ADM Masses
for Black Strings and $\mathcal{P}$-Branes

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Abstract

An ADM mass formula is derived for a wide class of black solutions with certain spherical symmetry. By applying this formula, we calculate the ADM masses for recently discovered black strings and p-branes in diverse dimensions. By this, the Bogolmol'nyi equation can be shown to hold explicitly. A useful observation is made for non-extremal black p-branes that only for $p = 0$, i.e. for a black hole, can its ADM mass be read directly from the asymptotic behaviour of the metric component g_{00}.

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1. Introduction

Black and extremal string and p-brane solutions have been found recently by a number of authors [1,2,3,4]. As the metric g_{MN} of a black configuration can be written as $g_{MN} = \eta_{MN} + h_{MN}$, with η_{MN} the flat Minkowski metric and h_{MN} not necessarily small everywhere, the total energy density has been given in [1], whose explicit form for extremal black strings has also been given. The explicit form of the total energy density for extremal black p-branes was given later, in [4], where $p = 1$, i.e. string as a special case. Then the ADM mass for an extremal black p-brane follows the integration of the corresponding explicit total energy density over the $(D - p - 1)$-dimensional transverse space. As is well known, the usual black solution with certain spherical symmetry is most conveniently cast in terms of some spherical coordinates. Black strings and p-branes fall into this category, where the formula given in [1] for the total energy density cannot be simply used. We will use the standard definition of the gravitational energy-momentum pseudo-tensor to derive the ADM mass in the next section. In this short note, we derive an ADM mass formula for a class of black solutions with some spherical symmetry, then calculate the ADM masses explicitly for the recently discovered black strings and p-branes. Finally, we use the calculated ADM mass to show that the Bogolmol'nyi equation is satisfied for each of the discovered black strings and p-branes, which provides a way to justify the stability of the corresponding solutions. An observation is made of when the ADM mass of a black configuration can be read directly from the asymptotic behaviour of the g_{00} component of the metric.

2. ADM mass formula

In general relativity, the local energy density of the gravitational field cannot be defined uniquely, even in the weak-field limit. We will adopt the standard definition of the gravitational energy-momentum pseudo-tensor to find the ADM mass for a black hole, the ADM mass per unit length for a black string, and in general the ADM mass per unit volume for a p-brane, in what follows. Let us write $g_{MN} = g_{MN}^{(0)} + h_{MN}$, where $g_{MN}^{(0)}$ is the flat limit of the corresponding space-time metric, for example, it could be Minkowski. In

$D > p+3$, h_{MN} is asymptotically zero but not necessarily small everywhere. In $D = p+3$, h_{MN} is asymptotically logarithmically divergent. Following the discussion for $p = 1$, i.e. string in [1], we pretend to take the above definition for those cases, too. To first order in h_{MN}, the Einstein equation looks like

$$R_{MN}^{(1)} - \frac{1}{2}g_{MN}^{(0)}R^{(1)} = \kappa^2\Theta_{MN}. \tag{2.1}$$

One can take this as the definition of the " total" energy-momentum tensor, to which the ADM mass per unit volume is defined as

$$M_d = \int d^{D-d}y\Theta_{00}, \tag{2.2}$$

for a black $(d-1)$-dimensional extended object. As $g_{MN}^{(0)} = \eta_{MN}$, i.e. the flat Minkowski, the general $R_{MN}^{(1)}$ has been given in [1] as

$$R_{MN}^{(1)} = \frac{1}{2}\left(\frac{\partial^2 h^P{}_M}{\partial x^P\partial x^N} + \frac{\partial^2 h^P{}_N}{\partial x^P\partial x^M} - \frac{\partial^2 h^P{}_P}{\partial x^M\partial x^N} - \frac{\partial^2 h_{MN}}{\partial x^P\partial x_P}\right), \tag{2.3}$$

where the indices are raised and lowered using the flat Minkowski metric. Using the above expression for $R_{MN}^{(1)}$, it is easy to calculate the total energy density for the following metric

$$ds^2 = e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{2B(r)}\delta_{mn}dy^m dy^n, \tag{2.4}$$

as

$$\Theta_{\mu\nu} = \frac{1}{2\kappa^2}\eta_{\mu\nu}\left[(d-1)\frac{\partial^2 e^{2A}}{\partial y^2} + (D-d-1)\frac{\partial^2 e^{2B}}{\partial y^2}\right], \tag{2.5}$$

where D is the space-time dimension; $d-1$ refers to the spatial dimension of the black extended object; $\mu, \nu = 0, 1, \cdots, d-1$; $m, n = d, d+1, \cdots, D-1$, and $r = \sqrt{\delta_{mn}y^m y^n}$. Formula (2.5) has been used to calculate the ADM mass per unit volume for elementary or solitonic (extremal black) p-branes in diverse dimensions in [4], for which the string case ($p = 1$) has been given before in [1]. We would like to stress that eq. (2.3) is not suitable to be used to calculate the total energy density for a metric such as

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 C(r)d\Omega_{D-d-1}^2 + D(r)\delta_{ij}dx^i dx^j, \tag{2.6}$$

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which is just the kind to describe a black $(d-1)$-dimensional extended object with a spherical symmetry $SO(D-d)$, and where $r > 0$, i runs from 1 to $d-1$. The $d\Omega^2_{D-d-1}$ is the line element on a unit $(D-d-1)$-sphere. It is the purpose of this paper to find the total energy density for the metric (2.6) by using eq. (2.1). Now $g^{(0)}_{MN}$ is: $g^{(0)}_{00} = -g^{(0)}_{rr} = -g^{(0)}_{ii} = -1$ and the remaining metric components are just r^2 times those of the unit $(D-d-1)$-sphere. After a rather tedious calculation, we obtain a very simple formula for Θ_{00}:

$$\Theta_{00} = -\frac{1}{2\kappa^2}\frac{1}{r^{\tilde{d}+1}}\partial_r\left[(d-1)r^{\tilde{d}+1}\partial_r D(r) + (\tilde{d}+1)r^{\tilde{d}+1}\partial_r C(r)\right.$$
$$\left. - (\tilde{d}+1)r^{\tilde{d}}\big(B(r)-C(r)\big)\right], \tag{2.7}$$

whose extremal limit goes back to (2.5) as $D = A \to e^{2A}$, $B = C \to e^{2B}$, and where $\tilde{d} = D - d - 2$. By using eq. (2.2), the corresponding ADM mass per unit volume is

$$M_d = -\frac{\Omega_{\tilde{d}+1}}{2\kappa^2}\left[(\tilde{d}+1)r^{\tilde{d}+1}\partial_r C(r) + (d-1)r^{\tilde{d}+1}\partial_r D(r) - (\tilde{d}+1)r^{\tilde{d}}\big(B(r)-C(r)\big)\right]_{r\to\infty}. \tag{2.8}$$

3. The ADM mass per unit volume for black strings and p-branes

We would like to calculate the ADM mass per unit volume for the discovered black strings and p-branes in [2,4], by using the formula developed in the last section. Before jumping to a calculation of those ADM masses, we wish to give a brief review of those black solutions. As discussed in detail in [2,4], a $(D-d-3)$-brane solution with magnetic charge $\frac{1}{\sqrt{2}\kappa}\int F_{d+1}$ can be found from the part of the bosonic sector of the D-dimensional supergravity action, which, in terms of canonical variable, is

$$I_D = \frac{1}{2\kappa^2}\int d^D x\sqrt{-g}\left[R - \frac{1}{2}(\partial\phi)^2 - e^{-\alpha(d)\phi}\frac{1}{(d+1)!}F^2_{d+1}\right], \tag{3.1}$$

where

$$\alpha^2(d) = 4 - \frac{2d\tilde{d}}{d+\tilde{d}}, \tag{3.2}$$

with $\tilde{d} = D - d - 2$. It is found that finding a $(D-d-3)$-brane solution from the above action is equivalent to finding a black-hole solution from the following action

$$I_{d+3} = \int d^{d+3}\hat{x}\sqrt{-\hat{g}}\left[\hat{R} - \frac{1}{2}(\hat{\nabla}\rho)^2 - \frac{1}{2}(\hat{\nabla}\sigma)^2 - \frac{1}{2(d+1)!}e^{\beta\rho}\hat{F}^2_{d+1}\right], \tag{3.3}$$

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through the following field redefinitions

$$\beta W = -\frac{d(\tilde{d}-1)}{(D-2)(d+1)}\rho + \sqrt{\frac{\tilde{d}-1}{2(D-2)(d+1)}}\alpha(d)\sigma,$$

$$\beta A = \frac{d}{D-2}\rho - \sqrt{\frac{d+1}{2(D-2)(\tilde{d}+1)}}\alpha(d)\sigma, \tag{3.4}$$

$$\beta\phi = -\alpha(d)\rho - \frac{2d}{\sqrt{2(D-2)}}\frac{\tilde{d}-1}{d+1}\sigma.$$

The above β is,

$$\beta = -\sqrt{\frac{2(d+2)}{d+1}}, \tag{3.5}$$

and W and A are defined through

$$ds^2 = e^{2W}d\hat{s}^2 + e^{2A}dx_i dx^i, \tag{3.6}$$

where i runs from 1 to $D-d-3$, the spatial dimension of the extended object, and $d\hat{s}^2$ is the rescaled metric of the remaining dimension, which is the one used in the action (3.3). A, W and $d\hat{s}^2$ are independent of x_i in order to have $D-d-3$ dimensional translation and rotation symmetries. The charged static black-hole solutions with spherical symmetry $SO(d+2)$ to the equations of motion of (3.3) are asymptotically flat and have a regular horizon. They are

$$F = Q\epsilon_{d+1},$$

$$d\hat{s}^2 = -\left[1-\left(\frac{r_+}{r}\right)^d\right]\left[1-\left(\frac{r_-}{r}\right)^d\right]^{1-\gamma d}dt^2$$

$$+\left[1-\left(\frac{r_+}{r}\right)^d\right]^{-1}\left[1-\left(\frac{r_-}{r}\right)^d\right]^{\gamma-1}dr^2$$

$$+r^2\left[1-\left(\frac{r_-}{r}\right)^d\right]^{\gamma}d\Omega_{d+1}^2, \tag{3.7}$$

$$e^{\beta\rho} = \left[1-\left(\frac{r_-}{r}\right)^d\right]^{\gamma d},$$

$$\sigma = 0,$$

where the exponent is

$$\gamma = \frac{d+2}{d(d+1)}. \tag{3.8}$$

Q is the charge of the black hole, and the ϵ_n is the volume element of the unit n-sphere. This is a two-parameter family of solutions labelled by r_+ and r_-. These two parameters are algebraically related to the charge and mass of the black hole. The charge Q is given by

$$Q = d(r_+ r_-)^{d/2},$$
$$(3.9)$$

and the mass is proportional to

$$M = -\frac{r_-^d}{d+1} + r_+^d,$$
$$(3.10)$$

with a convention-dependent proportionality constant. As discussed in [2], if $r_- = 0$, then $F = 0, \rho = 0$ and the above metric reduces to the $(d+3)$-dimensional Schwarzschild black hole. At $r = r_+$, the time-like Killing field becomes null, and there is an event horizon and the curvature is finite there. Since $\gamma > 0$ for $d \geq 1$, at $r = r_-$, the area of the sphere goes to zero and there is a curvature singularity. Thus these solutions describe black holes only as $r_+ > r_-$.

By using (3.4), (3.6) and (3.7), one obtains black $(D - d - 3)$-brane solutions of (3.1):

$$F = Q\epsilon_{d+1},$$

$$ds^2 = -\left[1 - \left(\frac{r_+}{r}\right)^d\right]\left[1 - \left(\frac{r_-}{r}\right)^d\right]^{\gamma_x - 1} dt^2$$

$$+ \left[1 - \left(\frac{r_+}{r}\right)^d\right]^{-1}\left[1 - \left(\frac{r_-}{r}\right)^d\right]^{\gamma_\Omega - 1} dr^2$$

$$+ r^2\left[1 - \left(\frac{r_-}{r}\right)^d\right]^{\gamma_\Omega} d\Omega_{d+1}^2$$

$$+ \left[1 - \left(\frac{r_-}{r}\right)^d\right]^{\gamma_x} dx_i dx_i,$$

$$e^{-2\phi} = \left[1 - \left(\frac{r_-}{r}\right)^d\right]^{\gamma_\phi},$$

$$(3.11)$$

where

$$\gamma_x = \frac{d}{D-2},$$

$$\gamma_\Omega = \frac{\alpha^2(d)}{2d},$$

$$\gamma_\phi = \alpha(d).$$

$$(3.12)$$

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The metric ds^2 in (3.11) fits the general expression (2.6), but $r > r_-$ rather than $r > 0$. In order to use the ADM mass formula developed in the last section, we must do the replacement $r^d \to r^d + r_-^d$ in (3.11). The resulting metric ds^2 looks as

$$ds^2 = -\left[1 + \frac{r_+^d}{r^d - (r_+^d - r_-^d)}\right]^{-1}\left[1 + \left(\frac{r_-}{r}\right)^d\right]^{\frac{\tilde{d}}{D-2}} dt^2$$
$$+ \left[1 + \left(\frac{r_-}{r}\right)^d\right]^{\frac{\tilde{d}}{D-2}}\left[\frac{1 + \frac{r_+^d}{r^d - (r_+^d - r_-^d)}}{1 + \left(\frac{r_-}{r}\right)^d} dr^2 + r^2 d\Omega_{d+1}^2\right] \qquad (3.13)$$
$$+ \left[1 + \left(\frac{r_-}{r}\right)^d\right]^{-\frac{d}{D-2}} dx^i dx^i.$$

Reading, from the above metric, the $B(r), C(r)$ and $D(r)$ in (2.8), we have the ADM mass per unit volume for the black $(D - d - 3)$-brane

$$M_{\tilde{d}} = \frac{\Omega_{d+1}}{2\kappa^2}\left[(d+1)r_+^d - r_-^d\right], \qquad (3.14)$$

where Ω_n is the volume of the unit n-sphere.

4. Discussion

From the brief review at the beginning of the last section, we know that a black $(D - d - 3)$-brane solution from the action (3.1) is essentially a black-hole solution from the effective action (3.3). Is there any relation between their ADM masses? Examining eqs. (3.10) and (3.14), we find that they are proportional to each other, so we expect that they are actually the same, since eq. (3.10) is determined only up to an overall factor. It is well known that an ADM mass for a black hole can be read directly from the asymptotic behaviour of the 00-component of the metric. One may ask: does this rule apply to black (non-extremal) p-branes with $p > 0$? The answer is simply no. One can check that only for $\tilde{d} = 1$, i.e. a black hole, the ADM mass from (3.14) agrees with what you read from (3.13). Therefore, finding a black extended object through a black hole provides also a simple way to determine its ADM mass per unit volume.

We have established a Bogolmol'nyi bound for an extremal black, so-called elementary (electric-like) or solitonic (magnetic-like) $(d - 1)$-brane in [3,4], which is

$$\kappa M_d \geq \frac{1}{\sqrt{2}}|e_d|, \qquad (4.1)$$

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for an elementary one, or

$$\kappa M_{\tilde{d}} \geq \frac{1}{\sqrt{2}} |g_{\tilde{d}}|, \tag{4.2}$$

for a solitonic one, where

$$e_d = \frac{1}{\sqrt{2}\kappa} \int e^{-\alpha(d)\star} F_{d+1}, \tag{4.3}$$

and

$$g_{\tilde{d}} = \frac{1}{\sqrt{2}\kappa} \int F_{d+1}. \tag{4.4}$$

We will show that the Bogolmol'nyi bound (4.2) does hold for the black strings and p-branes discussed in the last section. The magnetic charge for a black $(\tilde{d} - 1)$-brane, from (3.9) and (3.11), is

$$g_{\tilde{d}} = \frac{\Omega_{d+1}}{\sqrt{2}\kappa} d(r_+ r_-)^{d/2}. \tag{4.5}$$

By noticing that $r_+ \geq r_-$ and $(d+1)r_+^d - r_-^d \geq dr_+^d \geq d(r_+ r_-)^{d/2}$, we finish our proof.

In the previous sections, we have used only the canonical metric, which is the one used in the usual Einstein-Hilbert action, and probably the most suitable to be used to define the ADM mass. However, some authors may take, for example, the string σ-model metric to define the ADM mass. This actually happens in the literature, for example, Giddings and Strominger used this kind of ADM mass in discussing exact 5-branes in critical superstring theory [5]. In what follows, an ADM mass per unit $(\tilde{d} - 1)$-brane volume calculated in $(n-1)$-brane metric is given. The relation between the $(n-1)$-brane σ-model metric and the canonical metric is

$$g_{MN}(n) = e^{\frac{\alpha(n)\phi}{n}} g_{MN}(\text{canonical}), \tag{4.6}$$

where ϕ is the dilaton field and $\alpha(n)$ is given by (3.2) upon taking $d = n$. For example, taking $n = 2$, we have $\alpha(2) = 1$ from (3.2), eq. (4.6) gives just the familiar relation between the string σ-model metric and the canonical metric. By using this metric, repeating what we have done in the last two sections, we find

$$M_{\tilde{d}} = \frac{\Omega_{d+1}}{2\kappa^2} \left[\frac{d(D-2)\alpha(d)\alpha(n)}{2n} r_-^d + (d+1)r_+^d - r_-^d \right]. \tag{4.7}$$

Taking $n = 2$ and $D = 10$ in the above, we have the ADM mass for a black 5-brane: $M_6 = \frac{3\Omega_3}{2\kappa^2}[r_+^2 + r_-^2]$, which is the one used by Giddings and Strominger in [5].

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References

[1] A. Dabholkar, G. W. Gibbons, J. A. Harvey and F. Ruiz Ruiz, Nucl. Phys. **B340** (1990) 33.

[2] G. Horowitz and A. Strominger, Nucl. Phys. **B360** (1991) 197.

[3] M. J. Duff and J. X. Lu, Nucl. Phys. **B354** (1991) 141.

[4] M. J. Duff and J. X. Lu, "Black and super p-branes in diverse dimensions", CERN-TH.6675/92 and CTP–TAMU–54/92 (1992).

[5] S. B. Giddings and A. Strominger, Phys. Rev. Lett. **67** (1991) 2930.