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ON THE LARGEST ANALYTIC SET FOR CYCLIC OPERATORS

Abdellatif Bourhim[1]
Département de Mathématiques, Université Mohammed V,
B.P.1014, Rabat, Morocco
and
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

Abstract

In this paper, we prove that for every cyclic operator on a complex Hilbert space $\mathcal{H}$,

$$B_a(T) = \overline{S_{T^*}},$$

where $B_a(T)$ is the set of all analytic bounded point evaluations for T and S_{T^*} is the analytic residuum of T^*. Some consequences of this equality are discussed, along with several examples. Also, some results due to L. R. Williams are derived by shorter proofs. Furthermore, we show that two densely similar $\mathcal{DW}$-operators have equal spectra, compression spectra and approximate point spectra. We will also show that they have equal essential spectra under additional conditions.

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[1]E-mails: bourhim@ictp.trieste.it; abourhim@fsr.ac.ma

1. Introduction and main results

In the present paper all Banach spaces are complex. Let $\mathcal{X}$ be a Banach space and let $\mathcal{L}(\mathcal{X})$ denote the algebra of all linear bounded operators on $\mathcal{X}$. For an operator $T \in \mathcal{L}(\mathcal{X})$, let T^* denote its adjoint acting on the dual space $\mathcal{X}^*$, $\sigma(T)$ its spectrum, $\rho(T) := \mathbb{C}\backslash\sigma(T)$ its resolvent set, $\sigma_p(T)$ its point spectrum, $\sigma_{ap}(T)$ its approximate point spectrum, $\Gamma(T)$ its compression spectrum, $\sigma_e(T)$ its essential spectrum, $\rho_e(T) := \mathbb{C}\backslash\sigma_e(T)$ its Fredholm set, $\ker T$ its kernel and $\mathrm{ran}T$ its range. The *analytic residuum* S_T of an operator $T \in \mathcal{L}(\mathcal{X})$ is the open set of complex numbers $\lambda \in \mathbb{C}$ for which there exist a non-zero analytic function $\phi : \mathcal{V} \to \mathcal{X}$ on some open neighborhood $\mathcal{V}$ of λ so that

$$(T - \mu)\phi(\mu) = 0 \text{ for all } \mu \in \mathcal{V}.$$

For a subset F of $\mathbb{C}$, we let $\overline{F} := \{\overline{z} : z \in F\}$, $\mathrm{int}(F)$, $\mathrm{cl}(F)$ and $\mathrm{Fr}(F)$ denote the conjugate set, the interior, the closure and the boundary of F, respectively.

Let T be a cyclic linear bounded operator on a Hilbert space $\mathcal{H}$ with cyclic vector x that is the finite linear combinations of the vectors $x,\ Tx,\ T^2x, \dots$ are dense in $\mathcal{H}$. A complex number $\lambda \in \mathbb{C}$ is said to be a *bounded point evaluation* for T if there is a constant $M > 0$ such that

$$|p(\lambda)| \leq M\|p(T)x\|$$

for every polynomial p. The set of all bounded point evaluations of T will be denoted by $B(T)$. Note that it follows from the Riesz Representation Theorem that $\lambda \in B(T)$ if and only if there is a unique vector $k(\lambda) \in \mathcal{H}$ such that $p(\lambda) = \langle p(T)x, k(\lambda)\rangle$ for every polynomial p. An open subset O of $\mathbb{C}$ is said to be an *analytic set* for T if it is contained in $B(T)$ and if for every $y \in \mathcal{H}$, the complex valued function $\widehat{y}$ defined on $B(T)$ by $\widehat{y}(\lambda) = \langle y, k(\lambda)\rangle$, is analytic on O. The largest analytic set for T will be denoted by $B_a(T)$ and every point of it will be called an *analytic bounded point evaluation* for T.

Roughly speaking, the purpose of the present paper is to establish the following results.

Theorem 1.1. *Suppose that $\mathcal{H}$ is a Hilbert space. For every cyclic operator $T \in \mathcal{L}(\mathcal{H})$, we have $B_a(T) = \overline{S_{T^*}}$.*

The proof and some consequences of this theorem are the contents of Section 3. The next theorem generalizes many results namely, theorem 2 of [6] and theorem 2.1 of [19]. We shall need to introduce some notions from the local spectral theory. Suppose that $\mathcal{X}$ is a Banach space. Let $T \in \mathcal{L}(\mathcal{X})$; the *local resolvent set* $\rho_T(x)$ of T at a point $x \in \mathcal{X}$ is the union of all open subsets $U \subset \mathbb{C}$ for which there is an analytic $\mathcal{X}$-valued function ϕ on U such that

$$(T - \lambda)\phi(\lambda) = x \text{ for every } \lambda \in U.$$

The complement in $\mathbb{C}$ of $\rho_T(x)$ is called the *local spectrum* of T at x and will be denoted by $\sigma_T(x)$. The operator T is said to have the *single-valued extension property* if zero is the unique element

x of $\mathcal{X}$ for which $\sigma_T(x) = \emptyset$. For a closed subset F of $\mathbb{C}$, let $\mathcal{X}_T(F) := \{x \in \mathcal{X} : \sigma_T(x) \subset F\}$ be the corresponding analytic spectral subspace; it is a $T-$hyperinvariant subspace, generally non-closed in $\mathcal{X}$. The operator T is said to satisfy *Dunford's Condition* (C) if for every closed subset F of $\mathbb{C}$, the linear subspace $\mathcal{X}_T(F)$ is closed. Recall also that the operator T is said to possess *Bishop's property* (β) if for every open subset U of $\mathbb{C}$ and for every sequence $(f_n)_n$ of analytic $\mathcal{X}-$valued functions on U, the convergence $(T-\lambda)f_n(\lambda) \longrightarrow 0$ in the topology of uniform convergence on compact subsets of U should always entail the convergence to 0 of the sequence $(f_n)_n$ in the same topology. It is known that the Bishop's property (β) implies the Dunford's condition (C) and it turns out that the single-valued extension property follows from the Dunford's condition (C). J. G. Stampfli [16] and M. Radjabalipour [13] have shown that hyponormal operators satisfy the Dunford's Condition (C), and Putinar [11] has shown that hyponormal operators, $M-$hyponormal operators and more generally subscalar operators have Bishop's property (β). For thorough presentations of the local spectral theory, we refer to the monographs [7] and [10].

Theorem 1.2. *Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces. Let $T \in \mathcal{L}(\mathcal{X})$, $S \in \mathcal{L}(\mathcal{Y})$ and let $X : \mathcal{X} \to \mathcal{Y}$ be a bounded linear transformation with dense range such that $XT = SX$. If S has Dunford's condition (C), then $\sigma(S) \subset \sigma(T)$.*

The following result gives a complete description of the Fredholm set of cyclic Hilbert operators which have the single-valued extension property. We shall require some notations and definitions. Let $\mathcal{X}$ be a Banach space. Recall that an operator $T \in \mathcal{L}(\mathcal{X})$ is said to be *semi-Fredholm* if $\mathrm{ran} T$ is closed and at least one of the subspaces $\ker T$ and $\ker T^*$ is finite dimensional. The class of all semi-Fredholm operators will be denoted by $S\Phi(\mathcal{X})$. Note that if $T \in S\Phi(\mathcal{X})$ then $T^* \in S\Phi(\mathcal{X}^*)$. The non-empty compact set

$$\sigma_{lre}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin S\Phi(\mathcal{X})\}$$

is called the *Wolf spectrum* of T. Its complement in $\mathbb{C}$, $\rho_{s-F}(T) := \mathbb{C} \backslash \sigma_{lre}(T)$, is called the *semi-Fredholm domain* of T. If $T \in \mathcal{L}(\mathcal{X})$, we set

$$\gamma(T) := \{\lambda \in \sigma_p(T) \cap \Gamma(T) : \mathrm{ran}(T-\lambda) \text{ is closed and } \dim\ker\big(T-\lambda\big) = 1\}.$$

Proposition 1.3. *Suppose that $\mathcal{H}$ is a Hilbert space. For every cyclic operator $T \in \mathcal{L}(\mathcal{H})$ which has the single-valued extension property, we have*

$$\begin{aligned} \rho_e(T) &= \rho_{s-F}(T) = \{\lambda \in \mathbb{C} : \mathrm{ran}(T-\lambda) \text{ is closed}\} \\ &= \Gamma(T) \backslash \sigma_{ap}(T) \cup \rho(T) \cup \gamma(T). \end{aligned}$$

Moreover, if the linear subspace, generated by $\{k(\lambda) : \lambda \in B_a(T)\}$, is dense in $\mathcal{H}$ then

$$\rho_e(T) = \Gamma(T) \backslash \sigma_{ap}(T) \cup \rho(T).$$

Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces. Recall that two operators $T \in \mathcal{L}(\mathcal{X})$ and $S \in \mathcal{L}(\mathcal{Y})$ are said to be *densely similar* if there exist two bounded linear transformations $X : \mathcal{X} \to \mathcal{Y}$ and $Y : \mathcal{Y} \to \mathcal{X}$ having dense range such that

$$XT = SX \text{ and } TY = YS.$$

Moreover, if the operators X and Y are also injective then we say that T and S are *quasisimilar.* It is well known that the quasisimilarity preserves spectrum and essential spectrum of subnormal operators, hyponormal operators, $M-$hyponormal operators and operators having Bishop's property (β) (see [8], [12], [14] and [19]). However, the proofs in the case of cyclic subnormal operators, hyponormal operators and $M-$hyponormal operators follows from the fact that those operators answered positively the L. R. Williams Question A of [18] (see [5], [14] and [19]). On the other hand, it shown in [5] that if $T \in \mathcal{L}(\mathcal{H})$ is a cyclic operator and possess Bishop's property (β) then $B_a(T) = \Gamma(T)\backslash\sigma_{ap}(T)$ if and only if $B_a(T) \cap \sigma_p(T) = \emptyset$. However, M. Mbekhta and E. H. Zerouali proved recently that the answer of L. R. Williams question is positive for all cyclic Hilbert-operators which have Bishop's property (β). Those results and the preceding results motivate us to introduce the new so-called $\mathcal{DW}$-operators and show that the spectrum and some of its parts of $\mathcal{DW}$-operators are preserved under densely similarity.

Definition 1.4. *Suppose that $\mathcal{H}$ is a Hilbert space. A cyclic operator $T \in \mathcal{L}(\mathcal{H})$ is said to be a $\mathcal{DW}$-operator if the following hold.*

(a) *T satisfy the Dunford's condition (C).*
(b) *$B_a(T) = \Gamma(T)\backslash\sigma_{ap}(T)$.*

The class of $\mathcal{DW}$-operators on $\mathcal{H}$ will be denoted by $\mathcal{DW}(\mathcal{H})$.

Note that no one of the conditions (a) and (b) in the definition 1.4 implies the other (see [5]). On the other hand, it is shown in [4] that the condition (b) is not satisfied for every cyclic Hilbert operator. Immediate examples of $\mathcal{DW}$-operators are provided by all cyclic subnormal operators, hyponormal operators, $M-$hyponormal operators and more generally by all cyclic Hilbert-operators which have Bishop's property (β). However, we do not know whether $\mathcal{DW}$-operators have Bishop's property (β).

Proposition 1.5. *Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces. If $T \in \mathcal{DW}(\mathcal{H}_1)$ and $S \in \mathcal{DW}(\mathcal{H}_2)$ are densely similar then the following hold.*

(a) $\sigma(T) = \sigma(S)$.
(b) $\Gamma(T) = \Gamma(S)$.
(c) $\sigma_{ap}(T) = \sigma_{ap}(S)$.
(d) $\sigma_e(T) = \sigma_e(S)$ *if and only if* $\gamma(T) = \gamma(S)$.

2. Preliminary

In the sequel, we shall need the following elementary lemmas. The proofs of lemma 2.1 and lemma 2.2 can be found in [3], [8] and [17].

Lemma 2.1. *Let x be a cyclic vector of an operator T on a Hilbert space $\mathcal{H}$. The following are equivalent.*

(*a*) $\lambda \in B(T)$.

(*b*) $\lambda \in \Gamma(T)$.

(*c*) $\ker\big((T-\lambda)^*\big)$ *is one dimensional.*

Moreover, if $\lambda \in B(T)$ then for every eigenvector u of T^* corresponding to the eigenvalue $\overline{\lambda}$, we have $u = \overline{\langle x, u\rangle} k(\lambda)$.

From lemma 2.1, we note that for every cyclic Hilbert operator T, we have

$$\sigma_{_{lre}}(T) = \{\lambda \in \mathbb{C} : \operatorname{ran}(T-\lambda) \text{ is not closed } \}$$

since for every $\lambda \in \mathbb{C}$, $\dim\big(\ker(T^*-\lambda)\big) \leq 1$.

Lemma 2.2. *Let T be a cyclic operator on a Hilbert space $\mathcal{H}$ and let O be an open set of $\mathbb{C}$ contained in $B(T)$. Then $O \subset B_a(T)$ if and only if the function $\lambda \longmapsto \|k(\lambda)\|$ is bounded on compact subsets of O.*

Lemma 2.3. *Suppose that $\mathcal{X}$ is a Banach space. The analytic residuum $S_{_T}$ of an operator $T \in \mathcal{L}(\mathcal{X})$ is the set of complex numbers $\lambda \in \mathbb{C}$ for which there exists an analytic function $\phi : \mathcal{V} \to \mathcal{X}$ without zeros on some open neighborhood $\mathcal{V}$ of λ so that*

$$(T-\mu)\phi(\mu) = 0 \textit{ for all } \mu \in \mathcal{V}.$$

Proof. Let $\lambda \in S_{_T}$, then there is an open neighborhood $\mathcal{V}$ of λ and a non-zero analytic function $\phi : \mathcal{V} \to \mathcal{X}$ on $\mathcal{V}$ such that

$$(T-\mu)\phi(\mu) = 0 \text{ for all } \mu \in \mathcal{V}.$$

Since the zeros of ϕ are isolated, there is an open disc $\mathcal{W}$ centered at λ contained in $\mathcal{V}$, an integer $n \geq 0$ and an analytic function $\psi : \mathcal{W} \to \mathcal{X}$ such that $\phi(\mu) = (\mu-\lambda)^n\psi(\mu)$ and $\psi(\mu) \neq 0$ for every $\mu \in \mathcal{W}$. And so, for every $\mu \in \mathcal{W}$, we have

$$\begin{aligned} 0 &= (T-\mu)\phi(\mu) \\ &= (T-\mu)\Big[(\mu-\lambda)^n\psi(\mu)\Big] \\ &= (\mu-\lambda)^n(T-\mu)\psi(\mu). \end{aligned}$$

Therefore, for every $\mu \in \mathcal{W}$ such that $\mu \neq \lambda$, we have

$$(T-\mu)\psi(\mu) = 0.$$

By continuity we have $(T-\mu)\psi(\mu) = 0$ for every $\mu \in \mathcal{W}$. Thus, the proof is complete.

Remark 2.4. *For every operator $T \in \mathcal{L}(\mathcal{X})$, we have $S_T \subset \text{int}(\sigma_p(T))$. Moreover, T has the single-valued extension property if and only if $S_T = \emptyset$.*

Lemma 2.5. *Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces. Let $T \in \mathcal{L}(\mathcal{X})$, $S \in \mathcal{L}(\mathcal{Y})$ and let $X : \mathcal{X} \to \mathcal{Y}$ be an injective bounded linear transformation such that $XT = SX$. Then*

$$\sigma_p(T) \subset \sigma_p(S) \text{ and } S_T \subset S_S.$$

Proof. It is clear that $\sigma_p(T) \subset \sigma_p(S)$. Let $\mathcal{V}$ be a non-empty open set of $\mathbb{C}$ and let $\phi : \mathcal{V} \to \mathcal{X}$ be a non-zero analytic function on $\mathcal{V}$ such that

$$(T - \mu)\phi(\mu) = 0 \text{ for every } \mu \in \mathcal{V}.$$

For every $\lambda \in \mathbb{C}$, we have $X(T - \lambda) = (S - \lambda)X$ since $XT = SX$. And so,

$$(S - \mu)X\phi(\mu) = 0 \text{ for every } \mu \in \mathcal{V}.$$

On the other hand, $X\phi : \mathcal{V} \to \mathcal{Y}$ is a non-zero analytic function on $\mathcal{V}$ since X is injective. Therefore, $\mathcal{V} \subset S_S$. Thus, $S_T \subset S_S$.

We end this Section by quoting, without proof, the following theorem from [2].

Theorem 2.6. *Suppose that $\mathcal{X}$ is a Banach space. For every $T \in S\Phi(\mathcal{X})$, the following are equivalent.*

- (a) $0 \in S_T$.
- (b) 0 *is a limit point of* $\sigma_p(T)$.

3. Analytic bounded point evaluations for cyclic operators

We now prove theorem 1.1.

Proof of theorem 1.1. Let $\lambda \in S_{T^*}$; it follows from lemma 2.3 that there exist an open neighborhood $\mathcal{V}$ of λ and an analytic $\mathcal{H}-$valued function $\phi : \mathcal{V} \to \mathcal{H}$ without zeros such that

$$(T^* - \mu)\phi(\mu) = 0 \text{ for every } \mu \in \mathcal{V}.$$

Hence, $\mathcal{V} \subset \sigma_p(T^*) = \overline{B(T)}$. Therefore, $k(\overline{\mu}) = \frac{\phi(\mu)}{\langle x, \phi(\mu)\rangle}$ for every $\mu \in \mathcal{V}$. In particular, the function $k : \overline{\mathcal{V}} \to \mathcal{H}$ is continuous. So, the function $\lambda \longmapsto \|k(\lambda)\|$ is bounded on compact subsets of $\overline{\mathcal{V}}$. By lemma 2.2, $\overline{\mathcal{V}} \subset B_a(T)$. Therefore, $\overline{S_{T^*}} \subset B_a(T)$.

Conversely, let $O := \overline{B_a(T)}$. First let us see that the function $\phi : \lambda \longmapsto k(\overline{\lambda})$ is a non-zero analytic $\mathcal{H}$-valued function on O. To do this, we have to show that for every $y \in \mathcal{H}$, the function $\lambda \longmapsto \langle \phi(\lambda), y\rangle$ is differentiable on O. Indeed, let λ_0 be a fixed point of O. For every $y \in \mathcal{H}$, we have

$$\begin{aligned}
\lim_{\lambda\to\lambda_0}\frac{\langle\phi(\lambda),y\rangle-\langle\phi(\lambda_0),y\rangle}{\lambda-\lambda_0} &= \lim_{\lambda\to\lambda_0}\frac{\langle k(\overline{\lambda}),y\rangle-\langle k(\overline{\lambda_0}),y\rangle}{\lambda-\lambda_0}\\
&= \lim_{\lambda\to\lambda_0}\frac{\overline{\widehat{y}(\overline{\lambda})}-\overline{\widehat{y}(\overline{\lambda_0})}}{\lambda-\lambda_0}\\
&= \overline{\left[\lim_{\lambda\to\lambda_0}\frac{\widehat{y}(\overline{\lambda})-\widehat{y}(\overline{\lambda_0})}{\overline{\lambda}-\overline{\lambda_0}}\right]}\\
&= \overline{\widehat{y}\,'(\overline{\lambda_0})}.
\end{aligned}$$

On the other hand, it is obvious that

$$(T^*-\lambda)\phi(\lambda)=0 \text{ for every } \lambda\in O.$$

Hence, $O=\overline{B_a(T)}\subset S_{T^*}$. And the proof is complete.

Remark 3.1. *Let $T\in\mathcal{L}(\mathcal{H})$ be a cyclic operator. Since it is shown in the proof of theorem 1.1 that the function $\phi:\lambda\longmapsto k(\overline{\lambda})$ is analytic on $\overline{B_a(T)}$, it follows that the $\mathcal{H}$-valued function $k:\lambda\longmapsto k(\lambda)$ is continuous on $B_a(T)$.*

Remark 3.2. *Let $T\in\mathcal{L}(\mathcal{H})$ be a cyclic operator. Since the conjugate element of every $\lambda\in B(T)$ is a simple eigenvalue of T^* with a corresponding eigenvector $k(\lambda)$ (see lemma 2.1), it follows from [2, theorem 1.9] that*

$$\begin{aligned}
B_a(T) &= \{\lambda\in B(T):\sigma_{T^*-\overline{\lambda}}(k(\lambda))=\emptyset\}\\
&= \{\lambda\in B(T):\sigma_{T^*}(k(\lambda))=\emptyset\}.
\end{aligned}$$

As immediate consequences of theorem 1.1, we derive the following corollaries.

Corollary 3.3. *Let $T\in\mathcal{L}(\mathcal{H})$ be cyclic operator. Then $B_a(T)=\emptyset$ if and only if T^* has the single-valued extension property. In particular, if T is a cyclic normal operator then $B_a(T)=\emptyset$.*

Corollary 3.4. *Let $T\in\mathcal{L}(\mathcal{H})$ be a cyclic operator and let $\mathcal{V}$ be an open set of $\mathbb{C}$ such that $\mathcal{V}\subset B(T)$. The following are equivalent.*

(*a*) *$\mathcal{V}$ is an analytic set for T.*

(*b*) *The function $k:\mathcal{V}\to\mathcal{H}$ is continuous.*

Proof. Suppose that the function $k:\mathcal{V}\to\mathcal{H}$ is continuous, then it is clear that the function $\lambda\longmapsto\|k(\lambda)\|$ is bounded on compact sets of $\mathcal{V}$. Hence, by lemma 2.2, $\mathcal{V}$ is an analytic set for T. The converse follows from remark 3.1.

Here, we mention that theorem 1.1 allows us to derive by shorter proofs some results from [18].

Corollary 3.5. *Let $T\in\mathcal{L}(\mathcal{H})$ be a cyclic operator. Then $B_a(T)$ does not depend on the choice of the cyclic element for T.*

Proposition 3.6. *Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces, $T \in \mathcal{L}(\mathcal{H}_1)$, and $S \in \mathcal{L}(\mathcal{H}_2)$. If T and S are densely similar cyclic operators then*

$$B(T) = B(S) \text{ and } B_a(T) = B_a(S).$$

Proof. Since T and S are densely similar, there exist two bounded linear transformations $X : \mathcal{H}_1 \to \mathcal{H}_2$ and $Y : \mathcal{H}_2 \to \mathcal{H}_1$ having dense range such that

$$XT = SX \text{ and } TY = YS.$$

Hence, X^* and Y^* are injective bounded linear transformations and

$$T^*X^* = X^*S^* \text{ and } Y^*T^* = S^*Y^*.$$

By lemma 2.5, we have

$$\sigma_p(T^*) = \sigma_p(S^*) \text{ and } S_{T^*} = S_{S^*}.$$

And the proof is complete.

Proposition 3.7. *For every cyclic operator $T \in \mathcal{L}(\mathcal{H})$, we have*

$$\Gamma(T)\backslash\sigma_{ap}(T) \subset \text{int}(B(T)) \cap \rho_{s-F}(T) \subset B_a(T).$$

Proof. Note that for every $\lambda \notin \sigma_{ap}(T)$, the operator $T - \lambda$ is injective and has closed range. Hence, $\Gamma(T)\backslash\sigma_{ap}(T) \subset \rho_{s-F}(T)$. Since $\sigma(T) = \Gamma(T) \cup \sigma_{ap}(T)$ then $\sigma(T)\backslash\sigma_{ap}(T) = \Gamma(T)\backslash\sigma_{ap}(T)$. On the other hand, $\sigma(T)\backslash\sigma_{ap}(T) = \text{int}\big(\sigma(T)\big)\backslash\sigma_{ap}(T)$ since the boundary of the spectrum of T is contained in $\sigma_{ap}(T)$. So, $\Gamma(T)\backslash\sigma_{ap}(T)$ is open subset of $\mathbb{C}$ contained in $B(T)$. Hence, $\Gamma(T)\backslash\sigma_{ap}(T) \subset \text{int}(B(T))$. Therefore, $\Gamma(T)\backslash\sigma_{ap}(T) \subset \text{int}(B(T)) \cap \rho_{s-F}(T)$.

If $\lambda \in \text{int}(B(T)) \cap \rho_{s-F}(T)$, then $\overline{\lambda}$ is a limits point of $\sigma_p(T^*)$ and $(T^* - \overline{\lambda}) \in S\Phi(\mathcal{H})$. Hence, by theorem 2.6, $\overline{\lambda} \in S_{T^*}$. Therefore, $\text{int}(B(T)) \cap \rho_{s-F}(T) \subset \overline{S_{T^*}}$. By theorem 1.1, the desired result holds.

Following the proof of proposition II.7.12 of [8], we have

Proposition 3.8. *If $T \in \mathcal{L}(\mathcal{H})$ is a cyclic operator, then every connected component of $B_a(T)$ is simply connected. In particular, every connected component of S_{T^*} is simply connected.*

Proof. Let γ be a closed curve in $B_a(T)$, we show that the inside of γ, $\text{ins}(\gamma)$, is contained in $B_a(T)$. It follows from lemma 2.2 that there is a positive constant M such that $\|k(\lambda)\| \leq M$ for every $\lambda \in \gamma$. And so, for every polynomial p and for every $\lambda \in \gamma$, we have

$$|p(\lambda)| \leq M\|p(T)x\|.$$

By the Maximum Modulus Principal, for every polynomial p, we have

$$|p(\lambda)| \leq M\|p(T)x\| \text{ for every } \lambda \in \text{ins}(\gamma).$$

This implies that $\text{ins}(\gamma) \subset B(T)$ and $\sup\limits_{\lambda \in \text{ins}(\gamma)} \|k(\lambda)\| \leq M$. It follows from lemma 2.2 that $\text{ins}(\gamma) \subset B_a(T)$. And the proof is complete.

Remark 3.9. *If $T \in \mathcal{L}(\mathcal{H})$ is not a cyclic operator, then it is not true that every connected component of S_{T^*} is simply connected.*

Example 1. Let $\mathcal{H}$ be the Hilbert space of all bilateral sequences $x := (x_n)_{n\in\mathbb{Z}}$ such that $\|x\|^2 = \sum\limits_{n\in\mathbb{Z}} |x_n|^2 < \infty$. Let T be the injective bilateral weighted shift on $\mathcal{H}$ given by

$$Te_n = \begin{cases} 2e_{n+1} & \text{if } n \geq 0 \\ \frac{1}{2}e_{n+1} & \text{if } n < 0. \end{cases}$$

where $(e_n)_{n\in\mathbb{Z}}$ is the standard orthonormal basis of $\mathcal{H}$. It follows from theorem 9 of [15] that

$$\{\lambda \in \mathbb{C} : \frac{1}{2} < |\lambda| < 2\} \subset \sigma_p(T^*) \subset \{\lambda \in \mathbb{C} : \frac{1}{2} \leq |\lambda| \leq 2\}.$$

Hence, T has no cyclic elements (see proposition 42 of [15]) and

$$S_{T^*} \subset O := \{\lambda \in \mathbb{C} : \frac{1}{2} < |\lambda| < 2\}.$$

Conversely, consider the following non-zero analytic $\mathcal{H}-$ valued function ϕ defined on O by

$$\phi(\lambda) := \sum_{n\in\mathbb{Z}} 2^{-|n|}\lambda^n.$$

It is clear that $(T^* - \lambda)\phi(\lambda) = 0$ for every $\lambda \in O$. Hence,

$$O = \{\lambda \in \mathbb{C} : \frac{1}{2} < |\lambda| < 2\} \subset S_{T^*}.$$

Therefore, $S_{T^*} = \{\lambda \in \mathbb{C} : \frac{1}{2} < |\lambda| < 2\}$ is a connected open set but not a simply connected. In fact T is hyponormal operator and

$$S_{T^*} = \overline{S_{T^*}} = \Gamma(T)\backslash\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : \frac{1}{2} < |\lambda| < 2\} \text{ (see [15])}.$$

Example 2. Let $\mathcal{H} = L^2_a(G)$ denote the *Bergman space* of analytic functions on $G := \{\lambda \in \mathbb{C} : 1 < |\lambda| < 2\}$ that are square integrables with respect to area measure; it is a Hilbert space. The *Bergman operator* S for G is the operator multiplication by z on $L^2_a(G)$; i.e., $(Sf)(z) = zf(z)$ for every $f \in L^2_a(G)$. It is a bounded subnormal operator but not cyclic. We show that $S_{S^*} = G$. It follows from theorem II.8.5 part (a) of [8] that

$$\sigma_p(S^*) \subset \sigma(S^*) = \{\lambda \in \mathbb{C} : 1 \leq |\lambda| \leq 2\}.$$

Hence, $S_{S^*} \subset \text{int}(\sigma_p(S^*)) \subset G = \{\lambda \in \mathbb{C} : 1 < |\lambda| < 2\}$. On the other hand, it follows from theorem II.8.5 part (c) of [8] that $G \subset \sigma_p(S^*)$. By theorem 2.6 and theorem II.8.5 part (c) of [8], we deduce that $G \subset S_{S^*}$. Thus, $S_{S^*} = G$ is a connected open set but not simply connected.

4. Densely similarity of $\mathcal{DW}$-operators

In this Section, we begin by proving theorem 1.2.

Proof of theorem 1.2. We will first prove that $\sigma_S(Xx) \subset \sigma_T(x)$ for every $x \in \mathcal{X}$. Let $x \in \mathcal{X}$. If $Xx = 0$, then $\sigma_S(Xx) = \emptyset \subset \sigma_T(x)$. Thus, we may suppose that $Xx \neq 0$. Let $\lambda_0 \in \rho_T(x)$. So, there is an open neighborhood $\mathcal{V}$ of λ_0 and a non-zero analytic $\mathcal{X}$-valued function $\phi : \mathcal{V} \to \mathcal{X}$ such that

$$(T - \lambda)\phi(\lambda) = x \text{ for every } \lambda \in \mathcal{V}.$$

Since $X(T - \lambda) = (S - \lambda)X$ for every $\lambda \in \mathbb{C}$, it follows that

$$(S - \lambda)X\phi(\lambda) = Xx \text{ for every } \lambda \in \mathcal{V}.$$

It is clear that the analytic $\mathcal{Y}$-valued function $X\phi : \mathcal{V} \to \mathcal{Y}$ is without zeros since $Xx \neq 0$. Hence, $\mathcal{V} \subset \rho_S(Xx)$; thus, $\sigma_S(Xx) \subset \sigma_T(x)$ for every $x \in \mathcal{X}$. And so, for every $x \in \mathcal{X}$, we have $Xx \in \mathcal{Y}_S\Big(\sigma(T)\Big)$. Since X has dense range and S has Dunford's condition (C), it follows that $\mathcal{Y}_S\Big(\sigma(T)\Big) = \mathcal{Y}$. Therefore, $\sigma(S) = \bigcup\limits_{y \in \mathcal{Y}} \sigma_S(y) \subset \sigma(T)$.

Corollary 4.1. *Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces. Let $T \in \mathcal{L}(\mathcal{X})$ and $S \in \mathcal{L}(\mathcal{Y})$ are densely similar. If T and S have Dunford's condition (C), then $\sigma(S) = \sigma(T)$.*

Remark 4.2. *In theorem 1.2, the inclusion $\sigma(S) \subset \sigma(T)$ can be strict. Indeed, J. Agler, E. Franks and D. A. Herrero* [1] *gave an example of an operator T which is quasisimilar with the unweighted unilateral shift such that $\sigma(T)$ does not coincide with the unit disc. Hence, the Dunford's condition (C) or the Bishop's property (β) is not preserved under quasisimilarity.*

Proof of proposition 1.3. We have $\dim\big(\ker(T - \lambda)^*\big) \leq 1$ for every $\lambda \in \mathbb{C}$ since T is cyclic operator. On the other hand, for every $\lambda \in \Gamma(T)\backslash\sigma_{ap}(T)$, $\text{ran}(T - \lambda)$ is closed and $\dim\big(\ker(T - \lambda)\big) = 0$. Hence, $\Gamma(T)\backslash\sigma_{ap}(T) \subset \rho_e(T)$. And so,

$$\Gamma(T)\backslash\sigma_{ap}(T) \cup \rho(T) \cup \gamma(T) \subset \rho_e(T) \subset \rho_{s-F}(T) \subset \{\lambda \in \mathbb{C} : \text{ran}(T - \lambda) \text{ is closed}\}.$$

Conversely, let $\lambda \notin \Gamma(T)\backslash\sigma_{ap}(T) \cup \rho(T)$ such that $\text{ran}(T - \lambda)$ is closed. If $\lambda \notin \Gamma(T)$, then $T - \lambda$ is surjective. And so, it follows from proposition 1.2.10 of [10] that $T - \lambda$ is invertible since T has the single-valued extension property, contradiction with the assumption, and hence $\lambda \in \Gamma(T)$. On the other hand, by corollary 2.7 of [2], we have $\dim\ker\big(T - \lambda\big) \leq 1$. Since $\lambda \in \sigma_{ap}(T)$ and $\text{ran}(T - \lambda)$ is closed, $\lambda \in \sigma_p(T)$. Hence, $\dim\ker\big(T - \lambda\big) = 1$. Therefore, $\lambda \in \gamma(T)$. Thus,

$$\rho_e(T) = \Gamma(T)\backslash\sigma_{ap}(T) \cup \rho(T) \cup \gamma(T).$$

Now, suppose that the linear subspace, generated by $\{k(\lambda) : \lambda \in B_a(T)\}$, is dense in $\mathcal{H}$ and let $\lambda \in \Gamma(T)$ such that there is $y \in \mathcal{H}$ for which $Ty = \lambda y$; we show that $y = 0$. For every $\mu \in B_a(T)$, we have,

$$\begin{aligned}\lambda\widehat{y}(\mu) &= \langle Ty, k(\mu)\rangle \\ &= \langle y, T^*k(\mu)\rangle \\ &= \mu\widehat{y}(\mu).\end{aligned}$$

Hence, the analytic function $\widehat{y}$ is identically zero on $B_a(T)$; and so, $y = 0$. Therefore, $\gamma(T) = \emptyset$; thus, the proof is complete.

Corollary 4.3. *Suppose that T is a cyclic operator on a Hilbert space $\mathcal{H}$. If T has the single-valued extension property and $B_a(T) = \Gamma(T)\backslash\sigma_{ap}(T)$, then all points of $\gamma(T)$ are isolated and $\gamma(T) \subset \mathrm{Fr}\big(B_a(T) \cup \rho(T)\big)$.*

Proof. By theorem 2.6, every point of $\gamma(T)$ is not a limit point of $\sigma_p(T)$. In particular, all points of $\gamma(T)$ are isolated. Note that,

$$\gamma(T) \subset \mathrm{Fr}\big(B_a(T) \cup \rho(T)\big) \iff \gamma(T) \subset \mathrm{cl}\big(B_a(T) \cup \rho(T)\big),$$

since $\Big[B_a(T) \cup \rho(T)\Big] \cap \gamma(T) = \emptyset$. Suppose that,

$$\gamma(T) \not\subset \mathrm{cl}\big(B_a(T) \cup \rho(T)\big).$$

It follows that,

$$\rho_e(T) \cap \Big[\mathbb{C}\backslash\mathrm{cl}\big(B_a(T) \cup \rho(T)\big)\Big] = \gamma(T) \cap \Big[\mathbb{C}\backslash\mathrm{cl}\big(B_a(T) \cup \rho(T)\big)\Big]$$

is a non-empty open subset of $\mathbb{C}$ contained in $\sigma_p(T)$. By theorem 2.6, this is impossible since T has the single-valued extension property. And the proof is complete.

Proof of proposition 1.5. The properties (a) and (b) follow respectively from theorem 1.2 and proposition 3.6. It follows from proposition 3.6 and definition 1.4 that

$$\Gamma(T)\backslash\sigma_{ap}(T) = \Gamma(S)\backslash\sigma_{ap}(S).$$

Hence,

$$\sigma(T)\backslash\sigma_{ap}(T) = \sigma(S)\backslash\sigma_{ap}(S).$$

And so, $\sigma_{ap}(T) = \sigma_{ap}(S)$. And the part (c) is proved.

The last part follows from proposition 1.3 and the properties (a), (b) and (c).

Remark 4.4. (a) *Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces. Note that if $T \in \mathcal{L}(\mathcal{X})$ and $S \in \mathcal{L}(\mathcal{Y})$ are densely similar and possess Bishop's property (β) but not necessarily cyclic, then the following hold*

(a) $\sigma(T) = \sigma(S)$.

(b) $\Gamma(T) = \Gamma(S)$.

(c) $\sigma_e(T) = \sigma_e(S)$.

But T and S can have unequal approximate point spectra.

(b) Following the first part of the proof of theorem 1.1, we note that for every operator T not necessarily cyclic on a Hilbert space $\mathcal{H}$, we have $\Gamma(T)\backslash\sigma_{ap}(T) \subset \overline{S_{T^*}}$. This inclusion fails, in general, to be equality even for subnormal operator.

Example. Let $(e_n)_{n\geq 0}$ be an orthonormal basis of a separable Hilbert space $\mathcal{H}$. Let

$$Ue_n = e_{n+1} \ \forall n \geq 0,$$

be the unweighted unilateral shift on $\mathcal{H}$. For every integer $n \geq 0$, denote by S_n the unilateral weighted shift on $\mathcal{H}$ given by

$$S_n e_k = \begin{cases} \frac{1}{n+1} e_1 & \text{if } k = 0 \\ \\ e_{k+1} & \text{if } k > 0 \end{cases}$$

Denote by $T = U \oplus U \oplus U \oplus ... \in \mathcal{L}(\widehat{\mathcal{H}})$ and $S = S_0 \oplus S_1 \oplus S_2 \oplus ... \in \mathcal{L}(\widehat{\mathcal{H}})$ where

$$\widehat{\mathcal{H}} = \sum_{k=0}^{+\infty} \oplus \mathcal{H}_k$$

is the orthogonal direct sum of the Hilbert spaces $(\mathcal{H}_k)_{k\geq 0}$ with $\mathcal{H}_k = \mathcal{H}$ for each integer $k \geq 0$. It is clear that T and S are subnormal operators since U and every S_n are subnormal unilateral weighted shifts on $\mathcal{H}$. On the other hand, each S_n is similar to U (see theorem 2 [15]); it follows then from theorem 2.5 of [9] that T and S are quasisimilar subnormal operators. We claim that the following hold.

(a) $\sigma_{ap}(T) \neq \sigma_{ap}(S)$. Hence, quasisimilarity is s weaker relation than the similarity.

(b) $\Gamma(T)\backslash\sigma_{ap}(T) \subsetneqq \overline{S_{T^*}}$ or $\Gamma(S)\backslash\sigma_{ap}(S) \subsetneqq \overline{S_{S^*}}$. Hence, T and S are not cyclic operators.

Indeed,

(a) For every $k \geq 0$, let

$$\widehat{e_k} = (0, 0, ..., \underbrace{e_0}_{k+1^{th} \text{ position}}, 0, ...).$$

We have, $\|S\widehat{e_k}\| = \frac{1}{k+1}$ for every $k \geq 0$. Hence, $0 \in \sigma_{ap}(S)$. On the other hand, $0 \notin \sigma_{ap}(T)$ since T is an isometry. Thus, part (a) is proved.

(b) Suppose that

$$\Gamma(T)\backslash\sigma_{ap}(T) = \overline{S_{T^*}} \text{ and } \Gamma(S)\backslash\sigma_{ap}(S) = \overline{S_{S^*}}.$$

It follows from lemma 2.5 that

$$\Gamma(T)\backslash\sigma_{ap}(T) = \Gamma(S)\backslash\sigma_{ap}(S).$$

As before, we have $\sigma_{ap}(T) = \sigma_{ap}(S)$, contradiction with part (a).

Remark 4.5. *We quoted the above example from [6]. In fact one can show that*

$$S_{T^*} = S_{S^*} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}.$$

Hence, it is clear that $\Gamma(S)\backslash\sigma_{ap}(S) \subsetneqq \overline{S_{S^*}}$ *since* $0 \in \sigma_{ap}(S)$. *And so, the cyclicity of subnormal operators or hyponormal operators* R *is a necessary condition in* [5, *theorem 2.1*] *and in* [17, *theorem 1.1*] *in order to have*

$$\overline{S_{R^*}} = \Gamma(R)\backslash\sigma_{ap}(R).$$

We end this paper by two interesting problems.

Problem 1. Does every $\mathcal{DW}$-operator posses Bishop's property (β)?

Problem 2. Does two densely similar $\mathcal{DW}$-operators have equal essential spectra?

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References

[1] **J. Agler, E. Franks and D. A. Herrero**, *Spectral Pictures of Operators Quasisimilar to the Unilateral shift*, J. Reine Angew. Math. 422(1991) 1–20.

[2] **P. Aiena and O. Monsalve**, *Operators which do not have the single valued extension property*, J. Math. Anal. Appl. 250, no. 2(2000) 435–448.

[3] **A. Bourhim**, *Bounded Point Evaluations and Local Spectral Theory*, 1999-2000 DICTP Diploma thesis, (available on: http:/www.ictp.trieste.it/~pub_off (IC/2000/118)).

[4] **A. Bourhim, C. E. Chidume and E. H. Zerouali**, *Bounded Point Evaluations For Cyclic Operators and Local Spectra*, Proc. Amer. Math. Soc., (to appear).

[5] **A. Bourhim and E. H. Zerouali**, *Bounded Point Evaluations For Cyclic Hyponormal Operators* (submitted).

[6] **S. Clary**, *Equality of Spectra of Quasisimilar Hyponormal Operators*, Proc. Amer. Math. Soc., 53(1975) 88-90.

[7] **I. Colojoara and C. Foias**, *Theory of Generalized Spectral Operators*, Gordon and Breach, New York 1968.

[8] **J. B. Conway**, *The Theory of Subnormal Operators*, volume 36 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, R.I 1991.

[9] **T. B. Hoover**, *Quasi-similarity of Operators*, Illinois J. Math. 16(1972) 678–686.

[10] **K. B. Laursen and M. M. Neumann**, *An Introduction to Local Spectral Theory*, London Mathematical Society Monograph New Series 20 (2000).

[11] **M. Putinar**, *Hyponormal Operators are Subscalar*, J. Operator Theory, 12(1984) 385-395.

[12] **M. Putinar**, *Quasisimilarity of Tuples with Bishop's Property* (β), Integral Equations and Operator Theory 15(1992) 1047-1052.

[13] **M. Radjabalipour**, *Ranges of Hyponormal Operators*, Illinois J. Math., 21(1977) 70-75.

[14] **M. Raphael**, *Quasisimilarity and Essential Spectra for Subnormal Operators*, Indiana Univ. Math. J. 31, 243-246 (1982).

[15] **A. L. Shields**, *Weighted Shift Operators and Analytic Function Theory*, in Topics in Operator Theory, Mathematical Surveys, N0 13 (ed. C. Pearcy), pp. 49-128. American Mathematical Society, Providence, Rhode Island 1974.

[16] **J. G. Stampfli**, *A Local Spectral Theory for Operators. V: Spectral Subspaces for Hyponormal Operators*, Trans. Amer. Math. Soc., 21(1976) 285-296.

[17] **T. T. Trent**, $H^2(\mu)$ *Spaces and Bounded Point Evaluations*, Pac. J. Math., 80(1979) 279-292.

[18] **L. R. Williams**, *Bounded Point Evaluations and Local Spectra of Cyclic Hyponormal Operators*, Dynamic Systems and Applications 3(1994) 103-112.

[19] **L. Yang**, *Hyponormal and subdecomposable Operators*, J. Functional Anal. 112(1993) 204-217.