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GENERALIZED BOLTZMANN EQUATIONS FOR ON-SHELL PARTICLE PRODUCTION IN A HOT PLASMA

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ABSTRACT

A novel refinement of the conventional treatment of Kadanoff–Baym equations is suggested. Besides the Boltzmann equation another differential equation is used for calculating the evolution of the non-equilibrium two-point function. Although it was usually interpreted as a constraint on the solution of the Boltzmann equation, we argue that its dynamics is relevant to the determination and resummation of the particle production cut contributions. The differential equation for this new contribution is illustrated in the example of the cubic scalar model. The analogue of the relaxation time approximation is suggested. It results in the shift of the threshold location and in smearing out of the non-analytic threshold behaviour of the spectral function. Possible consequences for the dilepton production are discussed.

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1 Introduction

The real photon production and dilepton production in a hot plasma are intensively discussed subjects of the current literature. One aspect of this discussion is the behaviour of the production rate near zero invariant mass of the lepton pair. Numerical simulations [1] find a vanishing rate, while higher loop perturbative [2] and HTL-improved [3] calculations find diverging result in this limit. The possibility that the two-loop order enhancement may be a consequence of infrared (IR) singularities was studied in recent papers [4], where also an effective description was suggested to overcome the problem. In [5] a resummation method was worked out for ladder diagrams, and it was applied to the photon and gluon production rate.

Resummation in perturbation theory becomes unavoidable, in general, when we encounter IR divergences that enhance the contribution of higher order loops, thus overriding the coupling constant suppression. A well known example is the on-mass-shell IR divergence of the propagator due to a radiative mass shift. This divergence is cured by resumming 1PI diagrams with the help of Schwinger–Dyson (SD) equations. This example also illustrates that resummation may be effective only for some specific momentum range, while in other regions its effect might remain negligible. From this point of view, also finite-temperature Debye screening is a consequence of IR divergences.

In non-equilibrium systems there appear new sources of IR divergences. When working in real time one recognizes that higher loops may give contributions that grow as some power of the elapsed time ($\sim t^{\alpha n}$ where n is the loop order). This would clearly restrict the applicability of perturbation theory to rather short times, so we need a resummation. It turns out [6] that resummation leads to Boltzmann equations describing the temporal evolution of the propagator. In Fourier space the corresponding phenomenon is the appearance of pinch singularities [7, 8] in two-particle intermediate states with vanishing momentum difference. The resummation of these divergences, which involves ladder diagrams, leads, again, to Boltzmann equations [9].

In this paper we shall argue that there are further momentum ranges where we can observe pinch singularities. When these momentum ranges are relevant, we have to perform appropriate resummation also there. To see the origin of these divergences we have to analyse the corresponding pinch singularities, as will be done in Section 4. The core effect of pinch singularities is that in a pair of propagators one of them can force the other to its mass shell; for instance in $G_R(p)G_A(q) = G_R(p)i\varrho(q) + G_R(p)G_R(q)$ the first term yields the singular effect, the second turns out to be regular. If we parametrise the momenta carried by the particles as $p = Q+K/2$ and $q = Q - K/2$, then we can find for the contributions two formally equivalent expressions:

$$G_R(p)\varrho(q) \to \frac{\delta((Q-K/2)^2-m^2)}{(Q+K/2)^2-m^2} = \frac{\delta((Q-K/2)^2-m^2)}{2QK} = \frac{1}{2}\frac{\delta((Q-K/2)^2-m^2)}{Q^2+K^2/4-m^2}. \quad (1)$$

(The Landau prescription is omitted to simplify this introductory argument.) Thus the product of the propagators becomes divergent either when $2QK = 0$ (this case of the resummation leads to the Boltzmann equation), *or* when $Q^2 + K^2/4 - m^2 = 0$. This second is the new kinematical situation studied in the present paper.

After identifying the IR divergences, we have to design an appropriate resummation scheme for them. In order to perform ladder resummation we can use the linearized Kadanof–Baym (KB) equations [9]. These come from a procedure (for details see Sections 3.3 and 4), which consists of writing two SD equations for each of the Wigner–Fourier-transformed propagators, next taking their sum and difference, and finally linearizing them. They read, schematically:

$$2QK\bar{G}(Q,K) = (\mathcal{A}_- * \bar{G})(Q,K), \qquad 2(Q^2 + \frac{K^2}{4} - m^2)\bar{G}(Q,K) = (\mathcal{A}_+ * \bar{G})(Q,K), \tag{2}$$

where $\mathcal{A}_\pm$ denote the linear operators acting on the propagators. The equations are diagonal in K, which is the momentum associated with the average coordinate of the propagator. The coefficient of $\bar{G}$ vanishes on the left-hand side in regions of pinch singularity, while the expression is finite on the right-hand side. The perturbative solution diverges at every order in any such regime, but the complete solution of the differential equations is regular. This is the resummation realized with the help of SD equations. Using other resummation techniques will give the same result, at least to leading order of the pinch-singular terms [9].

Our new strategy is to suggest that in a general kinematical region where pinch singularities occur we have to solve simultaneously *both* KB equations for the propagator. It differs from the usual approach, where one treats the sum of the SD equations, for small K, as a constraint, which effectively puts Q on the mass shell. But how can one use two different solutions for the same propagator? As will be shown in Section 3, at one loop level these contributions are relevant to different kinematical regions: in Boltzmann equations we must have $k_0 < 2q_0$, in the other case (called normal cut regime in the following) $k_0 > 2q_0$ must be true. At one loop the exactly known complete solution is thus the sum of the respective KB equations. We propose to maintain this additivity feature also for the resummed propagator. We expect that the kinematical complementarity of the two solutions is a good approximation while the two-particle threshold is well separated from the regime where Landau-damping dominates.

While in our approach no constraint equations play any role, in the leading order pinch-singular terms we can solve the SD equations with an ansatz implying the missing constraints (see Sections 4.1 and 4.2). In this way the usual solutions for the Boltzmann equations survive, we just have supplementary contributions to the propagator from the complete solution.

Computing expectation values of composite operators involves integration over Q, which is the momentum associated with the relative coordinate. After this integration, more insight is offered into the physics of the different contributions. At one loop order the discontinuity of the Boltzmann term $\sim 1/(QK)$ gives a non-zero result for $K^2 < 0$ – this is the Landau damping regime. The discontinuity of the other contribution $\sim 1/(Q^2 + K^2/4 - m^2)$ is non-zero when $K^2 > 4m^2$ where $2m$ is the threshold value for particle production – this is a normal cut regime. The solution of the Boltzmann equation in the relaxation time approximation generates a finite imaginary part to k_0; this is why a quantity, conserved at tree level, can decay exponentially due to radiative effects. In a kind of similar relaxation time approximation, higher loop pinch singularities provide imaginary *and* real contributions to the denominator $Q^2+K^2/4-m^2$, which results in a shift of the threshold location and in a "smoothing" of the threshold singularity.

In this paper, the effects described above in qualitative terms will be demonstrated in the cubic scalar model. Beyond its simplicity it shows some similarity with fermion–gauge systems as far as the perturbation theory is concerned. We choose a simple observable, $\langle[\varphi^2(x), \varphi^2(0)]\rangle$; this is the simplified analogue of $\langle[j_\mu(x), j^\mu(0)]\rangle$, which plays an important role in the theory of dilepton production [10]. In Section 2 we introduce the model and our observable. In Section 3 we perform a thorough one-loop calculation, identify the different kinematical regions, and examine the way SD equations and KB equations reproduce the one-loop result. In Section 4 we turn to higher loops. We show the origin of pinch singularities, and apply KB equations to resum these singularities in different kinematic regions. We also solve the corresponding equations in relaxation time approximation. In Section 5 we make some comments on the relevance of our results to the dilepton production in fermion–gauge systems. Section 6 then closes the paper by presenting our conclusions and an outlook.

2 The model

The cubic scalar model has the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{h}{6}\varphi^3. \tag{3}$$

We are interested in the equilibrium expectation value of the composite spectral function:

$$\langle[\varphi^2(x), \varphi^2(0)]\rangle\Big|_{1PI,\langle\varphi\rangle=0}. \tag{4}$$

This quantity corresponds to the expectation value $\langle[j_\mu(x), j^\mu(0)]\rangle$ in the fermion–gauge systems, which plays a central role in dilepton production [10]. The subscripts 1PI and $\langle\varphi\rangle = 0$ will be omitted in the following. The quantity of phenomenological interest is the discontinuity of $iR(K)$, which is the Fourier transform of

$$iR(x) = \Theta(x_0)\langle[\varphi^2(x), \varphi^2(0)]\rangle. \tag{5}$$

$R(x)$, on the other hand, is the linear response function of $\langle\varphi^2(x)\rangle$ to the mass modification of the Lagrangian characterized by the linear operator $\mathcal{P}$:

$$\mathcal{L}_{\mathcal{P}} = \mathcal{L} - \frac{1}{2}\mathcal{P}\varphi^2 \qquad \Rightarrow \qquad \delta\left\langle\varphi^2\right\rangle(K) = \frac{1}{2}R(K)\mathcal{P}(K) + \dots, \tag{6}$$

where $\delta f = f - f|_{\mathcal{P}=0}$ for any quantity. We will define $\langle\varphi^2(x)\rangle$ via point splitting; in the Keldysh and in the R/A formalisms (cf. Appendix A) we should compute

$$\left\langle\varphi^2(x)\right\rangle = \frac{1}{2}\lim_{y\to x}\left[\langle\varphi(x)\varphi(y)\rangle + \langle\varphi(y)\varphi(x)\rangle\right] = \frac{1}{2}\lim_{y\to x}\left[iG_{21}(x,y) + iG_{12}(x,y)\right] = \lim_{y\to x} iG_{rr}(x,y). \tag{7}$$

In Fourier space

$$\left\langle\varphi^2\right\rangle(K) = \int dx\, e^{iKx}\lim_{y\to x}\int \text{đ}p\text{đ}q\, e^{-ipx+iqy} iG_{rr}(p,q) = \int \text{đ}Q\, i\bar{G}_{rr}(Q,K), \tag{8}$$

($\bar{G}(Q,K)$ denotes the Wigner–Fourier transform of $G(x,y)$, which has the usual two-variable Fourier-transform $G(p,q)$, cf. Appendix A)). We introduce the quantity $R(Q,K)$ defined through the relation

$$\delta i\bar{G}_{rr}(Q,K) = \frac{1}{2}R(Q,K)\mathcal{P}(K) \qquad \Rightarrow \qquad R(K) = \int \text{đ}Q\, R(Q,K). \tag{9}$$

3 One-loop perturbation theory

In leading order the only interaction we have to take into account is $-\mathcal{P}\varphi^2/2$. In the Keldysh and R/A formalism (cf. Appendix A) this piece of the Lagrangian has the form

$$-\frac{1}{2}\mathcal{P}\varphi^2 \longrightarrow -\frac{1}{2}\mathcal{P}\left(\varphi_1^2-\varphi_2^2\right) = -\mathcal{P}\varphi_r\varphi_a. \tag{10}$$

Linear response to $\mathcal{P}$ yields

$$\delta\bar{G}_{rr}(Q,K) = \left[\bar{G}^0_{ra}(Q+\frac{K}{2})\bar{G}^0_{rr}(Q-\frac{K}{2}) + \bar{G}^0_{ar}(Q-\frac{K}{2})\bar{G}^0_{rr}(Q+\frac{K}{2})\right]\mathcal{P}(K), \tag{11}$$

where G^0_{ab} means free equilibrium propagators. The coefficient of $\mathcal{P}$ identifies $-iR(Q,K)/2$. Its discontinuity reads

$$\operatorname*{Disc}_{k_0} iR(Q,K) = 2\left[\varrho^0(Q+\frac{K}{2})i\bar{G}^0_{rr}(Q-\frac{K}{2}) - \varrho^0(Q-\frac{K}{2})i\bar{G}^0_{rr}(Q+\frac{K}{2})\right]. \tag{12}$$

Inserting the free equilibrium expressions ($\omega_{\mathbf{q}}^2 = \mathbf{q}^2+m^2$)

$$\varrho^0(Q) = 2\pi\mathrm{sgn}(q_0)\delta(q_0^2-\omega_{\mathbf{q}}^2), \qquad iG^0_{rr}(Q) = 2\pi\mathrm{sgn}(q_0)\left(\frac{1}{2}+n(q_0)\right)\delta(q_0^2-\omega_{\mathbf{q}}^2) \tag{13}$$

into this equation yields

$$\operatorname*{Disc}_{k_0} iR(Q,K) = 8\pi^2\mathrm{sgn}(q_0+\frac{k_0}{2})\mathrm{sgn}(q_0-\frac{k_0}{2})(n_--n_+)\delta((Q+\frac{K}{2})^2-m^2)\delta((Q-\frac{K}{2})^2-m^2), \tag{14}$$

where $n_\pm = n(q_0\pm k_0/2)$.

We first analyse the product of the delta functions (for details,see Appendix B). One can write them in various forms:

$$\begin{aligned} D := \ & \delta((Q+\frac{K}{2})^2-m^2)\delta((Q-\frac{K}{2})^2-m^2) = \delta(2QK)\delta(Q^2+\frac{K^2}{4}-m^2) \\ = \ & \frac{1}{2\,\mathcal{Z}\,|k_0^2-k^2x^2|}\delta(q_0-\frac{kx}{2}\mathrm{sgn}(k_0)\mathcal{Z})\delta(q-\frac{|k_0|}{2}\mathcal{Z}), \end{aligned} \tag{15}$$

where $\mathcal{Z} = \sqrt{(K^2-4m^2)/(k_0^2-k^2x^2)}$, which is real when $K^2 > 4m^2$ or $K^2 < 0$ (see Appendix B).

The last form carries an important message concerning the relative magnitude of q_0 and k_0. For $K^2<0$ it requires that $|q_0| > |k_0|/2$, while for $K^2 > 4m^2$ we find $|q_0| < k/2 < |k_0|/2$ (see Appendix B). It has the consequence that in the present circumstances one can write:

$$\mathrm{sgn}(q_0\pm\frac{k_0}{2}) = \Theta(K^2-4m^2)(\pm)\mathrm{sgn}(k_0) + \Theta(-K^2)\mathrm{sgn}(q_0), \tag{16}$$

which implies, when inserting back into (14), that

$$\begin{aligned} \operatorname*{Disc}_{k_0} iR(Q,K) = \ & \frac{4\pi^2}{\sqrt{(k_0^2-k^2x^2)(K^2-4m^2)}}\delta(q_0-\frac{kx}{2}\mathrm{sgn}(k_0)\mathcal{Z})\delta(q-\frac{|k_0|}{2}\mathcal{Z}) \times \\ & \left[\Theta(K^2-4m^2)\left(1+n(\frac{k_0}{2}-q_0)+n(\frac{k_0}{2}+q_0)\right) + \Theta(-K^2)\left(n(q_0-\frac{k_0}{2})-n(q_0+\frac{k_0}{2})\right)\right] \end{aligned} \tag{17}$$

The Q integration is performed using this form (see Appendix B), and one finds

$$\int đQ \operatorname*{Disc}_{k_0} iR(Q,K) = \frac{\text{sgn}(k_0)}{4\pi k}\Bigg[\ \Theta(K^2-4m^2)\int\limits_{-k\mathcal{Z}_1/2}^{k\mathcal{Z}_1/2} dy\ \left(1+n(\frac{|k_0|}{2}-y)+n(\frac{|k_0|}{2}+y)\right)+$$
$$2\Theta(-K^2)\int\limits_{k\mathcal{Z}_1/2}^{\infty} dy\ \left(n(y-\frac{|k_0|}{2})-n(y+\frac{|k_0|}{2})\right)\Bigg], \tag{18}$$

which agrees with the results of earlier calculations [11].

The limiting case usually investigated is that the spatial variation of the external source is small, i.e. $k \ll m, T$. Then, in the first term of eq. (18), the variation of y is of order k, thus $y \ll |k_0|$ and we can neglect the y dependence of the integrand. In the second term $k \geq |k_0|$, so the small-k approximation yields $\mathcal{Z}_1 \approx 2m/|K| \gg 1$. Then $y \gg k \geq k_0$ and we can perform an expansion in power series with respect to k_0. Finally we obtain

$$\int đQ \operatorname*{Disc}_{k_0} iR(Q,K) \approx \frac{1}{4\pi}\Big[\Theta(K^2-4m^2)\sqrt{1-\frac{4m^2}{K^2}}\left(1+2n(\frac{k_0}{2})\right)+2\Theta(-K^2)\frac{k_0}{k}n(\frac{km}{|K|})\Big]. \tag{19}$$

3.1 The Landau damping region

The approximate expression (19) can be derived without the detailed computation outlined above. If $K^2 < 0$ and $k \ll m, T$, the plasma excitations behave as on-mass-shall particles because $Q^2 = m^2 - K^2/4 \approx m^2$. At high temperatures all components of Q are of the order of the temperature. In this regime it is a good approximation to neglect K in (14) and (15) whenever it is possible, and to perform an expansion in power series with respect to the components of K when it is necessary. This yields

$$\operatorname*{Disc}_{k_0} iR(Q,K) \approx -8\pi^2 k_0 \frac{dn(q_0)}{dq_0}\delta(2QK)\delta(Q^2-m^2) = -8\pi^2 k_0 \frac{dn}{d\omega}\delta(2QK)\delta(Q^2-m^2), \tag{20}$$

since dn/dq_0 is even in q_0. After the Q integration one readily reproduces the Landau damping piece of (19).

Although this is a straightforward consequence of our approximations, this result is still a bit surprising. The assumption we made is quite plausible: the variation of the background field K is small as compared to the typical scale of the loop momentum, which is T at high temperatures. This was implemented by requiring $Q \gg K$ for all components. This assumption, however, just reproduces the Landau cut region, while the normal cut is missing.

3.2 The normal cut region

Let us examine where we have lost the normal cut, and what are the appropriate assumptions that lead to its approximate form in (19), in a similarly direct way as the Landau cut is reproduced.

The answer is easy to find when we recall the earlier statement of the exact analysis that in the $K^2 > 4m^2$ regime $|q_0| < k/2 < |k_0|/2$: that means that the assumption $Q \gg K$ is not valid for all components. In the normal cut regime the time-like component of the loop momentum is small, even smaller than the spatial variation of the background field.

This observation suggests a complementary approximation scheme: we neglect the (spatial) k dependence, not assuming, however, that $k_0 < k$. In the product of the delta functions (15) we can then write

$$D = \delta(2QK)\delta(Q^2 + \frac{K^2}{4} - m^2) \longrightarrow \delta(2q_0 k_0)\delta(Q^2 + \frac{k_0^2}{4} - m^2) = \frac{1}{2|k_0|}\delta(q_0)\delta(\frac{k_0^2}{4} - m^2 - q^2). \quad (21)$$

Therefore, in the leading order, we can assume $q_0 \approx 0$ as well. We also see that we immediately obtain $k_0^2 > 4m^2$ from the second delta function. The approximation for the discontinuity of R then reads

$$\operatorname*{Disc}_{k_0} iR(Q,K) \approx \frac{4\pi^2}{|k_0|}\left(1 + 2n(\frac{k_0}{2})\right)\delta(q_0)\delta(\frac{k_0^2}{4} - m^2 - q^2). \quad (22)$$

This expression reproduces, after Q integration, the normal cut part of the approximated result (19).

It is interesting to observe that the approximations leading to Landau and normal cuts concern different factors of D in (15): using $\delta(Q^2 + K^2/4 - m^2)$ for small K yields that Q is on the mass shell, while using $\delta(2QK)$ for small k yields $q_0 \approx 0$. While both assumptions seem to be well justified, neither of them reproduces the complete discontinuity of R, just complementary parts of them. To understand this phenomenon better, we now turn to the Schwinger–Dyson equations.

3.3 Schwinger–Dyson equations

The general form of SD equations in Keldysh and R/A formalism can be found in Appendix A. At one loop perturbation theory we need the self-energies coming from the interaction Lagrangian (10):

$$\Pi_{ar}(Q,K) = \Pi_{ra}(Q,K) = \mathcal{P}(K), \qquad \Pi_{aa} = 0, \quad (23)$$

which has to be inserted into (81) of Appendix A. For a linear response we insert free propagators into its right-hand side. We are interested in the rr component (cf. (8))

$$\left((Q \pm \frac{K}{2})^2 - m^2\right)\bar{G}_{rr}(Q,K) = \mathcal{P}(K)\bar{G}^0_{rr}(Q \mp \frac{K}{2}). \quad (24)$$

The two equations, like the general ones, are related by complex conjugation and $K \to -K$ transformation.

Taking the sum and the difference of the two equations we find

$$\left(Q^2 + \frac{K^2}{4} - m^2\right)\bar{G}_{rr}(Q,K) = \mathcal{P}(K)\frac{\bar{G}^0_{rr}(Q + \frac{K}{2}) + \bar{G}^0_{rr}(Q - \frac{K}{2})}{2} \quad (25)$$

$$2QK\bar{G}_{rr}(Q,K) = \mathcal{P}(K)\left[\bar{G}^0_{rr}(Q - \frac{K}{2}) - \bar{G}^0_{rr}(Q + \frac{K}{2})\right]. \quad (26)$$

This yields two solutions for $\bar{G}_{rr}(Q,K)$. Since the two SD equations are not independent, the two solutions should imply the same physics. To see this we recall that $\bar{G}^0_{rr}(Q) \sim \delta(Q^2 - m^2)$, therefore

$$\frac{\bar{G}^0_{rr}(Q+\frac{K}{2})}{Q^2+\frac{K^2}{4}-m^2} = -\frac{\bar{G}^0_{rr}(Q+\frac{K}{2})}{2QK}. \tag{27}$$

On the other hand the equivalence is not complete, since we lose something from the analytical structure. If we treat these equations as differential equations with initial condition (i.e. we apply the Landau prescription), and we take the imaginary part, the two equations give different results. We use the following form of the free rr propagator ($\omega^2 = \mathbf{q}^2 + m^2$):

$$i\bar{G}^0_{rr}(Q) = 2\pi\left(\frac{1}{2}+n(\omega)\right)\delta(Q^2-m^2) \tag{28}$$

and write for the solution of (25) and (26)

$$\begin{aligned}
\operatorname*{Disc}_{k_0} i\bar{G}_{rr}(Q,K) &= 4\pi^2\mathcal{P}(K)\mathrm{sgn}(k_0)\left(1+n(\omega_+)+n(\omega_-)\right)\delta(Q^2+\frac{K^2}{4}-m^2)\delta(2QK),\\
\operatorname*{Disc}_{k_0} i\bar{G}_{rr}(Q,K) &= 4\pi^2\mathcal{P}(K)\mathrm{sgn}(q_0)\left(n(\omega_-)-n(\omega_+)\right)\delta(Q^2+\frac{K^2}{4}-m^2)\delta(2QK),
\end{aligned} \tag{29}$$

respectively, where $\omega_\pm = \omega(\mathbf{q}\pm\mathbf{k}/2)$. Here the first form corresponds to the normal cut regime, the second to the Landau-cut regime. Although formally there seems to be nothing that could prohibit applying, for example, the second equation to the $K^2 > 4m^2$ region, the result will be incorrect.

If we compare these formulae to the ones coming from direct perturbation theory (see (14) and (15)) we can see that the difference is that that in the exact formula there stands $\mathrm{sgn}(q_0 \pm k_0/2)$, which is substituted by $\mathrm{sgn}(k_0)$ and $\mathrm{sgn}(q_0)$ in the first and the second form of the above equations, respectively. The reason for this is simple: when we take for example the difference of the two equations, K^2 disappears, and so the corresponding $ik_0\varepsilon$ will be missing in the later calculations, too. Therefore we implicitly assume that the $2iq_0\varepsilon$ term (coming from $2QK$) dominates it: i.e. $q_0 > k_0/2$. This implicit assumption prevents the final result to be applicable everywhere.

Usually one considers this pair of equations as a differential equation ((26), the Boltzmann equation) and, assuming $K \ll Q$, a constraint (25). But this strategy, as we have seen, misses the normal cut part from the very beginning. From our analysis a slightly different picture seems to emerge: here both equations are differential equations to be solved, and the complete solution is the sum of the two solutions. We propose to apply this strategy also for the non-perturbative, non-linear response regime: we should solve not just the equation coming from the difference of the pair of SD equations, i.e. the Boltzmann equation. We have to treat the sum of the two equations as a differential equation, since this is the approach that makes explicit where the potentially important (especially in the zero-mass case) normal cut contribution comes from. The complete solution then can be composed as the sum of these partial solutions.

But where are the constraint equations? In our case the right-hand side, and so the solution was explicit, there was no need to add another equation. At higher loops or in the non-linear response regime, as we will see in the next section, concentrating on the IR sensitive pinch-singular region, we can still have a consistent ansatz that satisfies the (by now missing) constraint equation.

As a last remark in this section we emphasize the importance of the cut contributions in free systems. Let us consider the evolution of $\int d^3\mathbf{x}\langle\varphi^2\rangle(\mathbf{x},k_0) = \langle\varphi^2\rangle(\mathbf{k}=0,k_0)$ in a free system, without any background field, but with specified initial conditions. The relevant equations are the following (cf. (25), (26)):

$$2q_0 k_0 \bar{G}_{rr}(Q,k_0) = 0, \qquad \left(Q^2 + \frac{k_0^2}{4} - m^2\right)\bar{G}_{rr}(Q,K) = 0. \tag{30}$$

Taking into account only the first equation we would conclude that $\bar{G}_{rr}(Q,t)$ is constant, and thus $\langle\varphi^2\rangle(\mathbf{k}=0,t)$ is constant, too. This would mean that $\langle\varphi^2\rangle(\mathbf{k}=0,t)$ is a conserved quantity – which it is not, in reality. Analysing the second equation (cf. [11, 12]), we find that it yields a power law time dependence, that is $\langle\varphi^2\rangle(\mathbf{k}=0,t)$ goes to zero, with some power of time. This again shows that the Boltzmann equation, alone, cannot provide the complete treatment even in the simplest case.

4 Higher loops and pinch singularities

Higher loops are, of course, suppressed by powers of the coupling constant; therefore they become important only if the one-loop result is zero or there are IR divergences. The imaginary part of the one-loop result, as we have seen, is zero if $0 < K^2 < 4m^2$; at two loops, however, this restriction is relaxed and non-zero imaginary part is obtained for all K. As far as the IR divergences are concerned, in our case the pinch singularities [7, 8] are the primary sources of them.

If the product of a retarded and an advanced propagator appears in some objects with the same momentum, then the poles of the propagators pinch the real axis as $\varepsilon \to 0$ and yield a non-regularizable double pole. If the two momenta are not equal, we expect a singular behaviour of the object as the difference of the momenta vanishes. Using the fact that $G_R G_R$ is not singular and that $i(G_R - G_A) = \varrho$, we can write

$$G_R(Q+\frac{K}{2})G_A(Q-\frac{K}{2}) = G_R(Q+\frac{K}{2})\, i\varrho(Q-\frac{K}{2}) - \text{non-singular}. \tag{31}$$

If $K \to 0$ the free spectral function $\varrho(Q) \sim \delta(Q^2 - m^2)$ puts G_R on-shell: this is the core effect of the pinch singularities. On the other hand, $G_{rr} \sim \varrho$, thus similar singularities appear when we multiply $G_{R,A}$ and G_{rr}. We have to treat this case in complete analogy with the $G_R G_A$ product.

Pinch singularities appear in all higher order diagrams containing two-particle intermediate states [9]; to see the net effect we have to sum up all the diagrams of this type. In the linear response case, these diagrams form ladders.

Which are the kinematic ranges where we have to count with pinch singularities? From the above generic form

$$G_R(Q+\frac{K}{2})\, i\varrho(Q-\frac{K}{2}) = \frac{i\varrho(Q-\frac{K}{2})}{2QK} = \frac{1}{2}\,\frac{i\varrho(Q-\frac{K}{2})}{Q^2 + K^2/4 - m^2}. \tag{32}$$

There are therefore two sensitive regimes: the vanishing $2QK$ or the vanishing $Q^2+K^2/4-m^2$ regime.

To resum the pinch singular diagrams we can use any valid equation where the left-hand side is proportional to $2QK$ or $Q^2+K^2/4-m^2$, respectively, and the right-hand side is non-vanishing in the $2QK=0$ or $Q^2+K^2/4-m^2=0$ limit [9]; the difference in the exact choice of the equation just gives a subleading effect. In linear response theory we need a linearized right-hand side. Exactly of this form are the linearized SD equations (81) of Appendix A. Starting from it, and linearizing around equilibrium in the form

$$\bar G(Q,K)=G^{eq}(Q)\bar\delta(K)+g(Q,K),\quad \bar\Pi(Q,K)=\Pi^{eq}(Q)\bar\delta(K)+\pi(Q,K), \tag{33}$$

one finds the following equations for the deviations:

$$\begin{aligned}
\left(Q^2+\frac{K^2}{4}-m^2\right)2g_{rr} &= \pi_{aa}\left(G^-_{ar}+G^+_{ra}\right)+\Pi^+_{aa}g_{ar}+\Pi^-_{aa}g_{ra}+\\
&\quad \pi_{ar}G^-_{rr}+\pi_{ra}G^+_{rr}+\left(\Pi^+_{ar}+\Pi^-_{ra}\right)g_{rr}\\
2QK g_{rr} &= \pi_{aa}\left(G^-_{ar}-G^+_{ra}\right)+\Pi^+_{aa}g_{ar}-\Pi^-_{aa}g_{ra}+\\
&\quad \pi_{ar}G^-_{rr}-\pi_{ra}G^+_{rr}+\left(\Pi^+_{ar}-\Pi^-_{ra}\right)g_{rr}\\
\left(Q^2+\frac{K^2}{4}-m^2\right)2g_{ra} &= \pi_{ar}\left(G^-_{ra}+G^+_{ra}\right)+\left(\Pi^+_{ar}+\Pi^-_{ar}\right)g_{ra}\\
2QK g_{ra} &= \pi_{ar}\left(G^-_{ra}-G^+_{ra}\right)+\left(\Pi^+_{ar}-\Pi^-_{ar}\right)g_{ra},
\end{aligned} \tag{34}$$

where we have omitted the (Q,K) arguments of the functions and we have introduced the notations $F^\pm\equiv F^{eq}(Q\pm K/2)$. The $\mathcal{P}$ dependence is hidden in π_{ra} and π_{ar}

$$\pi_{ra}\to\mathcal{P}+\pi_{ra},\qquad \pi_{ar}\to\mathcal{P}+\pi_{ar}. \tag{35}$$

The self-energies (apart from the term containing $\mathcal{P}$) come from loop diagrams, i.e. they consist of propagator products supplemented with some integration. In a linear response, all propagators but one are in equilibrium, so they are functions of space-time coordinate difference. Therefore the general structure reads in Fourier space

$$\pi(p,q)\sim\int d\!\!\bar{}k\, d\!\!\bar{}\ell\,\mathcal{K}(p,-q,k,-\ell)g(k,\ell), \tag{36}$$

where we divided the momenta into incoming (p,k) and outgoing (q,ℓ) ones, but this is just a convention in the signs. Since $\mathcal{K}$ contains equilibrium propagators only, the momentum is conserved, and we can write

$$\pi(p,q)\sim\int d\!\!\bar{}\ell\,\mathcal{K}'(p,-q,\ell)g(p-q+\ell,\ell). \tag{37}$$

The Wigner transformed function reads

$$\bar\pi(Q,K)\sim\int d\!\!\bar{}\ell\,\bar{\mathcal{K}}'(Q,K,\ell)\bar g(K+\ell,K)=\int d\!\!\bar{}\ell\,\bar{\mathcal{K}}''(Q,K,\ell)\bar g(\ell,K). \tag{38}$$

In general, we have to sum over the R/A indexes, too. From this form we see that the linearized SD equations are diagonal in K.

4.1 The Boltzmann equations

Instead of using the full set of SD equations in (34), we can make some approximations in order to construct equations that can be treated more easily. According to our earlier analysis, we will concentrate on the summation of pinch singularities, which covers two regions: the small QK region and the small $Q^2 + K^2/4 - m^2$ region. The approximation procedure will be to power-expand everything with respect to the powers of these small parameters.

We start with the Boltzmann region: here $q_0 > k_0/2$ and the differential equations are the second and fourth of (34). First we examine the equations for g_{ra}:

$$2QKg_{ra} = \pi_{ar}\left(G^-_{ra} - G^+_{ra}\right) + \left(\Pi^+_{ar} - \Pi^-_{ar}\right) g_{ra}. \tag{39}$$

If $q_0 > k_0/2$ and in the small QK case the following approximation is valid

$$G^\pm_{ra} = \frac{1}{Q^2 + K^2/4 - m^2 \pm QK + iq_0\varepsilon} = \frac{1}{Q^2 + K^2/4 - m^2 + iq_0\varepsilon} + \mathcal{O}(QK), \tag{40}$$

therefore the difference of these functions is zero up to $\mathcal{O}(QK)$. Then the equation for g_{ra} is homogeneous (no mixing with g_{rr}), therefore a valid ansatz is to assume that $g_{ra} \equiv 0$. In a similar way we can assume that $g_{ar} \equiv 0$. This is a generally accepted approximation: the usual Boltzmann equation never leads to non-zero g_{ra} and g_{ar}.

Having said this, the equation for g_{rr} reads

$$2QKg_{rr} = \pi_{aa}\left(G^-_{ar} - G^+_{ra}\right) + \pi_{ar}G^-_{rr} - \pi_{ra}G^+_{rr} + \left(\Pi^+_{ar} - \Pi^-_{ra}\right) g_{rr}. \tag{41}$$

We approximate the free G_{rr} for small $2QK$ and $q_0 > k_0/2$ as (here $n_\pm = n(q_0 \pm k_0/2)$):

$$iG^\pm_{rr} = 2\pi\mathrm{sgn}(q_0 \pm \frac{k_0}{2})\left(\frac{1}{2} + n_\pm\right)\delta(Q^2 + \frac{K^2}{4} - m^2 \pm QK) \approx 2\pi\mathrm{sgn}(q_0)\left(\frac{1}{2} + n_\pm\right)\delta(Q^2 + \frac{K^2}{4} - m^2), \tag{42}$$

while the difference between G_{ra} and G_{ar}, as we have seen in (40), reads as

$$G^-_{ar} - G^+_{ra} = 2\pi i\mathrm{sgn}(q_0)\delta(Q^2 + \frac{K^2}{4} - m^2) + \mathcal{O}(QK). \tag{43}$$

Therefore all explicit terms in the equation are proportional to $\mathrm{sgn}(q_0)\delta(Q^2 + K^2/4 - m^2)$; a consistent solution, therefore, comes by writing the ansatz

$$ig_{rr}(Q, K) = 2\pi\mathrm{sgn}(q_0)\delta(Q^2 + \frac{K^2}{4} - m^2)\delta n(Q, K). \tag{44}$$

Therefore, without making use of the other equation arising from the sum of the two original SD equations, we are led to a form consistent with this constraint. This is now, however, somewhat weaker statement, it is just an ansatz, and there may exist other solutions that do not respect this ansatz.

If we are interested in the small $\mathbf{k}$ limit of the result, we can make other simplifications as well. Since in this region $K^2 < 0$, it also means a small k_0, and we can thus neglect K^2 on the right-hand side. We find finally

$$2QK\delta n = -\pi_{aa} + (\pi_{ar} - \pi_{ra})\left(\frac{1}{2} + n(q_0)\right) + (\Pi^0_{ar} - \Pi^0_{ra})\delta n. \tag{45}$$

This is the linearized Boltzmann equation when we expand π's in terms of δn.

In our concrete model (3), using the ansatz, we find

$$\pi_{aa} = -ih^2 \int \!\!d\!\!\!{}^-L\, \varrho(L)\varrho(Q-L) \left[\left(\frac{1}{2}+n(q_0-\ell_0)\right)\delta n(\ell_0) + \left(\frac{1}{2}+n(\ell_0)\right)\delta n(q_0-\ell_0)\right]$$
$$\pi_{ar}-\pi_{ra} = -ih^2 \int \!\!d\!\!\!{}^-L\, \varrho(L)\varrho(Q-L)[\delta n(\ell_0)+\delta n(q_0-\ell_0)]. \tag{46}$$

The equilibrium value of the self energy reads

$$\Pi^0_{ar} - \Pi^0_{ra} = -ih^2 \int \!\!d\!\!\!{}^-L\, \varrho(L)\varrho(Q-L)(1+n(\ell_0)+n(q_0-\ell_0)), \tag{47}$$

and thus for the integrand on the right hand side we find

$$\delta n(\ell_0)(n(q_0)-n(q_0-\ell_0)) + \delta n(q_0-\ell_0)(n(q_0)-n(\ell_0)) + \delta n(q_0)(1+n(\ell_0)+n(q_0-\ell_0)), \tag{48}$$

while the complete equation is

$$2QK\delta n = -ih^2 \int \!\!d\!\!\!{}^-L\, \varrho(L)\varrho(Q-L)\ \delta[\,(1+n(\ell_0))(1+n(q_0-\ell_0))n(q_0)$$
$$-n(\ell_0)n(q_0-\ell_0)(1+n(q_0))]. \tag{49}$$

This is, indeed, a linearized Boltzmann equation.

This expression is actually zero, since $Q^2=m^2$, $L^2=m^2$ and $(Q-L)^2=m^2$ cannot be true at the same time. In our model, therefore, there is no one-loop correction to the Boltzmann equation. To get a non-zero right-hand side, we can consider a model with different masses where $m_1 > m_2+m_3$ is true, or we go to higher loops in the present model.

4.2 The normal cut region

Let us now repeat the same analysis for the cut region, where $k_0 > 2q_0$ and the first and the third differential equations of (34) are relevant. First, we take the equation for g_{ra}

$$\left(Q^2+\frac{K^2}{4}-m^2\right)2g_{ra} = \pi_{ar}\left(G^-_{ra}+G^+_{ra}\right) + \left(\Pi^+_{ar}+\Pi^-_{ar}\right)g_{ra}. \tag{50}$$

In this region the following approximation is valid

$$G^{\pm}_{ra} = \frac{1}{Q^2+K^2/4-m^2 \pm QK \pm ik_0\varepsilon} = \frac{\pm 1}{QK+ik_0\varepsilon} + \mathcal{O}(Q^2+\frac{K^2}{4}-m^2), \tag{51}$$

and so the sum $G^-_{ra}+G^+_{ra}\approx 0$. The equation is approximately homogeneous in g_{ra}; we can therefore assume, as before, that $g_{ra}=g_{ar}=0$.

Then the equation for g_{rr} reads

$$\left(Q^2+\frac{K^2}{4}-m^2\right)2g_{rr} = \pi_{aa}\left(G^-_{ar}+G^+_{ra}\right) + \pi_{ar}G^-_{rr} + \pi_{ra}G^+_{rr} + \left(\Pi^+_{ar}+\Pi^-_{ra}\right)g_{rr}. \tag{52}$$

We now approximate G_{rr} for small $Q^2 + K^2/4 - m^2$ and for $k_0/2 > q_0$ as

$$\begin{aligned} iG^{\pm}_{rr} &= 2\pi \mathrm{sgn}(q_0 \pm \frac{k_0}{2}) \left(\frac{1}{2} + n(q_0 \pm \frac{k_0}{2})\right) \delta(Q^2 + \frac{K^2}{4} - m^2 \pm QK) \\ &\approx 2\pi \mathrm{sgn}(k_0) \left(\frac{1}{2} + n(\frac{k_0}{2} \pm q_0)\right) \delta(QK) \end{aligned} \tag{53}$$

and, according to the previous approximation:

$$G^-_{ar} + G^+_{ra} \approx \frac{-1}{QK - ik_0\varepsilon} + \frac{1}{QK + ik_0\varepsilon} = -2\pi i \mathrm{sgn}(k_0)\delta(QK). \tag{54}$$

So finally all known terms are explicitly proportional to $\mathrm{sgn}(k_0)\delta(QK)$, and we can have, similarly to the Boltzmann case, an ansatz that now reads as

$$ig_{rr}(Q,K) = 2\pi \mathrm{sgn}(k_0)\delta(QK) f(Q,K). \tag{55}$$

This agrees with the solution of the difference of the SD equations, if we have considered it as a constraint; but now it is just an ansatz.

In the small $\mathbf{k}$ region $\delta(QK) \approx 1/|k_0|\delta(q_0)$, we can therefore substitute $q_0 \approx 0$. Then the only non-zero momentum components are k_0 and $\mathbf{q}$, and we collect them in a single four-vector $S = (k_0/2, \mathbf{q})$. Using spatial rotation symmetry, $\Pi_{ra}(Q - K/2) = \Pi_{ar}(S)$, and we can write

$$(S^2 - m^2) 2f = \pi_{aa} + (\pi_{ar} + \pi_{ra}) \left(\frac{1}{2} + n(s_0)\right) + 2\Pi_{ar}(S) f. \tag{56}$$

This is a linear but non-local wave equation for f. On the right-hand side we have to use the fact that $Q^2 + K^2/4 - m^2$ is small, i.e. $S^2 \approx m^2$.

In our concrete model, at one-loop order $\pi_{aa}(k_0/2, \mathbf{q}) = \mathrm{Im}\,\Pi_{ar}(S) = 0$ at $S^2 = m^2$. The second equality was proved before. In order to prove the first one, we notice that, according to the ansatz:

$$\pi_{aa}(Q,K) = -2ih^2 \int đL\, \varrho(Q - L) \left(\frac{1}{2} + n(q_0 - \ell_0)\right) \mathrm{sgn}(k_0) 2\pi\delta(LK) f(L,K). \tag{57}$$

When $\mathbf{k} = q_0 = 0$ then $\delta(2LK) \sim \delta(\ell_0)$, and thus $\varrho(Q - L)$ would force $Q - L$ on the mass shell with $q_0 - \ell_0 = 0$; which is impossible. Thus, at the one-loop order, what remains in (56) (writing the $\mathcal{P}$ dependence explicitly) is

$$(S^2 - m^2) f = \Pi^0_{ar}(S) f - \left(\frac{1}{2} + n(s_0)\right) \frac{h^2}{s_0} \int \frac{d^3\ell}{(2\pi)^3} \frac{1}{(\mathbf{s} - \ell)^2 + m^2} f(\ell, s_0) + \left(\frac{1}{2} + n(s_0)\right) \mathcal{P}(2s_0). \tag{58}$$

We see that the treatment of the two regimes (the Boltzmann and the normal cut regions) is quite symmetric; the only difference comes from whether we consider the sum or the difference of the SD equations as the relevant dynamical differential equation. In the IR sensitive regions we have for the solution an ansatz that satisfies the other equation as a constraint.

4.3 Relaxation time approximation

Relaxation time approximation for Boltzmann equations is well-known. It means that we approximate the right-hand side of (49) as $-2iq_0\Gamma_B\delta n$. If this approximation is valid, then $\delta n(t) \sim \exp(-\Gamma_B t)$. In our illustrative model, as we have seen, $\Gamma_B = \mathcal{O}(h^4)$, since it is zero at the one-loop level.

This approximation scheme can be extended also to the normal cut regime. We assume that the operators acting on the right-hand side of the relevant eq. (58) can be represented as a diagonal linear operator. After rearrangement on the left-hand side, modified parameters (damping rate *and* mass) will appear. Then (58) reads as

$$\left[\left(\frac{k_0+i\Gamma_c}{2}\right)^2 - \mathbf{q}^2 - m_{eff}^2\right] f = \left(\frac{1}{2}+n(\frac{k_0}{2})\right)\mathcal{P}(K). \tag{59}$$

At one-loop order the imaginary part $\Gamma_c = 0$, just like in the Boltzmann case; but, in general, it is non-zero at higher loops. Its solution is almost the same as the one-loop level result, just the finite imaginary part requires some care. The ansatz (55) yields

$$ig_{rr}(Q,K) = \frac{2\pi}{k_0}\delta(q_0)\left(\frac{1}{2}+n(\frac{k_0}{2})\right)\frac{1}{(k_0+i\Gamma_c)^2/4-\mathbf{q}^2-m_{eff}^2}\mathcal{P}(k_0). \tag{60}$$

When compared with the definition $ig_{rr}(Q,K) = R(Q,K)\mathcal{P}(K)/2$, we find for $\int \! đQ\, iR$, after evaluating the q_0 integral,

$$\int đQ\, iR(Q,K) = \frac{1}{\pi^2 k_0}\left(\frac{1}{2}+n(\frac{k_0}{2})\right)\int_0^\infty dq\, \frac{q^2}{(k_0+i\Gamma_c)^2/4-m_{eff}^2-\mathbf{q}^2}. \tag{61}$$

After performing the integration and dropping the infinite piece (mass renormalization), we finally arrive at

$$\int đQ\, iR(Q,K) = \frac{i}{4\pi k_0}\left(1+2n(\frac{k_0}{2})\right)\sqrt{\frac{(k_0+i\Gamma_c)^2}{4}-m_{eff}^2}. \tag{62}$$

The discontinuity, calculated as twice the imaginary part, reads

$$\begin{aligned}\int đQ\, \operatorname*{Disc}_{k_0} iR(Q,K) &= \frac{1}{4\pi k_0}\left(1+2n(\frac{k_0}{2})\right) 2\,\mathrm{Re}\sqrt{\frac{(k_0+i\Gamma_c)^2}{4}-m_{eff}^2} \\ &= \frac{1}{4\pi}\left(1+2n(\frac{k_0}{2})\right)\frac{2}{k_0}F\left(\frac{k_0^2}{4}-M^2,\frac{k_0\Gamma_c}{2}\right)\end{aligned} \tag{63}$$

where $M^2 = m_{eff}^2 + \Gamma_c^2$ and

$$F(x,y) = \frac{1}{\sqrt{2}}\left(x+\sqrt{x^2+y^2}\right)^{1/2}. \tag{64}$$

Let us first analyse some properties of this result

- If $\Gamma_c = 0$ then, because of $F(x, y = 0) = \Theta(x)\sqrt{x}$, we get back the normal cut part of (19) with the substitution $m \to M$.

- Using perturbation theory without resummation, we see IR divergences (coming from higher loops) if $m = 0$. This is because $M^2 \sim h^2$, thus we can compute power series in M^2, which yields

$$\frac{2F}{k_0}\bigg|_{pert} = \frac{2}{k_0}\sqrt{\frac{k_0^2}{4} - M^2} = 1 - \frac{2M^2}{k_0^2} + \mathcal{O}(h^4). \tag{65}$$

- For $M^2 = 0$

$$\frac{2F}{k_0}\bigg|_{M^2=0} = \frac{1}{\sqrt{2}}\left(1 + \sqrt{1 + \frac{4\Gamma_c^2}{k_0^2}}\right)^{1/2} \xrightarrow{k_0 \to 0} \sqrt{\frac{\Gamma_c}{k_0}} \tag{66}$$

yields a divergent (although integrable) behaviour.

- At $k_0 = 0, M \neq 0$ we get

$$\frac{2F}{k_0}\bigg|_{k_0 \to 0} = \frac{\Gamma_c}{4M}\mathrm{sgn}(k_0). \tag{67}$$

This is a finite (and non-analytic) result.

- While the free result exhibited a non-analytic square-root behaviour near the threshold, higher-loop corrections smoothen out this non-analyticity. A non-analytic behaviour implies power-law real time dependence [11, 12], which now acquires an exponential damping multiplicative correction, just like the solution of the Boltzmann equation. On the other hand, at $k_0 = 0$, there appears a new non-analytic contribution.

In order to see these effects we plotted the function $2F/k_0$ in Fig. 1. Curve a shows the

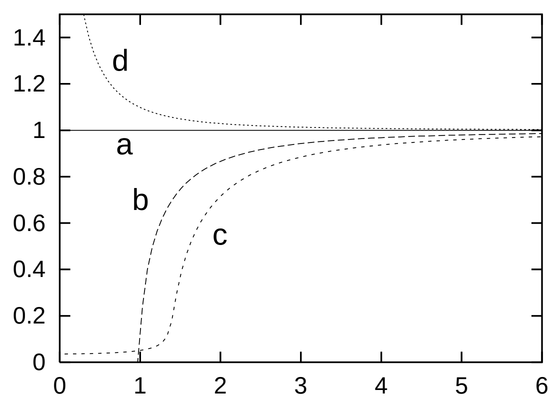

Figure 1: The function F/k_0 (cf. (64) for different parameters: *a:* $M = \Gamma_c = 0$, *b:* $M = 1, \Gamma_c = 0$, *c:* $M = 1.4, \Gamma_c = 0.1$, *d:* $M = 0, \Gamma_c = 0.1$.

$M = \Gamma_c = 0$ result; b is the free result with $m = 1$; c shows how it changes at higher loops: it acquires a finite shift in the mass as well as a damping that smears out the threshold non-analyticity (in the plot we have used $M = 1.4, \Gamma_c = 0.1$). Finally curve d shows the $M = 0, \Gamma_c \neq 0$ case.

5 Comments on the fermionic case

Let us make some comments on the fermionic case. In our simple model, in relaxation time approximation, the only change in the final result, compared to the free case, was a modification just in the threshold function $\Theta(x)\sqrt{x} \to F(x,y)$, cf. (63). This seems to be the consequence of the relaxation time approximation, not depending on the specific model, so we may assume that this feature generalizes to other theories, too. In case of dilepton production we have a fermionic system with approximately zero masses. The rate of the dilepton production is proportional to $\langle [j_\mu, j^\mu] \rangle(K)$ [10]; when computed perturbatively, this is very similar to our observable $\langle [\varphi^2, \varphi^2] \rangle(K) = \int d\!\!\!{}^-Q \; \mathrm{Disc}_{k_0}\, iR(Q,K)$. At $\mathbf{k} = 0$ the production rate is $\langle [j_\mu, j^\mu] \rangle(k_0)/k_0^2 \sim \tanh(\beta k_0/4)$; if we assume the same behaviour for the higher loop pinch singularities in the fermionic theory as they appear in our model, we obtain changes in the production rate qualitatively similar to Fig. 2. Here curve a shows the free result, curve b

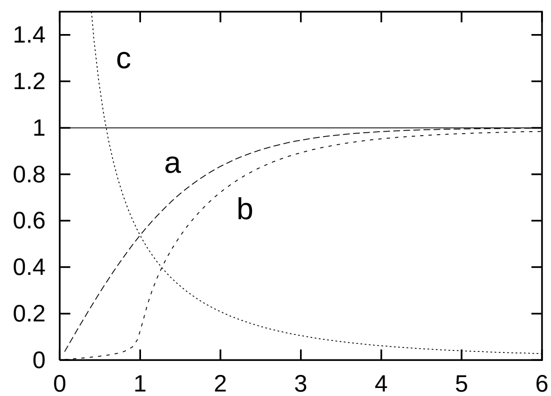

Figure 2: Expected result of $\langle [j_\mu, j^\mu] \rangle(k_0)/k_0^2$. Curve a is the one loop result, curve b is the result after resummation. Curve c indicates the size of the 2-loop corrections.

shows the expected result of the resummation, applying the present analysis. Curve c indicates the size of the 2-loop perturbative corrections. This figure suggests that, although the two-loop correction is divergent in the $k_0 \to 0$ limit, the resummed result can be convergent, or even zero. With the present approximation the discontinuity goes linearly to zero for vanishing momentum, and its slope is $\sim \Gamma_c/(4TM)$. Our approximate result is quite reminiscent of the numerical (MEM method) findings of Karsch et *al.* [1].

6 Conclusion

Let us summarize the main results of the paper:

- *Refinement of the conventional treatment of Kadanoff–Baym equations.* Starting from the pair of non-equilibrium Schwinger–Dyson equations, applying Wigner transformation and taking their sum and difference, we arrive at the KB equations. The conventional interpretation is the following: for mild variations of the average coordinates, one of the equations is understood as a conventional first-order partial differential equation: the quantum Boltzmann equation. In the other equation one neglects the dependence on the average coordinates, as a consequence of which it becomes a constraint.

We propose a somewhat different approach. Based on the thorough analysis of the one-loop contribution, and resumming the IR (pinch) singularities arising from higher loops, we argue that *both* equations have to be interpreted as dynamical differential equations, each one beeing relevant in complementary kinematical regimes. The complete solution is the sum of the two equations. Concentrating on the pinch-singular contributions, we suggested for both solutions an ansatz that satisfies the constraint equation of the conventional interpretation. In this way one of the equations is still a (quantum) Boltzmann equation; the other one, however, represents a new contribution, reproducing in a resummed form the particle production cut.

- *Features of the new contribution.* The new contribution is responsible at tree level for the appearance of the normal cut; it is IR (pinch) singular at higher loops. We have constructed the relevant equation near the IR-singular regime (see (56)). It is a wave equation for the quantity that is the analogue of the particle number variation in the Boltzmann case.

- *Solution in the relaxation time approximation.* We can try to diagonalize the linear differential equations and keep the eigenmode with the smallest damping rate: this is the relaxation time approximation in effect. In the Boltzmann regime this procedure changes the infinitesimal $i\varepsilon$ term of the Landau prescription to the finite imaginary part $i\Gamma_B$, associated with the presence of an effective term in the equation of motion, which is linear in the time derivative. In the normal cut regime there are two contributions: a real one, which is in fact a mass shift, and an imaginary one, which is analogous to the damping rate in the Boltzmann case. Computing the normal cut contribution with this approximation we can observe two effects: the shift of the threshold location and the smoothing of the threshold non-analyticity. If treated in finite orders of perturbation theory, both effects may lead to divergences.

- *Prospects for generalization.* The present computation was performed for the case of the cubic scalar theory, and direct computations in other more relevant models are required before one can have quantitative predictions: this is a project for the future. Still, some findings of the present calculation seem to be robust enough to conjecture similar features to be valid in other cases as well. The "dual" interpretation of the KB equations, the wave equation form of the equation describing the cut contributions, the imaginary *and* real parts in the relaxation time approximation probably generalize to the fermionic case as well. Then we conjecture for the quantity $\langle [j_\mu, j^\mu] \rangle (k_0)/k_0^2$, relevant to the determination of the dilepton production rate, the behaviour indicated on Fig. 2. While two-loop order perturbation theory gives a divergent contribution (curve c) at threshold, the resummed result shows a much milder behaviour (curve b). This qualitative prediction displays remarkable similarity to recent numerical data [1].

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A Appendix: The formalism

In the Keldysh formalism for a complex scalar field[2] we define the propagators

$$iG_{11}(t,t') = \Theta(t-t')\left\langle\Phi(t)\Phi^\dagger(t')\right\rangle + \Theta(t'-t)\left\langle\Phi^\dagger(t')\Phi(t)\right\rangle, \quad iG_{12}(t,t') = \left\langle\Phi^\dagger(t')\Phi(t)\right\rangle,$$
$$iG_{22}(t,t') = \Theta(t'-t)\left\langle\Phi(t)\Phi^\dagger(t')\right\rangle + \Theta(t-t')\left\langle\Phi^\dagger(t')\Phi(t)\right\rangle, \quad iG_{21}(t,t') = \left\langle\Phi(t)\Phi^\dagger(t')\right\rangle, \tag{68}$$

where the expectation value is defined in general as

$$\langle\hat{A}\rangle = \mathrm{Tr}\,\hat{\rho}\hat{A}, \tag{69}$$

where $\hat{\rho}$ is the density matrix. Formally we introduce two types of fields (Φ_1 and Φ_2), which are representations of the original field on the different branches of the closed time contour. Their expectation values are defined as $\langle\Phi_i\Phi_j\rangle = iG_{ij}$.

These propagators are not independent, there is a relation between them

$$G_{11} + G_{22} = G_{12} + G_{21}. \tag{70}$$

In fact, there are only two independent expectation values, e.g. G_{12} and G_{21}, since G_{11} and G_{22} are combinations of these two and the theta function. In equilibrium even these two are related through the KMS condition [7].

While these propagators determine the free theory, in case of interactions we have to modify the interaction Lagrangian as (cf. [7])

$$\mathcal{L}_I(\Phi) \longrightarrow \mathcal{L}_I(\Phi_1) - \mathcal{L}_I(\Phi_2). \tag{71}$$

In the R/A formalism [13] we introduce new fields

$$\Phi_r = \frac{\Phi_1+\Phi_2}{2}, \qquad \Phi_a = \Phi_1 - \Phi_2. \tag{72}$$

Their time-ordered products (denoting $\Phi_i(x)$ by Φ_i and $\Phi_i^\dagger(y)$ by Φ'_i) are expressible through the two-point functions as follows:

$$G_{rr} = \frac{1}{4i}\langle\Phi_1\Phi'_1 + \Phi_1\Phi'_2 + \Phi_2\Phi'_1 + \Phi_2\Phi'_2\rangle = \frac{1}{4}(G_{11}+G_{12}+G_{21}+G_{22}) = \frac{G_{12}+G_{21}}{2},$$
$$G_{ra} = \frac{1}{2i}\langle\Phi_1\Phi'_1 - \Phi_1\Phi'_2 + \Phi_2\Phi'_1 - \Phi_2\Phi'_2\rangle = \frac{1}{2}(G_{11}-G_{12}+G_{21}-G_{22}) = G_{11}-G_{12},$$
$$G_{ar} = \frac{1}{2i}\langle\Phi_1\Phi'_1 + \Phi_1\Phi'_2 - \Phi_2\Phi'_1 - \Phi_2\Phi'_2\rangle = \frac{1}{2}(G_{11}+G_{12}-G_{21}-G_{22}) = G_{11}-G_{21},$$
$$G_{aa} = -i\langle\Phi_1\Phi'_1 - \Phi_1\Phi'_2 - \Phi_2\Phi'_1 + \Phi_2\Phi'_2\rangle = G_{11}-G_{12}-G_{21}+G_{22} = 0. \tag{73}$$

[2]For fermionic fields the formalism is analogous, except for some appropriately placed minus signs, see [7].

In a non-equilibrium situation the propagators depend on the two space-time coordinates separately, and one can define several integral representations for the propagators. The double Fourier transform of a function of two arguments is defined as

$$f(p,q) = \int dxdy\, e^{ipx-iqy} f(x,y), \qquad f(x,y) = \int đpđq\, e^{-ipx+iqy} f(x,y), \tag{74}$$

where $đp = d^Dp/(2\pi)^D$ in D dimensions. Note that we define the momentum of the y variable with a relative negative sign! We use the same symbols for the function and its Fourier transform, the type of the argument defines which function we are dealing with. The Wigner transform $\bar{f}$ of a function f with two arguments is defined as

$$\bar{f}(u,X) = f(X+\frac{u}{2}, X-\frac{u}{2}), \qquad f(x,y) = \bar{f}(x-y, \frac{x+y}{2}), \tag{75}$$

i.e. it understands the function to depend on the relative and average coordinates. Fourier transformation with respect to these coordinates yields

$$\bar{f}(Q,K) = \int dudX\, e^{iQu+iKX} \bar{f}(u,X) = \int dxdy\, e^{iQ(x-y)+iK(x+y)/2} f(x,y) = f(Q+\frac{K}{2}, Q-\frac{K}{2}),$$
$$f(p,q) = \bar{f}(\frac{p+q}{2}, p-q). \tag{76}$$

The derivatives with respect to x and y transform as

$$i\partial_x f(x,y) \stackrel{Fourier}{\longrightarrow} pf(p,q) \stackrel{Wigner}{\longrightarrow} (Q+\frac{K}{2})\bar{f}(Q,K),$$
$$i\partial_y f(x,y) \stackrel{Fourier}{\longrightarrow} -qf(p,q) \stackrel{Wigner}{\longrightarrow} (-Q+\frac{K}{2})\bar{f}(Q,K). \tag{77}$$

Later on we will use the Schwinger-Dyson equations for propagators. It is used to resum bubble series

$$G_{ab} = G^0_{ab} + G^0_{ac}\Pi_{cd}G^0_{db} + \ldots = G^0_{ab} + G^0_{ac}\Pi_{cd}G_{db} = G^0_{ab} + G_{ac}\Pi_{cd}G^0_{db}. \tag{78}$$

The indices refer to space-time coordinates as well as to internal indices (e.g. Keldysh indices). To have a differential form we apply the free time evolution kernel on both sides. In the R/A formalism these kernels satisfy "twin" equations, which differ only in the variable to which the derivatives are referring to:

$$(\partial^2_{x,y}+m^2)G^0_{rr}(x,y) = 0 \quad \text{and} \quad (\partial^2_{x,y}+m^2)G^0_{ra}(x,y) = (\partial^2_{x,y}+m^2)G^0_{ar}(x,y) = -\delta(x-y). \tag{79}$$

Using these, we find the SD equations:

$$-(\partial^2_x+m^2)G_{rr} = \int dz\,(\Pi_{aa}G_{ar}+\Pi_{ar}G_{rr}), \quad -(\partial^2_y+m^2)G_{rr} = \int dz\,(G_{ra}\Pi_{aa}+G_{rr}\Pi_{ra})$$
$$-(\partial^2_x+m^2)G_{ra} = \delta(x-y) + \int dz\,\Pi_{ar}G_{ra}, \quad -(\partial^2_y+m^2)G_{ra} = \delta(x-y) + \int dz\, G_{ra}\Pi_{ar},$$
$$-(\partial^2_x+m^2)G_{ar} = \delta(x-y) + \int dz\,\Pi_{ra}G_{ar}, \quad -(\partial^2_y+m^2)G_{ar} = \delta(x-y) + \int dz\, G_{ar}\Pi_{ra}. \tag{80}$$

Here we have used the fact that $G_{aa} = 0$ is consistent with SD equations only if $\Pi_{rr} = 0$. After the Wigner–Fourier transformations the equations in (80) look a little more complicated:

$$\begin{aligned}
\left((Q+\tfrac{K}{2})^2 - m^2\right)\bar{G}_{rr}(Q,K) &= \int\!\bar{d}k\,\Big[\bar{\Pi}_{aa}(Q+\tfrac{k}{2},K-k)\bar{G}_{ar}(Q-\tfrac{K-k}{2},k) + \\
&\qquad + \bar{\Pi}_{ar}(Q+\tfrac{k}{2},K-k)\bar{G}_{rr}(Q-\tfrac{K-k}{2},k)\Big] \\
\left((Q-\tfrac{K}{2})^2 - m^2\right)\bar{G}_{rr}(Q,K) &= \int\!\bar{d}k\,\Big[\bar{\Pi}_{aa}(Q-\tfrac{k}{2},K-k)\bar{G}_{ra}(Q+\tfrac{K-k}{2},k) + \\
&\qquad + \bar{\Pi}_{ra}(Q-\tfrac{k}{2},K-k)\bar{G}_{rr}(Q+\tfrac{K-k}{2},k)\Big] \\
\left((Q+\tfrac{K}{2})^2 - m^2\right)\bar{G}_{ra}(Q,K) &= \bar{\delta}(K) + \int\!\bar{d}k\,\bar{\Pi}_{ar}(Q+\tfrac{k}{2},K-k)\bar{G}_{ra}(Q-\tfrac{K-k}{2},k) \\
\left((Q-\tfrac{K}{2})^2 - m^2\right)\bar{G}_{ra}(Q,K) &= \bar{\delta}(K) + \int\!\bar{d}k\,\bar{\Pi}_{ar}(Q-\tfrac{k}{2},K-k)\bar{G}_{ra}(Q+\tfrac{K-k}{2},k) \\
\left((Q+\tfrac{K}{2})^2 - m^2\right)\bar{G}_{ar}(Q,K) &= \bar{\delta}(K) + \int\!\bar{d}k\,\bar{\Pi}_{ra}(Q+\tfrac{k}{2},K-k)\bar{G}_{ar}(Q-\tfrac{K-k}{2},k) \\
\left((Q-\tfrac{K}{2})^2 - m^2\right)\bar{G}_{ar}(Q,K) &= \bar{\delta}(K) + \int\!\bar{d}k\,\bar{\Pi}_{ra}(Q-\tfrac{k}{2},K-k)\bar{G}_{ar}(Q+\tfrac{K-k}{2},k),
\end{aligned} \tag{81}$$

where $\bar{\delta}(p) = (2\pi)^D\delta(p)$ in D dimensions. The $\pm K$ equations are related to each other by complex conjugation and $K \to -K$ substitution.

B Appendix: Detailed computation of Section 3

In the product of delta functions (15) we observe that the first delta function $\delta(Q^2 + QK + K^2/4 - m^2)$ implies $QK = m^2 - Q^2 - K^2/4$, which yields $\delta(Q^2 - QK + K^2/4 - m^2) \to \delta(2QK)$. This, in turn, means that one can set $QK = 0$ in its coefficient, and therefore we can write $\delta(Q^2 + QK + K^2/4 - m^2) \to \delta(Q^2 + K^2/4 - m^2)$. We obtain in this way the second form of (15). In order to get the third form the following steps are done:

$$\begin{aligned}
D = {}& \delta(2q_0k_0 - 2qkx)\delta(q_0^2 - q^2 + \frac{K^2}{4} - m^2) = \frac{1}{2|k_0|}\delta(q_0 - \frac{qkx}{k_0})\delta(q^2(\frac{k^2x^2}{k_0^2} - 1) + \frac{K^2}{4} - m^2) = \\
= {}& \frac{|k_0|}{2}\frac{1}{|k_0^2 - k^2x^2|}\delta(q_0 - \frac{qkx}{k_0})\delta(q^2 - \frac{k_0^2}{4}\frac{K^2 - 4m^2}{k_0^2 - k^2x^2}) \\
= {}& \frac{|k_0|}{2}\frac{1}{|k_0^2 - k^2x^2|}\delta(q_0 - \frac{qkx}{k_0})\frac{1}{2q}\left[\delta(q - \frac{k_0}{2}\mathcal{Z}) + \delta(q + \frac{k_0}{2}\mathcal{Z})\right] \\
= {}& \frac{1}{2\,\mathcal{Z}\,|k_0^2 - k^2x^2|}\delta(q_0 - \frac{kx}{2}\mathrm{sgn}(k_0)\mathcal{Z})\delta(q - \frac{|k_0|}{2}\mathcal{Z}).
\end{aligned} \tag{82}$$

In the third line we have introduced the quantity $\mathcal{Z} = \sqrt{(K^2 - 4m^2)/(k_0^2 - k^2x^2)}$. In order to get the fourth line we used the fact that, since q is positive, only one of the two delta functions appearing on the third line can be satisfied, therefore $q = |k_0|\mathcal{Z}/2$. The expression is non-zero if $\mathcal{Z}$ is real, i.e. when $(K^2 - 4m^2)/(k_0^2 - k^2x^2) > 0$. To examine this condition we split the K^2 range into two regions:

- $K^2 > 4m^2$; then $k_0^2 > k^2 > k^2x^2$ (since $|x| < 1$), so the numerator and the denominator are positive;
- $K^2 < 4m^2$; then the numerator is negative, the denominator thus has to be negative as well. This yields $k_0^2 < k^2x^2$, which implies $1 > |x| > |k_0|/k$ and $k^2 > k_0^2$, i.e. $K^2 < 0$.

Therefore $\mathcal{Z}$ is real if $K^2 > 4m^2$ or $K^2 < 0$.

We can examine the relative size of $|q_0|$ and $|k_0|$ from the last form of (15). To find it we rewrite $\mathcal{Z}$ as

$$\mathcal{Z}^2 = \frac{K^2 - 4m^2}{k_0^2 - k^2x^2} = 1 - \frac{k^2(1-x^2) + 4m^2}{k_0^2 - k^2x^2}. \tag{83}$$

The numerator is always positive, thus

- if $K^2 > 4m^2$, then $|x| \le 1,\ \mathcal{Z} \le 1$, and one has $|q_0| = \frac{|kx|}{2}\mathcal{Z} \le \frac{k}{2} \le \frac{|k_0|}{2}$;
- if $K^2 < 0$ then $1 \ge |x| \ge |k_0|/k,\ \mathcal{Z} \ge 1$, giving $|q_0| = \frac{|kx|}{2}\mathcal{Z} \ge \frac{|k_0|}{2}$

For the Q-integration appearing in (17)

$$\underset{k_0}{\mathrm{Disc}}\, iR(Q,K) = \frac{4\pi^2}{\sqrt{(k_0^2 - k^2x^2)(K^2-4m^2)}}\delta(q_0 - \frac{kx}{2}\mathrm{sgn}(k_0)\mathcal{Z})\delta(q - \frac{|k_0|}{2}\mathcal{Z}) \times$$
$$\left[\Theta(K^2-4m^2)\left(1 + n(\frac{k_0}{2} - q_0) + n(\frac{k_0}{2} + q_0)\right) + \Theta(-K^2)\left(n(q_0 - \frac{k_0}{2}) - n(q_0 + \frac{k_0}{2})\right)\right] \tag{84}$$

we write the integration measure as

$$\frac{1}{8\pi^3}\int dq_0\, dq\, dx\, q^2. \tag{85}$$

The following change of variable is made:

$$x \to y = \frac{kx}{2}\mathcal{Z}, \quad \frac{dy}{dx} = \frac{k_0^2 k}{2}\sqrt{\frac{K^2 - 4^2}{(k_0^2 - k^2x^2)^3}}, \quad y \in \frac{1}{2}\begin{cases}[-k\mathcal{Z}_1,\, k\mathcal{Z}_1], & \text{if } K^2 > 4m^2\\ [-\infty, -k\mathcal{Z}_1] \cup [k\mathcal{Z}_1, \infty], & \text{if } K^2 < 0\end{cases} \tag{86}$$

where $\mathcal{Z}_1 = \mathcal{Z}(x=1) = \sqrt{1 - 4m^2/K^2}$. Then we have

$$\frac{1}{8\pi^3}\int_{-\infty}^{\infty} dq_0 \int_0^{\infty} dq \left[\Theta(K^2-4^2)\int_{-k\mathcal{Z}_1/2}^{k\mathcal{Z}_1/2} dy + \Theta(-K^2)\int_{-\infty}^{-k\mathcal{Z}_1/2} dy + \Theta(-K^2)\int_{k\mathcal{Z}_1/2}^{\infty} dy\right] \frac{2q^2}{k_0^2 k}\sqrt{\frac{(k_0^2 - k^2x^2)^3}{K^2 - 4m^2}}. \tag{87}$$

Putting the ingredients together we find the result quoted in (18):

$$\int \!\!d\!\!\bar{}\, Q\, \underset{k_0}{\mathrm{Disc}}\, iR(Q,K) = \frac{\mathrm{sgn}(k_0)}{8\pi k}\Big[\ \Theta(K^2-4m^2)\int_{-k\mathcal{Z}_1/2}^{k\mathcal{Z}_1/2} dy\left(1 + n(\frac{|k_0|}{2} - y) + n(\frac{|k_0|}{2} + y)\right) +$$
$$2\Theta(-K^2)\int_{k\mathcal{Z}_1/2}^{\infty} dy\left(n(y - \frac{|k_0|}{2}) - n(y + \frac{|k_0|}{2})\right)\Big]. \tag{88}$$

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