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# THE HOMOLOGY GROUPS OF MODULI SPACES OF NON-CLASSICAL KLEIN SURFACES

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**Abstract**

We describe the moduli space $\overrightarrow{\mathfrak{M}}^{\pm}(g,c)$ of non-classical directed Klein surfaces of genus $g = h - c - 1$ with $c \geq 0$ distinguished points as a configuration space $\mathfrak{B}^{\pm}(h,c)$ of classes of $h$-slit pairs in $\mathbb{C}$. Based on this model, we prove that $\overrightarrow{\mathfrak{M}}^{\pm}(g,c)$ is non-orientable for any $g$ and $c$ and we compute the homology groups of the moduli spaces $\overrightarrow{\mathfrak{M}}^{\pm}(g,c)$ for $g \leq 2$.

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## 1. Introduction

Denote $\overrightarrow{\mathfrak{M}}^{\pm}(g,c)$ a moduli space of non-classical directed Klein surfaces of genus $g$ with $c \geq 0$ distinguished points; in other words, the moduli space of non-orientable Riemann surfaces of genus $g$ with one boundary curve and $c \geq 0$ permutable punctures. The purpose of this article is to compute homology groups of the moduli spaces $\overrightarrow{\mathfrak{M}}^{\pm}(g,c)$ with $\mathbb{Z}_2$-coefficients. The starting point is the work of [B1] which gives a new description of the moduli spaces $\overrightarrow{\mathfrak{M}}(g)$ of directed Riemann surfaces of genus $g$. The homology groups of moduli spaces of Riemann (orientable) surfaces with rational coefficients (resp. integer coefficients) have been computed in [H] (resp. in [Eh]). The method used in these work do not apply to the non-orientable surfaes. The problem of computation of $H_*(\overrightarrow{\mathfrak{M}}(g))$ remained untouched. That is our motivation for the present work.

We first give a topological description for the model space $\mathfrak{B}^{\pm}(h,c)$ for $\overrightarrow{\mathfrak{M}}^{\pm}(g,c)$ in terms of certain classes of parallel slit $h$-pairs in the complex plane $\mathbb{C}$. We then describe the cell structure of $\mathfrak{B}^{\pm}(h,c)$. It is similar to the decomposition of other configuration spaces such as the classifying spaces of symmetric groups and braid groups. The boundary operator $\partial$ has two kinds of face operators. We point out that the cellular chain complex of the model space resembles formally with the Hochschild resolution of a noncommutative algebra without unity. This cyclic structure gives rise to a double complex analogous to that of [LQ] and hence to a Connes-Gysin long exact sequence relating Hochschild and cyclic homology. The incidence system on the cell complex gives a unique way to define the orientation of the cell complex. By an inductive construction of *closed galleries* which are closed chains of overlapping highest dimensional cells, we obtain a new result concerning the orientablity of the moduli spaces of non-classical Klein surfaces. $\overrightarrow{\mathfrak{M}}^{\pm}(g,c)$ *is non-orientable for any genus $g$ and $c \geq 0$ distinguished points.*

By computing the homology groups of the model complex for $h = 2, 3$, we obtain the homology groups of the moduli spaces $\overrightarrow{\mathfrak{M}}^{\pm}(g,c)$ of genus $g \leq 2$. Due to the large number of cells, the computation for $h = 3$ is done by the computer. Our new results in computing homology groups of the moduli space $\overrightarrow{\mathfrak{M}}^{\pm}(g,c)$ concern the following cases

- $\overrightarrow{\mathfrak{M}}^{\pm}(0,1)$, the moduli space of projective planes with a puncture
- $\overrightarrow{\mathfrak{M}}^{\pm}(0,2)$, the moduli space of projective planes with two punctures
- $\overrightarrow{\mathfrak{M}}^{\pm}(1,0)$, the moduli space of Klein bottles
- $\overrightarrow{\mathfrak{M}}^{\pm}(1,1)$, the moduli space of Klein bottles with a puncture
- $\overrightarrow{\mathfrak{M}}^{\pm}(2,0)$, the moduli space of non-classical Klein surfaces of genus 2.

We recall a brief description of $\overrightarrow{\mathfrak{M}}^{\pm}(g,c)$. Let $F^d$ be a double covering of a Klein (non-orientable) surface $F$. Then $F^d$ is a compact orientable surface without boundary. For each dianalytic structure $X$ of $F$ there exists a unique analytic structure $X^d$ of $F^d$ with the properties (i) it agrees with the orientation of $F^d$ (ii) $\sigma : X^d \to X^d$ is an anti-holomorphic involution and (iii) $\pi : X^d \to X$ is dianalytic cf.[AG]. The group $\mathrm{Diff}(F)$ of diffeomorphisms of $F$ acts on the set

$K(F)$ of dianalytic structures on $F$. Since the isotopy subgroup $\mathrm{Diff}_0(F)$ acts freely on $K(F)$ (cf. [EE]); the quotient $\mathcal{T}(F) := K(F)/\mathrm{Diff}_0(F)$ is a Teichmüller space of non-classical Klein surfaces $F$. The quotient $\mathcal{M}(F) := K(F)/\mathrm{Diff}(F)$ is the moduli space of non-classical Klein surfaces. By a non-classical directed Klein surface we mean a closed non-orientable Klein surface $F$ of some genus $g$, with a tangent direction $\mathfrak{X}$ at a given base point $\mathcal{O}$. Here a tangential direction $\mathfrak{X} = (x)$ is a non-zero tangent vector $x$, up to a positive multiple. The moduli space $\overrightarrow{\mathfrak{M}}^{\pm}(g,c)$ consists of dianalytic equivalent classes $[F, \mathfrak{X}, \mathcal{O}, \{S_1, S_2, \ldots, S_c\}]$ where $S_1, S_2, \ldots, S_c$ are distinguished permutable points on $F$; here a dianalytic equivalence is a dianalytic homeomorphism $f : F \longrightarrow F'$ such that

$$\begin{array}{ll} (i) & f(\mathcal{O}) = \mathcal{O}' \\ (ii) & df(\mathfrak{X}) = \mathfrak{X}' \\ (iii) & f\{S_1, S_2, \ldots, S_c\} = \{S_1', S_2', \ldots, S_c'\}. \end{array}$$

$\overrightarrow{\mathfrak{M}}^{\pm}(g,c)$ is a smooth, non-compact, non-orientable manifold of (real) dimension $3g - 3c$. For the description of moduli spaces of directed Riemann surfaces, we refer to [ADKP] and [B1].

## 2. Configurations of Slit Pairs and Associated Surfaces $F^{\pm}(L)$

*A parallel slit* $L_k$ is of the form $\{z = (x,y) \in \mathbb{C} | x \leq x_k, y = y_k\}$ for any given point $z_k = (x_k, y_k)$ in $\mathbb{C}$. Let $h > 0$ be an integer. An element $\lambda$ of a symmetric group $\Sigma_{2h}$ is said to be a *pairing* if it is a fixed point free involution. A signature of the pairing $\lambda$ is defined by either $+1$ or $-1$ for each pair $i$ and $\lambda(i)$. The index pair having the value $+1$ is called *type I* , otherwise *type II.*

By a *configuration of parallel slit pairs of type I and type II* we mean a collection consisting of the followings:

1. An ordered sequence $L_1, L_2, \ldots, L_{2h}$ of parallel slits in $\mathbb{C}$ such that $y_k \leq y_{k+1}$ and $x_k = x_{\lambda(k)}$ for $1 \leq k \leq 2h$,
2. a pairing $\lambda \in \Sigma_{2h}$,
3. the type sequence $T = (t_1, t_2, \ldots, t_{2h})$ with $t_i = t_{\lambda(i)}$ and $t_i = I$ or $II$.

The configuration of parallel slits is denoted by $L = (L_1, \ldots, L_{2h} | \lambda | T)$. The slits in $L$ are not necessarily to be distinct; some of them can be equal $L_i = L_{i+1}$, or contained in each other, $L_i \subset L_{i+1}$. If all slits in $L$ are disjoint, $L$ is called *generic.*

Identifying edges of slits associates to each $L$ a closed surface $F^{\pm}(L)$ of some genus $g$. Set $F_k = \{(x,y) \in \mathbb{C} | y_k \leq y \leq y_{k+1}\}$ for $k = 1, \ldots, 2h-1$, $F_0 = \{(x,y) \in \mathbb{C} | y \leq y_1\}$ and $F_{2h} = \{(x,y) \in \mathbb{C} | y_{2h} \leq y\}$. Then the $F_k$ are closed strips between the slits. Apart from the point at infinity, $F_k$ are disjoint. On the disjoint union of $F_0, \ldots, F_{2h}$ the points are identified by the following rules.

$$\begin{array}{ll} (1) & F_k \ni (x, y_k) \sim (x, y_k) \in F_{k+1} \text{for } x > x_k \\ (2)[typeI] & F_k \ni (x, y_k) \sim (x, y_{\lambda(k)}) \in F_{\lambda(k)-1} \\ & F_{k-1} \ni (x, y_k) \sim (x, y_{\lambda(k)}) \in F_{\lambda(k)} \text{ for } x \leq x_k \\ (3)[typeII] & F_k \ni (x, y_k) \sim (x, y_{\lambda(k)}) \in F_{\lambda(k)} \\ & F_{k-1} \ni (x, y_k) \sim (x, y_{\lambda(k)}) \in F_{\lambda(k-1)} \text{ for } x \leq x_k \end{array}$$

The identification rule(3) of a slit pair of type II reverses the orientation of an adjacent angle. Hence a surface $F^{\pm}(L)$ obtained by the configuration $L$ consisting of slit pairs of type II is always *non-orientable*. We now assume that any configuration $L$ contains at least one slit pair of type II. The quotient space $F_0(L) := \bigsqcup_{k=0}^{2h} F_k/\sim$ obtains the quotient topology. Later this space will be compactified. The resulting compact space will also be denoted by $F^{\pm}(L)$. By abuse of notation $F^{\pm}(L)$ will be used for both spaces.

## 3. The Non-classical Directed Klein Surface $F^{\pm}(\mathcal{L})$

**Lemma 3.1.** *Let $L = (L_1, \dots, L_{2h}|\lambda|T)$ be given. Then the pairing $\lambda \in \Sigma_{2h}$ induces a unique permutation $\sigma$ in $\Sigma_{4h}$ defined by*

$$\sigma(i^+) := \begin{cases} (\lambda(i+1))^+ & \text{if} \quad (i+1) \quad \text{is of type I} \\ (\lambda(i+1))^- & \text{if} \quad (i+1) \quad \text{is of type II} \end{cases}$$

$$\sigma(i^-) := \begin{cases} (\lambda(i-1))^- & \text{if} \quad (i-1) \quad \text{is of type I} \\ (\lambda(i-1))^+ & \text{if} \quad (i-1) \quad \text{is of type II} \end{cases}$$

Proof: For each index $i \in \lambda$, we have two copies of indices, say $i^+, i^-$. The $\sigma$ is well-defined by the following direct verification: Take any two indices $i \neq j$ from $\lambda$. Consider the cases: [Case 1:] If $\sigma(i^+)^+ = \sigma(j^-)^+$ , then by the formula of $\sigma$ $(i+1)$ must be of the type I and $(j-1)$ must be of type II; and $\lambda(i+1) = \lambda(j-1)$. Thus $i+1 = j-1$ having different types. Contradiction. [Case 2:] If $\sigma(i^+)^+ = \sigma(j^+)^+$, then again by the formula of $\sigma$, both $(i+1)$ and $(j+1)$ are of the same type I and $\lambda(i+1) = \lambda(j+1)$. Hence $i = j$. □

Such a $\sigma$ is called *$\lambda$-extended permutation.* It is straightforward to check that the permutation $\sigma$ has an even number of disjoint cycles.

The disjoint cycles of $\sigma$ can be equally seperated into two different parts; we call these the *forward permutation* (resp. *backward permutation*) and denote them by $\delta^\lambda$ (resp. $\delta_\lambda$). Signatures of $\delta^\lambda$ and $\delta_\lambda$ are defined by setting $+1$ for $i^+$ and $-1$ for $(i+1)^-$. Hence we have two associated vectors $\eta = (\eta_1, \dots, \eta_{2h})$ and $\zeta = (\zeta_1, \dots, \zeta_{2h})$ with coordinates $\eta_i = \pm 1$ and $\zeta_i = \pm 1$.

*Remark* 3.2. If $L$ is a configuration of slit pairs of type I only, then the associated surface $F^{\pm}(L)$ is orientable, and the $\delta^\lambda$ is the same as the permutation defined by the formula $\lambda\tau$ in [B1], where $\tau$ is the transposition $i \longmapsto i+1$ on $2h$ indices.

The *connectivity*, denoted by $c$, of $L = (L_1, \dots, L_{2h}|\lambda|T)$ is defined by the formula

$$c = 1/2(\text{number of disjoint cycles of } \sigma) - 1.$$

If $L$ contains only a slit pair of type II, then the connectivity $c = 0$. (If $L$ contained only a slit pair of type I, then the connectivity $c$ would be 1, which would not happen since any $L$ we consider contains at least a slit pair of type II). Hence the connectivity $c$ is strictly less than the number $h$ of slit pairs, at most $c = h - 1$.

The connectivity $c$ tells exactly distinguished points $S_1, \dots, S_c$ different from the base point $\mathcal{O}$ on the associated closed surface $F^{\pm}(L)$.

Let $S \subset \mathbb{C}$ be the smallest rectangle containing all endpoints $z_1, \dots, z_{2h}$ of the slits in $L$. The smallest means that the left lower corner of $S$ has the coordinate $\{\min(x_k), \min(y_k)\}$ and the right upper coordinate of $S$ is $\{\max(x_k), \max(y_k)\}$ for $1 \leq k \leq 2h$. Such a rectangle $S$ is called the *support* of the configuration $L$. For $x_0 \geq \min(x_k)$, the line $l = \{(x, y) \in \mathbb{C} | x = x_0\}$ is drawn parallel to the vertical boundary of $S$. It yields the closed intervals $Y_0, Y_1, \dots, Y_{2h}$ where $Y_0$ is below the cut of $L_1$ and $Y_K$ is between the cuts of $L_k, L_{k+1}$ and $Y_{2h}$ is above the cut of $L_{2h}$. Some intervals may be points. These intervals $Y_i$ are needed to re-glue in a certain permuted order since on meeting an edge of a slit, a path goes on at the corresponding edge in the same or the opposite direction until the path is closed. $\delta^\lambda$ prescribes the gluing way, $\eta$ gives the unique orientation for the $Y_i$. Therefore $Y_i$ is followed by $Y_{\delta^\lambda(i)}$; and the orientation of $Y_i$ is *preserved if* $\eta_i = 1$, unless $Y_i$ is re-glued in the reverse orientation; see the following figure. (In the figure a slit is depicted as a horizontal half-line unbounded to the left, a pair is depicted by an arc from an endpoint of a slit to the corresponding endpoint of another slit, and symbols I, II denote the type of slit pairs. )

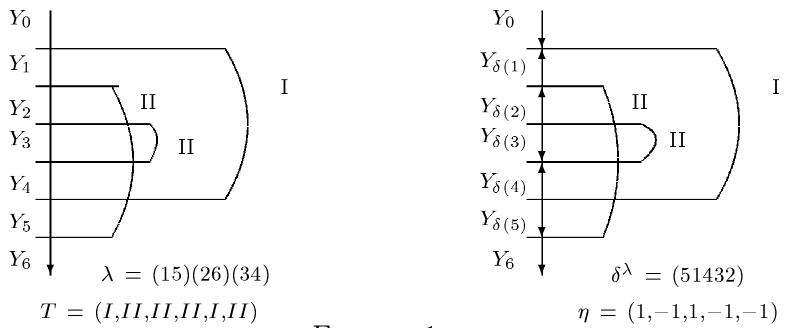



FIGURE 1

*Remark* 3.3. From the dynamical point of view the whole effect of re-gluing the intervals $Y_i$ is a discontinuous, orientation preserving and reversing, piecewise isometric self-map of the real line. It is a non-orientable, non-ergodic (if $c \neq 0$) interval exchange transformation.

Let $Y^0$ be the complex obtained by regluing $Y_i$. Then $Y^0$ is a subset $F_0(L)$ and has $(c + 1)$ path-connected components. For each $L$, there exists a distinguished component, namely the component containing $Y_{2h}$. Such a distinguished component of $Y^0$ is called the *principal component.* The existence of remaining components in $Y^0$ depends on $L$. Let $R^0$ be a set in $F^0(L)$ which is induced by the support $S$ in the construction for $F^0(L)$. The *complement of* $R^0$ in $F^0(L)$ has the same $(c+1)$ components as that of $Y^0$. It also has a principal component and another $c$ components. The principal component and the other components associated to points at infinity correspond to a base point $\mathcal{O}$ and to the distinguished points $S_1, \dots, S_c$ of the surface $F_0(L)$, respectively. We define

$$F^{\pm}(L) := F_0(L) \coprod \{\mathcal{O}, S_1, \dots, S_c\}.$$

The Euler-characteristic of $F^{\pm}(L)$ is $c - h + 2$ and hence the genus $g$ of $F^{\pm}(L)$ is $h - c - 1$.

Since slits in $L$ are not necessarily to be disjoint, they may touch. If so, then they jump. We now define jumps of slits as follows

1. $\rho := (\lambda(k+1)\dots(k+1)k)$ if $L_k \subset L_{k+1}$, $\lambda(k+1) > k+1$, $(k+1) = I$
2. $\rho := (\lambda(k+1)+1\dots k)$ if $L_k \subset L_{k+1}$, $\lambda(k+1) < k$, $(k+1) = I$
3. $\rho := (\lambda(k+1)-1\dots k)$ if $L_k \subset L_{k+1}$, $\lambda(k+1) > k+1$, $(k+1) = II$
4. $\rho := (\lambda(k+1)\dots k)$ if $L_k \subset L_{k+1}$, $\lambda(k+1) < k$, $(k+1) = II$

Here $\rho$ is a transposition in $\Sigma_{2h}$, $k+1 = I$ means $L_{k+1}$ is of type I.

After each jump, a new configuration $\tilde{L} = (\tilde{L}_1, \dots, \tilde{L}_{2h}|\tilde{\lambda}|\tilde{T})$ is obtained as follows. Set $\tilde{L}_{p(j)} = L_j$ for $j \neq k$, $j = 1, \dots, 2h$. For (1) and (2), set the endpoint of $\tilde{L}_{p(k)}$ as $(x_k, y_{\lambda(k+1)})$ and as $(x_k, y_{\lambda(k-1)})$ for the other cases. Define $\tilde{\lambda} = \rho\lambda\rho^{-1}$. For (3) and (4) the type of *index* $k$ in $\lambda$ is changed in $\tilde{\lambda}$. The jumps (1) and (2) are called *Jump I*, known as *Rauzy jumps* and the others are called *Jump II*. The relation generated by *Jump I* and *Jump II* is an equivalence relation on the set of configurations. The equivalence class of the configuration $L$ is denoted by $\mathcal{L} = [L_1, \dots, L_{2h}|\lambda|T]$. The following figure illustrates how a slit jumps and takes a new position.

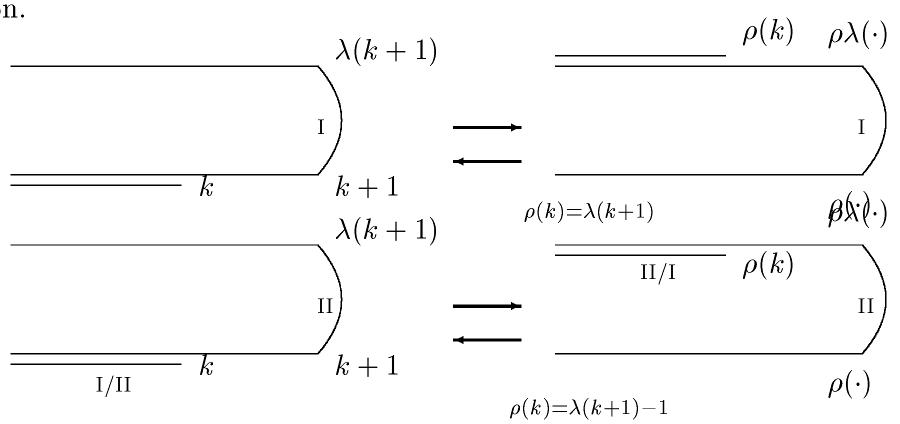



FIGURE 2

For the proof of the following Proposition, see [Z].

**Proposition 3.4.** *If $L$ and $\tilde{L}$ are equivalent, then $F^{\pm}(L)$ and $F^{\pm}(\tilde{L})$ are dianalytically equivalent.*

One of the crucial points is to exclude configurations which will lead to singular surfaces. By *a non-degenerate configuration* $L$ we mean a configuration $L$ which induces an associated surface $F^{\pm}(L)$ that is non-singular (smooth) at all points $z$ except at $\mathcal{O}$.

The criterion to determine the degeneracy is as follows: If $L$ contains a slit $L_K$ such that $L_{k+i} \subseteq L_k = L_{\lambda(k)}$ for any (if exists) index $i$ between $k$ and $\lambda(k)$ (assume $k < \lambda(k)$), then such a configuration $L$ is called *degenerate*. It is possible that in two equivalent configurations, one satisfies the above condition and another does not. Hence an equivalent class $\mathcal{L}$ is called *degenerate* if it contains a representative configuration $L \in \mathcal{L}$ that is degenerate. Since $F^{\pm}(L)$ depends only on the class of $L$, write the surface $F^{\pm}(\mathcal{L})$ instead $F^{\pm}(L)$. Hence $F^{\pm}(\mathcal{L})$ is a smooth surface away from $\infty$ if and only if $\mathcal{L}$ is non-degenerate.

Let $L = (L_1, \ldots, L_k, \ldots, L_{2h}|\lambda|T)$ be in a class $\mathcal{L}$ and let $L_k \subseteq L_{k\pm1}$. Then $k \leq \rho^n(k) = \underbrace{\rho \ldots \rho}_{n}(k)$ for finite $n \in \mathbb{N}$. If all such $k \in \mathbb{N}$, then $k < \rho^n(k)$ for all $n$. Such a configuration $L$ is called the *normal form* in $\mathcal{L}$. Here $\rho$ are the transpositions in $\Sigma_{2h}$.

**Theorem 3.5.** *Let $\mathcal{L}$ be a non-degenerate class of configurations. Then $F^{\pm}(\mathcal{L})$ is a non-classical directed punctured Klein surface of genus $g = h - c - 1$.*

PROOF: See [Z].

## 4. The Spaces of Parallel Slit Domains

Let $Conf(h,c)$ denote the space of all configurations $L$ with connectivity $c \neq 0$. Then $Conf(h,c) \subset \mathbb{C}^{2h} \times \Sigma_{2h}$, where $\Sigma_{2h}$ is regarded as a discrete space. Let $RegConf(h,c) \subset Conf(h,c)$ be the subset of all non-degenerate configurations. The quotient space $PSC^{\pm}(h,c) := RegConf(h,c)/\sim_{Jumps}$ is the space of all classes $\mathcal{L}$ of non-degenerate configurations. The elements $\mathcal{L}$ are called *parallel slit domains.* Let $Sim(\mathbb{C})$ be a group of similarities of $\mathbb{C}$. It is generated by translations and (positive) dilations, it is a subgroup of $GL(2,\mathbb{C})$ consisting of matrices of the forms:

$$M = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \; a \in \mathbb{R}^+, \; b \in \mathbb{C}.$$

The matrix $M$ is identified with the associated Möbius transformation $M(z) = (az + b)$. These transformations are automorphisms of the Klein surface, which fix $\infty$ and map horizontal lines to horizontal lines. The action is defined by

$$M \cdot \mathcal{L} := [M(L_1), \ldots, M(L_{2h})|\lambda|T].$$

The $M \cdot L$ is non-degenerate when $L$ is non-degenerate. Moreover, if $L \approx L'$, then $M \cdot L \approx M \cdot L'$ and the group $Sim(\mathbb{C})$ acts freely on $PSC^{\pm}(h,c)$. For each $M \in Sim(\mathbb{C})$, it is obvious that the associated surfaces $F^{\pm}(\mathcal{L})$ and $F^{\pm}(M \cdot \mathcal{L})$ are conformally equivalent. Hence we have

**Theorem 4.1.** *Let $M \in Sim(\mathbb{C}), \mathcal{L} \in PSC^{\pm}(h,c)$ and $\mathcal{L}' = M \cdot \mathcal{L}$, let $F^{\pm}(\mathcal{L})$ and $F^{\pm}(\mathcal{L}')$ be associated non-classical directed Klein surfaces. Then $M$ induces a conformal map*

$$\Phi_M : F^{\pm}(\mathcal{L}) \longrightarrow F^{\pm}(\mathcal{L}')$$

*such that $\Phi_M(\mathcal{O}) = \mathcal{O}', \overrightarrow{D}\Phi_M(\mathfrak{X}) = \mathfrak{X}'$ and $\Phi_M\{S_1, \ldots, S_c\} = \{S'_1, \ldots, S'_c\}$.*

We now define a normalization on $L$ as follows:

$$(i)\; y_1 = 0, \;\; (ii)\; y_{2h} = 1, \;\; (iii)\; \min(x_k) = 0.$$

Since these conditions are invariant under the jumps, and thus conditions on a class. Let $K^{\pm}(h,c)$ be the space of all classes $\mathcal{L}$ of normalized non-degenerate configurations $L$. We impose one more condition on $K^{\pm}(h,c)$:

$$\max(x_k) < 1.$$

This additional condition restricts the conformal type, and obviously selects a proper subspace denoted by $\mathfrak{B}^{\pm}(h,c)$ which is homeomorphic to $K^{\pm}(h,c)$; a homeomorphism is given by applying to the $x$-coordinates of the slit end points that reparametrizes $[0,\infty[$ as $[0,1[$ fixing 0.

By Theorem 3.5 and Theorem 4.1, we have a continuous map

$$\alpha : K^{\pm}(h,c) \longrightarrow \overrightarrow{\mathfrak{M}}^{\pm}(g,c)$$

defined by $\mathcal{L} \mapsto [F, \mathfrak{X}, \mathcal{O}, \{S_1, S_2, \ldots, S_c\}]$.

The inverse of $\alpha$ is obtained as follows. On a directed closed orientable surface $F$, there is a function $u :\to \bar{\mathbb{R}} = \mathbb{R} \cup \infty$ such that (1) $u$ is harmonic away from $\mathcal{O}$ (2) $u(z) - Re(1/z)$ is smooth and vanishes at $\mathcal{O}$ for any local parameter $z$ around $\mathcal{O}$ such that $z(\mathcal{O}) = 0$ and $z(\{S_1, \ldots, S_c\}) = \{1, 2, \ldots, c\}$ for $z$ around distinguished points and $dz(\mathfrak{X}) = -dx$. This characterizes $u$ uniquely up to an additive and a positive multiplicative constant. The gradient flow of $u$ determines the critical graph $K \subset F$ consisting of the dipole $\mathcal{O}$, all zeros of the flow and critical points as vertices and unstable submanifolds of the flow as edges. Since $F^0 = F - K$ is connected and simply-connected, there is a holomorphic map $w = u + iv : F^0 \longrightarrow \mathbb{C}$ which is unique up to another additive constant for harmonic conjugate $v$ of $u$.The complement of $w(F^0) \subseteq \mathbb{C}$ is described as the configuration $L$ of slit pairs in $\mathbb{C}$. Here $u$ transforms into the function $x$, and the gradient flow into the horizontal flow $-\partial/\partial x$.

As 3-dimensional contractible group $Sim(\mathbb{C})$ acts freely on $PSC^{\pm}(h,c)$, two normalization constants (the translations in $x$- resp. $y$-direction) correspond to the real additive integral constants of harmonic functions $u$ and $v$, and the dilations correspond to the undetermined length of the tangent vector representing the direction $\mathfrak{X}$. For the construction of the dipole function for non-classical Klein surface $F$, we consider the double covering $F^d$ of $F$ and transform results to $F$. The potential $w$ depends on the position $\mathcal{O}$ of $F$ only and not on the orientation of $F$ in the neighbourhood of $\mathcal{O}$. It follows that the complement of $w(F^0) \subseteq \mathbb{C}$ will comprise all moduli of the class $[F, \mathfrak{X}, \mathcal{O}, \{S_1, S_2, \ldots, S_c\}]$. Hence we have

**Theorem 4.2.** *The moduli space $\overrightarrow{\mathfrak{M}}^{\pm}(g,c)$ of non-classical directed Klein surfaces $F^{\pm}(L)$ is homeomorphic with the space $K^{\pm}(h,c)$ of all classes of normalized non-degenerate configurations $L$ and hence $\overrightarrow{\mathfrak{M}}^{\pm}(g,c) \cong \mathfrak{B}^{\pm}(h,c)$.*

For the compactification of $\overrightarrow{\mathfrak{M}}^{\pm}(g,c)$, we first take $R^{\pm}(h,c)$, a closure of the space of all normalized non-degenerate configurations with the condition max $(x_k) \leq 1$. Its topology is induced by the space $\mathbb{C}^{2h} \times \Sigma_{2h}$ which is homeomorphic with $2h$-disjoint copies of $\mathbb{C}^{2h}$. The condition $\max(x_k) \leq 1$ is clearly invariant under the jumps. The compactification of this moduli space is defined by $R^{\pm}(h,c)/\sim_{Jumps}$, denoted by $P^{\pm}(h,c)$.

We define subspaces of $P^{\pm}(h,c)$. Let $N^{\pm}(h,c)$ be the set of all $\mathcal{L}$ such that $\max(x_k) = 1$. Classes in $N^{\pm}(h,c)$ may be degenerate or not. The non-degenerate ones form a partial boundary to the manifold $\mathfrak{B}^{\pm}(h,c)$. Let $D^{\pm}(h,c)$ be the set of all degenerate classes. The union $W^{\pm}(h,c) := N^{\pm}(h,c) \cup D^{\pm}(h,c)$ is a *periphery* of $\mathcal{B}^{\pm}(h,c) = P^{\pm}(h,c) - W^{\pm}(h,c)$. $P^{\pm}(h,c)$

is a $3h-3$ dimensional pseudo-manifold and $(P^{\pm}(h,c), W^{\pm}(h,c))$ is a $3h-3$ dimensional real relative manifold. For more detail, we refer to [Z].

## 5. The Orientability of Moduli Spaces of Nonclassical Klein Surfaces

Let $L \in R^{\pm}(h,c)$ be such that the slits of $L$ lie on $(n+2)$-distinct $y$-levels $0 = v_0 < v_1 < \cdots < v_{n+1} = 1$ and let $a_i$ be the number of slits lying at the level $y = v_i$. Then $0 < a_i < 2h$, $\sum_{i=0}^{n+1} a_i = 2h$, where $0 \le n \le 2h-2$. The set of indices whose slits are lying at the level $y = v_i$ is denoted by $A_i \subset \{1, \ldots, 2h\}$. Assume that the endpoints of the slits of $L$ lie at $(m+1)$ distinct $x$-levels $0 = u_0 < u_1 < \cdots < u_m < 1$. (If $\max(x_i) = 1$, assume the endpoints lie at $(m+2)$-distinct levels $0 = u_0 < \cdots < u_m < u_{m+1} = 1$). The set of indices of the slits ending over $x = u_j$ is denoted by $B_j \subset \{1, \ldots, 2h\}$. Then $B_0 \sqcup B_1 \sqcup \cdots \sqcup B_{m+1} = \{1, \ldots, 2h\}$, $0 \le m \le h-1$. All these data are collected in a symbol $E = (a_0, \ldots, a_{n+1}|\lambda|B_0, \ldots, B_{m+1})$. We will also write $E = (a_0, \ldots, a_{n+1}|\lambda|B_0, \ldots, B_{m+1}; T)$ with the type sequence $T$.

Set $r_i = v_{i+1} - v_i, i = 0, \ldots, n$ and $s_j = u_{j+1} - u_j, j = 0, \ldots, m$. We have $\sum_{i=0}^{n} r_i = 1, \sum_{j=0}^{m} s_j = 1$. Hence $(r_0, r_1, \ldots, r_n)$ and $(s_0, s_1, \ldots, s_m)$ are barycentric coordinates in open simplices $\triangle^n$ and $\triangle^m$ respectively. By varying the coordinates $r_i$ and $s_j$, we get the configuration to which the same symbol $E$ is associated. To each symbol $E$ we define a map

$$f_E : \bar{\triangle}^n \times \bar{\triangle}^m \to R^{\pm}(h,c) \; ; (r_0, \ldots, r_n, s_0, \ldots, s_m) \longmapsto (L_1, \ldots, L_{2h}|\lambda|T) \; ,$$

where $\bar{\triangle}^n, \bar{\triangle}^m$ are closed simplices of dimensions $n$ and $m$, respectively. The map $f_E$ is a characteristic map for each $(n+m)$-cell $E$. The highest dimension of a cell in $R^{\pm}(h,c)$ is $3h-3$. Hence

**Theorem 5.1.** *The $(n+m)$ cell $E = (a_0, \ldots, a_{n+1}|\lambda|B_0, \ldots, B_{m+1})$ in $R^{\pm}(h,c)$ together with*

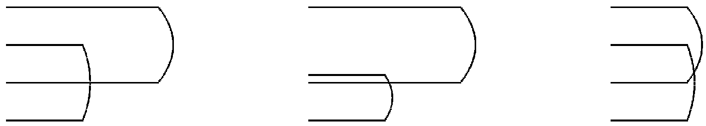
$$\sum_{i=0}^{n+1} a_i = 2h, \;\; B_0, \ldots, B_m \neq \emptyset, \;\; \bigsqcup_{j=0}^{m+1} B_j = \{1, \ldots, 2h\}$$

*where $0 \le n \le 2h-2$, $0 \le m \le h-1$ defines a finite cellular structure for $R^{\pm}(h,c)$. Hence $R^{\pm}(h,c)$ is a compact regular $(3h-3)$-dimensional cell complex.*

The boundary operator $\partial$ has two kinds of face operators $\partial^{'}, \partial^{''}$ for a $(n+m)$-cell $E$.

$$\begin{aligned} \partial^{'}_i E : &= (a_0, \ldots, a_i + a_{i+1}, \ldots, a_{n+1}|\lambda|B_0, \ldots, B_{m+1}) \\ \partial^{''}_j E : &= (a_0, \ldots, a_{n+1}|\lambda|B_0, \ldots, B_j \sqcup B_{j+1}, \ldots, B_{m+1}) \end{aligned}$$

for $i = 0, \ldots, n; \; j = 0, \ldots, m$.



$E$ $\partial_1' E$ $\partial_0'' E$

FIGURE 3

The boundary $\partial$ is defined by

$$\partial := \partial' + (-1)^n \partial'', \text{ where } \partial' := \sum_{i=0}^{n} (-1)^i \partial_i' \ , \ \partial'' := \sum_{j=0}^{m} (-1)^j \partial_j'' .$$

The face operators satisfy the following relations.

$$\begin{array}{llll} (1) & \partial_i' \partial_j' & = & \partial_{j-1}' \partial_i' & \text{for } 0 \le i < j \le n \\ (2) & \partial_i' \partial_i' & = & \partial_i' \partial_{i+1}' & \text{for } 0 \le i \le n \\ (3) & \partial_i'' \partial_j'' & = & \partial_{j-1}'' \partial_i'' & \text{for } 0 \le i < j \le n \\ (4) & \partial_j'' \partial_j'' & = & \partial_j' \partial_{j+1}'' & \text{for } 0 \le j \le n \\ (5) & \partial_i' \partial_j'' & = & \partial_j'' \partial_i' & \text{for } 0 \le i \le n, \ 0 \le j \le m \ . \end{array}$$

Assume that $E = (a_0, \dots, a_{n+1} | \lambda | B_0, \dots, B_{m+1})$ is a $(n+m)$-cell of $R^{\pm}(h,c)$ with indices $(k-1,k) \in A_{i_1}$ for $k \in \{1, \dots, 2h\}$ and $i_1 \in \{0, \dots, n+1\}$, $\lambda(k) \neq k-1$. Assume that $\lambda(k) \in A_{i_2}$, $k-1 \in B_{j_1}$ and $k, \lambda(k) \in B_{j_2}$ with $j_1 \le j_2$. (Such indices always exist whenever slits $L_{k-1} \subset L_k$ in the configuration associated to the cell $E$.) To each cell $E$, the jumps can be applied. The new cell $\rho E$ is of the form:

$$\rho E := (a_0, \dots, a_{i_1} - 1, \dots, a_{i_2} + 1, \dots, a_{n+1} | \rho \lambda \rho^{-1} | \rho B_0, \dots, \rho B_{m+1})$$

$$\begin{array}{lllllll} \text{where} & \rho = (\lambda(k) \dots k-1) & \text{if} & \lambda(k) > k & \text{and} & k \in B_{j_2} & \text{is of type I} \\ & \rho = (\lambda(k)+1 \dots k-1) & \text{if} & \lambda(k) < k & \text{and} & k \in B_{j_2} & \text{is of type I} \\ & \rho = (\lambda(k)-1 \dots k-1) & \text{if} & \lambda(k) > k & \text{and} & k \in B_{j_2} & \text{is of type II} \\ & \rho = (\lambda(k) \dots k-1) & \text{if} & \lambda(k) < k & \text{and} & k \in B_{j_2} & \text{is of type II} \ . \end{array}$$

The map $f : E \mapsto \rho E$ defines an identification on $R^{\pm}(h,c)$. The space $P^{\pm}(h,c)$ is a finite, connected cell complex of dimension $(3h-3)$. A cell of $P^{\pm}(h,c)$ is written by a symbol $E := [a_0, \dots, a_{n+1} | \lambda | B_0, \dots, B_{m+1}]$. Define the numbers $[E : \partial_i' E] := (-1)^i$ and $[E : \partial_j'' E] := (-1)^{n+j}$ where $i = 0, \dots, n$, $j = 0, \dots, m$. Set $[E : F] := 0$ for any other $(n+m-1)$-cells $F$ which are not faces of $E$.

If $E^{n+m-2} = \partial''_j \partial_i' E = \partial_i' \partial_j'' E$, then

$$\begin{aligned} & [E : \partial_i' E][\partial_i' E : E^{n+m-2}] + [E : \partial_j'' E][\partial_j'' E : E^{n+m-2}] \\ = \ & (-1)^i (-1)^{n+j-1} + (-1)^{n+j} (-1)^i \\ = \ & 0. \end{aligned}$$

A similar result is obtained for other possible $(n+m-2)$-cells

$$E^{n+m-2} = \partial'_{j-1} \partial_i' E = \partial_i' \partial_j' E; \ E^{n+m-2} = \partial''_{j-1} \partial_i'' E = \partial_i'' \partial_j'' E \ , \text{ for } i+1 \le j.$$

It is obvious that the incidence system is invariant under the identification induced by the jumps. Thus the incidence system determines a unique way an orientation for the cell $E = [a_0, \dots, a_{n+1} | \lambda | B_0, \dots, B_{m+1}]$ of $P^{\pm}(h,c)$. We take this orientation as the standard orientation.

Each cell $E$ can now be associated with a sign, $\epsilon(E) = \pm 1$ with respect to the chosen orientation. We write an *oriented cell* $E$ by $\epsilon(E)E$.

By *a chamber* we mean a cell $E$ of dimension $3h-3$ in $P^{\pm}(h,c)$. Two chambers $E$ and $E^{'}$ are called *adjacent* if they have a common face $F$ of co-dimension 1 in $P^{\pm}(h,c)$. By a *gallery* $\mathcal{G}$ joining $E_0$ and $E_q$ we mean a finite sequence of adjacent chambers $E_i$ together with boundary operators $d_i$ :

$$\mathcal{G} := (E_0, d_0, E_1, d_1, \ldots, d_{q-1}, E_q, d_q)$$

where $d_i E_i = d_{i+1} E_{i+1}$ and $d_i \in \{\partial'_0, \ldots, \partial^{'}_{2h-2}, \partial^{''}_0, \ldots, \partial^{''}_{h-2}\}$..

The oriented chambers $\epsilon(E)E$ and $\epsilon(E^{'})E^{'}$ are said to be *oriented coherently with respect to a common face* $F$ of co-dimension 1 if their incidence numbers satisfy the condition

$$[\epsilon(E)E : F] + [\epsilon(E^{'})E^{'} : F] = 0. \tag{5.1}$$

This condition is independent on the choice of the orientation for $F$. Hence we can choose a definite orientation on each chamber.

$(P^{\pm}(h,c), W^{\pm}(h,c))$ is said to be *orientable* if all chambers of $(P^{\pm}(h,c), W^{\pm}(h,c))$ can be simultaneously oriented so that any pair having a common face of co-dimension 1 are coherently oriented; i.e. all triple $(E, E^{'}, F)$ of chambers $E, E^{'}$ with a common face $F$ of co-dimension 1 satisfy 5.1. Otherwise it is *non-orientable.*

Let $\mathcal{G} = (E_0, d_0, \ldots, d_{q-1}, E_q, d_q)$. If we choose the sign $\epsilon(E_0)$ of $E_0$, we can then determine signs $\epsilon(E_i)$ of the other chambers $E_i$ in the gallery $\mathcal{G}$. Hence we get a gallery $\mathcal{G} := (\epsilon(E_0)E_0, d_0, \ldots, d_{q-1}, \epsilon(E_q)E_q, d_q)$ of coherently oriented chambers. If $E_0 = E_q$ in the gallery $\mathcal{G}$, then $\mathcal{G}$ is called *closed gallery* of the chamber $E_0$. The closed gallery $\mathcal{G}$ is said to be orientation preserving if $\epsilon(E_0) = \epsilon(E_q)$; otherwise orientation reversing. Hence chambers of $(P^{\pm}(h,c), W^{\pm}(h,c))$ are all simultaneously oriented coherently with respect to their common faces when *all possible closed galleries of chambers of* $(P^{\pm}(h,c),\ W^{\pm}(h,c))$ *are orientation preserving.* If some closed galleries of $(P^{\pm}(h,c), W^{\pm}(h,c))$ are orientation reversing, then coherent orientation for all chambers of $(P^{\pm}(h,c), W^{\pm}(h,c))$ is impossible.

**Lemma 5.2.** *The cell complex* $(P^{\pm}(h,c), W^{\pm}(h,c))$ *is orientable if all closed galleries of* $(P^{\pm}(h,c), W^{\pm}(h,c))$ *are orientation preserving. If there exist some closed galleries which are orientation reversing, then* $(P^{\pm}(h,c),\ W^{\pm}(h,c))$ *is non-orientable.*

**Theorem 5.3.** *The cell complex* $(P^{\pm}(h,c), W^{\pm}(h,c))$ *is always non-orientable independently on the parameters* $h$ *and* $c$.

PROOF: There are only two complexes $(P^{\pm}(2,0), W^{\pm}(2,0))$ and $(P^{\pm}(2,1), W^{\pm}(2,1))$ for $h=2$. From $(P^{\pm}(2,0), W^{\pm}(2,0))$, pick up the following chambers $E_1, E_2, E_3, E_4, E_5$ and $E_6$ ;

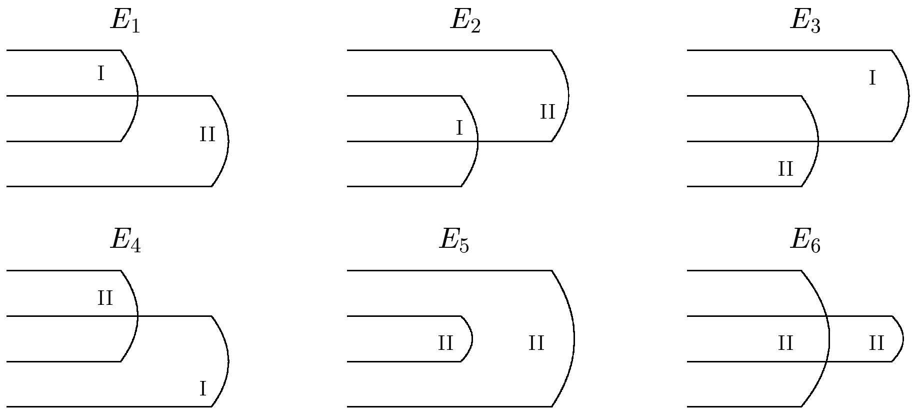



FIGURE 4

Take the face $\partial_2' E_1$ of $E_1$. Then $\partial_2' E_1 = \partial_0' E_5$, i.e. $E_1$ and $E_5$ are adjacent. Since $\partial_0'' E_5 = \partial_0'' E_6$, then $E_6$ is adjacent to $E_5$. We take $E_1, E_5, E_6$ into our constructing gallery. We now choose another one which is adjacent to $E_6$ and take it into our constructing gallery. We repeat this procedure until all the six chambers were taken up into our gallery. Finally we have a closed gallery $\mathcal{G}_{2,0} := (E_1, \partial_2', E_5, \partial_0'', E_6, \partial_2', E_2, \partial_0'', E_4, \partial_2', E_3, \partial_0'', E_1, \partial_0'')$.

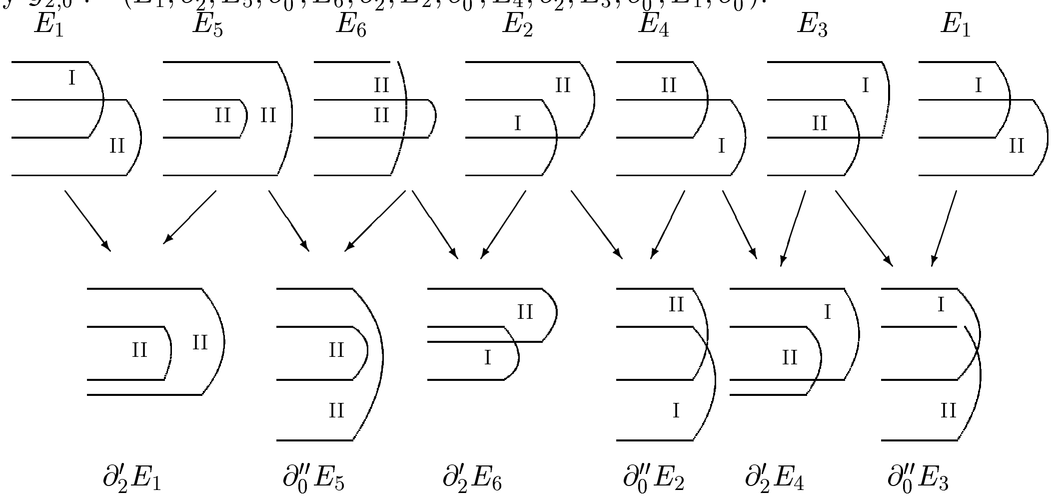



FIGURE 5

We now show that $\mathcal{G}_{2,0}$ is orientation-reserving. Choose $\epsilon(E_1) = +1$. Then the signs of other chambers in $\mathcal{G}_{2,0}$ have to be set as follows:

$$\epsilon(E_5) = -1,\ \epsilon(E_6) = +1,\ \epsilon(E_2) = +1,\ \epsilon(E_4) = -1,\ \epsilon(E_3) = +1,\ \epsilon(E_1) = -1.$$

Hence $\mathcal{G}_{2,0} := (+E_1, \partial_2', -E_5, \partial_0'', +E_6, \partial_2', E_2, \partial_0'', -E_4, \partial_2', +E_3, \partial_0'', -E_1, \partial_0'')$ is orientation reversing. Thus $(P^{\pm}(2,0), W^{\pm}(2,0))$ is a non-orientable cell complex.

Similarly, we can show that $(P^{\pm}(2,1), W^{\pm}(2,1))$ is also non-orientable.

We now give the proof for any $h$. We take a *hook* (i.e. a slit pair $(L_i, L_{i\pm1})$ with $\lambda(i \pm 1) = i$) $(L_5, L_6)$ of type II such that endpoints of slits have the form $x_5 < x_i,\ i = 1, 2, 3, 4$ and $y_6 > y_5 > \cdots > y_1$. We add this hook to each configuration associated to the chambers $E_1, \ldots, E_6$ off the closed gallery $\mathcal{G}_{2,0}$. It is obvious that adding a hook of type II leaves the connectivity invariant.

Since face operators $\partial_i'$, $i = 0, 1, 2$ and $\partial_0''$ make no effect on the added hooks, we have an orientation reversing gallery in $(P^{\pm}(3,0), W^{\pm}(3,0))$, i.e. we have a stabilization

$$Stab : (P^{\pm}(2,0), W^{\pm}(2,0)) \to (P^{\pm}(3,0), W^{\pm}(3,0))$$

which commutes with the boundary operator $\partial$. By induction on the number of slits this settles for any number $h$.

We now prove for any $c \geq 0$. We take a hook $(L_5, L_6)$ of type I. We add this hook to each configuration associated to chambers $E_1, \ldots, E_6$ of the closed gallery $\mathcal{G}_{2,0}$. Since adding a hook of type I increases one connectivity of the configurations, then we get a closed gallery in $(P^{\pm}(3,1), W^{\pm}(3,1))$. By the same argument, it is orientation reversing. By induction on $h$, the proof of the Theorem is completed. □

Hence we conclude our result concerning the orientability of $\overrightarrow{\mathfrak{M}}^{\pm}(g,c)$

**Theorem 5.4.** *The moduli space* $\overrightarrow{\mathfrak{M}}^{\pm}(g,c)$ *of non-classical directed Klein surfaces is non-orientable for any* $g$ *and* $c$.

*Remark* 5.5. It is well known that moduli spaces of Riemann surfaces with $c$ distinguished points are orientable for $c = 0, 1$ and are non-orientable for $c \geq 2$; see for instance [Mu].

## 6. The homology groups of the moduli spaces $\overrightarrow{\mathfrak{M}}^{\pm}(g,c)$

We now compute homology groups of the moduli spaces $\overrightarrow{\mathfrak{M}}^{\pm}(g,c)$ of non-classical Klein surfaces for genus $g \leq 2$. Due to the non-orientability, we restrict to $\mathbb{Z}_2$-coefficients to apply the following *Poincaré-Duality*,

$$\bar{H}_*((P^{\pm}(h,c), W^{\pm}(h,c)); \mathbb{Z}_2) \cong H^{3h-3-*}(\overrightarrow{\mathfrak{M}}^{\pm}(g,c); \mathbb{Z}_2) \ ,$$

where $g = h - (c+1)$.

Since $(P^{\pm}(h,c), W^{\pm}(h,c))$ is finite, its chain groups have finite ranks and its homology groups are finitely generated. By computing the incidence matrices of the cells $E$, we obtain the Betti numbers and torsion coefficients.

In $(P^{\pm}(2,0), W^{\pm}(2,0))$, there are eight 3-cells: $E_1, E_2, \ldots, E_8$, thirteen 2-cells: $E_9, \ldots, E_{21}$, five 1-cells: $E_{23}, \ldots, E_{26}$ and one 0-cell. The complete list of these cells is given below.

$$
\begin{array}{lllll}
E_1 = & [1,1,1,1| & (13)(24)| & \{2,4\},\{1,3\}; & T=(II,I,II,I)] \\
E_2 = & [1,1,1,1| & (13)(24)| & \{1,3\},\{2,4\}; & T=(I,II,I,II)] \\
E_3 = & [1,1,1,1| & (13)(24)| & \{1,3\},\{2,4\}; & T=(II,I,II,I)] \\
E_4 = & [1,1,1,1| & (13)(24)| & \{2,4\},\{1,3\}; & T=(I,II,I,II)] \\
E_5 = & [1,1,1,1| & (14)(23)| & \{2,3\},\{1,4\}; & T=(II,II,II,II)] \\
E_6 = & [1,1,1,1| & (14)(23)| & \{1,4\},\{2,3\}; & T=(II,II,II,II)] \\
E_7 = & [1,1,1,1| & (12)(34)| & \{1,2\},\{3,4\}; & T=(II,II,II,II)] \\
E_8 = & [1,1,1,1| & (12)(34)| & \{3,4\},\{1,2\}; & T=(II,II,II,II)] \\
E_9 = & [2,1,1,0| & (14)(23)| & \{2,3\},\{1,4\}; & T=(II,II,II,II)] \\
E_{10} = & [2,1,1,0| & (14)(23)| & \{1,4\},\{2,3\}; & T=(II,II,II,II)] \\
E_{11} = & [2,1,1,0| & (14)(23)| & \{1,4\},\{2,3\}; & T=(II,II,II,II)] \\
E_{12} = & [1,2,1,0| & (12)(34)| & \{1,2\},\{3,4\}; & T=(II,II,II,II)] \\
E_{13} = & [1,2,1,0| & (13)(24)| & \{1,3\},\{2,4\}; & T=(I,II,I,II)] \\
E_{14} = & [2,1,1,0| & (13)(24)| & \{1,3\},\{2,4\}; & T=(I,II,I,II)] \\
E_{15} = & [1,2,1,0| & (13)(24)| & \{1,3\},\{2,4\}; & T=(II,I,II,I)] \\
E_{16} = & [2,1,1,0| & (13)(24)| & \{1,3\},\{2,4\}; & T=(II,I,II,I)] \\
E_{17} = & [2,1,1,0| & (13)(24)| & \{2,4\},\{1,3\}; & T=(I,II,I,II)] \\
E_{18} = & [1,1,1,1| & (13)(24)| & \{1,2,3,4\}; & T=(II,I,II,I)] \\
E_{19} = & [1,1,1,1| & (13)(24)| & \{1,2,3,4\}; & T=(I,II,I,II)] \\
E_{20} = & [1,1,1,1| & (14)(23)| & \{1,2,3,4\}; & T=(II,II,II,II)] \\
E_{21} = & [1,1,1,1| & (12)(34)| & \{1,2,3,4\}; & T=(II,II,II,II)] \\
E_{22} = & [3,1,0,0| & (13)(24)| & \{1,3\},\{2,4\}; & T=(I,II,I,II)] \\
E_{23} = & [3,1,0,0| & (13)(24)| & \{1,3\},\{2,4\}; & T=(II,I,II,I)] \\
E_{24} = & [2,1,1,0| & (14)(23)| & \{1,2,3,4\}; & T=(II,II,II,II)] \\
E_{25} = & [2,1,1,0| & (13)(24)| & \{1,2,3,4\}; & T=(II,I,II,I)] \\
E_{26} = & [2,1,1,0| & (13)(24)| & \{1,2,3,4\}; & T=(I,II,I,II)]
\end{array}
$$

By computing the incident matrices of these cells, we obtain the homology groups of the cell complex $(P^{\pm}(2,0), W^{\pm}(2,0))$ with $\mathbb{Z}$-coefficients:

$$
H_*(P^{\pm}(2,0), W^{\pm}(2,0); \mathbb{Z}) \cong \begin{cases} 0 & , \ * \geq 3 \\ \mathbb{Z} \bigoplus \mathbb{Z}_2 & , \ * = 2 \\ \mathbb{Z} & , \ * = 1 \\ \mathbb{Z} & , \ * = 0 \ . \end{cases}
$$

By the Universal Coefficient Theorem, the homology groups of $(P^{\pm}(2,0), W^{\pm}(2,0))$ with $\mathbb{Z}_2$-coefficients are

$$
H_*(P^{\pm}(2,0), W^{\pm}(2,0); \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & , \ * = 3 \\ \mathbb{Z}_2 \bigoplus \mathbb{Z}_2 & , \ * = 2 \\ \mathbb{Z}_2 & , \ * = 1 \\ \mathbb{Z}_2 & , \ * = 0 \ . \end{cases}
$$

By Poincaré-Duality, the homology groups of moduli space $\overrightarrow{\mathfrak{M}}^{\pm}(1,0)$ of Klein bottles are

$$
H_*(\overrightarrow{\mathfrak{M}}^{\pm}(1,0); \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & , \ * = 0 \\ \mathbb{Z}_2 \bigoplus \mathbb{Z}_2 & , \ * = 1 \\ \mathbb{Z}_2 & , \ * = 2 \\ 0 & , \ * \geq 3 \ . \end{cases}
$$

**Note.** The homology classes $\{E_{15} + E_{17} + E_{20}\}$ and $\{E_9 - E_{10}\}$ represent basis elements for $H_2(P^{\pm}(2,0), W^{\pm}(2,0))$ and the homology class $\{E_{22}\}$ represents a basis element for $H_1(P^{\pm}(2,0), W^{\pm}(2,0))$.

The complex $(P^{\pm}(2,1), W^{\pm}(2,1))$ consists of ten 3-cells, fourteen 2-cells, four 1-cells and one 0-cell. By computing the rank and the elementary divisors from their incidence matrices, we obtain

$$H_*(P^{\pm}(2,1), W^{\pm}(2,1); \mathbb{Z}) \cong \begin{cases} 0 & , \quad * \geq 3 \\ \mathbb{Z}_2 & , \quad * = 2 \\ 0 & , \quad * = 1 \\ \mathbb{Z} & , \quad * = 0 \ . \end{cases}$$

The homology groups with $\mathbb{Z}_2$ of the moduli space $\overrightarrow{\mathfrak{M}}^{\pm}(0,1)$ of projective planes with a puncture are

$$H_*(\overrightarrow{\mathfrak{M}}^{\pm}(0,1); \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & , \quad * = 0 \\ \mathbb{Z}_2 & , \quad * = 1 \\ 0 & , \quad * \geq 2 \end{cases}$$

From the computation of the complex $(P^{\pm}(3,0), W^{\pm}(3,0))$, we obtain the following:

| **Dim.** | **No. of Cells** | $\mathbf{H_*((P^{\pm}(3,0), W^{\pm}(3,0)); \mathbb{Z})}$ | $\mathbf{H_*((P^{\pm}(3,0), W^{\pm}(3,0)); \mathbb{Z}_2)}$ |
|---|---|---|---|
| 6 | 246 | 0 | $\mathbb{Z}_2$ |
| 5 | 786 | $\mathbb{Z}_2$ | $\mathbb{Z}_2 \bigoplus \mathbb{Z}_2$ |
| 4 | 907 | $\mathbb{Z}_2$ | $\mathbb{Z}_2 \bigoplus \mathbb{Z}_2 \bigoplus \mathbb{Z}_2$ |
| 3 | 440 | $\mathbb{Z}_2 \bigoplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \bigoplus \mathbb{Z}_2$ |
| 2 | 73 | 0 | 0 |
| 1 | 0 | 0 | 0 |
| 0 | 1 | $\mathbb{Z}$ | $\mathbb{Z}_2$ |

We have the homology groups of the moduli space $\overrightarrow{\mathfrak{M}}^{\pm}(2,0)$ of non-classical Klein surfaces of genus $g = 2$:

$$H_*(\overrightarrow{\mathfrak{M}}^{\pm}(2,0); \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & , \quad * = 0 \\ \mathbb{Z}_2 \bigoplus \mathbb{Z}_2 & , \quad * = 1 \\ \mathbb{Z}_2 \bigoplus \mathbb{Z}_2 \bigoplus \mathbb{Z}_2 & , \quad * = 2 \\ \mathbb{Z}_2 \bigoplus \mathbb{Z}_2 & , \quad * = 3 \\ 0 & , \quad * \geq 4 \ . \end{cases}$$

From the computation for $h = 3, c = 1$, we obtain the following

| **Dim.** | **No. of Cells** | $\mathbf{H_*((P^{\pm}(3,1), W^{\pm}(3,1)); \mathbb{Z})}$ | $\mathbf{H_*((P^{\pm}(3,1), W^{\pm}(3,1); \mathbb{Z}_2)}$ |
|---|---|---|---|
| 6 | 252 | 0 | $\mathbb{Z}_2$ |
| 5 | 747 | $\mathbb{Z} \bigoplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \bigoplus \mathbb{Z}_2$ |
| 4 | 786 | $\mathbb{Z}$ | $\mathbb{Z}_2 \bigoplus \mathbb{Z}_2$ |
| 3 | 339 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| 2 | 48 | 0 | 0 |
| 1 | 0 | 0 | 0 |
| 0 | 1 | $\mathbb{Z}$ | $\mathbb{Z}_2$ |

Hence the homology groups of the moduli space $\overrightarrow{\mathfrak{M}}^{\pm}(1,1)$ of Klein bottles with a puncture are

$$H_*(\overrightarrow{\mathfrak{M}}^{\pm}(1,1); \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & , \quad * = 0 \\ \mathbb{Z}_2 \bigoplus \mathbb{Z}_2 & , \quad * = 1 \\ \mathbb{Z}_2 \bigoplus \mathbb{Z}_2 & , \quad * = 2 \\ \mathbb{Z}_2 & , \quad * = 3 \\ 0 & , \quad * \geq 4 \ . \end{cases}$$

The computation for $h = 3, c = 2$ gives the following

| Dim. | No. of Cells | $\mathbf{H}_*((\mathbf{P}^{\pm}(\mathbf{3},\mathbf{2}),\mathbf{W}^{\pm}(\mathbf{3},\mathbf{2}));\mathbb{Z})$ | $\mathbf{H}_*((\mathbf{P}^{\pm}(\mathbf{3},\mathbf{2}),\mathbf{W}^{\pm}(\mathbf{3},\mathbf{2}));\mathbb{Z}_{\mathbf{2}})$ |
|---|---|---|---|
| 6 | 132 | 0 | $\mathbb{Z}_2$ |
| 5 | 357 | $\mathbb{Z}_2$ | $\mathbb{Z}_2 \bigoplus \mathbb{Z}_2$ |
| 4 | 330 | $\mathbb{Z}_4$ | $\mathbb{Z}_2$ |
| 3 | 117 | 0 | 0 |
| 2 | 12 | 0 | 0 |
| 1 | 0 | 0 | 0 |
| 0 | 1 | $\mathbb{Z}$ | $\mathbb{Z}_2$ |

The homology groups of the moduli space $\overrightarrow{\mathfrak{M}}^{\pm}(0,2)$ of projective planes with two punctures are

$$H_*(\overrightarrow{\mathfrak{M}}^{\pm}(0,2);\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & , \quad * = 0 \\ \mathbb{Z}_2 \bigoplus \mathbb{Z}_2 & , \quad * = 1 \\ \mathbb{Z}_2 & , \quad * = 2 \\ 0 & , \quad * \geq 3 \; . \end{cases}$$

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