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BOUNDED POINT EVALUATIONS FOR CYCLIC HYPONORMAL OPERATORS

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Abstract

It is shown in [1] that the Question A of L. R. Williams *Dynamic systems and Applications* 3(1994) 103-112 has a negative answer for arbitrary cyclic operators. In this paper, we prove that the answer to this question is affirmative in the case of cyclic hyponormal operators. Furthermore, we discuss the fat local spectra property for cyclic hyponormal operators.

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1. INTRODUCTION

Throughout this paper, $\mathcal{L}(H)$ will denote the algebra of all linear bounded operators on a complex separable Hilbert space H. For an operator $T \in \mathcal{L}(H)$, let T^* denote its adjoint, $\sigma(T)$ its spectrum, $\sigma_p(T)$ its point spectrum, $\sigma_{ap}(T)$ its approximate point spectrum, $\Gamma(T)$ its compression spectrum, $\sigma_e(T)$ its essential spectrum, $\ker T$ its kernel, ran$T$ its range, $r(T)$ its spectral radius, $m(T) = \inf\{\|Tx\| \ : \ \|x\| = 1\}$ its lower bound and let $r_1(T)$ denote $\sup\limits_{n\geq 1}[m(T^n)]^{\frac{1}{n}}$ which equals $\lim\limits_{n\to+\infty}[m(T^n)]^{\frac{1}{n}}$. For a subset M of H, let $\bigvee M$ denote the closed linear subspace generated by M and let $M^\perp$ denote its orthogonal space in H. For any subset Y of a topological space X we shall denote cl(Y) its closure in X and for a complex number $\lambda \in \mathbb{C}$ we shall denote $\overline{\lambda}$ its conjugate element.

Let $T \in \mathcal{L}(H)$ be a cyclic operator on H with cyclic vector $x \in H$ i.e., $H = \bigvee\{T^n x \ : \ n \geq 0\}$. A complex number $\lambda \in \mathbb{C}$ is said to be a *bounded point evaluation* of T if there is a constant $M > 0$ such that for every complex polynomial p,

$$|p(\lambda)| \leq M\|p(T)x\|.$$

The set of all bounded point evaluations of T will be denoted by $B(T)$. Note that it follows from Riesz Representation Theorem that $\lambda \in B(T)$ if and only if there is a unique vector denoted $k(\lambda) \in H$ such that $p(\lambda) = \langle p(T)x \ , \ k(\lambda)\rangle$ for every complex polynomial p. An open subset O of $\mathbb{C}$ is said to be an *analytic set* for T if it is contained in $B(T)$ such that for every $y \in H$, the complex function $\widehat{y}$ defined on $B(T)$ by $\widehat{y}(\lambda) = \langle y \ , \ k(\lambda)\rangle$, is analytic on O; equivalently, if the function $\lambda \longmapsto \|k(\lambda)\|$ is bounded on compact subsets of O (see [14, Lemma 1.2]). The largest analytic set for T will be denoted by $B_a(T)$ and its points will be called *analytic bounded point evaluations* for T.

In the following results, T will denote a cyclic operator in $\mathcal{L}(H)$. The proofs can be found in [3], [13] and [14].

Proposition 1.1. *The following are equivalent.*

(i) $\lambda \in B(T)$.
(ii) $\lambda \in \Gamma(T)$.
(iii) $\ker\big((T-\lambda)^*\big)$ *is one dimensional.*

Proposition 1.2. $\Gamma(T)\backslash\sigma_{ap}(T) \subset B_a(T)$.

An operator $T \in \mathcal{L}(H)$ is said to be *subnormal* if it has a normal extension and it is said to be *hyponormal* if $\|T^*x\| \leq \|Tx\|$ for every $x \in H$. Note that every subnormal operator is hyponormal. Using the analytic function theory, Tavan Trent proved in [13] that

$$B_a(T) = \Gamma(T)\backslash\sigma_{ap}(T)$$

for every cyclic subnormal operator $T \in \mathcal{L}(H)$; L. R. Williams asked ([14] Question A) if this equality remains valid for arbitrary cyclic operators. However, it is shown in [1] that the containment in Proposition 1.2 fails to be equality for arbitrary cyclic operators. In the present paper, we shall prove that the answer to Question A of L. R. Williams is affirmative for arbitrary cyclic hyponormal operators using a different proof of Tavan Trent; our proof is inspired by the papers of A. Bourhim, C. E. Chidume and E. H. Zerouali [1] and T. L. Miller and V. G. Miller [6].

2. STATEMENT AND PROOFS OF MAIN RESULTS

Before outlining the statement of our main results, let us introduce some definitions. For an open subset U of $\mathbb{C}$, let $\mathcal{H}(U,H)$ denote the space of analytic $H-$valued functions on U. Equipped with the topology of uniform convergence on compact subsets of U, the space $\mathcal{H}(U,H)$ is a Fréchet space. Note that every operator $T \in \mathcal{L}(H)$ induces a continuous mapping T_U on $\mathcal{H}(U,H)$ defined by $T_U f(\lambda) = (T-\lambda)f(\lambda)$ for $f \in \mathcal{H}(U,H)$ and $\lambda \in U$. An operator $T \in \mathcal{L}(H)$ is said to have Bishop's property (β) provided that for every open subset U of $\mathbb{C}$ the mapping T_U is injective and has a closed range; equivalently, if for every open subset U of $\mathbb{C}$ and for every sequence $(f_n)_n$ of $\mathcal{H}(U,H)$, the convergence $(T-\lambda)f_n(\lambda) \longrightarrow 0$ in $\mathcal{H}(U,H)$ should always entail the convergence to 0 of the sequence $(f_n)_n$ in $\mathcal{H}(U,H)$. M. Putinar [7] has shown that hyponormal operators have Bishop's property (β).

Throughout this section, T will be a cyclic operator of $\mathcal{L}(H)$ with cyclic vector $x \in H$. We have the following result.

Theorem 2.1. *If the operator T is hyponormal then $B_a(T) = \Gamma(T)\backslash\sigma_{ap}(T)$.*

This theorem follows from the following more general result.

Theorem 2.2. *If the operator T has Bishop's property (β) then $B_a(T) = \Gamma(T)\backslash\sigma_{ap}(T)$ if and only if $B_a(T) \cap \sigma_p(T) = \emptyset$.*

Proof. If $B_a(T) = \Gamma(T)\backslash\sigma_{ap}(T)$ then it is clear that $B_a(T) \cap \sigma_p(T) = \emptyset$ since $\sigma_p(T) \subset \sigma_{ap}(T)$. Conversely, suppose that $B_a(T) \cap \sigma_p(T) = \emptyset$. Since $B_a(T) \subset B(T) = \Gamma(T)$ and $\Gamma(T)\backslash\sigma_{ap}(T) \subset B_a(T)$ (see Proposition 1.2) then it suffices to prove that $B_a(T) \cap \sigma_{ap}(T) = \emptyset$. Suppose that there is $\lambda \in B_a(T) \cap \sigma_{ap}(T)$. It follows then that the range $\mathrm{ran}(T-\lambda)$ of $(T-\lambda)$ is not closed. Let $y \in \mathrm{cl}\Big(\mathrm{ran}(T-\lambda)\Big)\backslash\mathrm{ran}(T-\lambda)$, then

$$y \notin \mathrm{ran}(T-\lambda) \text{ and } \langle y\ ,\ k(\lambda)\rangle = 0.$$

Therefore, there is a sequence of polynomials $(p_n)_n$ vanishing at λ such that $p_n(T)x$ converges to y in H. Define on $U := B_a(T)$ the following analytic $H-$valued functions by $f(\mu) = y - \widehat{y}(\mu)x$ and $f_n(\mu) = p_n(T)x - p_n(\mu)x$. Since, $f(\lambda) = y \notin \mathrm{ran}(T-\lambda)$ then, $f \notin \mathrm{ran}(T_U)$. On the other hand, it is easy to see that $f_n \in \mathrm{ran}(T_U)$ for every $n \geq 0$. Now, let K be a compact subset of U, we have

$$\begin{aligned}\sup_{\mu\in K}\|f_n(\mu)-f(\mu)\| &\le \|p_n(T)x-y\|+\sup_{\mu\in K}\|\left[p_n(\mu)-\widehat{y}(\mu)\right]x\|\\ &\le \|p_n(T)x-y\|+\|x\|\sup_{\mu\in K}|p_n(\mu)-\widehat{y}(\mu)|\\ &\le \left[1+\|x\|\sup_{\mu\in K}\|k(\mu)\|\right]\|p_n(T)x-y\|.\end{aligned}$$

Therefore, $f_n \longrightarrow f$ in $\mathcal{H}(U,H)$. Thus, the range $\text{ran}(T_U)$ of T_U is not closed which contradicts T has Bishop's property (β) and the proof is complete.

The following is immediate.

Corollary 2.3. *If the operator T has Bishop's property (β) such that $\sigma_p(T)=\emptyset$, then $B_a(T)=\Gamma(T)\backslash\sigma_{ap}(T)$.*

We also have.

Corollary 2.4. *If the operator T has Bishop's property (β) such that $H=\bigvee\{k(\lambda)\ :\ \lambda\in B_a(T)\}$ then $B_a(T)=\Gamma(T)\backslash\sigma_{ap}(T)$.*

Proof. Let $\lambda\in B_a(T)$. Suppose that there is $y\in H$ such that $Ty=\lambda y$. For every $\mu\in B_a(T)$, we have,

$$\begin{aligned}\lambda\widehat{y}(\mu) &= \langle Ty\ ,\ k(\mu)\rangle\\ &= \langle y\ ,\ T^*k(\mu)\rangle\\ &= \mu\widehat{y}(\mu).\end{aligned}$$

Hence, the analytic function $\widehat{y}$ is identically zero on $B_a(T)$; and so, $y=0$ since $H=\bigvee\{k(\lambda)\ :\ \lambda\in B_a(T)\}$. Therefore, $B_a(T)\cap\sigma_p(T)=\emptyset$. Thus the proof is complete.

Proof of Theorem 2.1. Let $\lambda\in B_a(T)$. Suppose that there is $y\in H$ such that $Ty=\lambda y$. It follows that the analytic function $\widehat{y}$ is identically zero on $B_a(T)$. On the other hand, $y=\alpha k(\lambda)$, for some $\alpha\in\mathbb{C}$, since $T^*y=\overline{\lambda}y$. And so, $\widehat{y}(\lambda)=\alpha\|k(\lambda)\|^2=0$; it follows that $y=0$. Therefore, $B_a(T)\cap\sigma_p(T)=\emptyset$. Thus the proof is complete.

Remark 2.5. *Recall that the operator is said to be M-hyponormal if there is a positive constant M such that*

$$\|T^*y\|\le M\|Ty\| \text{ for every } y\in H.$$

Recall that M-hyponormal operators posses Bishop's property (β) (see [7] and [17]). Using the same proof of theorem 2.1, we deduce that

$$B_a(T)=\Gamma(T)\backslash\sigma_{ap}(T)$$

for every cyclic M-hyponormal operator T.

In [17], L. Yang has proved that quasi-similar hyponormal operators have equal essential spectra. In his paper [14], L. R. Williams mentioned that if the answer to his Question A is affirmative for arbitrary cyclic hyponormal operators, then, using Raphael's techniques (see [9]) and [14, Theorem 1.5], one has a simple proof of this following theorem.

Theorem 2.6. *Two quasi-similar cyclic hyponormal operators have equal essential spectra.*

3. ON A QUESTION OF L. R. WILLIAMS

We first recall a few basic notions and properties from the local spectral theory which will be needed in the sequel. Let $T \in \mathcal{L}(H)$ be a bounded operator on H. For an element $x \in H$, let $\sigma_T(x)$ be its local spectrum and $\rho_T(x)$ be its local resolvent (see [2] and [5]). For a closed subset F of $\mathbb{C}$, let $H_T(F) = \{x \in H \ : \ \sigma_T(x) \subset F\}$ be the corresponding analytic spectral linear subspace. The operator T is said to have the *single valued extension property* (SVEP) if zero is the unique vector $x \in H$ such that $\sigma_T(x) = \emptyset$; equivalently, if the mapping T_U is injective for every open set $U \subset \mathbb{C}$. Note that if T has SVEP, then for every $x \in H$ there exists a unique maximal analytic solution $\widetilde{x}$ on $\rho_T(x)$ for which $(T - \lambda)\widetilde{x}(\lambda) = x$ for all $\lambda \in \rho_T(x)$, and satisfies $\widetilde{x}(\lambda) = -\sum_{n\geq 0} \frac{T^n x}{\lambda^{n+1}}$ on $\{\lambda \in \mathbb{C} \ : \ |\lambda| > r_T(x)\}$ where $r_T(x) = \limsup_{n\to+\infty} \|T^n x\|^{\frac{1}{n}}$ is the *local radius* of T at x; if in addition T is invertible then $\widetilde{x}(\lambda) = \sum_{n\geq 1} \lambda^{n-1} T^{-n} x$ on $\{\lambda \in \mathbb{C} \ : \ |\lambda| < \frac{1}{r_{T^{-1}}(x)}\}$. The operator T is said to satisfy *Dunford's Condition* (*C*) (DCC) if for every closed subset F of $\mathbb{C}$, the linear subspace $H_T(F)$ is closed. It is known that every operator which has Bishop's property (β) satisfies DCC (see [5]), and every operator which satisfies DCC has the single valued extension property (see [2] and [5]). The operator T is said to have *fat local spectra* if $\sigma_T(x) = \sigma(T)$ for all non-zero $x \in H$. It is clear that every operator with fat local spectra has DCC. The operator T is said to be pure if the only reducing subspace M of T such that $T_{|M}$ is normal is $M = \{0\}$.

Next, we recall the open Question B of L. R. Williams [14].

Question. Does a pure cyclic hyponormal operator have fat local spectra?

Note that the answer is affirmative in the case of hyponormal weighted shifts (see [1, Theorem 3.7] and [15, Theorem 2.5]). At present, however, we do not have neither a counterexample nor an affirmative answer. The following result gives the necessary conditions for the fat local spectra property.

Lemma 3.1. *Let $T \in \mathcal{L}(H)$ be an operator which has fat local spectra. The following properties hold.*

(i) *T has* DCC.

(ii) *$\sigma_p(T) = \emptyset$ whenever $\sigma(T)$ is not a singleton.*

(iii) $r_{_T}(x) = r(T)$ *for every non-zero element* $x \in H$.

(iv) *If* T *is invertible then* $r_{_{T^{-1}}}(x) = r(T^{-1})$ *for every* $x \in H\backslash\{0\}$.

(v) $\sigma(T)$ *is connected set.*

Proof. The first four properties are trivial. Suppose that $\sigma(T)$ is disconnected then $\sigma(T) = \sigma_1 \cup \sigma_2$, where σ_1 and σ_2 are two non-empty disjoint compacts. Let $x \in H$, $x \neq 0$ then using local version of Riesz′s functional calculus, one shows that there are two non-zero elements x_1, x_2 of H such that $x = x_1 + x_2$ and $\sigma_T(x_i) \subset \sigma_i$, $i = 1, 2$ which contradicts T have fat local spectra.

Remark 3.2. *It follows from Lemma 3.1 that every pure cyclic hyponormal operator on* H *satisfies the first two properties of Lemma 3.1. A pure cyclic hyponormal operator in* $\mathcal{L}(H)$ *such that its spectrum is disconnected provides a negative answer to the question above.*

After submitting the first version of the present paper, we received from L. R. Williams the reference [16], where the author produced a pure cyclic subnormal operator with disconnected spectrum; namely, the Bergman shift operator on $L^2_a(G)$ where G is a union of two open discs with disjoint closures. We are thankful to L. R. Williams for pointing out to us this reference [16].

Following example 4.1 of [16], one can give a large class of such operators which provide a negative answer to L. R. Williams's question. Let $T \in \mathcal{L}(H)$ be a pure cyclic hyponormal operator, by setting $T_\lambda = T \oplus (T - \lambda) \in \mathcal{L}(H \oplus H)$ for $\lambda \in \mathbb{C}$, $|\lambda| > \|T\|$. It is easy to show that T_λ is a pure cyclic hyponormal operator with disconnected spectrum.

Following the proof of Theorem 2.4 of [14].

Proposition 3.3. *Let* $T \in \mathcal{L}(H)$ *be a cyclic hyponormal operator. If* T *has fat local spectra then* $H = \bigvee\{k(\lambda)\ :\ \lambda \in G\}$ *for every connected component* G *of* $B_a(T)$.

Proof. Since T is a hyponormal operator then $\bigcap\limits_{\lambda \in G} \operatorname{ran}(T - \lambda) = H_T(\mathbb{C}\backslash G)$. On the other hand, T has fat local spectra then $H_T(\mathbb{C}\backslash G) = \{0\}$. And so,

$$\begin{aligned} \bigvee\{k(\lambda)\ :\ \lambda \in G\} &= \bigvee_{\lambda \in G} \ker\big((T-\lambda)^*\big) = \Big(\bigcap_{\lambda \in G} \operatorname{cl}\big(\operatorname{ran}(T-\lambda)\big)\Big)^{\perp} \\ &= \Big(\bigcap_{\lambda \in G} \operatorname{ran}(T-\lambda)\Big)^{\perp} = H. \end{aligned}$$

Conversely, we have the following.

Proposition 3.4. *Let* $T \in \mathcal{L}(H)$ *be a cyclic operator. If* $H = \bigvee\{k(\lambda)\ :\ \lambda \in G\}$ *for some connected component* G *of* $B_a(T)$ *then* $\operatorname{cl}(G) \subset \sigma_{_T}(x)$ *for every non-zero* $y \in H$.

Proof. Let G be a connected component of $B_a(T)$. Suppose that there is $y \in H$ such that $G \cap \rho_T(y) \neq \emptyset$. And so, there is an analytic $H-$ valued function f such that

$$(T-\lambda)f(\lambda) = y \text{ for } \lambda \in V,$$

where $V = G \cap \rho_T(y)$. Hence, for every $\lambda \in V$, we have

$$\begin{aligned} \widehat{y}(\lambda) &= \langle y \ , \ k(\lambda)\rangle \\ &= \langle (T-\lambda)f(\mu) \ , \ k(\lambda)\rangle \\ &= \langle f(\lambda) \ , \ (T-\lambda)^* k(\lambda)\rangle \\ &= 0. \end{aligned}$$

Therefore, $\widehat{y} \equiv 0$ on G. Since, $H = \bigvee\{k(\lambda) \ : \ \lambda \in G\}$, it follows then $y = 0$. Thus the desired result holds.

Corollary 3.5. *If $T \in \mathcal{L}(H)$ is a cyclic hyponormal operator such that $\sigma(T) = \mathrm{cl}\big(B_a(T)\big)$, then T has fat local spectra if and only if $H = \bigvee\{k(\lambda) \ : \ \lambda \in G\}$ for every connected component G of $B_a(T)$.*

4. EXAMPLES AND COMMENTS

The weighted shift operators are interesting for solving a lot of problems in operator theory, they are a rich source of examples and counterexamples to illustrate many properties of operators. We shall fix some terminology and recall some basic notions concerning their spectral theory; the survey of A. Shields [11] contains further information. Let S be a unilateral weighted shift on a Hilbert space H with a positive bounded weight sequence $(\omega_n)_{n\geq 0}$, that is

$$Se_n = \omega_n e_{n+1},$$

where $(e_n)_{n\geq 0}$ is an orthonormal basis of H. Let W be the following sequence given by:

$$W_n = \begin{cases} \omega_0 \ldots \omega_{n-1} & \text{if } n > 0 \\ 1 & \text{if } n = 0 \end{cases}$$

Note that,

$$r_1(S) = \lim_{n\to\infty}\left[\inf_{k\geq 0}\frac{W_{n+k}}{W_k}\right]^{\frac{1}{n}} \text{ and } r(S) = \lim_{n\to\infty}\left[\sup_{k\geq 0}\frac{W_{n+k}}{W_k}\right]^{\frac{1}{n}};$$

and define,

$$r_2(S) = \liminf_{n\to\infty}\left[W_n\right]^{\frac{1}{n}} \text{ and } r_3(S) = \limsup_{n\to\infty}\left[W_n\right]^{\frac{1}{n}}.$$

Note that, $r_1(S) \leq r_2(S) \leq r_3(S) \leq r(S)$ and recall that for every three non-negative numbers $a, \ b$ and c with $a \leq b \leq c$ there is a weighted shift operator S such that

$$r_1(S) = a, \ r_2(S) = b \text{ and } r(S) = c.$$

On the other hand, one can find a weighted shift operator S such that

$$r_1(S) = a, \ r_3(S) = b \text{ and } r(S) = c.$$

For details see W. C. Ridge [10].

We begin first with an example of a weighted shift operator which shows us that the converse of corollary 2.3 is not valid. Let S be a weighted shift such that $r_1(S) = r_2(S) \leq r_3(S) < r(S)$. We have $B_a(S) = \Gamma(S)\backslash\sigma_{ap}(S)$ (see [1, Theorem 2.2]). Let $F = \sigma_S(e_0)$, the linear subspace $H_S(F)$ is not closed, otherwise, $H_S(F) = H$ since it contains the cyclic vector e_0 for S, and so, it follows from [5, Proposition 1.3.2] that $\sigma(S) = \bigcup_{x\in H} \subset F = \sigma_S(e_0)$ which is impossible since $r_S(e_0) = r_3(S) < r(S)$. Thus, S does not have neither DCC nor Bishop's property (β).

It remains to give an example of a weighted sequence $(\omega_n)_n$ so that the corresponding weighted shift S satisfies $r_1(S) = r_2(S) \leq r_3(S) < r(S)$. Let $(C_k)_{k\geq 0}$ be a sequence of successive disjoint segments covering $\mathbb{N}$ and such that each segment C_k contains k^2 elements. Let $k \in \mathbb{N}$, for $n \in C_k$ we set

$$\omega_n = \begin{cases} 2 & \text{if } n \text{ is one of the first } k^{th} \text{ terms of } C_k \\ 1 & \text{otherwise} \end{cases} \tag{4.1}$$

For every $n \geq 1$, there are two unique integers $r = r(n)$ and $s = s(n)$ such that $n = s + \sum_{k=1}^{r} k^2$ with $0 \leq s \leq (r+1)^2$ (i.e., $n \in C_{r+1}$). Hence,

$$W_{n+1} = \Big(\prod_{k=1}^{r} \prod_{j\in C_k} \omega_j\Big) \times \begin{cases} 2^s & \text{if } s \leq r+1 \\ 2^{r+1} & \text{otherwise} \end{cases}$$

Therefore, it is easy to see that for every $n \in \mathbb{N}$,

$$\sup_k \frac{W_{n+k}}{W_k} = 2^n \quad \text{and} \quad \inf_k \frac{W_{n+k}}{W_k} = 1.$$

And so, $r(S) = 2$ and $r_1(S) = 1$. On the other hand, for every $n \geq 1$, we have

$$W_{n+1} \leq 2^{\sum_{k=1}^{r+1} k} = 2^{\frac{(r+1)(r+2)}{2}}.$$

Since, $n \geq \sum_{k=1}^{r} k^2 = \frac{(2r+1)(r+1)r}{6}$, then

$$[W_{n+1}]^{\frac{1}{n+1}} \leq 2^{\frac{3(r+2)}{(2r+1)r}}.$$

And so, this inequality implies that $r_3(S) \leq 1$. Therefore,

$$r_1(S) = r_2(S) = r_3(S) = 1 \quad \text{and} \quad r(S) = 2.$$

Now let us see that in Theorem 2.2, one cannot omit the assumption that the operator T has Bishop's property (β) nor suppose that it satisfies only DCC neither suppose that it has fat local spectra. Let S be a weighted shift such that $r_1(S) < r_2(S) = r(S)$. It is shown in [1] that $\{\lambda \in \mathbb{C} \;:\; |\lambda| \leq r_2(S)\} \subset \sigma_S(x)$ for every non-zero element $x \in H$; and so, S has fat local spectra therefore it has DCC. On the other hand, $\Gamma(S)\backslash\sigma_{ap}(S) \subsetneq B_a(S)$ (see [1, Theorem 2.2]) and so, according to Theorem 2.2, S is without Bishop's property (β).

As before, it also remains to produce an example of a such weighted shift. If we modify the last example and we change in (4.1) 2 by 1 and 1 by $\frac{1}{2}$, we get by the same computation that

$$r_1(S) = \frac{1}{2} \quad \text{and} \quad r(S) = r_2(S) = 1.$$

It would be interesting to give an example of a pure cyclic hyponormal operator with fat local spectra which is not a weighted shift hyponormal operator. Let us consider the following operator $T = S^* + 2S$ where S is the unweighted shift on H i.e. $Se_n = e_{n+1}$ for every $n \in \mathbb{N}$ with $(e_n)_n$ is an orthonormal basis of H. A simple computation shows that for every $x = \sum_n \alpha_n e_n \in H$ we have

$$\big\langle (T^*T - TT^*)x \ , \ x\big\rangle = 3\alpha_0^2 \geq 0.$$

Hence, T is a hyponormal operator. On the other hand, for $x = e_0 - 2e_2$ we have

$$\|{(T^2)}^* x\| = \|{T^*}^2 x\| = \sqrt{89} > \sqrt{80} = \|T^2 x\|.$$

It follows that the operator T is not subnormal since every power of a subnormal operator is subnormal.

Proposition 4.1. *In considering the above operator T, the following properties hold.*

(i) *T is a pure cyclic hyponormal operator.*
(ii) $\sigma_{ap}(T) = \{a + ib \in \mathbb{C} \ : \ (\frac{a}{3})^2 + b^2 = 1\}$.
(iii) $\sigma(T) = \{a + ib \in \mathbb{C} \ : \ (\frac{a}{3})^2 + b^2 \leq 1\}$.
(iv) *T has fat local spectra.*

In the proof of (ii) of Proposition 4.1 we shall require the following elementary result. We begin by recalling that an operator $A \in \mathcal{L}(H)$ is said to be *Fredholm* if its range $\text{ran}(A)$ is closed, and $\ker(A)$ and $\ker(A^*)$ are finite dimensional. The essential spectrum $\sigma_e(A)$ of A is the set of all $\lambda \in \mathbb{C}$ such that $A - \lambda$ is not Fredholm, in fact, The essential spectrum $\sigma_e(A)$ of A is exactly the spectrum $\sigma\big(\pi(A)\big)$ in the Calkin algebra $\mathcal{L}(H)/C$ of $\pi(A)$, where π is the natural quotient map of $\mathcal{L}(H)$ onto $\mathcal{L}(H)/C$, here C denotes the ideal of all compact operators in $\mathcal{L}(H)$.

Lemma 4.2. *Let $A \in \mathcal{L}(H)$ be cyclic operator such that $\sigma_p(A) = \emptyset$, then $\sigma_{ap}(A) = \sigma_e(A)$.*

Proof. Since $\dim\Big(\ker(A - \lambda)^*\Big) \leq 1$ for every $\lambda \in \mathbb{C}$ (see Proposition 1.1) and $\sigma_p(A) = \emptyset$ then

$$\begin{aligned} \lambda \in \sigma_e(A) \quad &\Longleftrightarrow \quad \text{ran}\big(A - \lambda\big) \text{ is not closed} \\ &\Longleftrightarrow \quad \lambda \in \sigma_{ap}(A). \end{aligned}$$

Proof of Proposition 4.1.(i) It is easy to see that e_0 is a cyclic vector of the operator T. Now, let M be a reducing subspace of T. Since,

$$S = \frac{2T - T^*}{3},$$

then M is reducing subspace of S. And so, $M = \{0\}$ or $M = H$. Hence, T is pure operator.

(ii) Since, $\sigma\big(\pi(S)\big) = \mathbb{T}$, where $\mathbb{T}$ is the unit circle of $\mathbb{C}$, and $\pi(S)$ is a normal element in the Calkin algebra then, it follows from the Spectral Mapping Theorem that $\sigma\big(\pi(T)\big) = \{\overline{\lambda} + 2\lambda \;:\; \lambda \in \mathbb{T}\}$. And so, from Lemma 4.2 the desired result holds.

(iii) Note that the operator T is Toeplitz operator T_ϕ with associated function $\phi = \overline{z} + 2z$. It follows then from [12, Theorem 5] and [8, Theorem 1] that $\sigma(T) = \{a + ib \in \mathbb{C} \;:\; (\frac{a}{3})^2 + b^2 \leq 1\}$.

(iv) According Theorem 2.1 and the preceding assertions, we have $\sigma(T) = \mathrm{cl}\big(B_a(T)\big)$; on the other hand, it follows from [14, Theorem 2.6] that $H = \bigvee\{k(\lambda) \;:\; \lambda \in B_a(T)\}$. And so, the desired result follows from Proposition 3.4.

Remark 4.3. *The example considered above is not a weighted shift since its spectrum is not a disc. In fact, this example is given in [4, Problem 209] to show that the square of hyponormal operator fails to be a hyponormal.*

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