(F.R.W) Equations and Quantum Cosmology

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Abstract:

We give an equivalent hamiltonian form of the three (F.R.W) fundamental equations of cosmology, for a perfect cosmic fluid in a γ-state.It leads to an effective potential depending on the (2-3γ) power of the R(t) scale factor. The quantum vacuum describes a null total energy,associated with a zero spatial curvature. An inflationary era occurs then before the radiative period. The use of the correspondance principle gives a Schrödinger-like equation describing the cosmic fluid in its different states.

The Big-Bang model of cosmology, based on the fundamental equations of General Relativity, extensively uses the so-called (F.R.W) equations [1].

To recall briefly their content and formulation,we shall notice first that they start from the Cosmological Principle hypothesis, setting that the energy density ρ depends only on the time coordinate ,since at a large scale observation ,our Universe appears to be homogeneous and isotropic. The definition of co-moving coordinates for an observer in motion with the cosmic fluid, leads to the **cosmological time** concept ,indispensable for a global description of our universe. Such an hypothesis allows indeed to get out from the local aspect inherent to General Relativity, to reach a global vision of the universe.

The (F.R.W) Friedman-Robertson-Walker's metric reads:

$$ds^2 = - dt^2 + R^2(t) [dr^2 /(1- k r^2) + r^2 d\Omega^2] \qquad (1)$$

and shows effectively that the $g_{00}(x)$ coefficient of the time coordinate in the metric, is independant on the observer's position ,and thus gives a time identical for any observer, a global time , usually called **cosmological time** .

The R(t) function representes a scale factor, improperly called « radius » ,while the k parameter denotes the spatial curvature of the universe, (+1 value for an elliptic,-1 for an hyperbolic, and 0 for an euclidian model).

The fundamental equations of cosmology [1], are very often written with the θ **expansion factor**, related to the derivative of the scale factor with:

$$\theta = 3 (\partial_t R)/R = d /dt (\log R^3) \qquad (2)$$

The γ-state equation of the cosmic fluid:

$$P(t) = (\gamma - 1)\, \rho(t) \qquad (3)$$

links the ρ energy density, to the P pressure (in a natural-unit system in wich G = c =1), and the different states of the cosmic-fluid are associated with the following values of the γ-coefficient:

$\gamma = 0$	$P = -\rho$	quantum vacuum	
$\gamma = 4/3$	$P = 1/3\,\rho$	radiative era	(4)
$\gamma = 1$	$P = 0$	matter era	

Einstein's fundamental equations of general relativity,lead then to the three (F.R.W), fundamental equations of cosmology [1] :

* energy conservation $$\partial_t \rho + \gamma \rho \theta = 0 \qquad (5)$$

*evolution of the expansion factor (Raychaudhuri's equation)

$$\partial_t \theta + 1/3\, \theta^2 + 1/2(3\gamma - 2)\, \rho = 0 \qquad (6)$$

* spatial curvature definition (FRW equation)

$$1/3\, \theta^2 - \rho = -3k / R^2 \qquad (7)$$

We must notice here [1], that this last equation can be considered as the E total energy conservation ,of a particle with mass m, in interaction in a newtonian classical,non-relativistic gravitational potential $V(R) = - G M m / R$, when we set :

$$k = - 2 E / m \qquad (8)$$

for a cosmic fluid with uniform density ρ,and null pressure P .

Using the major part of a paper already published [2], let us determine a hamiltonian form of the preceding equations by setting:

$$H = ½\, p^2 + A\, q^n \qquad (9)$$

The A and n coefficients, and a relation between variables (p,q) and (θ,ρ) remain to be established.. The p and q variables are choosen as the monomial forms:

$$p = a\, \theta^{\alpha} \rho^{\beta}$$

$$q = b\, \rho^{\sigma} \qquad (10)$$

The first Hamilton's equation $\partial_t q = \partial_p H$ gives ,with the above choice on p and q, the energy density derivative

$$\partial_t \rho = a/\sigma b \;\; \theta^{\alpha} \rho^{\beta - \sigma + 1} \qquad (11)$$

and a comparison with (5) gives the coefficient values:

$$\alpha = 1 \qquad a/\sigma b = -\gamma \tag{12}$$

The second Hamilton 's equation $\partial_t\ p = -\ \partial_q H$ gives ,with the chosen form of the V(q) potential ,and the above determined coefficients:

$$\partial_t \theta + a/b\ \theta^2 + n A b^{n-1}/a\ \rho^{\beta (n-2)} = 0 \tag{13}$$

A comparison with (6) fixes the A and n parameters:

$$a/b = 1/3 = -\gamma\beta$$
$$\beta(n-2) = 1 \quad \text{and} \quad n = (2 - 3\gamma) \tag{14}$$
$$A = -1/6\ b^{2-n}$$

We have thus obtained a hamiltonian form of the (F.R.W) equations of the cosmic fluid in a γ-state, basis of the modern cosmology since:

$$H = ½\, p^2 - 1/6\ b^{3\gamma}\ q^{2-3\gamma} \tag{15}$$

with the conjuguate variables :

$$p = 1/3\ b\ \theta\ \rho^{-1/3\gamma} \tag{16}$$

$$q = b\ \rho^{-1/3\gamma} \tag{17}$$

The ratio of the conjugate variables, eliminates the b free parameter ,and gives the Hubble parameter:

$$p/q = 1/3\ \theta = \partial_t \log R \tag{18}$$

If one demands q to represent a length dimension, and p a moment one ,one must give a dimension to the free b parameter. Integration of energy conservation equation (5) defines $\rho\ (t)$ with respect to the scale factor R (t) :

$$\rho(t) = \text{Const}\ [R(t)]^{-3\gamma} = M_\gamma\ [R(t)]^{-3\gamma} \tag{19}$$

During the quantum vacuum era ($\gamma = 0$) ,the constant energy density ρ_0 , is nothing but the M_0 constant, while in each of the following eras,

$$M_\gamma = (\rho R^{3\gamma}) \tag{20},$$

constant during the considered period, even if the scale factor is increasing. Bringing $\rho(t)$ into definition (17) of the q variable ,it comes that:

$$q = (b\ M_\gamma^{-1/3\gamma})\ R \tag{21}$$

If one fixes the b arbitrary constant to the value $M_\gamma^{1/3\gamma}$, the q variable becomes simply the scale factor, while relation (18) shows that p becomes its first time derivative:

$$q = R \quad \text{leads to} \quad p = \partial_t q = \partial_t R \tag{22}$$

This hamiltonian approach of a γ-state fluid, indicates that one can modelize the force acting on that fluid in its different states, since the effective interacting potential $V_\gamma(q)$ reads:

$$V_\gamma(R) = - 1/6\, M_\gamma\, R^{2-3\gamma} \qquad (23)$$

This leads to a newtonian effective force $F_\gamma = (1/3 -1/2\gamma)\, M_\gamma\, R^{1-3\gamma}$,

attractive $F_1(R) = -1/6\, M_1\, R^{-2}$ for a cosmic fluid with null pressure

$F_{4/3}(R) = - 1/3\, M_{4/3}\, R^{-3}$ for a pure radiative cosmic fluid.

The quantum vacuum is classically modelized by a **repulsive force** proportionnal to the scale factor $F_0(R) = 1/3\ M_0 R$. One can imagine that it just exhibits the cosmological constant. One must notice however,that such a repulsive force is not constant, since the scale factor depends on the cosmological time. It's very tricky when one thinks about the **quintessence** hypothesis , recently introduced by some cosmologists [3], in order to explain the acceleration of the universe expansion, as measured in 1998 by two astro-physicists teams [4].

Let us now rewrite the H hamiltonian ,with the θ and ρ variables with the use of eq. (15),(16),(17),and the (7) spatial curvature definition :

$$H = 1/6\ b^2 \rho^{-2/3\gamma} [\ 1/3\theta^2 - \rho] = -1/2\, (b\, M_\gamma^{-1/3\gamma})^2\, k = - 1/2\, k \qquad (24)$$

A comparison with the non-relativistic classical case, shows that the constant appearing on the right-hand side ,is the the total energy of the cosmic fluid in its γ-state , now modelized by a unit mass particle ,with p momentum, in a $V_\gamma(q)$ interaction potential. One can thus set:

$$H = E_\gamma \qquad (25)$$

It shows that such an hamiltonian is equal to zero, when the cosmic fluid describes the quantum vacuum,in which case $\gamma = 0$ et $P = - \rho = - \rho_0$ This is the usual considered hypothesis in Quantum Cosmology. In such a case, the k spatial cuvature is equal to zero according to (24), in this era. This implies that the ρ_0 vacuum energy density, is the critical density $\rho c = 3\theta^2 /8\pi G = \rho_0$ .One can thus obtain the evolution of the scale factor during this period,since by integration we get

$$R (t) = R (0) \exp[(\sqrt{8\pi G /3\, \rho_0})\, t] = R (0) \exp[t / t_P] = R (0) \exp[2\ 10^{43}\, t]$$

With t expressed in seconds , $t \geq t_P$ (26).

The initial value of the scale factor cannot be equal to zero,avoiding thus the singularity problem. If on admits that the quantum vacuum is charasteristic of the Planck era, the scale factor at the beginning of the radiative era ,will be different from zero equal to

$$R_P(t_P) = R (0)\ \exp [1] \qquad (27)$$

Roughfly speaking,the R (0) initial scale-factor is thus equal to the Planck's lenght (10^{-33} cm). One finds that

$$R(t) = R_P(t_P) \exp(t / t_P) = 10^{-33} \exp(2\ 10^{43}\ t) \quad , \quad t \geq t_P \qquad (28)$$

when the scale factor is expressed in cm and t in seconds. Even if one admits that such a transition period is very short,of the order of 10^{-32} s, one gets an **inflatory** behaviour since

$$R(t) = 10^{-33} \exp(10^{12}). \qquad (29)$$

It thus appears that,if in the constant density energy of the quantum vacuum, a Planck domain occurs, it will inflate and leads after infflation, to the classical Big-Bang evolution of our universe,described within the General Relativity theory.

During the radiative and matter eras, the E_γ energy, corresponding to a constant value of the hamiltonian, is proportionnal to the spatial curvature .

The hamiltonian form of the (F.R.W) equations for γ-state cosmic fluid is thus :

$$H = ½\, p^2 - 1/6\, b^{3\gamma}\, q^{2-3\gamma} = E_\gamma \qquad (30)$$

This leads to a new approach of the quantum universe, completely different,and so simpler, than Wheeler and DeWitt's method !

It becomes indeed quite easy to consider now the H hamiltonian as an operator acting on the $\psi_\gamma(q)$ wave function,by introducing the wave-mechanics correspondance principle

$p \Rightarrow -i\,\partial q$ getting thus a Schrödinger-like equation:

$$[\partial^2 / \partial q^2 + 1/3\, b^{3\gamma} q^{2-3\gamma}]\, \psi_\gamma(q) = E_\gamma\ \psi_\gamma(q) \qquad (31)$$

Going back to the R variable with relation (21) and the previous choice of the b parameter ,one obtains the **fundamental equation of quantum cosmology,for a γ-state cosmic fluid** :

$$[\partial^2 / \partial R^2 + 1/3\ M_\gamma\ R^{2-3\gamma}]\, \psi_\gamma(R) = E_\gamma\ \psi_\gamma(R) \qquad (32)$$

These Schrödinger-like equations do not dependexplicitely on time.The cosmological time appears through the scale factor R(t).

The vacuum quantum state ($\gamma = 0$) ,usual application domain of quantum cosmology will be determined by the simple wave-equation:

$$[\partial^2 / \partial R^2 + 1/3\ \rho_0 R^2]\, \psi_0(R) = E_0\ \psi_0(R) \qquad (33)$$

The fundamental equation (31) of cosmology has been used by T.R.Mongan [5] ,for describing the radiative era, showing that the solution is spherical Hankel 's kind:

$$\psi_{4/3} = A\, R\, h_{ip-1/2}(i\kappa R) \text{ avec } \kappa^2 = E_{4/3}$$

where $p = (1/3\, M_{4/3} - 1/4)^{1/2}$ and A is a normalization constant. He thus find that the wave-function $\psi_{4/3}$ vanishes for R = 0. This author does'nt take into account that the radiative era takes place just after the quantum era described by equation (33).

Contrarily to usual quantum cosmology approaches, our Schrödinger-like equation, allows to follow the wave-function of the cosmic fluid in its different states, from the quantum vacuum, to the present matter era, leading finally to a classical description of our Universe. By matching the wave-funcions and their first derivative at each change of γ-state one can hope getting a better understanding of our universe, since its creative Big-Bang.The interpretation problem for the wave-function, and of the transition from the quantum to the classical eras remains as in the Wheeler-DeWitt's approach.

Références

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