United Nations Educational Scientific and Cultural Organization
and
International Atomic Energy Agency

THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

CONTROLLABILITY OF AFFINE SYSTEMS
FOR THE GENERALIZED HEISENBERG LIE GROUPS

Ayse Kara[1]
Department of Mathematics, Yildiz Technical University,
Davutpasa, 34120 Istanbul, Turkey
and
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy

and

Luiz A.B. San Martin[2]
Instituto de Matematica, Universidade Estadual de Campinas,
Cx. Postal 6065, 13081-970 Campinas SP, Brasil.

Abstract

Affine control systems on the generalized Heisenberg Lie group are studied. Controllability of this kind of class of systems on the generalized Heisenberg Lie group is established by relating to their associated bilinear control systems.

MIRAMARE – TRIESTE

August 2001

[1]E-mail: kara@yildiz.edu.tr
[2]E-mail: smartin@ime.unicamp.br

1 Introduction

In this paper, we study the controllability properties of affine control systems on the generalized Heisenberg Lie group, relating them to the controllability of the bilinear control system. In [2], the authors Jurdjevic and Sallet studied the controllability of affine systems in Euclidean spaces. They consider the controllability of associated bilinear system on $\mathbb{R}^n$ to get a condition for controllability of the affine system. In this work, the approach of [2] is extended to the study of affine control systems on a generalized Heisenberg Lie group.

The class of affine control systems is not studied so much and it is possible to see the controllability results for this class of systems in the recent book by Jurdjevic, [3].

In [1], the authors Ayala and San Martin had studied the controllability problem through the subalgebra of the Lie algebra of the affine group of G generated by the vector fields of the linear control system where the drift vector field X is an infinitesimal automorphism, i.e., $(X_t)_{t \in \mathbb{R}}$ is a one-parameter subgroup of $\text{Aut}(G)$; lifted the system itself to a right-invariant control system in the Lie group $\text{Af}(G)$ and given controllability results for compact connected and noncompact semi-simple Lie groups cases.

This paper is organized as follows. In section 2, the affine control systems on Lie groups and in section 3, the generalized Heisenberg Lie group, which is an important model of nilpotent simply connected Lie groups, are presented. In the third section, we show the existence of an automorphism of the Lie algebra of the generalized Heisenberg Lie group such that it shrinks the elements of the Lie algebra to the neutral element and that the $\text{Aut}(H)$-orbit of the generalized Heisenberg Lie group is dense; and the main result of this paper which generalizes the result of [2], is that the characterization of the controllability of affine control systems on the generalized Heisenberg Lie group associated to their bilinear control systems is given. By the controllability property, we mean that the positive orbit is equal to the state space which is the generalized Heisenberg Lie Group in this work, i.e., it is possible to reach every state via the trajectories with positive time.

2 Affine Control Systems on Lie Groups

Let G be a connected Lie group with Lie algebra $L(G)$. The affine group $\text{Af}(G)$ of G is the semi-direct product of $\text{Aut}(G)$ with G itself, i.e.,
$\text{Af}(G) = \text{Aut}(G) \times_s G$. The multiplication in $\text{Af}(G)$ is defined by

$$(\phi, g_1) \cdot (\psi, g_2) = (\phi \circ \psi, g_1 \phi(g_2)).$$

Denote by 1 the identity element of $\text{Aut}(G)$ and by e the identity element of G. The group identity of $\text{Af}(G)$ is $(1, e)$ and $(\phi^{-1}, \phi^{-1}(g^{-1}))$ is the inverse of $(\phi, g) \in \text{Af}(G)$. Then, $g \to (1, g)$ and $\phi \to (\phi, e)$ embed G into $\text{Af}(G)$ and $\text{Aut}(G)$ into $\text{Af}(G)$, respectively. Therefore, G and

$\mathrm{Aut}(G)$ are subgroups of $\mathrm{Af}(G)$. There is a natural action

$$\mathrm{Af}(G) \times G \to G$$

defined by

$$(\phi, g_1) \cdot g_2 \to g_1 \phi(g_2),$$

where $(\phi, g_1) \in \mathrm{Af}(G)$ and $g_2 \in G$. This action is transitive. Indeed, if it is taken $g_2 = e$, then $(\phi, g_1) \cdot e = g_1$ since $\phi(g_2) = e$.

The groups G and $\mathrm{Aut}(G)$ are closed subgroups of $\mathrm{Af}(G)$. Denote by $\mathrm{Aut}(L(G))$ the automorphism group of $L(G)$ which is a Lie group and its Lie algebra is $\mathrm{Der}(L(G))$, the Lie algebra of derivations of $L(G)$. If G is simply connected, then $\mathrm{Aut}(L(G))$ and $\mathrm{Aut}(G)$ are isomorphic. In fact, there is an isomorphism Φ which assigns to each authomorphism ϕ of G its differential $d\phi \mid_1$ at the identity. Any authomorphism of $L(G)$ extends to an authomorphism of G, therefore Φ is indeed an isomorphism between $\mathrm{Aut}(L(G))$ and $\mathrm{Aut}(G)$. Thus, in this case, the Lie algebra of $\mathrm{Aut}(G)$ is $\mathrm{Der}(L(G))$.

The Lie algebra $\mathrm{af}(G)$ of $\mathrm{Af}(G)$ is the semi-direct product $\mathrm{Der}(L(G)) \times_s L(G)$. Its Lie bracket is given by

$$[(D_1, X_1), (D_2, X_2)] = ([D_1, D_2], D_1 X_2 - D_2 X_1 + [X_1, X_2])$$

An affine control system $\Sigma = (G, \mathcal{D})$ on a Lie group $G \subset \mathrm{Af}(G)$ is determined by the family of differential equations :

$$\dot{x} = (D + X)(x) + \sum_{j=1}^{d} u_j(t)(D^j + Y^j)(x)$$

parametrized by U, a family of piecewise constant real valued funtions, where
$x \in G$; $D, D^1, \ldots, D^d \in \mathrm{Der}(L(G))$ and $X, Y^1, \ldots, Y^d \in L(G)$. Then, the dynamic is given by

$$\mathcal{D} = \{D + X + \sum_{j=1}^{d} u_j(D^j + Y^j) \mid u \in \mathbb{R}^d\}.$$

If the affine control system is considered on an abelian Lie group, then it becomes a linear control system. In fact, for the abelian Lie group case since any bracket between the elements of the Lie algebra is null, the elements of the affine system turns to the form of the elements of the linear control system.

If $X = 0$ and $Y^1 = Y^2 = \ldots = Y^d = 0$ are considered for the affine control system on Lie group, then it becomes a bilinear control system. In general, affine control systems define a richer class of systems than the bilinear class, and their controllability properties on generalized Heisenberg Lie group are essentially governed by their bilinear parts $D, D^1, \ldots, D^d$ as will be seen in the next section.

3 The Generalized Heisenberg Lie Group

The Heisenberg Lie Groups are obtained by the following construction, see [4] by Stroppel : Let V and Z be finite dimensional real vector spaces and $\beta : V \times V \to Z$ be a symplectic map (i.e., a nondegenerate skew-symmetric bilinear map). The vector space $V \times Z$ becomes a Lie algebra $L(H) =: h(V, Z, \beta)$ with bracket $[(v_1, z_1), (v_2, z_2)] = (0, \beta(v_1, v_2))$. The corresponding simply connected Lie group $H =: H(V, Z, \beta)$ is $V \times Z$ endowed with the Campbell-Hausdorff multiplication

$$(v_1, z_1) * (v_2, z_2) = (v_1 + v_2, z_1 + z_2 + \frac{1}{2}\beta(v_1, v_2)).$$

For the proof of our controllability result the following lemma is required.

Lemma 1 : For the generalized Heisenberg Lie group $H =: H(V, Z, \beta)$, the map $\varphi_\lambda = \sqrt{\lambda}\mathrm{Id} \times \lambda\mathrm{Id}$, i.e., $\varphi_\lambda(v, z) = (\sqrt{\lambda}v, \lambda z)$ is an automorphism.

Proof. That the mapping φ_λ is 1-1 and onto its image is obvious. Now, we check that φ_λ is a homomorphism, that is,

$$\varphi_\lambda((v_1, z_1) * (v_2, z_2)) = \varphi_\lambda(v_1 + v_2, z_1 + z_2 + \frac{1}{2}\beta(v_1, v_2))$$

$$= (\sqrt{\lambda}\mathrm{Id}v_1 + \sqrt{\lambda}\mathrm{Id}v_2, \lambda\mathrm{Id}z_1 + \lambda\mathrm{Id}z_2 + \frac{\lambda\mathrm{Id}}{2}\beta(v_1, v_2))$$

by bilinearity of β

$$= (\sqrt{\lambda}\mathrm{Id}v_1 + \sqrt{\lambda}\mathrm{Id}v_2, \lambda\mathrm{Id}z_1 + \lambda\mathrm{Id}z_2 + \frac{1}{2}\beta(\sqrt{\lambda}v_1, \sqrt{\lambda}v_2))$$

$$= (\sqrt{\lambda}\mathrm{Id}v_1, \lambda\mathrm{Id}z_1) * (\sqrt{\lambda}\mathrm{Id}v_2, \lambda\mathrm{Id}z_2)$$

$$= \varphi_\lambda(v_1, z_1) * \varphi_\lambda(v_2, z_2).$$

Corollary : In H there exists a 1-parameter family of automorphisms φ_λ such that $\varphi_\lambda(v, z) \to 0$ as $\lambda \to 0$.

Lemma 2 : Let H be a generalized Heisenberg Lie group. Then there exits a dense $\mathrm{Aut}(H)$-orbit.

Proof. The set

$$\mathcal{O} =: exp(L(H) - [L(H), L(H)]) = H - [H, H]$$

is an $\mathrm{Aut}(H)$-orbit of H. In fact, the exponential map is a global diffeomorphism for simply connected nilpotent Lie groups. Moreover, taken any two elements $X, Y \in \mathcal{O}$ such that the line segment $\mid XY \mid$ is parallel to $[H, H]$, they can be connected via a line segment by taking once X as an initial point so that the function of that connection $f_p : \mathcal{O} \to \mathcal{O}$, defined by $X \to t_1 X + t_2 = Y$, where $t_1, t_2 \in \mathbb{R}$, is an automorphism. Actually, it is possible to connect these segments with the perpendicular segments to each other in the same way. That $\mathrm{Aut}(H)$-orbit of H is $\mathcal{O}$ is open. In fact, if $dimZ = 1$ the center $[H, H]$ forms a line. Indeed, for any

Heisenberg group $[X, Y] = Z$, $X, Y, Z \in L(H)$. To show the density of the Aut(H)-orbit of H, for any $x \in [H, H]$ and every ball $B(x, \delta)$

$$B(x, \delta) \cap H - [H, H] \neq \emptyset.$$

Thus, $\overline{H - [H, H]} = H$.

Theorem : An affine control system $\Sigma = (H, \mathcal{D})$ on the generalized Heisenberg Lie group H is controllable if Σ does not have an equilibrium point and the induced bilinear control system $\Sigma_b = (H, \mathcal{D}_b)$ where

$$\mathcal{D}_b = \{D + \sum_{j=1}^{d} u_j D^j \mid D, D^j \in \text{Der}(L(H)); u \in \mathbb{R}^d\},$$

is controllable in the Aut(H)-orbit of H.

Proof. For the affine system, to not have an equilibrium point is a necessary condition for controllability, since the set of reachable points from a fixed point consists of a single point. Define the automorphism $\xi_\lambda : \text{af}(H) \to \text{af}(H)$ such that $\xi_\lambda(D + X) = D + \varphi_\lambda(X)$, where $\varphi_\lambda = (\sqrt{\lambda}\text{Id}, \lambda\text{Id})$ and so $\varphi_\lambda(X) \to 0$ as $\lambda \to 0$, for all $X \in L(H)$. Then, $\xi_\lambda(\mathcal{D}) \to \mathcal{D}_b$ as $\lambda \to 0$. Therefore, $\xi_\lambda(\Sigma) \to \Sigma_b$ as $\lambda \to 0$. By the hypothesis, Σ_b is controllable on the Aut(H)-orbit of H and by Lemma 2, the Aut(H)-orbit of H is dense. Hence, Σ_b is controllable on every orbit.

Consider the unit sphere $S(1, 1)$ centered at 1 which is the boundary of the unit ball $B(1, 1)$ centered at 1. For λ sufficiently small, $\xi_\lambda(\Sigma)$ is controllable on $S(1, 1) - [H, H]$ since complete controllability is preserved under small perturbations, [6] by Sussmann. Then $\xi_\lambda(\Sigma)$ is controllable on $B(1, 1) - [H, H]$. Indeed, finite systems normally controllable on $S(1, 1)$ are open, [6] by Sussmann. Therefore, Σ is controllable on $B(1_{\varphi^{-1}}, 1) - [H, H]$, where $1_{\varphi^{-1}} = (1, (\frac{\text{Id}}{\sqrt{\lambda}}e, \frac{\text{Id}}{\lambda}e))$. Then, the positive orbit of an affine system through $(1, (\frac{\text{Id}}{\sqrt{\lambda}}e, \frac{\text{Id}}{\lambda}e))$ is open and the interior is nonempty since it contains $B(1_{\varphi^{-1}}, 1) - [H, H]$. Then, the system Σ is normally accessible from $1_{\varphi^{-1}}$. Since the state space is connected, Σ is controllable on H.

Acknowledgments

References

[1] Ayala, V. and San Martin, L.A.B., "*Controllability Properties of a Class of Control Systems on Lie Groups*", Proceedings of Second Nonlinear Control Network (NCN) Workshop, France, June 5-8, 2000.

[2] Jurdjevic, V. and Sallet, G., "*Controllability Properties of Affine Systems*", SIAM J.Control Opt. 22:501-8, 1984.

[3] Jurdjevic, V., "*Geometric Control Theory*", Cambridge University Press, 1997.

[4] Stroppel, M., "*Homogeneous Symplectic Maps and Almost Homogeneous Heisenberg Groups*", Forum Math. 11, 659-672, 1999.

[5] San Martin, L.A.B., "*Algebras de Lie*", Editora de Unicamp, 1999.

[6] Sussmann, H., "*Some Properties of Vector Field Systems That are not Altered by Small Perturbations*", Journal of Differential Equations 20, 292-315, 1976.