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**COHERENT AND GENERALIZED INTELLIGENT STATES
FOR INFINITE SQUARE WELL POTENTIAL AND
NONLINEAR OSCILLATORS**

A.H. El Kinani

*L.P.T., Physics Department, Faculty of Sciences, University Mohammed V,
P.O. Box 1014, Rabat, Morocco*

and

The Abdus Salam International Centre for Theoretical Physics, Trieste Italy

and

M. Daoud

*L.P.M.C., Physics Department, Faculty of Sciences, University Ibn Zohr,
P.O. Box 28/S, Agadir, Morocco.*

Abstract

This article is an illustration of the construction of coherent and generalized intelligent states which has been recently proposed by us for an arbitrary quantum system [1]. We treat the quantum system submitted to the infinite square well potential and the nonlinear oscillators. By means of the analytical representation of the coherent states à la Gazeau-Klauder and those à la Klauder-Perelomov, we derive the generalized intelligent states in analytical ways.

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1 Introduction

The concept of coherent states (CS) has been successfully used in the last decade in many different contexts of theoretical and experimental physics, in particular quantum optics [2 – 4]. They were first introduced for the harmonic oscillator (described by the Weyl-Heisenberg algebra) by Schrödinger [5].

It is well known that, for the harmonic oscillator case, there are three equivalent definitions of the coherent states $|z\rangle$:

D₁ : The elements of the set $\{|z\rangle, z \in \mathbf{C}\}$ are the eigenstates of the annihilation operator a^-

$$a^- |z\rangle = z |z\rangle \quad (1)$$

D₂ : The coherent state $|z\rangle$ is the orbit of the ground state $|0\rangle$ under the Weyl-Heisenberg displacement operator

$$|z\rangle = D(z) |0\rangle = \exp(z a^+ - \bar{z} a^-) |0\rangle \quad (2)$$

Here $[a^-, a^+] = \mathbf{1}$ and $(a^-)^\dagger = a^+$. In view of this commutation relation, both the definitions **D₁** and **D₂** are equivalent.

D₃ : Finally, the (CS) $|z\rangle$ saturate the Heisenberg uncertainty relation

$$2\Delta X \Delta P = 1 \quad (3)$$

with the position X and momentum P operators are given as usual by

$$X = \frac{1}{\sqrt{2}} (a^+ + a^-) \quad \text{and} \quad P = \frac{i}{\sqrt{2}} (a^+ - a^-). \quad (4)$$

Furthermore, the (CS) $|z\rangle$ satisfy the following properties:

P₁ : The map $z \in \mathbf{C} \rightarrow |z\rangle \in L^2(\mathbf{R})$ is continuous.

P₂ : The family of coherent states resolve the unity. Indeed, we have

$$\int |z\rangle \langle z| d\mu(z) = I \quad d\mu(z) = \frac{d^2 z}{\pi} = \frac{1}{\pi} d\text{Re}(z) d\text{Im}(z) \quad (5)$$

This property provides the useful analytic representation, known as the Fock-Bargmann analytic representation, in which a^- and a^+ are represented respectively by ∂_z and z and the arbitrary state $|\psi\rangle$ is represented by the function $\psi(z) = \exp\left(\frac{|z|^2}{2}\right) \langle \bar{z} | \psi \rangle$ where \bar{z} is the complex conjugate of z .

P₃ : The coherent states family is temporally stable. Indeed

$$e^{-iHt} |z\rangle = |ze^{-i\omega t}\rangle \quad (6)$$

P₄ : They provide the classical-quantum correspondence.

The generalization of the above definitions for other potentials different from the harmonic oscillator, was proposed recently by Gazeau and Klauder. They give a general scheme leading to coherent states for an arbitrary quantum system [6] (see also [7, 8]) by using the definition \mathbf{D}_1 . A direct illustration of this construction was given in [9] for a particle trapped in the infinite square well and in Pöschl-Teller potentials. Using the definition \mathbf{D}_2 , the coherent states à la Klauder-Perelomov for the Pöschl-Teller potentials was given in [10]. To extend the third definition, we have solved the eigenvalue equation of states minimizing the Robertson-Schrödinger uncertainty relation (which extend the Heisenberg one) for an arbitrary quantum system [1, 11]. The resulting states are called the generalized intelligent states. We have shown that these states include the Gazeau-Klauder coherent one.

Recently, we gave the extension of the above three definitions for an arbitrary quantum system and as an application, we treated a quantum system evolving in the Pöschl-Teller potentials [12].

The main purpose of this paper is to give other illustrations for the construction of coherent and generalized intelligent states for quantum systems, trapped in the infinite square well potentials and nonlinear oscillators.

We start by introducing in section 2, the Gazeau-Klauder coherent states (definition \mathbf{D}_1), Klauder-Perelomov (definition \mathbf{D}_2) and generalized intelligent states (definition \mathbf{D}_3) for an exact solvable quantum system. Sections 3 and 4 are respectively, devoted to the infinite square well potential and the x^4 -anharmonic oscillator. Concluding remarks are given at the end of this work.

2 General considerations

In this section, we give the general scheme to follow in order to construct the coherent states à la Gazeau-Klauder, à la Klauder-Perelomov and the one's called generalized intelligent states for an arbitrary quantum system.

2.1 Gazeau-Klauder coherent states

To begin, we choose a Hamiltonian H admitting nondegenerate discrete infinite spectrum e_n , such that the fundamental energy $e_0 = 0$ and the others $\{e_1, e_2, \dots, e_n\}$ are in increasing order i.e., $e_{n+1} > e_n$. The eigenstates $|\psi_n\rangle$ of H are orthogonal and satisfying

$$H |\psi_n\rangle = e_n |\psi_n\rangle. \quad (7)$$

The Hamiltonian H can be factorized as

$$H = A^+ A^- \quad (8)$$

where A^+ and A^- are the creation and annihilation operators respectively. They act on the states $|\psi_n\rangle$ as

$$A^- |\psi_n\rangle = \sqrt{e_n} e^{i\alpha(e_n - e_{n-1})} |\psi_{n-1}\rangle \quad (9)$$

$$A^+ |\psi_n\rangle = \sqrt{e_{n+1}} e^{-i\alpha(e_{n+1} - e_n)} |\psi_{n+1}\rangle, \quad (10)$$

where α is a parameter belonging to \mathbf{R} .

The exponential factor appearing in the above equations produces only a phase factor, and ensures, as we will see, the temporal stability of the states which will be constructed in what follows.

The commutator of A^- and A^+ takes the form

$$[A^-, A^+] = G(N), \quad (11)$$

where $G(N)$ acts on $|\psi_n\rangle$ as

$$G(N) |\psi_n\rangle = (e_{n+1} - e_n) |\psi_n\rangle. \quad (12)$$

Note that the operator $N \neq A^+ A^- = H$ and it satisfies the following commutation relations

$$[A^-, N] = A^- \quad , \quad [A^+, N] = -A^+ \quad (13)$$

The so-called Gazeau-Klauder coherent states [6, 7] are defined as the eigenstates of the annihilation operators A^- . Let us denote them by $|z, \alpha\rangle$. They satisfy the eigenvalue equation

$$A^- |z, \alpha\rangle = z |z, \alpha\rangle \quad , \quad z \in \mathbf{C}. \quad (14)$$

The solutions of (14) are given by

$$|z, \alpha\rangle = \mathcal{N}(|z|) \sum_{n=0}^{+\infty} \frac{z^n e^{-i\alpha e_n}}{\sqrt{E(n)}} |\psi_n\rangle \quad (15)$$

where the function $E(n)$ is defined by

$$E(n) = \begin{cases} 1 & \text{for } n = 0 \\ e_1 e_2 \dots e_n & \text{for } n \neq 0 \end{cases} \quad (16)$$

and $\mathcal{N}(|z|)$ the normalization constant, which can be computed by using the normalization condition $\langle z, \alpha | z, \alpha \rangle = 1$. We obtain

$$\mathcal{N}(|z|) = \left(\sum_{n=0}^{\infty} \frac{|z|^{2n}}{E(n)} \right)^{-\frac{1}{2}}. \quad (17)$$

It is then clear that the coherent states equation (15) are continuous in $z \in \mathbf{C}$ and $\alpha \in \mathbf{R}$ and are temporally stable under the evolution operator. Indeed

$$e^{-iHt} |z, \alpha\rangle = |z, \alpha + t\rangle. \quad (18)$$

This property is ensured by the presence of the phase factor in equations (9) and (10).

In order to prove that the Gazeau-Klauder states resolves the identity, one must find a measure $d\mu(z)$ such that

$$\int |z, \alpha\rangle \langle z, \alpha| d\mu(z) = I_{\mathcal{H}} = \sum_{n=0}^{\infty} |\psi_n\rangle \langle \psi_n|, \quad (19)$$

where the integral is over the disk $\{z \in \mathbf{C}, |z| < \mathcal{R}\}$, and \mathcal{R} the radius of convergence defined as

$$\mathcal{R} = \lim_{n \rightarrow \infty} \sqrt[n]{E(n)}. \quad (20)$$

Writing $d\mu(z)$ as

$$d\mu(z) = [\mathcal{N}(|z|)]^{-2} h(r^2) r dr d\phi, \quad z = r e^{i\phi}, \quad (21)$$

and integrating over the whole plane, the resolution of the identity is then equivalent to the determination of the function $h(u)$ satisfying

$$\int_0^{+\infty} h(u) u^{n-1} du = \frac{E(n-1)}{\pi}. \quad (22)$$

So, it is clear that $h(u)$ is the inverse Mellin transform [13] of the function $\pi^{-1}E(n-1)$

$$h(u) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{E(s-1)}{\pi} u^{-s} ds, \quad c \in \mathbf{R}. \quad (23)$$

Then it is obvious that the explicit computation of the function $h(u)$ requires the explicit knowledge of the spectrum of the system under study. Two applications of such computation will be given in sections 3 and 4.

Using Eq (14), one obtains the main value of the Hamiltonian H in the states $|z, \alpha\rangle$

$$\langle z, \alpha| H |z, \alpha\rangle = |z|^2, \quad (24)$$

this relation is known as the action identity.

Finally, we remark that the coherent states $|z, \alpha\rangle$ can be written as an operator $U(z)$ acting on the ground state (up to normalization constant) as

$$|z, \alpha\rangle = U(z) |\psi_0\rangle \quad \text{where} \quad U(z) = \exp\left(z \frac{N}{H} A^+\right). \quad (25)$$

Note that the operator $U(z)$ is not unitary and cannot be seen as a displacement operator in the Klauder-Perelomov's sense. Hence, the resulting coherent states, also, cannot be interpreted as the Klauder-Perelomov's one.

2.2 Klauder-Perelomov coherent states

Following the definition \mathbf{D}_2 , the coherent states of Klauder-Perelomov type for an arbitrary quantum system are defined by

$$|z, \alpha\rangle = \exp(zA^+ - \bar{z}A^-) |\psi_0\rangle, \quad z \in \mathbf{C}. \quad (26)$$

Using the action of the annihilation and creation operators on the Hilbert space $\mathcal{H} = \{|\psi_n\rangle, n = 0, 1, 2, \dots\}$ one can show that the states $|z, \alpha\rangle$ can be written as

$$|z, \alpha\rangle = \sum_{n=0}^{+\infty} z^n c_n(|z|) \sqrt{E(n)} e^{-i\alpha e_n} |\psi_n\rangle. \quad (27)$$

The quantities $c_n(|z|)$ are defined by

$$c_n(|z|) = \sum_{j=0}^{+\infty} \frac{(-|z|^2)^j}{(n+2j)!} \pi(n+1, j) \quad (28)$$

where

$$\pi(n+1, j) = \sum_{i_1=1}^{n+1} e_{i_1} \sum_{i_2=1}^{i_1+1} e_{i_2} \dots \sum_{i_j=1}^{i_{j-1}+1} e_{i_j}, \quad \text{and} \quad \pi(n+1, 0) = 1 \quad (29)$$

One can verify that the π 's satisfy the following relation

$$\pi(n+1, j) - \pi(n, j) = e_{n+1} \pi(n+2, j-1). \quad (30)$$

Using this recurrence formula, it is not difficult to show that the $c_n(|z| = r)$'s satisfy the following differential equation

$$r \frac{dc_n(r)}{dr} = c_{n-1}(r) - nc_n(r) - e_{n+1} r^2 c_{n+1}(r). \quad (31)$$

The solution of this differential equation leads to the explicit expression of the coherent states of Klauder-Perelomov's type for an arbitrary quantum system. Here also, we note that the solutions are intimately related to the spectrum of the system under study. Solutions in the cases corresponding to nonlinear oscillators (with an x^4 -interaction) and the infinite square well potential will be treated in the sequel of this paper.

2.3 Generalized intelligent states

These states are known as those minimizing the so-called Robertson-Schrödinger uncertainty relation [14, 15] (for more details see [1] where they were constructed for an exact solvable system) and generalize the Gazeau-Klauder ones. In what follows, we give a short review of their main characteristics.

We introduce two hermitian operators defined in terms of the operators A^- and A^+ as follows

$$W = \frac{1}{\sqrt{2}}(A^- + A^+) \quad , \quad P = \frac{i}{\sqrt{2}}(A^+ - A^-) \quad (32)$$

which satisfy the commutation relation

$$[W, P] = iG(N) \equiv iG \quad (33)$$

where the operator G is defined by (12).

It is well known that for two hermitian operators W and P satisfying the non-canonical commutation relation (33), the variances $(\Delta W)^2$ and $(\Delta P)^2$ satisfy the Robertson-Schrödinger uncertainty relation

$$(\Delta W)^2 (\Delta P)^2 \geq \frac{1}{4} (\langle G \rangle^2 + \langle F \rangle^2) \quad (34)$$

where the operator F is defined by

$$F = \{W - \langle W \rangle, P - \langle P \rangle\} \quad (35)$$

and its mean value is expressed by

$$\langle F \rangle = i [(\Delta A^+)^2 - (\Delta A^-)^2] \quad (36)$$

in terms of the variances of A^- and A^+ .

The symbol $\{, \}$, appearing in equation (35), stands for the anticommutator and the uncertainty relation (34) is a generalization of the well-known Heisenberg one.

The so-called generalized intelligent states are obtained when the equality in the Robertson-Schrödinger uncertainty relation is realized [16] (see also [17, 18]). They satisfy the eigenvalue equations

$$(W + i\lambda P) |z, \lambda, \alpha\rangle = z\sqrt{2} |z, \lambda, \alpha\rangle \quad , \quad \lambda, z \in \mathbf{C}. \quad (37)$$

Using equation (32), the above equation can be rewritten as

$$[(1 - \lambda) A^+ + (1 + \lambda) A^-] |z, \lambda, \alpha\rangle = 2z |z, \lambda, \alpha\rangle. \quad (38)$$

For the generalized intelligent states, solutions of (38), the variances of W and P are

$$(\Delta W)^2 = |\lambda| \Delta \quad , \quad (\Delta P)^2 = \frac{1}{|\lambda|} \Delta \quad (39)$$

with

$$\Delta = \frac{1}{2} \sqrt{\langle G \rangle^2 + \langle F \rangle^2}. \quad (40)$$

The main values of G and F , in the generalized intelligent states can be expressed in terms of the variances of P as follows

$$\langle G \rangle = 2 \operatorname{Re}(\lambda) (\Delta P)^2 \quad (41)$$

$$\langle F \rangle = 2 \operatorname{Im}(\lambda) (\Delta P)^2. \quad (42)$$

Clearly, for $|\lambda| = 1$, we have

$$(\Delta W)^2 = (\Delta P)^2 \quad (43)$$

The states satisfying (38), with $|\lambda| = 1$, are called the generalized coherent states. The complete classification of the solutions of (38) was considered in [1] for an exact solvable quantum system.

In what follows we give the main results of this classification which will be adopted to the quantum systems considered in this work.

Solutions of equation (38) for $\lambda \neq -1$, are given by

$$|z, \lambda, \alpha\rangle = \sum_{n=0}^{+\infty} a_n(z) |\psi_n\rangle \quad (44)$$

where

$$a_n(z) = a_0 \frac{(2z)^n}{(1+\lambda)^n \sqrt{E(n)}} \left[\sum_{h=0(1)\lfloor \frac{n}{2} \rfloor} (-1)^h \frac{(1-\lambda^2)^h}{(2z)^{2h}} \Delta(n, h) \right] e^{-iaen} \quad (45)$$

The symbol $\lfloor \frac{n}{2} \rfloor$ stands for the integer part of $\frac{n}{2}$ and the function $\Delta(n, h)$ is defined by

$$\Delta(n, h) = \sum_{j_1=1}^{n-(2h-1)} e_{j_1} \left[\sum_{j_2=j_1+2}^{n-(2h-3)} e_{j_2} \dots \left[\dots \left[\sum_{j_h=j_{h-1}+2}^{n-1} e_{j_h} \right] \dots \right] \right]. \quad (46)$$

The states $|z, \lambda, \alpha\rangle$ can be written, in a compact form, as the action of the operator $U(\lambda, z)$ on the ground state $|\psi_0\rangle$ of H as

$$|z, \lambda, \alpha\rangle = U(\lambda, z) |\psi_0\rangle \quad (47)$$

where $U(\lambda, z)$ is defined (up to normalization constant) as

$$U(z, \lambda) = \sum_{n=0}^{+\infty} \left[\left(\frac{2z}{1+\lambda} \right) \frac{1}{H} A^+ + \left(\frac{\lambda-1}{\lambda+1} \right) \frac{1}{H} (A^+)^2 \right]^n \quad (48)$$

For more details, we invite the reader to see reference [1]. It is clear that the generalized intelligent states (44) obtained by the minimization of the Robertson-Schrödinger uncertainty relation are different from the others introduced before, for an arbitrary quantum system, the definitions \mathbf{D}_1 , \mathbf{D}_2 and \mathbf{D}_3 lead to inequivalent families of states except, of course, for the harmonic oscillator case. All these matters will be adapted in what follows to two interesting quantum mechanical systems: the infinite square well potential and the anharmonic oscillators.

3 Infinite square well potential

In this section, we will give the coherent and generalized intelligent states for a quantum system trapped in an infinite square well potential by exploiting the results of the previous section.

3.1 Gazeau-Klauder coherent states

Let us recall the eigenvalues and eigenvectors of the Hamiltonian H corresponding to a quantum system submitted to the infinite square well potential. Indeed, H acts on $|\psi_n\rangle$ as

$$H |\psi_n\rangle = e_n |\psi_n\rangle \quad \text{where} \quad e_n = n(n+2). \quad (49)$$

The lowering and raising operators A^- and A^+ act on $|\psi_n\rangle$ now as follows

$$A^- |\psi_n\rangle = \sqrt{n(n+2)} e^{i\alpha(2n+1)} |\psi_{n-1}\rangle \quad (50)$$

$$A^+ |\psi_n\rangle = \sqrt{(n+1)(n+3)} e^{-i\alpha(2n+3)} |\psi_{n+1}\rangle \quad (51)$$

and the Hamiltonian H can be factorized as

$$H = A^+ A^-. \quad (52)$$

The number operator ($N \neq A^+ A^- = H$) acts on $|\psi_n\rangle$ as follows

$$N |\psi_n\rangle = n |\psi_n\rangle \quad (53)$$

and the commutation relation between A^- and A^+ is given by

$$[A^-, A^+] = G(N) \quad (54)$$

where $G(N)$ is defined as

$$G(N) = (2N + 3) \quad (55)$$

The Hilbert space \mathcal{H} for the infinite square well potential is easily constructed in the same way as the standard harmonic oscillator. This space is spanned by the states

$$|\psi_n\rangle = \frac{(A^+)^n}{\sqrt{E(n)}} e^{i\alpha e_n} |\psi_0\rangle \quad (56)$$

where

$$E(n) = \frac{n!(n+2)!}{2}. \quad (57)$$

The Gazeau-Klauder coherent states equation (15) becomes

$$|z, \alpha\rangle = \mathcal{N}(|z|) \sum_{n=0}^{+\infty} \frac{z^n \sqrt{2} e^{-i\alpha n(n+2)}}{\sqrt{\Gamma(n+1)\Gamma(n+3)}} |\psi_n\rangle \quad (58)$$

where the normalization constant is

$$\mathcal{N}(|z|) = \left[{}_0F_1\left(3, |z|^2\right) \right]^{-\frac{1}{2}}. \quad (59)$$

The Gazeau-Klauder coherent states for the system under study are normalized but they are not orthogonal to each other. Indeed, we have

$$\langle z, \alpha | z', \alpha \rangle = \frac{{}_0F_1(3, \bar{z}z')}{\sqrt{{}_0F_1(3, |z|^2) {}_0F_1(3, |z'|^2)}} \quad (60)$$

The set of states (58) are overcomplete in respect to the measure

$$d\mu(z) = \frac{2}{\pi} I_2(2r) K_1(2r) r dr d\phi \quad , \quad z = r e^{i\phi} \quad (61)$$

where $I_\nu(x)$ and $K_\nu(x)$ are respectively the modified Bessel functions of the first and second kinds.

By using this last property, one can represent the state space as the Hilbert space of analytic function in the whole plane. So, for a normalized state

$$|\Psi\rangle = \sum_{n=0}^{+\infty} b_n |\psi_n\rangle \text{ one gets}$$

$$\begin{aligned} \Psi(z, \alpha) &\equiv \sqrt{{}_0F_1(3, |z|^2)} \langle \bar{z}, \alpha | \Psi \rangle \\ &= \sum_{n=0}^{+\infty} b_n \frac{z^n \sqrt{2} e^{i\alpha n(n+2)}}{\sqrt{\Gamma(n+1) \Gamma(n+3)}}. \end{aligned} \quad (62)$$

Then, it is obvious that for the state $|\psi_n\rangle$ we associate

$$\psi_n(z, \alpha) = \frac{z^n \sqrt{2} e^{i\alpha n(n+2)}}{\sqrt{\Gamma(n+1) \Gamma(n+3)}} \quad (63)$$

The operators A^- , A^+ and $G(N)$ act on the Hilbert space of analytic functions as first order differential operators

$$A^+ = z \quad , \quad A^- = z \frac{d^2}{dz^2} + 3 \frac{d}{dz} \quad \text{and} \quad G(N) = 2z \frac{d}{dz} + 3 \quad (64)$$

It is easy to verify that the actions of A^+ , A^- and $G(N)$ on $\psi_n(z, \alpha)$ lead to

$$A^+ \psi_n(z, \alpha) = \sqrt{(n+1)(n+3)} e^{-i\alpha(2n+3)} \psi_{n+1}(z, \alpha) \quad (65)$$

$$A^- \psi_n(z, \alpha) = \sqrt{n(n+2)} e^{i\alpha(2n+1)} \psi_{n-1}(z, \alpha) \quad (66)$$

$$G(N) \psi_n(z, \alpha) = (2n+3) \psi_n(z, \alpha). \quad (67)$$

It is obvious that the coherent states constructed here are temporally stable and satisfy Eq (24).

Finally, we remark that the coherent states $|z, \alpha\rangle$ can be written as an operator $U(z)$ acting on the ground state $|\psi_0\rangle$ (up to normalization constant)

$$U(z) = \exp\left(z \frac{1}{N+2} A^+\right) \quad (68)$$

such that we have

$$|z, \alpha\rangle = U(z) |\psi_0\rangle. \quad (69)$$

As we mentioned, in section 1, the operator $U(z)$ Eq. (68) is not unitary. The analytic representation of the coherent states à la Gazeau-Klauder introduced here is important since it will be used to derive the generalized intelligent states in an analytical way.

3.2 Klauder-Perelomov coherent states

The coherent states à la Klauder-Perelomov for an arbitrary quantum system are defined in the subsection (2.2). We have shown that their explicit expressions depend on the spectrum structure of the system. Here, we will solve the differential equation (31) for the spectrum of the infinite square well potential. Indeed for $e_n = n(n+2)$ the c_n 's coefficient admit the solution

$$c_n(r) = \frac{1}{n!} (\cosh(r))^{-3} \left(\frac{\tanh(r)}{r} \right)^n. \quad (70)$$

The coherent states of Klauder-Perelomov's type takes the form

$$|z, \alpha\rangle = \left(1 - \tanh^2(|z|)\right)^{\frac{3}{2}} \sum_{n=0}^{+\infty} \left(\frac{z \tanh(|z|)}{|z|} \right)^n \left[\frac{(n+1)(n+2)}{2} \right]^{\frac{1}{2}} e^{-i\alpha n(n+2)} |\psi_n\rangle. \quad (71)$$

By setting $\zeta = \frac{z \tanh(|z|)}{|z|}$, we obtain

$$|\zeta, \alpha\rangle \equiv \left(1 - |\zeta|^2\right)^{\frac{3}{2}} \sum_{n=0}^{+\infty} \zeta^n \left[\frac{(n+1)(n+2)}{2} \right]^{\frac{1}{2}} e^{-i\alpha n(n+2)} |\psi_n\rangle. \quad (72)$$

We note that the parameter ζ belongs to the unit disk $D = \{\zeta \in \mathbf{C}, |\zeta| < 1\}$.

The states $|\zeta, \alpha\rangle$, are temporally stable. Indeed we have

$$e^{-iHt} |\zeta, \alpha\rangle = |\zeta, \alpha + t\rangle. \quad (73)$$

From Eq (72), one can see that the Klauder-Perelomov coherent states are normalized but not orthogonal to each other

$$\langle \zeta, \alpha | \zeta', \alpha' \rangle = \sqrt{(1 - |\zeta|^2)^3 (1 - |\zeta'|^2)^3} \sum_{n=0}^{+\infty} \frac{(\bar{\zeta} \zeta')^n}{n!} \frac{\Gamma(n+3)}{2} e^{-i(\alpha' - \alpha)n(n+2)}. \quad (74)$$

The measure ensuring the identity resolution of $|\zeta, \alpha\rangle$ takes the form

$$d\mu(\zeta) = \frac{2}{\pi} \frac{d^2\zeta}{(1 - |\zeta|^2)^2}. \quad (75)$$

Then we can express any coherent states in terms of the others

$$|\zeta', \alpha'\rangle = \int |\zeta, \alpha\rangle \langle \zeta, \alpha | \zeta', \alpha' \rangle d\mu(\zeta). \quad (76)$$

For any state $|\Phi\rangle = \sum_{n=0}^{+\infty} c_n |\psi_n\rangle$ in the Hilbert space, one can construct the analytic function:

$$\Phi(\zeta, \alpha) = \left(1 - |\zeta|^2\right)^{-\frac{3}{2}} \langle \bar{\zeta}, \alpha | \Phi \rangle = \sum_{n=0}^{+\infty} \zeta^n \sqrt{\frac{(n+1)(n+2)}{2}} c_n e^{i\alpha n(n+2)}. \quad (77)$$

Here, the $|\psi_n\rangle$ state is represented by the function

$$\psi'_n(\zeta, \alpha) = \zeta^n \sqrt{\frac{(n+1)(n+2)}{2}} e^{i\alpha n(n+2)}. \quad (78)$$

The operators A^\pm and $G(N)$ act on the Hilbert space of analytic functions $\Phi(\zeta, \alpha)$ as first-order differential operators

$$A^+ = \zeta^2 \frac{d}{d\zeta} + 3\zeta \quad , \quad A^- = \frac{d}{d\zeta} \quad \text{and} \quad G(N) = 2\zeta \frac{d}{d\zeta} + 3 \quad (79)$$

By a simple computation, one can verify

$$A^+ \psi'_n(\zeta, \alpha) = \sqrt{(n+1)(n+3)} e^{-i\alpha(2n+3)} \psi'_{n+1}(\zeta, \alpha) \quad (80)$$

$$A^- \psi'_n(\zeta, \alpha) = \sqrt{n(n+2)} e^{i\alpha(2n+1)} \psi'_{n-1}(\zeta, \alpha) \quad (81)$$

$$G(N) \psi'_n(\zeta, \alpha) = (2n+3) \psi'_n(\zeta, \alpha). \quad (82)$$

Finally, we note that the above analytic representations will be the main tool by means of which we can get the analytic solutions of generalized intelligent states.

3.3 Generalized intelligent states

In this part, we will use the analytic representation of the coherent states introduced before in subsections (3.1) and (3.2), in order to obtain the generalized intelligent states in an analytical way.

3.3.1 Gazeau-Klauder analytic representation

In this representation, we define the Hilbert space as a space of functions S which are holomorphic in the complex plane. The scalar product is given by

$$\langle f | g \rangle = \int \overline{f(z)} g(z) d\mu(z) \quad (83)$$

where $d\mu(z)$ is the measure defined by Eq (61).

By introducing the analytic function

$$\Psi_{(z', \lambda, \alpha)}(z) = \sqrt{{}_0F_1(3, |z|^2)} \langle \bar{z}, \alpha | z', \lambda, \alpha \rangle \quad (84)$$

we can convert the eigenvalue equation

$$[(1-\lambda) A^+ + (1+\lambda) A^-] |z', \lambda, \alpha\rangle = 2z' |z', \lambda, \alpha\rangle \quad (85)$$

into the second-order linear homogeneous differential equation

$$\left[(1+\lambda) \left(z \frac{d^2}{dz^2} + 3 \frac{d}{dz} \right) + (1-\lambda) z - 2z' \right] \Psi_{(z', \lambda)}(z) = 0. \quad (86)$$

First, we consider the general case $\lambda \neq \pm 1$. Setting

$$\Psi_{(z', \lambda)}(z) = \exp\left(\pm \sqrt{\frac{\lambda-1}{\lambda+1}} z\right) F_{(z', \lambda)}(z). \quad (87)$$

Equation (86) can be transformed into the Kummer equation

$$\left[Z \frac{d^2}{dZ^2} + (3 - Z) \frac{d}{dZ} - \left(\frac{3}{2} \mp \frac{z'}{\sqrt{\lambda^2 - 1}} \right) \right] F_{(z', \lambda)}(z) = 0 \quad (88)$$

where $Z = \mp 2\sqrt{\frac{\lambda-1}{\lambda+1}}z$.

Then the solutions of equation (86) are given by

$$\Psi_{(z', \lambda)}(z) = \exp\left(\pm\sqrt{\frac{\lambda-1}{\lambda+1}}z\right) {}_1F_1\left(\frac{3}{2} \mp \frac{z'}{\sqrt{\lambda^2-1}}, 3; \mp 2\sqrt{\frac{\lambda-1}{\lambda+1}}z\right) \quad (89)$$

or

$$\Psi_{(z', \lambda)}(z) = \exp\left(\pm\sqrt{\frac{\lambda-1}{\lambda+1}}z\right) z^{-2} {}_1F_1\left(-\frac{1}{2} \mp \frac{z'}{\sqrt{\lambda^2-1}}, -1; \mp 2\sqrt{\frac{\lambda-1}{\lambda+1}}z\right). \quad (90)$$

The first solution (89) is always analytic, but the second (90) is not. Because the hypergeometric function ${}_1F_1(a; b; z)$ satisfies the equation

$${}_1F_1(a; b; z) = e^z {}_1F_1(b - a; b; -z) \quad (91)$$

the upper and lower signs in equation (89) are equivalent.

By using the properties of this hypergeometric functions, we conclude that the squeezing parameter λ obeys to the condition

$$\sqrt{\left| \frac{\lambda - 1}{\lambda + 1} \right|} < 1 \Leftrightarrow \text{Re}(\lambda) > 0 \quad (92)$$

which traduce the restriction on λ imposed by the positivity of the commutator $[A^-, A^+] = 2N + 3$ (see equations (53) and (54)).

We consider now the degenerate cases $\lambda = \pm 1$. For the case $\lambda = -1$ equation (86) does not have any normalized analytic solution (the operator A^+ does not have any eigenstate). For $\lambda = 1$, using the power series of ${}_1F_1(a, b; z)$, we get

$$\Psi_{(z', \lambda=1)}(z) = {}_0F_1(3; zz'). \quad (93)$$

The result (93) coincides with the solution (58) (up to normalization constant). Then we recover the infinite square well coherent states defined as the A^- eigenstates (Gazeau-Klauder coherent states).

3.3.2 Klauder-Perelomov analytic representation

In this representation the Hilbert space is equipped with the following scalar product

$$\langle f | g \rangle = \int \overline{f(\zeta)} g(\zeta) d\mu(\zeta). \quad (94)$$

Note that the integration is over the unit disk $D = \{\zeta \in \mathbf{C}, |\zeta| < 1\}$ and the measure is defined by equation (75).

To solve the eigenvalue equation (38) we introduce the analytic function

$$\Phi_{(\zeta', \lambda)}(\zeta) = \sqrt{(1 - |\zeta|^2)^{-3}} \langle \bar{\zeta}, \alpha | \zeta', \lambda, \alpha \rangle. \quad (95)$$

The equation (38) is then converted to the following differential equation

$$\left[[(1 - \lambda)\zeta^2 + (1 + \lambda)] \frac{d}{d\zeta} + 3(1 - \lambda)\zeta - 2\zeta' \right] \Phi_{(\zeta', \lambda)}(\zeta) = 0. \quad (96)$$

Admissible values of λ and ζ' are determined by the requirements that the functions $\Phi_{(\zeta', \lambda)}(\zeta)$ should be analytic in the unit disk. The solutions of Eq (96) are

$$\Phi_{(\zeta', \lambda)}(\zeta) = \mathcal{A}(|\zeta|) \left(1 + \left(\frac{\lambda - 1}{\lambda + 1} \right)^{\frac{1}{2}} \zeta \right)^{\alpha_+} \left(1 - \left(\frac{\lambda - 1}{\lambda + 1} \right)^{\frac{1}{2}} \zeta \right)^{\alpha_-} \quad (97)$$

where

$$\alpha_{\pm} = -\frac{3}{2} \pm \frac{\zeta'}{\sqrt{\lambda^2 - 1}} \quad (98)$$

and $\mathcal{A}(|\zeta|)$ is a normalization constant. The condition of analyticity requires

$$\left| \frac{\lambda - 1}{\lambda + 1} \right| < 1 \Leftrightarrow \text{Re } \lambda > 0. \quad (99)$$

If $\text{Re } \lambda < 0$, the functions $\Phi_{(\zeta', \lambda)}(\zeta)$ cannot be analytic in the unit disk.

The decomposition of the generalized intelligent states $|\zeta', \lambda, \alpha\rangle$ over the Hilbert orthonormal basis $\{|\psi_n\rangle\}$ can be obtained by expanding the function $\Phi_{(\zeta', \alpha)}(\zeta)$ into a power series in ζ . This can be done by using the following relations

$$\prod_{l=\pm 1} \left(1 + \left(\frac{\lambda - 1}{\lambda + 1} \right)^{\frac{1}{2}} \zeta \right)^{-\frac{3}{2} + l \frac{\zeta'}{\sqrt{\lambda^2 - 1}}} = \sum_{n=0}^{+\infty} \zeta^n \left(2\sqrt{\frac{\lambda - 1}{\lambda + 1}} \right)^n P_n^{(\alpha_+ - n, \alpha_- - n)}(0). \quad (100)$$

Then, the function $\Phi_{(\zeta', \alpha)}(\zeta)$ can be expanded in terms of the Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$. Using the relation between the hypergeometric function and Jacobi polynomials [19] we can show that

$$\begin{aligned} |\zeta', \lambda, \alpha\rangle &= \mathcal{A}(|\zeta|) \sum_{n=0}^{+\infty} \left[\frac{n!}{(n+2)!} \right]^{\frac{1}{2}} \left[\frac{n! \Gamma(\alpha_+ - n + 1)}{\Gamma(\alpha_+ + 1)} \right] \left(2\sqrt{\frac{\lambda - 1}{\lambda + 1}} \right)^n \times \\ & {}_2F_1(-n, -n - 2; \alpha_+ - n + 1; \frac{1}{2}) e^{-i\alpha n(n+2)} |\psi_n\rangle \end{aligned} \quad (101)$$

or

$$|\zeta', \lambda, \alpha\rangle = \mathcal{A}(|\zeta|) \sum_{n=0}^{+\infty} \left[\frac{n!}{(n+2)!} \right]^{\frac{1}{2}} \left(2\sqrt{\frac{\lambda - 1}{\lambda + 1}} \right)^n P_n^{(\alpha_+ - n, \alpha_- - n)}(0) e^{-i\alpha n(n+2)} |\psi_n\rangle. \quad (102)$$

The generalized intelligent states $\Phi_{(\zeta',\lambda)}(\zeta)$ (Eq (97)) and $\Psi_{(\zeta',\lambda)}(z)$ (Eq (89)) are related through a Laplace transform [20]. In fact, equation (96) can be written as

$$\left[\left[(1+\lambda)\zeta^2 + (1-\lambda) \right] \frac{d}{d\zeta} - 3 \frac{(1-\lambda)}{\zeta} + 2\zeta' \right] \Phi_{(\zeta',\lambda)} \left(\frac{1}{\zeta} \right) = 0 \quad (103)$$

Using

$$\Phi_{(\zeta',\lambda)} \left(\frac{1}{\zeta} \right) = \frac{\zeta^{-3}}{\sqrt{2}} \int_0^{+\infty} z^2 \Psi_{(\zeta',\lambda)}(z) e^{-\frac{z}{\zeta}} dz, \quad (104)$$

it is easy to see that the eigenvalue equation (103) becomes

$$\left[(1+\lambda) \left(z \frac{d^2}{dz^2} + 3 \frac{d}{dz} \right) + (1-\lambda)z - 2\zeta' \right] \Psi_{(\zeta',\lambda)}(z) = 0. \quad (105)$$

We note that this last differential equation coincides for $\zeta' = z'$ with that given by equation (86).

4 x^4 -anharmonic oscillator

Let us recall briefly the general structure of the Hamiltonian eigenvalues and eigenvectors for the one-dimensional nonlinear oscillators. Indeed, we are interested to the Hamiltonian which has the form

$$H = a^+ a^- + \frac{\varepsilon}{4} (a^- + a^+)^4 - c_0 \quad (106)$$

where a^+ and a^- are the creation and annihilation operators for the harmonic oscillator and the parameter ε is positive. The quantity c_0 is defined as follows

$$c_0 = \frac{3}{4}\varepsilon - \frac{21}{8}\varepsilon^2. \quad (107)$$

The Hamiltonian H can be factorized in the following form [21]

$$H = A_\varepsilon^+ A_\varepsilon^- \quad (108)$$

where $(A_\varepsilon^-)^\dagger = A_\varepsilon^+$ and the operator A_ε^- is defined as a function of a^- and a^+ , for more details see [21].

The energy levels are given by [21] (see also [22])

$$e_n = n + \frac{3}{2}\varepsilon (n^2 + n). \quad (109)$$

The Hilbert space \mathcal{H} is spanned by the states

$$|n, \varepsilon\rangle = \frac{(A_\varepsilon^+)^n}{\sqrt{E(n)}} e^{i\alpha e_n} |0, \varepsilon\rangle, \quad n \in \mathbf{N} \quad (110)$$

where $|0, \varepsilon\rangle$ is the ground state and the function $E(n)$ is defined by

$$E(n) = \begin{cases} 1 & \text{if } n = 0 \\ \left(\frac{3\varepsilon}{2}\right)^n \frac{\Gamma(n+1)\Gamma(n+2+\frac{2}{3\varepsilon})}{\Gamma(2+\frac{2}{3\varepsilon})} & \text{if } n \neq 0 \end{cases} \quad (111)$$

The action of the annihilation and creation operators are defined as follows:

$$A_\varepsilon^+ |n, \varepsilon\rangle = \sqrt{\left(\frac{3\varepsilon}{2}\right) (n+1) \left(n+2+\frac{2}{3\varepsilon}\right)} e^{-i\alpha(e_{n+1}-e_n)} |n+1, \varepsilon\rangle \quad (112)$$

$$A_\varepsilon^- |n, \varepsilon\rangle = \sqrt{\left(\frac{3\varepsilon}{2}\right) n \left(n+1+\frac{2}{3\varepsilon}\right)} e^{i\alpha(e_n-e_{n-1})} |n-1, \varepsilon\rangle. \quad (113)$$

We define the number operator N as

$$N |n, \varepsilon\rangle = n |n, \varepsilon\rangle \quad (114)$$

The operator N is different from the product $A_\varepsilon^+ A_\varepsilon^- (= H)$.

4.1 Gazeau-Klauder coherent states

Following the construction introduced before, the Gazeau-Klauder coherent states obey to the eigenvalue equation (14).

A simple computation leads to

$$|z, \alpha\rangle = \mathcal{N}(|z|) \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma\left(2+\frac{2}{3\varepsilon}\right)}{(3\varepsilon)^n \Gamma(n+1) \Gamma\left(n+2+\frac{2}{3\varepsilon}\right)}} (z\sqrt{2})^n e^{-i\alpha e_n} |n, \varepsilon\rangle \quad (115)$$

where

$$\mathcal{N}(|z|) = \left[{}_0F_1\left(2+\frac{2}{3\varepsilon}, \frac{2}{3\varepsilon}|z|^2\right) \right]^{-\frac{1}{2}}. \quad (116)$$

Remark that the coherent states $|z, \alpha\rangle$ are continuously labelled by z and α , and the radius of convergence is infinite. The measure in respect which we have an overcomplete set of coherent states is

$$d\mu(z) = \frac{4}{3\pi\varepsilon} I_{\left(1+\frac{2}{3\varepsilon}\right)}\left(2\sqrt{\frac{2}{3\varepsilon}}r\right) K_{\left(\frac{1}{2}+\frac{1}{3\varepsilon}\right)}\left(2\sqrt{\frac{2}{3\varepsilon}}r\right) r dr d\phi \quad z = re^{i\phi}. \quad (117)$$

The overlapping between two x^4 -anharmonic oscillator coherent states is given by

$$\langle z, \alpha | z', \alpha \rangle = \frac{{}_0F_1\left(2+\frac{2}{3\varepsilon}, \frac{2}{3\varepsilon}\bar{z}z'\right)}{\sqrt{{}_0F_1\left(\frac{2}{3\varepsilon}+2, \frac{2}{3\varepsilon}|z|^2\right) {}_0F_1\left(\frac{2}{3\varepsilon}+2, \frac{2}{3\varepsilon}|z'|^2\right)}}. \quad (118)$$

The Gazeau-Klauder coherent states provide a representation of any state

$$|\phi\rangle = \sum_{n=0}^{+\infty} d_n |n, \varepsilon\rangle \text{ in the Hilbert space by an entire function}$$

$$\begin{aligned}
\phi(z, \alpha) &= \sqrt{{}_0F_1\left(2 + \frac{2}{3\varepsilon}, \frac{2}{3\varepsilon} |z|^2\right)} \langle \bar{z}, \alpha | \phi \rangle \\
&= \sum_{n=0}^{+\infty} \sqrt{\frac{\Gamma\left(2 + \frac{2}{3\varepsilon}\right)}{(3\varepsilon)^n \Gamma(n+1) \Gamma\left(n+2 + \frac{2}{3\varepsilon}\right)}} (z\sqrt{2})^n d_n e^{i\alpha e_n}.
\end{aligned} \tag{119}$$

The state $|n, \varepsilon\rangle$ is represented by

$$\begin{aligned}
\phi_{(n,\varepsilon)}(z, \alpha) &= \sqrt{{}_0F_1\left(2 + \frac{2}{3\varepsilon}, \frac{2}{3\varepsilon} |z|^2\right)} \langle \bar{z}, \alpha | n, \varepsilon \rangle \\
&= \sqrt{\frac{\Gamma\left(2 + \frac{2}{3\varepsilon}\right)}{(3\varepsilon)^n \Gamma(n+1) \Gamma\left(n+2 + \frac{2}{3\varepsilon}\right)}} (z\sqrt{2})^n e^{i\alpha e_n}
\end{aligned} \tag{120}$$

The operators A_ε^\pm and $G_\varepsilon(N) = [A_\varepsilon^-, A_\varepsilon^+] = 3\varepsilon(N+1) + 1$ act in the Hilbert space of analytic functions $\phi(z, \alpha)$ as linear differential operators

$$A_\varepsilon^+ = z, \quad A_\varepsilon^- = \frac{3\varepsilon}{2} z \frac{d^2}{dz^2} + (1+3\varepsilon) \frac{d}{dz} \quad \text{and} \quad G_\varepsilon(N) = 3\varepsilon z \frac{d}{dz} + (1+3\varepsilon) \tag{121}$$

with the following actions

$$A_\varepsilon^+ \phi_{(n,\varepsilon)}(z, \alpha) = \sqrt{\left(\frac{3\varepsilon}{2}\right) (n+1) \left(n+2 + \frac{2}{3\varepsilon}\right)} e^{-i\alpha(e_{n+1}-e_n)} \phi_{(n+1,\varepsilon)}(z, \alpha) \tag{122}$$

$$A_\varepsilon^- \phi_{(n,\varepsilon)}(z, \alpha) = \sqrt{\left(\frac{3\varepsilon}{2}\right) n \left(n+1 + \frac{2}{3\varepsilon}\right)} e^{i\alpha(e_n-e_{n-1})} \phi_{(n-1,\varepsilon)}(z, \alpha) \tag{123}$$

$$G_\varepsilon(N) \phi_{(n,\varepsilon)}(z, \alpha) = (1+3\varepsilon(n+1)) \phi_{(n,\varepsilon)}(z, \alpha). \tag{124}$$

Using the relation (25), we can write

$$|z, \alpha\rangle = \exp\left(z \frac{2}{3\varepsilon(N+1)+2} A_\varepsilon^+\right) |0, \varepsilon\rangle. \tag{125}$$

Note that when $\varepsilon \rightarrow 0$, it is obvious that x^4 -anharmonic oscillator leads to the harmonic oscillator Hamiltonian. Indeed, using the formula

$$\lim_{\varepsilon \rightarrow 0} \exp\left(\frac{3\varepsilon}{2}\right)^n \frac{\Gamma\left(n+2 + \frac{2}{3\varepsilon}\right)}{\Gamma\left(2 + \frac{2}{3\varepsilon}\right)} = 1, \tag{126}$$

the Gazeau-Klauder coherent states Eq. (115) becomes

$$|z, \alpha\rangle = \exp\left(-\frac{|z|^2}{2}\right) \sum_{n=0}^{+\infty} \frac{z^n}{\sqrt{n!}} e^{-i\alpha n} |n, 0\rangle. \tag{127}$$

We remark also that for the special case when $\varepsilon = \frac{2}{3}$ we get all the results obtained for the infinite square well potential.

4.2 Klauder-Perelomov coherent states

Following the definition \mathbf{D}_2 the coherent states of Klauder-Perelomov type for the nonlinear oscillators are defined as

$$|z, \alpha\rangle = \exp(zA_\varepsilon^+ - \bar{z}A_\varepsilon^-)|0, \varepsilon\rangle \quad , \quad z \in \mathbf{C}. \quad (128)$$

After a more or less complicated manipulation, or by applying the results of section 1 one has

$$|z, \alpha\rangle = \sum_{n=0}^{+\infty} Z^n c_n(|Z|) \sqrt{F(n)} e^{-i\alpha e_n} |n, \varepsilon\rangle \quad (129)$$

where

$$Z = \sqrt{\frac{3\varepsilon}{2}} z \quad \text{and} \quad F(n) = \frac{\Gamma(n+1)\Gamma(n+2+\frac{2}{3\varepsilon})}{\Gamma(2+\frac{2}{3\varepsilon})}. \quad (130)$$

The quantities $c_n(|Z|)$ are defined by

$$c_n(|Z|) = \sum_{j=0}^{+\infty} \frac{(-|Z|^2)^j}{(n+2j)!} \pi(n+1, j) \quad (131)$$

where

$$\pi(n+1, j) = \sum_{i_1=1}^{n+1} e'_{i_1} \sum_{i_2=1}^{i_1+1} e'_{i_2} \dots \sum_{i_j=1}^{i_{j-1}+1} e'_{i_j} \quad , \text{ and } \quad \pi(n+1, 0) = 1 \quad (132)$$

with

$$e'_n = \frac{2}{3\varepsilon} e_n. \quad (133)$$

We can verify that the π 's satisfy the following relation

$$\pi(n+1, j) - \pi(n, j) = (n+1) \left(n+2+\frac{2}{3\varepsilon} \right) \pi(n+2, j-1). \quad (134)$$

Remark that here we have the same relations that one obtained in section 1 (see (30)) with a minor modification. Using the recurrence formula (134), it is not difficult to show that the $c_n(|Z|)$'s satisfy the following differential equation

$$|Z| \frac{dc_n(|Z|)}{d|Z|} = c_{n-1}(|Z|) - nc_n(|Z|) - (n+1) \left(n+2+\frac{2}{3\varepsilon} \right) |Z|^2 c_{n+1}(|Z|). \quad (135)$$

Setting

$$c_n(|Z|) = \frac{1}{n! |Z|^n} \mathcal{A}_n(|Z|) \quad (136)$$

the differential equation (135) takes the simple form

$$\frac{d\mathcal{A}_n(|Z|)}{d|Z|} = n\mathcal{A}_{n-1}(|Z|) - \left(n+2+\frac{2}{3\varepsilon} \right) \mathcal{A}_{n+1}(|Z|). \quad (137)$$

It comes that the solutions of (137) are

$$\mathcal{A}_n(|Z|) = [\cosh(|Z|)]^{-n-2-\frac{2}{3\varepsilon}} [\sinh(|Z|)]^n \quad (138)$$

Finally the coherent states Eq. (128) are given by

$$|\zeta, \alpha\rangle \equiv (1 - |\zeta|^2)^{1+\frac{1}{3\varepsilon}} \sum_{n=0}^{+\infty} \frac{\zeta^n}{\sqrt{n!}} \left[\frac{\Gamma\left(n+2+\frac{2}{3\varepsilon}\right)}{\Gamma\left(2+\frac{2}{3\varepsilon}\right)} \right]^{\frac{1}{2}} e^{-i\alpha e_n} |n, \varepsilon\rangle \quad (139)$$

where $\zeta = \frac{Z \tanh(|Z|)}{|Z|}$, and they form an overcomplete set in respect to measure

$$d\mu(\zeta) = \left(\frac{1}{\pi} + \frac{2}{3\varepsilon\pi} \right) \frac{d^2\zeta}{(1 - |\zeta|^2)^2}. \quad (140)$$

The kernel $\langle \zeta, \alpha | \zeta', \alpha' \rangle$ is easily evaluated from (139)

$$\begin{aligned} \langle \zeta, \alpha | \zeta', \alpha' \rangle &= \sqrt{(1 - |\zeta|^2)^{1+\frac{1}{3\varepsilon}} (1 - |\zeta'|^2)^{1+\frac{1}{3\varepsilon}}} \sum_{n=0}^{+\infty} \frac{(\bar{\zeta}\zeta')^n}{n!} \times \\ &\quad \left[\frac{\Gamma\left(n+2+\frac{2}{3\varepsilon}\right)}{\Gamma\left(2+\frac{2}{3\varepsilon}\right)} \right] e^{-i(\alpha' - \alpha)n(n+2)}. \end{aligned} \quad (141)$$

For an arbitrary state $|\varphi\rangle = \sum_{n=0}^{+\infty} f_n |n, \varepsilon\rangle \in \mathcal{H}$, one can construct the analytic function

$$\varphi(\zeta, \alpha) = (1 - |\zeta|^2)^{-1-\frac{1}{3\varepsilon}} \langle \bar{\zeta}, \alpha | \varphi \rangle = \sum_{n=0}^{+\infty} \zeta^n \left[\frac{\Gamma\left(n+2+\frac{2}{3\varepsilon}\right)}{\Gamma\left(2+\frac{2}{3\varepsilon}\right)} \right]^{\frac{1}{2}} f_n e^{i\alpha e_n} \quad (142)$$

with

$$|\varphi\rangle = \int |\zeta, \alpha\rangle (1 - |\zeta|^2)^{1+\frac{1}{3\varepsilon}} \varphi(\bar{\zeta}, \alpha) d\mu(\zeta). \quad (143)$$

In particular, for the states $|n, \varepsilon\rangle$ we associate the monomial

$$\phi'_{(n, \varepsilon)}(\zeta, \alpha) = \zeta^n \left[\frac{\Gamma\left(n+2+\frac{2}{3\varepsilon}\right)}{\Gamma\left(2+\frac{2}{3\varepsilon}\right)} \right]^{\frac{1}{2}} e^{i\alpha e_n}. \quad (144)$$

The creation A_ε^+ annihilation A_ε^- and $G_\varepsilon(N)$ operators act on the Hilbert space of analytic functions $\varphi(\zeta, \alpha)$ as follows

$$A_\varepsilon^+ = \sqrt{\frac{3\varepsilon}{2}} \left[\zeta^2 \frac{d}{d\zeta} + \left(2 + \frac{2}{3\varepsilon}\right) \zeta \right], \quad A_\varepsilon^- = \sqrt{\frac{3\varepsilon}{2}} \frac{d}{d\zeta} \quad (145)$$

and

$$G_\varepsilon(N) = 3\varepsilon \left[\zeta \frac{d}{d\zeta} + 1 + \frac{1}{3\varepsilon} \right]. \quad (146)$$

One can verify that

$$A_\varepsilon^+ \phi'_{(n,\varepsilon)}(\zeta, \alpha) = \sqrt{\left(\frac{3\varepsilon}{2}\right) (n+1) \left(n+2 + \frac{2}{3\varepsilon}\right)} e^{-i\alpha(e_{n+1}-e_n)} \phi'_{(n+1,\varepsilon)}(\zeta, \alpha) \quad (147)$$

$$A_\varepsilon^- \phi'_{(n,\varepsilon)}(\zeta, \alpha) = \sqrt{\left(\frac{3\varepsilon}{2}\right) n \left(n+1 + \frac{2}{3\varepsilon}\right)} e^{i\alpha(e_n-e_{n-1})} \phi'_{(n-1,\varepsilon)}(\zeta, \alpha) \quad (148)$$

$$G_\varepsilon(N) \phi'_{(n,\varepsilon)}(\zeta, \alpha) = (1+3\varepsilon(n+1)) \phi'_{(n,\varepsilon)}(\zeta, \alpha). \quad (149)$$

Now, let us discuss the limit $\varepsilon \rightarrow 0$.

By using the formula (126), the coherent states Eq. (139) takes the form

$$|z, \alpha\rangle = \exp\left(-\frac{|z|^2}{2}\right) \sum_{n=0}^{+\infty} \frac{z^n}{\sqrt{n!}} e^{-i\alpha n} |n, 0\rangle \quad (150)$$

and the measure, Eq. (140), becomes

$$\lim_{\varepsilon \rightarrow 0} d\mu(\zeta) = d\mu(z) = \frac{1}{\pi} d^2 z, \quad (151)$$

we note also that for $\varepsilon \rightarrow 0$ the representation of the operators A_ε^+ and A_ε^- reduces to creation and annihilation ones of the harmonic oscillator (Weyl-Heisenberg algebra). Indeed, we have

$$A_\varepsilon^+ \rightarrow a^+ \equiv z, \quad A_\varepsilon^- \rightarrow a^- \equiv \frac{d}{dz} \quad \text{and} \quad G_\varepsilon(N) \rightarrow I \quad (152)$$

4.3 Generalized intelligent states

Having the analytical representation of the Gazeau-Klauder and Klauder-Perelomov coherent states, we can derive the generalized intelligent states, for the x^4 -anharmonic oscillator, in an analytical way.

4.3.1 Gazeau-Klauder analytic representation

The Hilbert space, in this representation, is the space of analytic function $\{\phi(z, \alpha)\}$ equipped with the scalar product Eq. (83), where the measure $d\mu(z)$ is given by (117).

To solve the eigenvalue equation (38) in the case of x^4 -anharmonic oscillators, we introduce the function

$$\phi_{(z',\lambda,\alpha)}(z, \alpha) = \sqrt{{}_0F_1\left(2 + \frac{2}{3\varepsilon}, \frac{2}{3\varepsilon} |z|^2\right)} \langle \bar{z}, \alpha | z', \lambda, \alpha \rangle. \quad (153)$$

Then the eigenvalue equation (38), is converted to the second-order homogenous differential equation

$$\left\{ \left((1+\lambda) \frac{3\varepsilon}{2} \right) \left[\frac{2}{3\varepsilon} (1+3\varepsilon) \frac{d}{dz} + z \frac{d^2}{dz^2} \right] + (1-\lambda) z \right\} \phi_{(z',\lambda)}(z) = 2z' \phi_{(z',\lambda)}(z). \quad (154)$$

By means of simple substitutions, the above equation is reduced to the Kummer equation for the confluent hypergeometric function ${}_1F_1(a; b; z)$ [19] and we arrive to the following solution

$$\phi_{(z', \lambda)}(z) = \exp(cz) {}_1F_1(a; b; -2cz) \quad (155)$$

where

$$a = 1 + \frac{1}{3\varepsilon} \mp \frac{2z'}{\sqrt{(\lambda^2 - 1)6\varepsilon}}, \quad b = \frac{2}{3\varepsilon} + 2 \quad \text{and} \quad c = \pm \sqrt{\left(\frac{\lambda - 1}{\lambda + 1}\right) \frac{2}{3\varepsilon}}. \quad (156)$$

We note that the generalized intelligent states for the harmonic oscillator can be obtained from the equation (155) in the limit $\varepsilon \rightarrow 0$ (or from the differential equation (154) by setting $\varepsilon = 0$).

Thus, we have

$$\phi_{(z', \lambda)}(z) = \phi_{(z', \lambda)}(0) \exp\left(\frac{2z'}{1 + \lambda}z + \left(\frac{\lambda - 1}{\lambda + 1}\right)\frac{z^2}{2}\right) \quad (157)$$

where $\phi_{(z', \lambda)}(0)$ is the normalization constant.

4.3.2 Klauder-Perelomov analytic representation

In this representation, the Hilbert space is equipped with the scalar product given by (94) and the measure $d\mu(\zeta)$ is given by (140).

By introducing the analytic function

$$\varphi_{(\zeta', \lambda)}(\zeta) = \left(1 - |\zeta|^2\right)^{-1 - \frac{1}{3\varepsilon}} \langle \bar{\zeta}, \alpha | \zeta', \lambda, \alpha \rangle \quad (158)$$

we convert the eigenvalue equation (38) into the differential equation

$$\left[\left[(1 - \lambda)\zeta^2 + (1 + \lambda) \right] \frac{d}{d\zeta} + (1 - \lambda) \left(2 + \frac{2}{3\varepsilon} \right) \zeta - 2\sqrt{\frac{2}{3\varepsilon}}\zeta' \right] \varphi_{(\zeta', \lambda)}(\zeta) = 0. \quad (159)$$

In the general case where $\lambda \neq \pm 1$, the solution of the equation (159) is

$$\varphi_{(\zeta', \lambda)}(\zeta) = \mathcal{B}(|\zeta|) \left(1 + \left(\frac{\lambda - 1}{\lambda + 1} \right)^{\frac{1}{2}} \zeta \right)^{\alpha_+} \left(1 - \left(\frac{\lambda - 1}{\lambda + 1} \right)^{\frac{1}{2}} \zeta \right)^{\alpha_-} \quad (160)$$

where

$$\alpha_{\pm} = -1 - \frac{1}{3\varepsilon} \pm \frac{2\zeta'}{\sqrt{(\lambda^2 - 1)6\varepsilon}} \quad (161)$$

and $\mathcal{B}(|\zeta|)$ is a normalization constant.

The condition of the analyticity of the solution $\varphi_{(\zeta', \lambda)}(\zeta)$ in the unit disk is satisfied when we have

$$\left| \frac{\lambda - 1}{\lambda + 1} \right| < 1 \Leftrightarrow \text{Re } \lambda > 0. \quad (162)$$

In order to obtain the decomposition of the generalized intelligent states $|\zeta', \lambda, \alpha\rangle$ over the Hilbert orthonormal basis $\{|n, \varepsilon\rangle\}$, we expand the function $\varphi_{(\zeta', \alpha)}(\zeta)$ into a power series in ζ in the same way that was discussed previously for the infinite square well potential. This can be done by using the following relations

$$\prod_{l=\pm 1} \left(1 + \left(\frac{\lambda - 1}{\lambda + 1} \right)^{\frac{1}{2}} \zeta \right)^{-1 - \frac{1}{3\epsilon} + l \frac{2\zeta'}{\sqrt{(\lambda^2 - 1)6\epsilon}}} = \sum_{n=0}^{+\infty} \zeta^n \left(2\sqrt{\frac{\lambda - 1}{\lambda + 1}} \right)^n P_n^{(\alpha_+ - n, \alpha_- - n)}(0) \quad (163)$$

where $P_n^{(\alpha, \beta)}(x)$ is the Jacobi polynomials [19].

Using the relation between the hypergeometric function and the Jacobi polynomials [19], one can show that

$$|\zeta', \lambda, \alpha\rangle = \mathcal{B}(|\zeta|) \sum_{n=0}^{+\infty} \left[\frac{n!}{(n+2)!} \right]^{\frac{1}{2}} \left[\frac{n! \Gamma(\alpha_+ - n + 1)}{\Gamma(\alpha_+ + 1)} \right] \times \left(2\sqrt{\frac{\lambda - 1}{\lambda + 1}} \right)^n {}_2F_1(-n, -n - 2; \alpha_+ - n + 1; \frac{1}{2}) e^{-i\alpha e_n} |\psi_n\rangle. \quad (164)$$

The two generalized intelligent states $\phi_{(z', \lambda)}(z)$ (Eq (155)) and $\varphi_{(\zeta', \lambda)}(\zeta)$ (Eq (160)) are related as in the previous case (i.e., the infinite square well potential) through the Laplace transform.

5 Concluding remarks

In this paper, we constructed the coherent states (à la Gazeau-Klauder and à la Klauder-Perelomov) and the generalized intelligent states for an arbitrary quantum systems. As an illustration of our construction, we treated the system of a free particle in the infinite square well potential and the x^4 -nonlinear oscillators. We showed the advantage of the analytic representations of Gazeau-Klauder as well as Klauder-Perelomov coherent states in obtaining the generalized intelligent states in analytical ways. Finally, one can see that our results could be extended to other exactly solvable quantum systems like, for instance, Coulomb, hyperbolic Rosen-Morse, Eckart and trigonometric Rosen-Morse potentials. This extension is under study.

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