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ON RECURSIVE RELATIONS AND MOMENT PROBLEMS

B. El Wahbi[1]
Département de Mathématiques et Informatique,
Faculté des Sciences de Tétouan, Tétouan, Morocco
and
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy,

M. Rachidi[2]
Département de Mathématiques et Informatique,
Faculté des Sciences de Rabat, Rabat, Morocco

and

E.H. Zerouali[3]
Département de Mathématiques et Informatique,
Faculté des Sciences de Rabat, Rabat, Morocco
and
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

Abstract

We investigate in this paper the link between the moment problem for sequences (1), the associated Jacobi matrices and the Padé approximants of the associated analytic functions. We generalize some classical results by providing simple proofs that use functional calculus.

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[1]E-mail: elwahbi@hotmail.com
[2]E-mail: rachidi@fsr.ac.ma
[3]Regular Associate of the Abdus Salam ICTP. E-mail: zerouali@fsr.ac.ma

1. Introduction

Let $a_0, \cdots, a_{r-1}$ ($r \geq 1$, $a_{r-1} \neq 0$) be real numbers and let $\mathcal{T} = \{T_n\}_{n \geq 0}$ be the sequence defined by the following recursive relation of order r

$$T_{n+1} = a_0 T_n + a_1 T_{n-1} + \cdots + a_{r-1} T_{n-r+1}, \ \text{for } n \geq r-1, \tag{1}$$

where $T_0, T_1, \cdots, T_{r-1}$ are the given initial values (or conditions). We will refer to such sequences $\mathcal{T} = \{T_n\}_{n \geq 0}$ as sequences (1). The polynomial $P(X) = X^r - a_0 X^{r-1} - a_1 X^{r-2} - \ldots.. - a_{r-1}$, called the characteristic polynomial of (1), together with the initial values are said to define the sequence $\mathcal{T}$. Note that if Q is any multiple of P, then Q also defines $\mathcal{T}$ provided that $T_0, T_1, \cdots, T_{degQ-1}$ are taken as initial conditions. As observed in [3] among all polynomials defining $\mathcal{T}$, there exists a unique polynomial denoted by $P_{\mathcal{T}}$ of minimal degree. This later is called the minimal polynomial of $\mathcal{T}$.

Let $\gamma = \{\gamma_n\}_{n \geq 0}$ be a sequence of complex numbers and K a closed subset of the complex plane. The purpose of the $K-$moment problem associated with γ is to find a positive measure μ such that

$$\gamma_n = \int_K t^n d\mu(t). \tag{2}$$

Since its introduction by Stieltjes in [14] for $K = \mathbb{R}^+$, the moment is a subject of an extensive literature. Particularly, Hamburger and Hausdorff had studied it for $K = \mathbb{R}$ and $K = [0, 1]$ respectively. The main idea in computing the measure μ, solution of (2) for a given sequence $\gamma = \{\gamma_n\}_{n \geq 0}$ is to extend the linear form defined on polynomials by

$$S_\gamma(X^n) = \gamma_n, \tag{3}$$

to a positive linear form on some Hilbert completion and to use the L^2-representation of Hilbert spaces. The construction of S_γ motivated different approaches to treat the moment problem. The continued fractions, the positivity of Hankel matrices and the decomposition of positive polynomials play a crucial role in this treatment [1, 4, 6, 7, 8, 12, 13, 14].

Let $\mathcal{H}$ be a separable Hilbert space and let $\gamma = \{\gamma_n\}_{n \geq 0}$ be a sequence of real numbers. The linear moment problem associated with γ entails finding a self-adjoint operator A and a non-vanishing vector $x \in \mathcal{H}$ satisfying

$$\gamma_n = < A^n x, x >, \ \text{for } n \geq 0. \tag{4}$$

Using the spectral representation of self-adjoint operators, one can easily show that the moment problems (2) and (4) are equivalent (see [5] for example).

The study of the moment problem for sequences (1) is motivated by the so-called "Truncated moment problems" treated by R. Curto and L. Fialkow in [6, 7]. It is known that the moment problem for sequence (1) is equivalent to the truncated moment problem and that a necessary condition for (2) (or for (4)) to have a solution is that P_γ has only simple roots. The moment problem (2) for sequences (1) correspond to the case where K is a finite set (see [3, 9, 10] for example). We will omit any reference to the set K in this paper.

We investigate in this paper the case of sequences (1). Section 2 is devoted to Jacobi matrices associated with moment sequences (1). We show that (4) has solution in finite dimensional spaces and that the associated Jacobi matrices are of finite order. The link with continued fractions is studied in section 3. Particularly, we prove that these fractions are terminating in this case. In section 4, we introduce the analytic function associated with a moment sequence (1). We give its Padé approximants and use the analytic functional calculus to provide some generalizations of results from [8]. We discuss in section 5 some moment problems arising from continued fractions and we give a new formula of the linear form associated with a terminating fraction.

2. Jacobi matrices associated with moment problems for sequences (1)

2.1. **Jacobi matrices associated with moment problems.** Let $\mathcal{T} = \{T_n\}_{n\geq 0}$ be a given sequence of real numbers. Define on $\mathbb{C}[X]$, the space of all polynomials, the bilinear form

$$\ll P, Q \gg = \sum_{n,m} \alpha_n \bar{\beta}_m T_{n+m}$$

with $P = \sum_n \alpha_n X^n$ and $Q = \sum_m \beta_m X^m$. (We suppose the upper limits in the sums are equal by completing by some zero coefficients if necessary.)
Observe that $\ll P, Q \gg = < P, HQ >$ where $<,>$ is the usual Euclidean inner product, $H = [T_{i+j}]_{i,j\geq 0}$ the Hankel matrix associated with $\mathcal{T}$.

If $H \geq 0$ then $\ll P, P \gg = < P, HP > \geq 0$ for all $P \in \mathbb{C}[X]$ and the bilinear form $\ll,\gg$ is an inner product on $\mathbb{C}[X]$. This defines a norm on $\mathbb{C}[X]$ when H is positive definite. Denote $\mathcal{H}^{\mathcal{T}}$ the Hilbert completion of $(\mathbb{C}[X], \| \, . \, \|)$ and $\bar{A}$ the unique extension to $\mathcal{H}^{\mathcal{T}}$ of the densely defined operator A on $\mathbb{C}[X]$ by $AX^n = X^{n+1}$. If $\bar{A}$ is self-adjoint, A is called essentially self-adjoint and $\bar{A}$ answers positively to (4). Otherwise, $\bar{A}$ has self-adjoint extensions and (4) is again solved (see [15]). In the orthonormal basis obtained by Gram-Schmidt process from $\{1, X, X^2, ...\}$, the self-adjoint extension $A^{\mathcal{T}}$ of A, solution of (4) has a semi-infinite Jacobi matrix of the form,

$$J_{\mathcal{T}} = \begin{pmatrix} b_0 & a_0 & 0 & 0 & . & . & . & . \\ a_0 & b_1 & a_1 & 0 & . & . & . & . \\ 0 & a_1 & b_2 & a_2 & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & a_{r-2} & . & . \\ . & . & . & . & a_{r-2} & b_{r-1} & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{pmatrix}$$

Hence the Hamburger moment problem and the theory of semi-infinite Jacobi matrices coincide.

2.2. **Finite Jacobi matrices.** Let $A \in \mathcal{L}(\mathcal{H})$ be a solution of the moment problem (4) associated with the sequence $\mathcal{T}$, where $\mathcal{H}$ is a given Hilbert space. For $x \in \mathcal{H}$ satisfying (4), set $\mathcal{H}_0 = Span\{x, Ax, ..., A^n x, ...\}$ the invariant subspace generated by x. By the recursive relation (1) we have $< P_{\mathcal{T}}(A)x, A^n x >= 0$ for every $n \geq 0$, particularly $\|P_{\mathcal{T}}(A)x\| = 0$. Hence $A^n x \in Span\{x, Ax, ..., A^{r-1}x\}$ for every $n \geq r$ and $\mathcal{H}_0$ is of finite dimensional. The study of moment problem for sequences (1) is then reduced to the case of finite dimensional Hilbert spaces. Such link has been observed and studied in [9]. More precisely, we have

Proposition 2.1. *Let $\mathcal{T}$ be a sequence (1). Then, (4) has a solution $A \in \mathcal{L}(\mathcal{H})$ for some Hilbert space if and only if it has a solution A on some r-dimensional Hilbert space.*

It is known that H is positive definite if and only if $det(H_n) > 0$ for all $n \geq 0$ where $H_n = [T_{i+j}]_{0 \leq i,j \leq n-1}$. In the case of sequences (1), we have $det(H_n) = 0$ whenever $n \geq r+1$. The process used in [15] is hence obstructed.

We provide in this section an alternative method to avoid this obstruction. We begin by proving an auxiliary result.

Lemma 2.1. *If $\mathcal{T} = \{T_n\}_{n \geq 0}$ is a sequence (1) with $P_{\mathcal{T}}$ the characteristic polynomial then $\ll P, Q \gg = 0$ for every $Q \in \mathbb{C}[X]$ if and only if $P \in (P_{\mathcal{T}})$, where $(P_{\mathcal{T}})$ is the ideal of $\mathbb{C}[X]$ generated by $P_{\mathcal{T}}$.*

Proof. The reverse implication is a direct consequence of the relation (1).
Suppose that $\ll P, Q \gg = 0$ for any $Q \in \mathbb{C}[X]$, then by writing $P = QP_{\mathcal{T}} + R$ and $R = \sum_{i=0}^{p} \alpha_i X^i$ where $\alpha_p \neq 0$ and $p < r$, we obtain $\ll R, X^n \gg = \sum_{i=0}^{p} \alpha_i T_{n+i} = 0$ for every $n \geq 0$. Hence $T_{n+1} = \sum_{i=0}^{p-1} a_i T_{n-p+i-1}$ with $a_i = (-\frac{\alpha_i}{\alpha_p})$, which implies that R is a characteristic polynomial of $\mathcal{T}$ with degree less than $r-1$. Contradiction.

An immediate consequence is the following corollary.

Corollary 2.1. *Let $P_1 = Q_1 P_{\mathcal{T}} + R_1, P_2 = Q_2 P_{\mathcal{T}} + R_2 \in \mathbb{C}[X]$, then*
$$\ll P_1, P_2 \gg = \ll R_1, R_2 \gg .$$

Set $\mathcal{H}^{(\mathcal{T})} = \mathbb{C}[X]/(P_{\mathcal{T}})$ and π the canonical surjection of $\mathbb{C}[X]$ onto $\mathbb{C}[X]/(P_{\mathcal{T}})$. Seeking simplicity, we will write $P = \pi(P)$. If H_r is positive definite, then the bilinear form $< P, Q >_{\mathcal{T}} := \ll \pi(P), \pi(Q) \gg$ for $P, Q \in \mathbb{C}[X]$, is an inner product on $\mathcal{H}^{(\mathcal{T})}$.

Let $A \in \mathcal{L}(\mathcal{H}^{(\mathcal{T})})$ given by $AX^j = X^{j+1}$ for $j = 1, 2, ..., r-1$. We have
$$< P, AQ >_{\mathcal{T}} = < P, S_r Q >_{\mathcal{T}}$$
where $S_r = [T_{i+j+1}]_{0 \leq i,j \leq r-1}$ and in particular,
$$< A^n 1 | 1 >= T_n \text{ for } n = 0, 1, ..., r-1.$$

On the other hand, $A^r 1 = X^r = \sum_{j=0}^{r-1} a_j X^{r-j-1}$, consequently we have

$$
\begin{aligned}
< A^r 1|1 > &= \sum_{j=0}^{r-1} a_j < X^{r-j-1}|1 > \\
&= \sum_{j=0}^{r-1} a_j T_{r-j-1} = T_r.
\end{aligned}
$$

By induction we establish that $< A^n 1|1 >= T_n$, for $n \geq 0$.

Thus, we have

Proposition 2.2. *Let $\mathcal{T} = \{T_n\}_{n\geq 0}$ be a sequence (1) with positive definite Hankel matrix H_r and $P_{\mathcal{T}}$ as a characteristic polynomial. Then there exist a $(degP_{\mathcal{T}})-$dimensional Hilbert space $\mathcal{H}^{(\mathcal{T})}$ and a self-adjoint operator A on $\mathcal{H}^{(\mathcal{T})}$, which provide a solution of the Stieltjes moment problem (4). Moreover, if S_r is positive definite, then $A \geq 0$, that yields a solution of the Hamburger moment problem.*

Let $\{P_0, P_1, ..., P_{r-1}\}$ be the orthonormal basis of $\mathcal{H}^{(\mathcal{T})}$, obtained from the basis $\{1, X, X^2, ..., X^{r-1}\}$ by the Gram-Schmidt process of the form

$$P_i(X) = X^i + lower\ \ order, \quad \text{for } i = 0, 1, ..., r-1.$$

The polynomial $XP_i(X)$ has an expansion in terms of $P_0, P_1, ..., P_{i+1}$. Therefore, we have $< XP_i, P_j >=< P_i, XP_j >= 0$, for $j > i+1$ and $j < i-1$. It follows that for suitable sequences, $\{a_n\}_{0\leq n\leq r-1}$ and $\{b_n\}_{0\leq n\leq r-1}$ (with $P_{-1}(X) = 0$ and $P_r(X) = 0$), we have

$$XP_n(X) = a_n P_{n+1}(X) + b_n P_n(X) + a_{n-1} P_{n-1}(X), \quad \text{for} \quad n = 0, 1, ..., r-1.$$

Thus, given $\mathcal{T} = \{T_n\}_{n\geq 0}$ a sequence (1), with positive definite Hankel matrix H_r, we can find a finite dimensional Hilbert space $\mathcal{H}^{(\mathcal{T})}$ (with dim $\mathcal{H}^{(\mathcal{T})} = r$), an orthonormal basis $\{P_0, P_1, ..., P_{r-1}\}$ some real numbers $b_0, b_1, ..., b_{r-1}$ and some positive numbers $a_0, a_1, ..., a_{r-2}$, such that the moment problem (4) is associated to the self-adjoint operator A on $\mathcal{H}^{(\mathcal{T})}$ with Jacobi matrix

$$
J_{\mathcal{T}} = \begin{pmatrix}
b_0 & a_0 & 0 & 0 & . & . \\
a_0 & b_1 & a_1 & 0 & . & . \\
0 & a_1 & b_2 & a_2 & . & . \\
. & . & . & . & . & . \\
. & . & . & . & . & a_{r-2} \\
. & . & . & . & a_{r-2} & b_{r-1} \\
. & & & & &
\end{pmatrix}
$$

Note that the matrix $J_{\mathcal{T}}$ determines uniquely the moments, since from the expansion $A^k P_0 = X^k = \sum_{j=0}^{k} c_{kj} P_j(X)$, for $k \geq 0$, it follows that

$$m_k =< A^k P_0|P_0 >= c_{k0}.$$

Let $x = \sum_{j=0}^{r-1} x_j e_j \in \mathcal{H}^{(\mathcal{T})}$ be an eigenvector of the matrix $J_{\mathcal{T}}$ associated with the eigenvalue λ. We obtain the following system of r linear equations.

$$
(5) \qquad \begin{cases} b_0x_0 + a_0x_1 & = \lambda x_0 \\ a_0x_0 + b_1x_1 + a_1x_2 & = \lambda x_1 \\ \quad . & \\ \quad . & \\ \quad . & \\ a_{r-3}x_{r-3} + b_{r-2}x_{r-2} + a_{r-1}x_{r-1} & = \lambda x_{r-2} \\ a_{r-2}x_{r-2} + b_{r-1}x_{r-1} & = \lambda x_{r-1}. \end{cases}
$$

By induction we derive,

$$
(6) \qquad x_j = P_j(\lambda)x_0, \ \ (j = 0, 1, ..., r-1)
$$

where $\{P_j\}_{0 \le j \le r-1}$ is the family of polynomials defined by $P_0 = 1$, $P_1(X) = \dfrac{X - b_0}{a_0}$ and the recursive relation

$$
a_jP_{j+1}(u) = (u - b_j)P_j(u) - a_{j-1}P_{j-1}(u), \ \ (j = 1, ..., r-2).
$$

To the system of equations (5), we associate the terminating fraction given by

$$
(7) \qquad \frac{1|}{|u - b_0} - \frac{a_0^2|}{|u - b_1} - \frac{a_1^2|}{|u - b_2} - ..... - \frac{a_{r-2}^2|}{|u - b_{r-1}},
$$

and the j^{th} convergent

$$
(8) \qquad \frac{A_j(u)}{B_j(u)} := \frac{1|}{|u - b_0} - \frac{a_0^2|}{|u - b_1} - \frac{a_1^2|}{|u - b_2} - ..... - \frac{a_{j-2}^2|}{|u - b_{j-1}},
$$

for $1 \le j \le r$. The family $\{B_j\}_{1 \le j \le r-1}$ of polynomials satisfies

$$
B_j(u) = a_0a_1...a_{j-1}P_j(u), \quad \text{for} \quad j = 1, 2, ..., r-1
$$

By setting $B_0 := 1$ and using the recursive relation involving the $P_j's$, we obtain

$$
(9) \qquad B_{j+1}(u) = (u - b_j)B_j(u) - a_j^2B_{j-1}(u),
$$

for $1 \le j \le r-2$. The denominator of the terminating fraction (7) is

$$
B_r(u) = (u - b_{r-1})B_{r-1}(u) - a_{r-2}^2B_{r-2}(u).
$$

The $B_j's$ (resp. $A_j's$) are defined by (8) provided to take $B_0 = 1, B_1(u) = u - b_0$ (resp. $A_0 = 0, A_1(u) = \frac{1}{a_0}$) as initial conditions. They are called the polynomials of the first kind (respectively the second kind).

Replacing x_{r-1} by the expression (6) in the last line of the system (5), we obtain that $B_r(\lambda) = 0$ for any λ in the spectrum of A. Hence B_r is the characteristic polynomial of the matrix $J_{\mathcal{T}}$ (see also [8], for example). On the other hand, from (1) easy computations give $P_{\mathcal{T}}(A) = 0$. As $degP_{\mathcal{T}} = degB_r$ and they are unital we obtain $P_{\mathcal{T}} = B_r$. Thus, we have the following proposition.

Proposition 3.1. *Under the preceding notations, B_r is the characteristic polynomial of the operator A. Particularly,*

- *B_r has only simple roots.*
- *$A \geq 0$ if and only if $Z(P_{\mathcal{T}}) \subset \mathbb{R}^+$, where $Z(P_{\mathcal{T}})$ is the set of zeros of $(P_{\mathcal{T}})$.*

Proposition 3.1 can be regarded as the solution of the Stieltjes moment problem.

4. Analytic function associated with moment problems

4.1. **Analytic functional calculus for sequences** (1)**.** For a moment sequence $\gamma = (\gamma_n)$, the formal series $f_\gamma(z) = \sum\limits_{n\geq 0}(-1)^n\gamma_n z^n$, that is associated canonically to the moment sequence γ, is called the Hamburger series in the case of the Hamburger moment problem. It is easy to check that

$$f_\gamma(z) = \int \frac{d\mu(t)}{1+tz}, \tag{10}$$

where μ is the measure solution of (2)(see [2], p. 208 for details).

Proposition 4.1. *Let γ be a moment sequence. Then, γ is a sequence (1) if and only if f_γ is a rational function. More precisely, we have $f_\gamma = \frac{P}{Q}$, where Q is a polynomial of degree r with only simple roots.*

Proof. Suppose that γ is a sequence (1). By [3] or [9], we have $\mu = \sum\limits_{n=0}^{r-1}\rho_n\delta_{z_n}$. Hence,

$$f_\gamma(z) = \int \frac{d\mu(t)}{1+tz} = \sum_{n=0}^{r-1}\frac{\rho_n}{1+z_n z} = \frac{P(z)}{Q(z)}, \tag{11}$$

with $Q(z) = \prod\limits_{n=0}^{r-1}(1+z_n z)$ and f_γ is a rational function. Conversely, write $f_\gamma = \frac{P}{Q}$ and set $Q(z) = 1 + a_0 z + ... + a_{r-1}z^r$. Using an Euclidean division, one can suppose without loss of generality that $deg(P) < deg(Q)$, we get

$$P(z) = \sum_{n\geq 0}(-1)^n\gamma_n z^n(1 + a_0 z + ... + a_{r-1}z^r).$$

Thus, we have

$$(-1)^n\gamma_n + (-1)^{n-1}a_0\gamma_{n-1} + (-1)^{n-2}a_1\gamma_{n-2} + ... + (-1)^{n-r}a_{r-1}\gamma_{n-r} = 0,$$

for $n \geq r$, or equivalently

$$\gamma_n = a_0\gamma_{n-1} - a_1\gamma_{n-2} + ... + (-1)^r a_{r-1}\gamma_{n-r}. \tag{12}$$

The desired result is obtained.

Corollary 4.1. *Under the notations of Proposition 4.1, we have*

$$\frac{1}{z}f_\gamma(\frac{1}{z}) = \frac{A_r(-z)}{B_r(-z)}$$

Proof. Proposition 4.1 implies that $\frac{1}{z}f_\gamma(\frac{1}{z})$ is rational. By writing $\frac{A_j(z)}{B_j(z)} = \sum_{p=0}^{\infty} \frac{(-1)^p c_p^j}{z^{p+1}}$ at infinity for $1 \le j \le r$, we have by [8], $c_p^j = \gamma_p$ for $p \le j$. Particularly, $c_p^r = \gamma_p$ for $p \le r$. Therefore, γ and $(c_p^r)_{p\ge 0}$ are sequences (1), associated with the same initial conditions and with the same characteristic polynomial, the required assertion is proved.

4.2. **Padé approximants and analytic functional calculus.** Given $f(z) = \sum_{n\le 0} \gamma_n z^n$ a power serie. We denote by $[L/M]$ the Padé approximant to f given by,

$$[L/M](f) = \frac{P_L}{Q_M},$$

where P_L and Q_M are polynomials of degree at most L and M (respectively), satisfying

$$f(z) - \frac{P_L(z)}{Q_M(z)} = o(z^{L+M+1}).$$

It is known that Padé approximant, if it exists, is unique under the assumption that P_L and Q_M have no common roots and $Q_M(0) = 1$ (For further information, see [2]).

If γ is a sequence (1), f_γ is rational and we have $f_\gamma = [r-1/r](f) := \frac{P_{r-1}}{Q_r} = [L/M](f)$ for every $L \ge r-1$ and $M \ge r$. The relation between Padé approximants of f_γ and the terminating fraction associated with γ is given by,

Proposition 4.2. *Let B_r and A_r given by (8). Then, we have*

- $B_r(-z) = z^r Q_r(\frac{1}{z})$.
- $A_r(-z) = z^{r-1} P_{r-1}(\frac{1}{z})$.

Proof. By Proposition 3.1 the polynomial $B_r(z)$ is associated with γ and by corollary 4.1, $(-z)^r Q_r(\frac{-1}{z})$ also defines γ. As $B_r(z)$ and $z^r Q_r(\frac{-1}{z})$ are unital with the same degree, we get the first assumption. The second assertion is derived from

$$\begin{aligned} \frac{A_r(-z)}{B_r(-z)} &= \frac{1}{z} f_\gamma(\frac{1}{z}) \\ &= \frac{1}{z}\frac{P_{r-1}(z)}{Q_r(z)} \\ &= \frac{z^{r-1}P_{r-1}(\frac{1}{z})}{z^r Q_r(\frac{1}{z})}. \end{aligned}$$

The following lemma will be used to prove the main result on functional calculus.

Lemma 4.1. *Let A be as in (4) and $z \in \mathbb{C}$ such that $|z| > \|A\|$, then*

$$< (A - zI)^{-1}x, x > = \frac{-1}{z} f_\gamma(\frac{-1}{z}) = \frac{A_r(z)}{B_r(z)} \tag{13}$$

Proof.

$$\begin{aligned} < (A - zI)^{-1}x, x > &= \frac{1}{z} < (\frac{1}{z}A - I)^{-1}x, x > \\ &= \frac{-1}{z} \sum_{n\ge 0} < A^n x, x > (\frac{1}{z})^n \\ &= \frac{-1}{z} \sum_{n\ge 0} \gamma_n (\frac{1}{z})^n = \frac{1}{z} f_\gamma(\frac{1}{z}) \end{aligned}$$

The second equality is trivial from Proposition 4.2.

Using this lemma we obtain.

Proposition 4.3. *For any entire function f, denote $f(A)$ the operator defined by the Riesz functional calculus. Then*

$$< f(A)x, x >= \sum_{z_j \in \sigma(A)} f(z_j) \frac{A_r(z_j)}{{B_r}'(z_j)},$$

where $\sigma(A)$ is the spectrum of A.

Proof. For $R > \|A\|$, let $\Gamma_R = \{z \in \mathbb{C} \,:\, |z| = R\}$. We have

$$f(A) = \frac{1}{2i\pi} \int_{\Gamma_R} f(z)(A - zI)^{-1} dz.$$

Then

$$\begin{aligned} < f(A)x, x > &= \tfrac{1}{2i\pi} \int_{\Gamma_R} < f(z)(A - zI)^{-1}x, x > dz \\ &= \tfrac{1}{2i\pi} \int_{\Gamma_R} f(z) < (A - zI)^{-1}x, x > dz \\ &= \tfrac{1}{2i\pi} \int_{\Gamma_R} f(z) \tfrac{A_r(z)}{B_r(z)} dz & (by (13)) \\ &= \sum_{z_j \in Z(B_r) = \sigma(A)} f(z_j) \tfrac{A_r(z_j)}{{B_r}'(z_j)} & (\text{ by the residue theorem}). \end{aligned}$$

Lemma 4.2. *Let S_γ be the associated linear form with the linear moment sequence γ, then for any entire function f, we have*

$$S_\gamma(f) =< f(A)x, x > \tag{14}$$

where A and x are given by (4).

Proof. It is clear that (14) is valid for polynomials, the formula is obtained by density.

For f holomorphic, we denote by $L_u(f)$ the holomorphic function defined as follows,

$$L_u(f)(z) = \begin{cases} \frac{f(z) - f(u)}{z - u} & \text{if } \; z \neq u \\ f'(u) & \text{if } \; z = u \end{cases}$$

The following proposition unifies some results of [8].

Proposition 4.4. *For any holomorphic function f, we have*

$$S_\gamma(L_u(fB_r)) = f(u)A_r(u)$$

Proof. As in the proof of proposition 4.3, we have

$$\begin{aligned} S_\gamma(L_u(fB_r)) &= < (L_u(fB_r)(A)x, x > \\ &= \tfrac{1}{2i\pi} \int_{\Gamma_R} \tfrac{f(u)B_r(u) - f(z)B_r(z)}{u - z} \tfrac{A_r(z)}{B_r(z)} dz \\ &= \tfrac{1}{2i\pi} \int_{\Gamma_R} \tfrac{f(u)B_r(u)}{u - z} \tfrac{A_r(z)}{B_r(z)} dz - \tfrac{1}{2i\pi} \int_{\Gamma_R} \tfrac{f(z)}{u - z} A_r(z) dz \\ &= f(u)A_r(u) + \tfrac{f(u)B_r(u)}{2i\pi} \int_{\Gamma_R} \tfrac{A_r(z)}{(u - z)B_r(z)} dz \\ &= f(u)A_r(u) - f(u)B_r(u)([\tfrac{A_r(u)}{B_r(u)} - \sum_{z_j \in Z(B_r)} \tfrac{A_r(z_j)}{{B_r}'(z_j)} \tfrac{1}{u - z_j}] = 0) \\ &= f(u)A_r(u). \end{aligned}$$

We derive the two following corollaries. For $f \equiv 1$ in proposition 4.4, we have

Corollary 4.2. *([8] Theorem 1 (17)) Under the same notations of Proposition 4.4, we have*

$$S_\gamma(L_u(B_r)) = A_r(u).$$

Combining Proposition 4.3, Lemma 4.2 and the above Corollary, we obtain,

Corollary 4.3. *([8] page 6) For any polynomial P, we have*

$$S_\gamma(P) = \sum_{z_j \in Z(B_r)} S_\gamma(L_{z_j}(B_r)) \frac{P(z_j)}{B_r'(z_j)} = \sum_{z_j \in Z(B_r)} A_r(z_j) \frac{P(z_j)}{B_r'(z_j)}.$$

5. Moment problems associated with limited continued fractions

In this section, we use the preceding section to shed some light on the moment problem arising from the terminating fraction (7).

Consider the limited Jacobi fraction,

$$\frac{1|}{|u-b_0} - \frac{a_0^2|}{|u-b_1} - \frac{a_1^2|}{|u-b_2} - \cdots\cdots - \frac{a_{r-2}^2|}{|u-b_{r-1}}, \tag{15}$$

where $b_0, b_1, ..., b_{r-1}$ are reals and $a_0, a_1, ..., a_{r-2}$ are non-vanishing real numbers. Let

$$J = \begin{pmatrix} b_0 & a_0 & 0 & 0 & . & . \\ a_0 & b_1 & a_1 & 0 & . & . \\ 0 & a_1 & b_2 & a_2 & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & a_{r-2} \\ . & . & . & . & a_{r-2} & b_{r-1} \end{pmatrix},$$

be the Jacobi matrix associated with (15) and consider the operator A associated with the matrix J defined on a $r-$dimensional Hilbert space $\mathcal{H}$. For $x \in \mathcal{H}$ a non-vanishing vector, the sequence $T_n(x) =< A^n x, x >$ is clearly a moment sequence (1) defined by its r initial conditions and the characteristic polynomial of the matrix J.

Given $T_0, ..., T_{r-1}$ some real numbers, does it exist $x \in \mathcal{H}$ such that $T_j = T_j(x)$ for $0 \le j \le r-1$?

Suppose such x exists and write $x = \sum_{j=0}^{r-1} \rho_j x_j$, where $\{x_j : j = 0, ..., r-1\}$ is the orthonormal basis of eigenvectors of A associated with the eigenvalues $\{z_j : j = 0, ..., r-1\}$. Therefore, we have

$$< A^n x, x >= \sum_{j=0}^{r-1} \rho_j^2 z_j^n = T_n.$$

Complete $\{T_n\}_{0 \le n \le r-1}$ to a sequence (1) $\gamma = \{\gamma_n\}_{n \ge 0}$ defined by $\gamma_j = T_j$ for $j = 0, 1, ..., r-1$ and the characteristic polynomial $P(X) = \prod_{j=0}^{r-1} (X - z_j)$. By Theorem 3 of [8], the sequence γ is associated with a positive linear form if and only if the Hankel matrix $H_r = [\gamma_{i+j}]_{0 \le i,j \le r-1}$ is positive definite.

Let $A_j(u)$ and $B_j(u)$ defined as in (8). It is known that there exists a linear functional S, in the ring $\mathbb{R}[u]$, which orthogonalizes the B_j's. That is

$$S(B_iB_j) = 0, \quad \text{for} \quad 0 \leq i < j \leq r-1.$$

Under the additional assumption

$$S(B_n^2) = a_0a_1...a_n, \quad \text{for} \quad 0 \leq n \leq r-1,$$

S is unique and satisfies the following property.

Theorem 5.1. *There exist $\lambda_0, ..., \lambda_{r-1}$ positives such that*

$$S(P) = \sum_{j=0}^{r-1} \lambda_j P(z_j),$$

for every polynomial P.

In view of Corollary 4.3 we have $\lambda_j = \frac{A_r(z_j)}{B_r{}'(z_j)} > 0$. Thus

Proposition 5.1. *Under the same notations above, we have*

$$S(X^n) =< A^n x, x >= \int_{I\!R} t^n d\mu(t),$$

with $x = \sum_{j=0}^{r-1} \sqrt{\frac{A_r(z_j)}{B_r{}'(z_j)}} x_j$ *and* $\mu = \sum_{j=0}^{r-1} \frac{A_r(z_j)}{B_r{}'(z_j)} \delta_{z_j}$.

Moreover, B_r is the characteristic polynomial of $\{S(X^n)\}_{n \geq 0}$.

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