamma^3)\right)^{1/2}$.

It is desirable to find the stationary solution to this equation (when $Z'' = 0$) in the smooth approximation, so that one can predict the equilibrium half-length of a bunched beam with given beam and machine parameters. Unlike the well-known radial envelope equation of Kapchinskij and Vladimirskij,[2] the longitudinal envelope equation does not yield a simple analytic solution. While the equation can be solved numerically, an analytic solution provides more physical insight. We derive the solution as an infinite sequence and show that the third term, and for many cases the second term, provides an accurate equilibrium solution.

The longitudinal envelope equation with $Z'' = 0$ can be solved easily for two special cases. When the space-charge term is negligible ($K_L = 0$), the solution is $Z = (\epsilon_z/k_{z0})^{1/2}$. When the emittance term is negligible ($\epsilon_z = 0$) the solution is $Z = (K_L/k_{z0}^2)^{1/3}$. We use the former to write Z in units of $Z_0 = (\epsilon_z/k_{z0})^{1/2}$ for the general case, and Equation (1) with $Z'' = 0$ becomes

$$\zeta^4 - \alpha\zeta - 1 = 0, \tag{2}$$

where $\zeta = Z/Z_0$ and $\alpha = K_L/(\epsilon_z^{3/2} k_{oz}^{1/2})$. This equation has one positive real solution,

$$\zeta = \frac{U^{1/2}}{2} + \left[\left(\frac{U^2}{4} + 1\right)^{1/2} - \frac{U}{4}\right]^{1/2},$$

where

$$U = \left[\frac{\alpha^2}{2} + \left(\frac{64}{27} + \frac{\alpha^4}{4}\right)^{1/2}\right]^{1/3} + \left[\frac{\alpha^2}{2} - \left(\frac{64}{27} + \frac{\alpha^4}{4}\right)^{1/2}\right]^{1/3}, \tag{3}$$

in which the resulting expression for Z is quite complicated. We seek a simpler approximate solution in which the effects of the emittance and space charge terms can be clearly seen.

Solving Equation (2) is equivalent to finding a fixed point of $\zeta_{i+1} = T(\zeta_i)$, where $T(\zeta) = (1/\zeta + \alpha)^{1/3}$. $T(\zeta)$ represents a contraction near the (positive) fixed point, so we choose a starting value of $\zeta_1 = 1$ and write the solution as

$$\zeta = \lim_{i\to\infty} \zeta_i \quad \text{with} \quad \zeta_i = T(\zeta_{i-1}). \tag{4}$$

This sequence converges rapidly towards the exact solution. The second term, $\zeta_2 = (1+\alpha)^{1/3}$, is always within 3.4% of (and always greater than) the exact solution; the third term, $\zeta_3 = \left((1+\alpha)^{-1/3} + \alpha\right)^{1/3}$, is always within 0.52% of (and always less than) the exact solution. In terms of the parameters given in Equation (1), the envelope can be approximated by the second term as

$$Z \approx Z_2 = \left(\frac{\epsilon_z^{3/2}}{k_{z0}^{3/2}} + \frac{K_L}{k_{z0}^2}\right)^{1/3}, \tag{5}$$

which shows the scaling with respect to the beam parameters ($\epsilon_z,\ K_L$) and the applied focusing wave constant (k_{z0}). Depending on the desired accuracy, therefore, either the second or third term in the sequence given in Equation (3) can be used as a simple analytic solution to the longitudinal envelope equation for a bunched beam.

REFERENCES

1. D. Neuffer, IEEE Trans. Nucl. Sci. **NS-26**, 3, (1979), 3031.
2. I. M. Kapchinskij and V. V. Vladimirskij, in *Proceedings of the International Conference on High Energy Accelerators*, (CERN 1959), p.274.
3. The geometry factor represents the constant of linearity in the bunch self-field, including the image field due to the presence of a conducting pipe. It is typically of order unity. A more detailed discussion can be found in M. Reiser, *Theory and Design of Charged Particle Beams* (John Wiley & Sons, New York, in press, Ch. 5.4, and in C. Allen, N. Brown and M. Reiser, "Image effects for bunched beams in axisymmetric systems" (to be published).