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NON-HOMOGENEOUS RECURRENCE RELATIONS AND MOMENT PROBLEMS

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Abstract

We study here the linear moment problem for a perturbed recursive sequences. We give a link between the truncated and the full moment problems for those sequences.

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1. Introduction

Let $\mathcal{H}$ be a real separable Hilbert space and $\gamma = \{\gamma_n\}_{0\leq n\leq p}$ $(p \leq +\infty)$ be a sequence of real numbers. The linear moment problem associated to γ consists of finding a self-adjoint operator A and a non-vanishing vector $x \in \mathcal{H}$ satisfying,

$$\gamma_n =< A^n x|x >, \quad \text{for } \ 0 \leq n \leq p. \tag{1}$$

The problem (1) has been studied in [10, 14, 16]. Let $a_0, \cdots, a_{r-1}$ ($r \geq 2$, $a_{r-1} \neq 0$) be some real numbers, and let $\mathcal{C} = \{C_n\}_{n\geq r}$ be a sequence in $I\!R$ (or $\mathbb{C}$).
Let $\mathcal{T} = \{T_n\}_{n\geq 0}$ be the sequence defined by the following nonhomogeneous recurrence relation of order r,

$$T_{n+1} = a_0 T_n + a_1 T_{n-1} + \cdots + a_{r-1} T_{n-r+1} + C_{n+1}, \ \text{for } n \geq r-1, \tag{2}$$

where $T_0, \cdots, T_{r-1}$ are the specified initial values (or conditions). In the sequel we refer to such sequence $\mathcal{T}$ as the solution of the recurrence relation (2), and $P(X) = X^r - a_0 X^{r-1} - a_1 X^{r-2} - \ldots.. - a_{r-1}$ is the characteristic polynomial supposed with minimal degree (see [4] for example).

The solution of (2) has been studied by various methods and techniques for C_n polynomial and factorial polynomial (see [1, 2, 3, 5, 13, 15, 18] for example). Recently, a matrix method has been considered in [13], for studying solutions of (2) in the general setting. The method of [13] consists of considering equation(2) under an equivalent nonlinear matrix equation, where appears a companion matrix.

When $\mathcal{C}$ is a vanishing sequence, the sequence $\mathcal{T}$ is called r-generalized Fibonacci sequence ($r-$GFS, for short). If $\mathcal{H}$ is of finite dimension, a connection between the full and truncated linear moment problem for r-GFS, has been studied by the authors in [10].

The purpose of this paper is to study the moment problems (1) (and also (4)) in connection with the solutions (2) for a general sequence $\mathcal{C}$.

This paper is organized as follows. In section 2, we establish the connection between sequence (2) and moments of operators. Section 3 is devoted to the case when the sequence $\mathcal{C}$ is an s-GFS. In section 4 we study solutions of (2) in terms of the spectral measures of self-adjoint extensions.

2. Solutions of (2) and linear moments problem (1)

Consider the polynomial $P(X) = X^r - a_0 X^{r-1} - a_1 X^{r-2} - \ldots.. - a_{r-1}$, and let $\{Q_n\}_{n\geq r}$ be the family of polynomials given by $Q_n(X) = X^{n-r}P(X)$. Let $x \neq 0$ be a non-vanishing element of $\mathcal{H}$. For every operator A on $\mathcal{H}$, The sequence of moments $\{< A^n x, x >\}_{n\geq 0}$ is a sequence (2) with $C_n =< Q_n(A)x, x >$, for every $n \geq 0$.

As a consequence, we have the following proposition,

Proposition 2.1. *Let A be an operator on $\mathcal{H}$ and $x \neq 0$ in $\mathcal{H}$. For every monic polynomial P, there exists a sequence $\{C_n\}_{n\geq 0}$ such that the sequence of moments $\mathcal{T} = \{< A^n x|x >\}_{n\geq 0}$, is a solution of the recurrence relation (2), whose characteristic polynomial is P.*

Therefore, the question of studying the converse of the result of Proposition 2.1 arises.

Theorem 2.1. *Let $\mathcal{T} = \{T_n\}_{n\geq 0}$ be a sequence (2), whose characteristic polynomial is $P(X) = X^r - a_0X^{r-1} - a_1X^{r-2} - ..... - a_{r-1}$ ($a_{r-1} \neq 0$). Let A be an operator of $\mathcal{H}$ and $x \neq 0 \in \mathcal{H}$. Then $T_n =< A^n x|x >$, for any $n \geq 0$, if and only if $T_n =< A^n x|x >$ for $n = 0, 1, .., r-1$ and $C_n =< A^{n-r}P(A)x|x >$, for every $n \geq r$.*

Proof. For every $k \geq r$, we have,

$$C_k = T_k - \sum_{j=0}^{r-1} a_j T_{k-j-1} =< (A^k - \sum_{j=0}^{r-1} a_j A^{j-k-1})x|x > .$$

Therefore, $C_k =< A^{k-r}P(A)x|x >$.

Conversely, suppose that $T_n =< A^n x|x >$, for $n = 0, 1, .., r-1$ and $C_n =< A^{n-r}P(A)x|x >$, for every $n \geq 0$. Then $T_r = \sum_{j=0}^{r-1} a_j < A^{r-j-1}x|x > + < P(A)x|x > =< A^r x|x >$. By induction, we prove that $\mathcal{T}$ satisfies $T_n =< A^n x|x >$, for every $n \geq 0$.□

Proposition 2.2. *With notations of Theorem 2.1, if A is a self-adjoint operator then the following statements are equivalent,*
(i) $T_n =< A^n x|x >$ for every $n \geq 0$.
(ii) $T_n =< A^n x|x >$, for $n = 0, 1, .., 2r-1$, and $C_n = \sum_{j=0}^{r-1} a_j C_{n-j-1} + < A^{n-2r}z|z >$, for $z = P(A)x$ and every $n \geq 2r$.

Proof. It suffices to establish the equivalence between (ii) and the second statement of Theorem 2.1. Let A be a self-adjoint operator. Suppose that $T_n =< A^n x|x >$ for $n = 0, 1, .., r-1$ and $C_n =< A^{n-r}P(A)x|x >$, for $n \geq r$. Then, for every $n \geq 2r$, we have,

$$\begin{aligned} < A^{n-2r}z, z > &= < A^{n-r}x, P(A)x > - \sum_{j=0}^{r-1} a_j < A^{n-r-j-1}x, P(A)x > \\ &= < A^{n-r}P(A)x|x > - \sum_{j=0}^{r-1} a_j < A^{n-r-j-1}P(A)x|x > \\ &= C_n - \sum_{j=0}^{r-1} a_j C_{n-j-1}. \end{aligned}$$

Conversely, suppose that (ii) holds . It is easy to show that $C_n =< A^{n-r}P(A)x|x >$, for $n = r, r+1, ..., 2r-1$. On the other hand, we have,

$$\begin{aligned} C_{2r} &= \sum_{j=0}^{r-1} a_j C_{2r-j-1} + < P(A)x, P(A)x > \\ &= \sum_{j=0}^{r-1} a_j C_{2r-j-1} + < A^r P(A)x, x > - \sum_{j=0}^{r-1} a_j < A^{r-j-1}P(A)x, x > \\ &= < A^r P(A)x, x > . \end{aligned}$$

And by induction we prove that

$$C_n =< A^{n-r}P(A)^x, x >, \quad \text{for any} \quad n \geq 2r$$

It follows that (ii) and (iii) are equivalent.□

Let $\mathcal{T}$ be a sequence (2), whose characteristic polynomial is $P(X) = X^r - a_0X^{r-1} - a_1X^{r-2} - ..... - a_{r-1}$. Suppose that $\mathcal{T}$ is a solution of the linear moment problem (1). For every $k \geq r$, we have

$$C_{2k} - \sum_{j=0}^{r-1} a_j C_{2k-j-1} = \|A^{k-r}P(A)x\|^2.$$

Remark that if $C_n \neq 0$, for some $n \geq r$, then $C_{2k} > \sum_{j=0}^{r-1} a_j C_{2k-j-1}$ (for any $k \geq r$) is a necessary condition for $\mathcal{T}$ to be a solution of the linear moment problem (1). More precisely, we have the following corollary.

Corollary 2.1. *Let $\mathcal{T}$ be a sequence (2) with $P(X) = X^r - a_0X^{r-1} - a_1X^{r-2} - ..... - a_{r-1}$ as characteristic polynomial. If $\mathcal{T}$ satisfies (1) then,*

- *$\mathcal{C} = \{C_n\}_{n\geq r}$ is an r-GFS sequence with characteristic polynomial P if and only if $C_n = 0$, for every $n \geq 2r$.*
- *$C_{2k} > \sum_{j=0}^{r-1} a_j C_{2k-j-1}$ for every $k \geq r$ if and only if there exists $k_0 > r$ such that $C_{2k_0} \neq 0$.*

3. Finite dimensional case

Let $\mathcal{H}$ be a finite dimensional Hilbert space (dim($\mathcal{H}$) = m) and let $\mathcal{T} = \{T_n\}_{n\geq 0}$ be a sequence (2). By theorem 2.1, $\mathcal{T} = \{T_n\}_{n\geq 0}$ is a sequence of moments of a self-adjoint operator A on a non-vanishing vector $x \in \mathcal{H}$ if and only if $T_n = \sum_{j=1}^{s} {\lambda_j}^n \|x_j\|^2$ for $n = 0, 1, ..., r-1$ and $C_n =< A^{n-r}P(A)x, x >$. If $x_j = \Pi_j x$ $(0 \leq j \leq s \leq m)$ are the eigenvectors of A corresponding to the eigenvalues λ_j, then a straightforward computation allows us to see that $\mathcal{T} = \{T_n\}_{n\geq 0}$ is a sequence of moments of a self-adjoint operator A on a non-vanishing vector $x \in \mathcal{H}$ if and only if $T_n = \sum_{j=1}^{s} {\lambda_j}^n \|x_j\|^2$, for $n = 0, 1, ...r-1$ and

$$C_n = \sum_{j=1}^{s} \frac{P(\lambda_j)}{\lambda_j^r} \|x_j\|^2 {\lambda_j}^n \text{ for every } n \geq r \tag{3}$$

Expression (3) is nothing else but the Binet formula of an s-generalized Fibonacci sequence. More precisely, (3) implies that $\{C_n\}_{n\geq r}$ is an s-generalized Fibonacci sequence, whose characteristic polynomial is $P(x) = \prod_{j=1}^{s}(x - \lambda_j)$.

Proposition 3.1. *Let $\mathcal{T}$ be a sequence (2). If $\mathcal{T}$ is a sequence of moments of some operator A on some finite dimensional Hilbert space $\mathcal{H}$, then the nonhomogeneous part $\mathcal{C}$ is an s-generalized Fibonacci sequence ($s \leq dim(\mathcal{H})$). More precisely the characteristic polynomial of $\mathcal{C}$ is $P(X) = \prod_{i=0}^{s}(X - \lambda_i)$, where the ${\lambda_i}'s$ are the eigenvalues of A.*

In the special case of dimension 1 (where $\mathcal{H} = I\!R$), we obtain the following,

Let $\mathcal{H} = I\!R$ and $\mathcal{T} = \{T_n\}_{n\geq 0}$ be a sequence (2). By theorem 2.1 $\mathcal{T} = \{T_n\}_{n\geq 0}$ is a sequence of moments of an operator A on a non-vanishing vector $x \in I\!R$ if and only if $x = \pm\sqrt{T_0}$, $T_n = \lambda^n T_0$

for $n = 0, 1, ..., r-1$ and $C_n =< A^{n-r}P(A)x, x >$, where $\lambda = A.1$ and P is the characteristic polynomial of the homogeneous part of $\mathcal{T}$. An easy computation shows that,

$$C_n = \lambda^{n-r}P(\lambda)x^2 = \lambda^{n-r}C_r, \text{ for every } n \geq r.$$

Therefore, the following result is obtained.

Corollary 3.1. *Let $\mathcal{T} = \{T_n\}_{n\geq 0}$ be a sequence (2). Then $\mathcal{T} = \{T_n\}_{n\geq 0}$ is a sequence of moments of an operator A on $\mathbb{R}$ if and only if the two sequences $\{T_n\}_{0\leq n\leq r-1}$ and $\{C_n\}_{n\geq r}$ are a geometric sequences with $\nu = A.1$ as reason and $C_r = P(\nu)T_0$.*

Therefore, the only sequences (2) who are moment sequences of some operator on $\mathbb{R}$ are the geometric sequences.

4. Problems (1) in the case when $\mathcal{C}$ is an s-GFS

4.1. **Linear moment problem.** We start this section by the following proposition obtained by a straightforward computation.

Proposition 4.1. *Let $\mathcal{T} = \{T_n\}_{n\geq 0}$ be a sequence (2) and let $H_{r+n} = [T_{i+j}]_{0\leq i,j\leq r+n-1}$ and $S_{r+n} = [T_{i+j+1}]_{0\leq i,j\leq r-1}$ be the Hankel matrices associated with $\mathcal{T}$. Then*

$$det H_{r+n} = \begin{vmatrix} T_0 & . & . & . & T_{r-1} & C_r & . & . & . & C_{r+n-1} \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ T_{r-1} & . & . & . & T_{2r-2} & C_{2r-1} & . & . & . & C_{2r+n-2} \\ T_r & . & . & . & T_{2r-1} & C_{2r} & . & . & . & C_{2r+n-1} \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ T_{r+n-1} & . & . & . & T_{2r+n-2} & C_{2r+n-1} & . & . & . & C_{2r+2n-2} \end{vmatrix}$$

and

$$det S_{r+n} = \begin{vmatrix} T_1 & . & . & . & T_r & C_{r+1} & . & . & . & C_{r+n} \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ T_r & . & . & . & T_{2r-1} & C_{2r} & . & . & . & C_{2r+n-1} \\ T_{r+1} & . & . & . & T_2 r & C_{2r+1} & . & . & . & C_{2r+n} \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ T_{r+n} & . & . & . & T_{2r+n-1} & C_{2r+n} & . & . & . & C_{2r+2n-1} \end{vmatrix}.$$

Let $\mathcal{T} = \{T_n\}_{n\geq 0}$ be a sequence (2) and suppose that the nonhomogeneous part $\mathcal{C}$ is an s-GFS whose characteristic polynomial is $Q(X) = X^s - b_0X^{s-1} - b_1X^{s-2} - ... - b_{s-1}$. Let $R(X) = X^r - a_0X^{r-1} - a_1X^{r-2} - ... - a_{r-1}$ be the characteristic polynomial of the homogeneous part of (2). It follows that $\mathcal{T}$ is an (r+s)-GFS with $T_0, T_1, ..., T_{r+s-1}$ and $P = Q.R$ as initial conditions and characteristic polynomial (not necessarily with minimal degree). The remark 2.1 of [12] allows us to suppose that $\mathcal{H}$ is of finite dimension $(r+s)$.

Lemma 4.1. *Let $\mathcal{T} = \{T_n\}_{n\geq 0}$ be a sequence (2) and let $H_{r+n} = [T_{i+j}]_{0\leq i,j\leq r+n-1}$ be the Hankel matrices associated with $\mathcal{T}$. If $\mathcal{C}$ is an s-GFS, then $det(H_{r+n}) = 0$, for every $n > s$.*

By Proposition 2.3 of [12] we have,

Proposition 4.2. *Let $\mathcal{T} = \{T_n\}_{n\geq 0}$ be a sequence (2), with positive definite Hankel matrix H_{r+s} and $P_{\mathcal{T}}$ as a characteristic polynomial. If $\mathcal{C}$ is an s-GFS associated with the characteristic polynomial $Q_{\mathcal{C}}$, then there exist a $(degP_{\mathcal{T}} + degQ_{\mathcal{C}})-$dimensional Hilbert space $\mathcal{H}^{(\mathcal{T})}$ and a self-adjoint operator A on $\mathcal{H}^{(\mathcal{T})}$ solution of the moment problem (1). Moreover, if S_{r+s} is positive definite, then there exists a nonnegative self-adjoint operator A, that is a solution of the moment problem (1).*

4.2. **Stieltjes and Hamburger moment problems.** Through this subsection, we suppose that $\mathcal{T} = \{T_n\}_{n\geq 0}$ is a sequence (2) where $\mathcal{C}$ is considered as an s-GFS and H_{r+s} is a positive definite Hankel matrix.

Recall that the purpose of the K-moment problem associated to a given sequence of real numbers $\gamma = \{\gamma_n\}_{0\leq n\leq p}$ ($p \leq +\infty$), where K is a closed set of real numbers is the following. Find a positive measure μ such that,

$$\gamma_n = \int_K t^n d\mu(t) \quad , \quad \text{for every} \quad 0 \leq n \leq p. \tag{4}$$

There is a large amount of literature on the moment problems and its different formulation. Therefore, it has been studied by various methods and techniques. The problem (4) is called the *full moment problem* when $p = +\infty$ and the *truncated moment problem* for $p < +\infty$ (see [6, 7, 8, 11] for example).

Let $\tilde{A}$ be a self-adjoint extension of the densely defined operator A on $\mathbb{C}[X]$ given by $AX^n = X^{n+1}$. By the spectral theorem, there is a spectral measure $d\tilde{\mu}$ for $\tilde{A}$ with vector $[1] \in \mathcal{H}^T$, that is, so that for any bounded function of $\tilde{A}$,

$$< 1, f(\tilde{A})1 >= \int_K f(x)d\tilde{\mu}(x). \tag{5}$$

where $K = supp(\tilde{\mu})$. Since $[1] \in D(A^n) \subset D(\tilde{A}^n)$, expression (5) extends to polynomially bounded functions and by (3) we have,

$$T_n = \int_K x^n d\tilde{\mu}(x), \text{for any} \quad n = 0, 1, ..., r-1.$$

Therefore, we see that a self-adjoint extension of A yields a solution of the Hamburger moment problem. Moreover, a nonnegative extension of A has $supp(\tilde{\mu}) \subset [0, +\infty[$ and so yields a solution of the Stieltjes moment problem.

Using Proposition 1.2 of [17], we obtain.

Theorem 4.1. *1) A necessary and sufficient condition that there exists a measure μ solution of the truncated Hamburger moment problem associated with a sequence $\mathcal{T}$ is that the Hankel matrix H_{r+s} is positive definite or equivalently $detH_n > 0$ for $n = 0, 1, ..., r+s$.*

2) A necessary and sufficient condition that there exists a measure μ solution of the truncated Stieltjes moment problem associated with a sequence $\mathcal{T}$ is that the two matrices H_{r+s} and S_{r+s} are positive definite or equivalently $detH_n > 0$ and $detS_n > 0$ for $n = 0, 1, ..., r + s$.

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