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ISOMETRIC C^1-IMMERSIONS FOR PAIRS OF RIEMANNIAN METRICS

Giuseppina D'Ambra[1]
Department of Mathematics, University of Cagliari, Cagliari, Italy

and

Mahuya Datta[2]
Department of Mathematics, University of Calcutta, Calcutta, India.

Abstract

Let h_1, h_2 be two Euclidean metrics on $\mathbb{R}^q$, and let V be a C^∞-manifold endowed with two Riemannian metrics g_1 and g_2. We study the existence of C^1-immersions $f : (V, g_1, g_2) \longrightarrow (\mathbb{R}^q, h_1, h_2)$ such that $f^*(h_i) = g_i$ for $i = 1, 2$.

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[1]E-mail: dambra@vaxca1.unica.it
[2]E-mail: mahuyad@hotmail.com

1. Introduction

In 1954 Nash [4] proved his C^1-isometric Embedding Theorem saying that if an n-dimensional Riemannian manifold (V, g) admits a C^1-immersion (resp. embedding) in $\mathbb{R}^q$, where $q \geq \dim V + 2$, then it admits a C^1-isometric immersion (resp. embedding) in $\mathbb{R}^q$. Kuiper [3] improved this result by showing that it is true even when $q \geq \dim V + 1$. The key idea in the proof is to start with a g-short C^∞ map and to approach g through a sequence of g-short C^∞-maps — improving the approximation in every successive step and keeping control over the first derivative of the map – so that finally the required isometric C^1-map comes as the C^1-limit of the sequence.

The problem of C^1-isometric immersions in a general Riemannian Manifold has been treated by Gromov through the Convex Integration Theory [2] which accommodates the essence of Nash-Kuiper techniques.

The aim of this paper is to generalize the result of Nash to the case where the manifolds come with pairs of Riemannian metrics. The problem we consider can be formulated as follows: Let h_1 be the standard Euclidean metric on $\mathbb{R}^q$ with canonical coordinates $(x_1, x_2, \ldots, x_q)$ and $h_2 = \sum_{i=1}^q \lambda_i^2 dx_i^2$, where we assume that $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_q$. Consider a C^∞-manifold V with two C^∞-Riemannian metrics g_1 and g_2. The problem is to determine when there exists a C^1-immersion $f : (V, g_1, g_2) \longrightarrow (\mathbb{R}^q, h_1, h_2)$ such that

$$f^*(h_1) = g_1 \quad \text{and} \quad f^*(h_2) = g_2.$$

It is evident that not all pairs (g_1, g_2) can be induced from (h_1, h_2) by a map. In fact, (g_1, g_2) must satisfy the inequality

$$\lambda_1^2 g_1 \leq g_2 \leq \lambda_q^2 g_1.$$

The above condition in general is not sufficient. For example we consider the linear situation, when $V = \mathbb{R}^n$, $g_1 = \sum_{i=1}^n dy_i^2$, and $g_2 = \sum_{i=1}^q \mu_i^2 dy_i^2$ with $\mu_i > 0$ for all $i = 1, \ldots, n$. It may be proved by simple calculation that if there are $2n$ indices $\{i_1, i_2, \ldots, i_{2n}\}$ such that

$$\lambda_{i_{2k-1}} < \mu_k < \lambda_{i_{2k}}$$

for $k = 1, 2, \ldots, n$, then we obtain an injective linear map $\mathbb{R}^n \longrightarrow \mathbb{R}^q$ inducing (g_1, g_2) from (h_1, h_2). In particular, let $a = \min_k \mu_k$ and $b = \max_k \mu_k$ be two positive numbers. If there are n many λ_i's smaller than a and an equal number of λ_i's greater than b, then there exists an injective linear map which is isometric with respect to both the pairs (g_1, h_1) and (g_2, h_2).

On the other hand if we take $g_1 = g_2$ then the existence of such a linear map necessarily implies the existence of an injective linear map $L : \mathbb{R}^n \longrightarrow \mathbb{R}^q$ which is $(h_2 - h_1)$-isotropic, that is $L^*(h_2 - h_1) = 0$. If $q_+(h)$ and $q_-(h)$ respectively denote the positive and the negative ranks of $h = h_2 - h_1$ then such an isotropic map exists only if $q_+ \geq n$ and $q_- \geq n$, for the set of h-isotropic vectors contains at the most an m-dimensional subspace for $m = \min(q_+, q_-)$.

We now state the main result of this paper and an immediate corollary to it.

Theorem 1.1. *Let $(\mathbb{R}^q, h_1, h_2)$ be as above and let V be an n-dimensional C^∞-manifold. Let a, b be two positive numbers, $a < b$, such that $\lambda_i < a$ for all $i \leq 3n+2$ and $\lambda_i > b$ for $i \geq q - 3n - 1$, and none of the λ_i's lies between a and b. If g_1 and g_2 are two Riemannian metrics on V satisfying the relation $a^2 g_1 < g_2 < b^2 g_1$ then there exists a C^1-immersion $f : V \longrightarrow \mathbb{R}^q$ such that*

$$f^*(h_1) = g_1 \quad \text{and} \quad f^*(h_2) = g_2.$$

Corollary 1.2. *Let (V, g) be a smooth Riemannian manifold of dimension n, and let $(\mathbb{R}^q, h_1, h_2)$ be as in Theorem 1.1. If $h_1 - h_2$ is non-singular, and if the positive rank q_+ and the negative rank q_- of $h_1 - h_2$ satisfy $q_+ \geq 3n + 2$ and $q_- \geq 3n + 2$ then there exists a C^1-immersion $f : V \longrightarrow \mathbb{R}^q$ such that $f^*(h_1) = g = f^*(h_2)$.*

Observe that the map f in the corollary is, in particular, an isotropic immersion relative to $h_1 - h_2$. One may compare the above corollary with a result of Gromov ([2], 2.4.9(B)) which says that if (W, h) is a pseudo-Riemannian manifold with $q_+, q_- \geq 2n$ then an arbitrary C^∞-manifold V admits a isotropic C^1-immersion in W.

We shall closely follow Nash in our proof of Theorem 1.1. We explain the basic ideas of Nash's techniques [4] in Section 2. Through Section 3 to Section 5 we shall make preparation for the adaptation of Nash's techniques in our context. Finally in Section 6 and 7 we prove Theorem 1.1

2. Preliminaries

Here we briefly discuss Nash's proof of C^1-Isometric Embedding Theorem. For a detailed treatment we refer the reader to [1] [4].

Theorem 2.1. (*Nash*) *If a Riemannian manifold (V, g) admits a smooth immersion (or embedding) in $\mathbb{R}^q$, then there exists an isometric C^1-immersion (respectively, a C^1-embedding) $f : (V, g) \longrightarrow (\mathbb{R}^q, h)$, provided $\dim V \leq q - 2$, where h denotes the canonical Euclidean metric on $\mathbb{R}^q$.*

To prove the theorem Nash starts with a C^∞-immersion $f_0 : V \longrightarrow (\mathbb{R}^q, h)$ which is strictly short relative to the C^∞-Riemannian metric g on V. Recall that a C^1-map $f_0 : V \longrightarrow \mathbb{R}^q$ is strictly short if the difference $g_1 - f_0^* h_1$ is positive definite. This is expressed by the notation $g_1 - f_0^* h_1 > 0$ or $g_1 > f_0^* h_1$. Short immersions $V \longrightarrow \mathbb{R}^q$ exist whenever V admits an immersion in $\mathbb{R}^q$.

The basic idea is to construct a sequence of strictly short C^∞-immersions $f_j : (V, g) \longrightarrow (\mathbb{R}^q, h)$ satisfying the two properties:

1. $\|g - f_j^*(h)\|_0 < \frac{2}{3}\|g - f_{j-1}^*(h)\|_0$,
2. $\|f_j - f_{j-1}\|_1 < c(n)\|g - f_{j-1}^*(h)\|_0^{\frac{1}{2}}$,

where $\|.\|_r$ denotes the norm in the C^r topology and $c(n)$ is a constant depending on the dimension n of the manifold V. It then follows that the sequence $\{f_j\}$ is a Cauchy sequence in the C^1-topology and hence it converges to some C^1-function $f : V \longrightarrow \mathbb{R}^q$ such that $f^*(h) = g$. The problem therefore boils down to the following:

Lemma 2.2. *Assume* $q \geq \dim V + 2$ *and let* $f_0 : V \longrightarrow (\mathbb{R}^q, h)$ *be a strictly short* C^∞*-immersion. Then there exists a* C^∞*-immersion* $f_1 : V \longrightarrow (\mathbb{R}^q, h)$ *such that:*

(*i*) f_1 *is strictly short,*

(*ii*) $\|g - f_1^*(h)\|_0 < \frac{2}{3}\|g - f_0^*(h)\|_0$,

(*iii*) $\|f_1 - f_0\|_1 < c(n)\|g - f_0^*(h)\|_0^{1/2}$.

Proof. The construction of f_1 involves a sequence of successive 'stretching' and 'twisting' on f_0. We fix a locally finite covering $\mathcal{U} = \{U_p : p \in \mathbb{Z}\}$ of V by contractible coordinate neighbourhoods U_p in V. Nash noted in his paper that a positive definite C^∞-metric g on a manifold V admits a decomposition as stated below:

$$g = \sum_{i=1}^{\infty} \phi_i^2 d\psi_i^2, \tag{1}$$

where ϕ_i and ψ_i are C^∞-functions on V with support of ϕ_i compact and contained in some $U_p \in \mathcal{U}$. Further, almost all ϕ_i's vanish on a U_p, and there are at most N many ϕ_i's which are simultaneously non-zero at any point of V, where N is a constant depending on the dimension of V. This observation plays a crucial role in his proof of Theorem 2.1. Since f_0 is strictly short, $\delta = g - f_0^*(h)$is positive definite. Hence $\delta/2$ can be decomposed as $\delta/2 = \sum_{i=1}^{\infty} \phi_i^2 d\psi_i^2$, where ψ_i and ϕ_i are C^∞-functions on V as in equation (1).

We first construct a sequence of C^∞-immersions, $\{\bar{f}_i\}$ with $\bar{f}_0 = f_0$. In the i-th successive step we deform $\bar{f}_{i-1}$ on some $U_p \in \mathcal{U}$ where $U_p \supset \mathit{supp}\,\phi_i$, to obtain $\bar{f}_i$ such that the induced metric is approximately augmented by the factor $\phi_i^2 d\psi_i^2$.

Fix any $\varepsilon_1 > 0$. We describe how f_0 can be deformed within its ε_1-neighbourhood (relative to the C^0 topology) to a C^∞-immersion $\bar{f}_1$ such that the induced metric $\bar{f}_1^*(h) = f_0^*(h) + \phi_1^2 d\psi_1^2 + O(\varepsilon_1)$.

Normal Extension(stretching): From the above decomposition of $\delta/2$ we obtain an $U_p \in \mathcal{U}$ such that $\mathit{supp}\,\phi_1 \subset U_p$. We extend the map f_0 to $\tilde{f}_0 : U_p \times \mathbb{R}^2 \longrightarrow \mathbb{R}^q$ in such a way that

$$\tilde{f}_0^*(h) = f_0^*(h) \oplus dt^2 \oplus ds^2$$

on $T\mathbb{R}^q|_{U_p}$, where (t, s) denote the global coordinates on $\mathbb{R}^2$. To get the extension we merely have to choose two vector fields namely τ_1 and τ_1' on U_p which are mutually orthogonal and also normal to TV. Since f_0 is an immersion and $q \geq \dim V + 2$ such vector fields always exist on contractible sets. Then we define $\tilde{f}_0 : U_p \times \mathbb{R}^2 \longrightarrow \mathbb{R}^q$ by the formula: $(y, t, s) \mapsto f_0(y) + t\tau_1 + s\tau_1'$. The difference $\tilde{f}_0^*(h) - f_0^*(h) \oplus dt^2 \oplus ds^2$ on $U_p \times D^2_{\varepsilon_1}$ is of the order of ε_1, where $D^2_{\varepsilon_1}$ denotes the ε_1-ball in $\mathbb{R}^2$ centered at the origin.

Adding $\phi_1^2 d\psi_1^2$(twisting): Consider the C^∞-map $\alpha : \mathbb{R} \longrightarrow \mathbb{R}^2$ defined by $u \mapsto (\varepsilon_1 \cos \varepsilon_1^{-1} u, \varepsilon_1 \sin \varepsilon_1^{-1} u)$. Note that the map α is an isometry with respect to the canonical

Euclidean metrics on $\mathbb{R}$ and $\mathbb{R}^2$. Moreover, it takes $\mathbb{R}$ into the ε_1-ball D^2_ε in $\mathbb{R}^2$ centered at the origin.

Define a map $\alpha_1 : U_p \longrightarrow U_p \times \mathbb{R}^2$ as follows:

$$u \mapsto (u,\, \varepsilon_1\phi_1(u)\cos\varepsilon_1^{-1}\psi_1(u),\, \varepsilon_1\phi_1(u)\sin\varepsilon_1^{-1}\psi_1(u)).$$

It can be shown that $\alpha_1^*(dt^2 \oplus ds^2) = \phi_1^2 d\psi^2 + \varepsilon_1^2 d\phi^2$. Since ϕ_1 has its support contained in U_p, the composite map

$$\bar{f}_1 : U_p \xrightarrow{\alpha_1} U_p \times \mathbb{R}^2 \xrightarrow{\tilde{f}_0} \mathbb{R}^q$$

extends to a C^∞-map on all of V such that it coincides with f_0 outside U_1. We shall denote this extension also by $\bar{f}_1$. Then

$$\bar{f}_1^*(h) = f_0^*(h) + \phi_1^2 d\psi_1^2 + O(\varepsilon_1).$$

The difference $\bar{f}_1^*(h) - f_0^*(h) - \phi_1^2 d\psi_1^2$ can be made arbitrarily small in the C^0 topology by controlling ε_1.

On the other hand it is clear from the construction that the C^0-distance between $\bar{f}_1$ and $\bar{f}_0 = f_0$ is less than ε_1. Moreover, if $\{y_1, y_2, \dots, y_n\}$ is a coordinate system on U_p then

$$\left|\frac{\partial \bar{f}_1}{\partial y_j} - \frac{\partial \bar{f}_0}{\partial y_j}\right| = \phi_1 \left|\frac{\partial \psi_1}{\partial y_j}\right| + O(\varepsilon_1)$$

for all $j = 1, 2, \dots, n$.

Repeating the above two steps for $\phi_2^2 d\psi_2^2$ on $\bar{f}_1$ and so on in succession, we obtain a sequence of maps $\bar{f}_1, \bar{f}_2, \dots$. Since $\mathcal{U}$ is locally finite and supports of only finitely many ϕ_i's intersect an U_p, the sequence $\{\bar{f}_i\}$ is eventually constant on each U_p. Therefore, $\lim \bar{f}_i$ exists uniformly on it. Let $f_1 = \lim \bar{f}_i$. Then, f_1 is a C^∞ map and $f_1^*(h) = f_0^*(h) + \frac{\delta}{2} - \sum_i O(\varepsilon_i)$ which gives

$$g - f_1^*(h) = \tfrac{1}{2}(g - f_0^*(h)) - \textstyle\sum_i O(\varepsilon_i),$$

where $\sum_i O(\varepsilon_i)$ can be made arbitrarily small in the fine C^0 topology.

Also, on any $U_p \in \mathcal{U}$ with coordinate functions $\{y_1, \dots, y_n\}$ we have,

$$\left|\frac{\partial f_1}{\partial y_j} - \frac{\partial \bar{f}_0}{\partial y_j}\right| = \sum_i \left|\frac{\partial \bar{f}_i}{\partial y_j} - \frac{\partial \bar{f}_{i-1}}{\partial y_j}\right| = \sum_i \phi_i \left|\frac{\partial \psi_i}{\partial y_j}\right| + \sum_i O(\varepsilon_i).$$

Since at most N many ϕ_i's (N depending on n) are non-vanishing at a point, we can apply the Cauchy-Schwartz inequality on the first sum on the right hand side expression to obtain

$$\begin{array}{rcl} (\sum_i |\phi_i \frac{\partial \psi_i}{\partial y_j}|)^2 & \leq & N \sum_i \phi_i^2 (\frac{\partial \psi_i}{\partial y_j})^2 \\ & \leq & N \sum_i \phi_i^2 d\psi_i^2 (\frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_j}) \\ & \leq & N \sum_i \phi_i^2 d\psi_i^2)(\frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_j}) \\ & = & \frac{N}{2}\delta(\frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_j}) \end{array},$$

Therefore,

$$\|f_1 - f_0\|_1 \leq c(n)\|\delta\|_0^{\frac{1}{2}} + \textstyle\sum_i O(\varepsilon_i).$$

on each U_p, where $c(n)$ is a constant which depends on n.

We can choose ε_i at the i-th step for $i = 1, 2, \ldots$ in such a way that f_1 is strictly short, and that the term $\sum_i O(\varepsilon_i)$ is strictly less than $\frac{1}{6}\|g - f_0^*(h)\|_0$. This gives relation (ii) and (iii). $\square$

We shall modify Nash's techniques to give a proof of Theorem 1.1. The initial map $f_0 : V \longrightarrow \mathbb{R}^q$ in our context should be such that:

(a) $g_1 - f_0^* h_1$ and $g_2 - f_0^* h_2$ are positive definite, and they have simultaneous decomposition as $g_1 - f_0^* h_1 = \sum_i \phi_i^2 d\psi_i^2$ and $g_2 - f_0^* h_2 = \sum_i c_i^2 \phi_i^2 d\psi_i^2$ for some C^∞-functions ϕ_i and ψ_i as above and some constants c_i.

(b) f_0 admits a 'normal extension' $\tilde{f}_0$; that is, $\tilde{f}_0^*(h_1) = f_0^*(h_1) \oplus dt^2 \oplus ds^2$ $\tilde{f}_0^*(h_2) = f_0^*(h_2) \oplus c^2 dt^2 \oplus c^2 ds^2$ on V for some specific constant c which appear in the decomposition.

3. Decomposition Lemma

In this section and in the two subsequent ones we shall make the preparation for the Nash process. Here we state and prove the decomposition lemma in our context.

We first fix a countable covering $\mathcal{U} = \{U_p | p \in \mathbb{Z}\}$ of the manifold V which has the following properties:

(a) each U_p is a coordinate neighbourhood in V and

(b) for any p_0, U_{p_0} intersects atmost m different U_p's including itself (m is an integer depending on $n = \dim V$).

Let $\{\phi_p\}$ be a smooth partition of unity on V subordinate to this covering $\{U_p\}$ such that support of ϕ_p, denoted $supp\, \phi_p$, is compact for each p. Let $K_p = supp\, \phi_p \subset U_p$.

Lemma 3.1. *Let g_1 and g_2 be two Riemannian metrics on V such that $a^2 g_1 < g_2 < b^2 g_1$, where a, b are two numbers with $0 < a < b$. Then there exists a decomposition of g_1 and g_2 as:*

$$g_1 = \sum_{i=1}^{\infty} \phi_i^2 (d\psi_i)^2, \quad g_2 = \sum_{i=1}^{\infty} c_i^2 \phi_i^2 (d\psi_i)^2,$$

where for each i, ϕ_i and ψ_i are C^∞ functions on V with $supp\, \phi_i$ contained in some U_p and c_i's are real numbers such that $a < c_i < b$. Moreover, almost all ϕ_i's vanish on an U_p and there are at most N many ϕ_i which are simultaneously non-zero at any point of V, where N is an integer depending on n.

Proof. Let P denote the vector space of quadratic forms on $\mathbb{R}^n$. Then $\dim P$ is $n(n+1)/2$. Let $\bar{P}$ be the subset of $P \times P$ consisting of ordered pairs of positive definite quadratic forms (α, β) such that $a^2 \alpha < \beta < b^2 \alpha$. Then $\bar{P}$ is an open convex subset of $P \times P$. We cover $\bar{P}$ by geometrical open simplexes which lie completely within it. The collection of these gemetrical simplexes is locally finite and is such that no point lies in more than N_1 simplexes, where N_1 is a number depending on n.

Now, since $a^2 g_1 < g_2 < b^2 g_1$, (g_1, g_2) maps K_p onto a compact set in $\bar{P}$ and hence the image intersects only a finite number of simplexes, say $\sigma_1, \sigma_2, \ldots, \sigma_{n_p}$. Denote the vertices of

σ_μ, $\mu = 1, 2, \ldots, n_p$, by $(\alpha_{\mu 1}, \beta_{\mu 1}), \ldots, (\alpha_{\mu N_2}, \beta_{\mu N_2})$ where $N_2 = n(n+1)+1$. If $(g_1(x), g_2(x))$ lies in the interior to some simplex σ_i then we can express (g_1, g_2) at x as $(g_1(x), g_2(x)) = \sum_\nu c_{\mu\nu}(x)(\alpha_{\mu\nu}, \beta_{\mu\nu})$, where $c_{\mu\nu}(x) > 0$ for each ν and $\sum_\nu c_{\mu\nu}(x) = 1$.

We introduce, for each $\mu = 1, 2, \ldots n_p$, a weight function ω_μ by the following formula:

$$\omega_\mu = \frac{\exp\{-\sum_k 1/c_{\mu k}\}}{\sum_j \exp\{-\sum_k 1/c_{jk}\}}.$$

Observe that ω_μ is a smooth function on a neighbourhood of K_p and $\sum_\mu \omega_\mu = 1$. These functions distribute a point over all the simplices in which it lies in the interior. Let $c^*_{\mu\nu} = c_{\mu\nu}\omega_\mu$, $\mu = 1, \ldots n_p$ and $\nu = 1, \ldots, N_2$. Then it may be easily seen that $g_1 = \sum_{\mu,\nu} c^*_{\mu\nu}\alpha_{\mu\nu}$ and $g_2 = \sum_{\mu,\nu} c^*_{\mu\nu}\beta_{\mu\nu}$ on a neighbourhood of K_p.

Now for each pair $(\alpha_{\mu\nu}, \beta_{\mu\nu})$ we can obtain a set of n-vectors $V^1_{\mu\nu}, \ldots, V^n_{\mu\nu}$ with respect to which $\alpha_{\mu\nu}$ and $\beta_{\mu\nu}$ can be simultaneously diagonalised in such a way that

$$\alpha_{\mu\nu} = \textstyle\sum_{r=1}^n (d\psi^r_{\mu\nu})^2 \text{ and } \beta_{\mu\nu} = \sum_{r=1}^n (v^r_{\mu\nu})^2 (d\psi^r_{\mu\nu})^2,$$

where $v^1_{\mu\nu}, \ldots, v^n_{\mu\nu}$ are n numbers, and $\psi^r_{\mu\nu}$ is the linear functional defined on $(U_p; y_1, y_2, \ldots, y_n)$ by $\psi^r_{\mu\nu} = V^r_{\mu\nu}.(y_1, \ldots, y_n)$. Clearly $a < v^r_{\mu\nu} < b$ for all $r = 1, 2, \ldots, n$.

Now $(g_1, g_2) = (\sum_p \phi_p g_1, \sum_p \phi_p g_2)$. Since *supp* $\phi_p = K_p$ the following expressions make sense all over V:

$$\begin{aligned}
\phi_p g_2 &= \phi_p \textstyle\sum_{\mu,\nu} c^*_{\mu\nu}\beta_{\mu\nu} \\
&= \phi_p \textstyle\sum_{\mu,\nu} c^*_{\mu\nu} \sum_{r=1}^n (v^r_{\mu\nu})^2 (d\psi^r_{\mu\nu})^2 \\
&= \textstyle\sum_{\mu,\nu} \phi_p c^*_{\mu\nu} \sum_{r=1}^n (v^r_{\mu\nu})^2 (d\psi^r_{\mu\nu})^2 \\
&= \textstyle\sum_{r=1}^n \sum_{\mu,\nu} (v^r_{\mu\nu})^2 (\phi^r_{\mu\nu})^2 (d\psi^r_{\mu\nu})^2
\end{aligned}$$

Similarly,

$$\phi_p g_1 = \textstyle\sum_{r=1}^n \sum_{\mu,\nu} (\phi^r_{\mu\nu})^2 (d\psi^r_{\mu\nu})^2$$

where $\phi^r_{\mu\nu} = (\phi_p.c^*_{\mu\nu})^{1/2}$ are smooth functions on V which are supported in U_p. Also, note that the functions $\psi^r_{\mu\nu}$ in the above can be replaced by smooth functions which are defined all over V.

Reindexing and then summing over p we get:

$$g_1 = \sum_{p=1}^{\infty} \sum_{r=1}^{nn_pN_2} (\phi^r_p)^2 (d\psi^r_p)^2 \qquad g_2 = \sum_{p=1}^{\infty} \sum_{r=1}^{nn_pN_2} (v^r_p)^2 (\phi^r_p)^2 (d\psi^r_p)^2$$

where supports of the functions ϕ^r_p are compact and contained in U_p and $a < v^r_p < b$ for all $r = 1, 2, \ldots, nn_pN_2$, $p = 1, 2, \ldots$.

Further this decomposition is such that there can be at the most mnN_1N_2 terms which are non-vanishing at a given point of V. This proves that the above decomposition has all the required features. $\square$

Remark 3.2. We assume g_1, g_2 to be C^∞ for convenience only. If they are C^0 we can approximate them inside each U_p by C^∞ metrics which are sufficiently C^0 close to g_1, g_2. It can be shown following Nash that the accumulated error at the end of each stage arising from this preliminary approximation can be controlled according to our requirement.

4. Existence of (g_1, g_2)-short maps

Definition 4.1. Let V be a manifold with two Riamannian metrics g_1 and g_2. A C^1-map $f_0 : V \longrightarrow (\mathbb{R}^q, h_1, h_2)$ is (g_1, g_2)-short if the metrics $g_1 - f_0^*(h_1)$ and $g_2 - f_0^*(h_2)$ on V are positive definite. This will be expressed by $g_i - f_0^*(h_i) > 0$ or $g_i > f_0^*(h_i)$, $i = 1, 2$.

Proposition 4.2. *Let V be a C^∞-manifold with two Riemannian metrics g_1 and g_2 which are related by $a^2 g_1 < g_2 < b^2 g_1$ for some constants $0 < a < b$. Then there exists a (g_1, g_2)-short C^∞-immersion $f_0 : V \longrightarrow (\mathbb{R}^q, h_1, h_2)$ which also satisfies the following inequalities:*

$$a^2(g_1 - f_0^* h_1) < (g_2 - f_0^* h_2) < b^2(g_1 - f_0^* h_1). \tag{1}$$

Proof. Here we prove the existence of f_0 when V is a compact manifold. This is done by scaling a given C^∞-immersion with a suitable scalar.

Let $f_.$ be any immersion from V to $\mathbb{R}^q$. Consider the following four maps: $F_1, F_2 : TV \times \mathbb{R} \longrightarrow \mathbb{R}$ defined respectively by

$$(v, t) \mapsto (g_1 - t^2 f^* h_1) \text{ and } (v, t) \mapsto (g_2 - t^2 f^* h_2)$$

$G_1, G_2 : TV \times \mathbb{R} \longrightarrow \mathbb{R}$ defined respectively by

$$\begin{aligned} &(v, t) \mapsto (g_2 - a^2 g_1)(v, v) - t^2(f^* h_2 - a^2 f^* h_1)(v, v) \\ \text{and } &(v, t) \mapsto (b^2 g_1 - g_2)(v, v) - t^2(b^2 f^* h_1 - f^* h_2)(v, v) \end{aligned}$$

for $v \in TV_x$, $x \in V$, $t \in \mathbb{R}$.

Let SV denote the sphere bundle associated to TV. Since by our hypothesis $g_1, g_2, g_2 - a^2 g_1$ and $b^2 g_1 - g_2$ are positive-definite, all the above maps, when restricted to $SV \times \{0\}$ are strictly positive. Therefore, there exists a real-valued continuous function $\lambda : V \longrightarrow \mathbb{R}_+$ such that $F_i(v, t) > 0$, $G_i(v, t) > 0$ for all $v \in TV_x$ and $t \in (0, \lambda(x))$, $x \in V$, $i = 1, 2$. If V is compact, the function λ has a global minima. Let it be called λ_0 which is strictly positive. Then the function $f_0 = \lambda_0 . f_.$ has all the desired property.

The open case can be treated as in Nash[4] combining the above observation with a partition of unity argument. □

Remark 4.3. The set of (g_1, g_2)-short immersions which also satisfy the inequality (1) is an open set in the fine C^∞-topology.

5. Regular Maps

In this section we identify a class of maps which admit 'normal extension' in our context (Section 6). We start with the following definition.

Definition 5.1. An immersion $f : V \longrightarrow \mathbb{R}^q$ is said to be (h_1, h_2) *regular* or simply *regular* (when the Euclidean metrics h_1 and h_2 on $\mathbb{R}^q$ are well understood from the context), if the h_1-orthogonal complement of $df_x(TV_x)$ is transversal to the h_2-orthogonal complement of $df_x(TV_x)$ at each $x \in V$.

Let A denote the linear transformation $\mathbb{R}^q \longrightarrow \mathbb{R}^q$ defined by the relation: $h_2(w, w') = h_1(Aw, w')$ for all $w, w' \in \mathbb{R}^q$. Then it can be shown easily that a map $f : V \longrightarrow \mathbb{R}^q$ is regular if and only if $df_x(TV_x) \oplus A(df_x(TV_x))$ has the maximum dimension at every $x \in V$.

Proposition 5.2. *A generic map* $f : V \longrightarrow \mathbb{R}^q$ *is regular if* $q > 3 \dim V - 1$.

Proof: Let $\dim V = n$. Let Σ be a subset of the Grassmannian manifold $Gr_n(\mathbb{R}^q)$ which consists of all n-planes T in $\mathbb{R}^q$ such that $T \cap A(T) \neq 0$. Then Σ is the disjoint union of two sets Σ' and Σ'' where

1. Σ' consists of all n-planes T in $\mathbb{R}^q$ which contains an eigenvector of A,
2. Σ'' consists of all n-planes T in $\mathbb{R}^q$ which contains a 2-dimensional subspace spanned by a pair $\{v, Av\}$.

The subset Σ' of $Gr_n(\mathbb{R}^q)$ is of dimension $(n-1)(q-n)$ since λ_i's are all distinct. On the other hand Σ'' is of dimension $(q-1) + (n-2)(q-n)$. Therefore, $\dim \Sigma = \max(\dim \Sigma', \dim \Sigma'') = (q-1) + (n-2)(q-n)$.

Let $\mathcal{R}$ denote the open subset of $J^1(V, \mathbb{R}^q)$ consisting of 1-jets of germs of immersions from V to $\mathbb{R}^q$. There is a canonical projection map $p : \mathcal{R} \longrightarrow Gr_n(\mathbb{R}^q)$ which maps an 1-jet $j^1_f(x)$, $x \in V$, onto the n-subspace $\operatorname{Im} df_x$ in $\mathbb{R}^q$. Let $f : V \longrightarrow \mathbb{R}^q$ be a C^1-immersion and let $j^1_f : V \longrightarrow J^1(V, \mathbb{R}^q)$ denote the 1-jet map of f. Then f is regular if $p \circ j^1_f$ misses the set Σ.

If $q > 3n - 1$ then $\operatorname{codim} \Sigma > n$, and hence by the Thom Transversality Theorem $p \circ j^1_f$ misses Σ for a generic f. Thus a generic map $f : V \longrightarrow \mathbb{R}^q$ is regular if $q > 3 \dim V - 1$. $\square$

Notation 5.3. For brevity we shall use the symbol T_x for the subspace $df_x(TV_x)$ in $\mathbb{R}^q$. T will denote the subbundle of $f^*(T\mathbb{R}^q)$ whose fibre over x is T_x. The notation for the orthogonal complement of the subspace T_x relative to the metric h_i will be $T_x^{\perp_i}$, $i = 1, 2$.

Definition 5.4. Let $f : V \longrightarrow \mathbb{R}^q$ be a regular map. Then the dimension of $T_x^{\perp_1} \cap T_x^{\perp_2}$ is $q - 2n$ for every $x \in V$, where $T_x = df_x(TV_x)$. Therefore, $\bigcup_{x \in V} T_x^{\perp_1} \cap T_x^{\perp_2}$ defines a vector subbundle of $f^*(T\mathbb{R}^q)$ of rank $q - 2n$.

In the following, we denote this subbundle by BN and we call it the (h_1, h_2)-*normal bundle* of f.

Remark 5.5. It follows from Remark 4.3 and Proposition 5.2 that any (g_1, g_2)-short immersion $f : V \longrightarrow \mathbb{R}^q$ satisfying the inequality $a^2(g_1 - f_0^* h_1) < (g_2 - f_0^* h_2) < b^2(g_1 - f_0^* h_1)$ can be approximated in the fine C^∞-topology by a regular (g_1, g_2)-short immersion which satisfies the same inequality.

6. Normal Extension

Let $f : V \longrightarrow (\mathbb{R}^q, h_1, h_2)$ be a C^∞-immersion which is regular on an open set U of V, and let a, b be two numbers satisfying the hypothesis of Theorem 1.1. We shall show that f admits a normal extension $\tilde{f} : U \times \mathbb{R}^2 \longrightarrow W$ over U such that

$$\tilde{f}^*(h_1) = f^*(h_1) \oplus dt^2 \oplus ds^2 \text{ and } \tilde{f}^*(h_2) = f^*(h_2) \oplus c^2dt^2 \oplus c^2ds^2$$

on $U \times \{0\}$, for any constant c with $a < c < b$.

Observe that to obtain such an extension we require two smooth vector fields τ and τ' along f in $\mathbb{R}^q$ with the following properties:

P1. τ and τ' are h_1- as well as h_2-orthogonal to T (see Notation 5.3);

P2. $\|\tau\|_1 = \|\tau'\|_1 = 1$ and $\|\tau\|_2 = \|\tau'\|_2 = c$;

P3. τ and τ' are mutually orthogonal relative to both h_1 and h_2.

Indeed if such vector fields exist then we can define $\tilde{f} : U \times \mathbb{R}^2 \longrightarrow \mathbb{R}^q$ by the formula $\tilde{f}(x, t, s) = f(x) + s\tau + t\tau'$.

Now, P1 amounts to saying that τ and τ' are sections of the vector bundle BN over U. If U is contractible such fields always exist.

Let $\bar{h}$ denote the pseudo-Riemannian metric $c^2h_1 - h_2$ on $\mathbb{R}^q$, for $a < c < b$. The hypothesis of Theorem 1.1 implies that

1. $\bar{h}$ is non-singular;
2. the positive and the negative ranks of $\bar{h}$ are greater than or equal to $3n + 2$.

Let h denote the quadratic form on BN induced from $\bar{h}$, and let h_x be the restriction of h to the fibre BN_x. If we denote the positive and the negative ranks of h_x by $q_+(x)$ and $q_-(x)$ respectively, then by our hypothesis $q_+(x) \geq n + 2$ and $q_-(x) \geq n + 2$ for all $x \in V$.

Then the existence of τ and τ' satisfying P2 and P3 means

P2$'$. the vector fields τ and τ' are h-isotropic; that is, $h(\tau, \tau) = 0$ and $h(\tau', \tau') = 0$;

P3$'$. $h(\tau, \tau') = 0$.

Hence τ and τ' define in each fibre of BN an h-isotropic 2-subspace.

Before we proceed further, we set our terminology and recall some basic results.

Definition 6.1. Let h be an indefinite quadratic form on $\mathbb{R}^q$. Recall that the kernel or the null space of h consists of all those vectors v for which $h(v, .) \equiv 0$. A subspace L in $\mathbb{R}^q$ is called *h-regular* (or simply *regular*) if $L \cap \ker h = 0$ ([2], 2.4.9(B)). A subspace of $\mathbb{R}^q$ which is regular as well as isotropic will be referred to as *regular isotropic* subspace.

An isotropic vector which does not lie in the kernel of h will be called a *regular isotropic* vector.

Given an indefinite quadratic form h on $\mathbb{R}^q$, it may be recalled that the space of regular isotropic vectors has the same homotopy type as $S^{q_+-1} \times S^{q_--1}$. Hence this space is k-connected for $k = \min(q_+(h) - 2, q_-(h) - 2)$. In general, the space of regular isotropic m-subspaces in $\mathbb{R}^q$ has the homotopy type of $V_m(\mathbb{R}^{q_+}) \times V_m(\mathbb{R}^{q_-})$ and is therefore $(k-1)$-connected for $k \leq \min(q_+ - m, q_- - m)$ ([2], 3.3.1(C)).

Now we consider in the total space BN the subset which consists of all h_x-regular isotropic vectors for all $x \in U$ and call it Iso. If Iso has a section, then normalizing this section relative to h_1 we get a section τ satisfying conditions P1 and P3.

To obtain a pair of such sections which also satisfy P2 we consider the subset Iso_2 in $BN \oplus BN$ defined as follows:

$$Iso_2 = \left\{ (v,w) \in BN \oplus BN : \begin{array}{c} \text{span of } v, w \text{ is a regular} \\ h- \ \text{isotropic 2-subspace} \end{array} \right\}.$$

Lemma 6.2. *Iso_2 is a submanifold in $BN \oplus BN$ and $p \oplus p|_{Iso_2}$ is a submersion.*

Proof. Consider the 2-frame bundle of BN and denote it by $V_2(BN)$. An element (v,w) in $V_2(BN)$ which lies over x, will be denoted by (x,v,w). Further consider all those 2-frames (v,w) in the fibres which span a h-regular subspace. We denote this set of vectors by $V_2^{reg}(BN)$. Clearly this is an open subset in the total space $BN \oplus BN$.

Let $\phi : V_2^{reg}(BN) \longrightarrow \mathbb{R}^3$ be the smooth map defined by

$$\phi(x,v,w) = (h_x(v,v), h_x(w,w), h_x(v,w)).$$

Note that $\phi^{-1}(0)$ is precisely the subset Iso_2. We shall show that ϕ is a submersion.

Let $(x,v,w) \in V_2^{reg}(BN)$. In order to prove that ϕ is a submersion at this point it would be enough to verify that the linear map $\phi' : BN_x \oplus BN_x \longrightarrow \mathbb{R}^3$ defined by

$$(v',w') \mapsto d\phi_{(x,v,w)}(0,v',w')$$

is surjective, since $d\phi_{(x,v,w)}(u,v',w') = d\phi_{(x,v,w)}(u,0,0) + d\phi_{(x,v,w)}(0,v',w')$.

To compute $d\phi_{(x,v,w)}(0,v',w')$ we need to calculate the derivative of the map $(v,w) \mapsto (h_x(v,v), h_x(w,w), h_x(v,w))$. Observing that h_x is a bilinear pairing we easily obtain: $d\phi_{(x,v,w)}(0,v',w') = (2h_x(v,v'), 2h_x(w,w'), h_x(v,w') + h_x(w,v'))$.

Now, the linear map $(v',w') \mapsto (2h_x(v,v'), 2h_x(w,w'))$ has maximum rank if and only if v,w are regular vectors and $\ker h_x(v,\ ) \neq \ker h_x(w,\ )$. Note that if v and w are regular then, $\ker h_x(v,\ ) = \ker h_x(w,\ )$ if and only if there exists a $\lambda \in \mathbb{R}$ such that $\lambda v - w$ is in $\ker h_x$, that is the span of v and w is not a regular subspace. Thus, if $(x,v,w) \in V_2^{reg}(BN)$ then $(v',w') \mapsto (2h_x(v,v'), 2h_x(w,w'))$ is surjective. On the other hand since $\ker h_x(v,\ ) \neq \ker h_x(w,\ )$, there exists a $v' \in BN_x$ such that $h_x(v,v') = 0$ and $h_x(w,v') \neq 0$. Therefore, ϕ' is surjective at each $(x,v,w) \in V_2^{reg}(BN)$. Hence, ϕ is a submersion, and consequently $\phi^{-1}(0) = Iso_2$ is a submanifold of $V_2^{reg}(BN)$.

Finally, note that the tangent space to Iso_2 at a point (x,v,w) is given by the equation $d\phi_{(x,v,w)}(u,v',w') = 0$. Since $d\phi_{(x,v,w)}$ restricted $BN_x \oplus BN_x$ is surjective, for any given $u \in TV_x$, there exists a $(v',w') \in BN_x \oplus BN_x$ such that $d\phi_{(x,v,w)}(0,v',w') = -d\phi_{(x,v,w)}(u,0,0)$. Clearly then (u,v',w') belongs to the tangent space of Iso_2. This proves that $p \oplus p|_{Iso_2}$ is a submersion. $\square$

The fibre of Iso_2 over x consists of all 2-frames which span a h_x-regular isotropic subspace in the fibre BN_x. Hence it has the homotopy type of $V_2(\mathbb{R}^{q+(x)}) \times V_2(\mathbb{R}^{q-(x)})$. By our hypothesis this is at least $n-1$ connected. Thus we obtain a section (η, η') of Iso_2 over $U \subset V$ ([2], 3.3.1(A)). Then clearly η and η' satisfy the properties P1, P2$'$ and P3$'$. Moreover, span of η and η' defines a field σ of two dimensional isotropic subspaces over U. We normalize the vector field η with respect to h_1 and call it τ. Since h_1 is positive definite, $\tau^{\perp_1}$ in $\mathbb{R}^q$ intersects σ in a 1-dimensional subspace in each fibre. This uniquely determines τ'. Indeed, if τ lies in σ and has unit norm relative to h_1, then its h_2 norm is c. Also, if τ and τ' lie in σ and if they are orthogonal relative to h_1 then they are orthogonal relative to h_2 also.

Thus we obtain vector fields τ and τ' over U which satisfy the properties P1, P2 and P3.

7. Proof of the Main Theorem

We are now in a position to prove Theorem 1.1. We start with a (h_1, h_2)-regular C^∞-immersion $f_0 : V \longrightarrow \mathbb{R}^q$ which is (g_1, g_2)-short and satisfies the inequality

$$a^2(g_1 - f_0^*h_1) < g_2 - f_0^*h_2 < b^2(g_1 - f_0^*h_1)$$

It follows from our discussion in Section 4 and 5 that such a map exists under the hypothesis $a^2 g_1 < g_2 < b^2 g_1$ and $q \geq 6n + 4$.

We shall obtain a sequence of regular (g_1, g_2)-short C^∞-immersions $\{f_j\}$ which satisfy the inequality

$$a^2(g_1 - f_j^*h_1) < g_2 - f_j^*h_2 < b^2(g_1 - f_j^*h_1) \tag{1}$$

and are such that

$$\begin{aligned} \|g_1 - f_j^*h_1\|_0 &< \tfrac{2}{3}\|g_1 - f_{j-1}^*h_1\|_0, \\ \|f_j - f_{j-1}\|_{C^1} &< c(n)\|g_1 - f_{j-1}^*h_1\|_0^{\frac{1}{2}}, \end{aligned} \tag{2}$$

Then, the above three conditions would together imply that $\{f_j\}$ is a Cauchy sequence in the C^1-topology and hence converges to a C^1-map $f : V \longrightarrow \mathbb{R}^q$ such that $f^*h_1 = g_1$ and $f^*h_2 = g_2$. Thus Theorem 1.1 will be proved.

Observe that the sequence $\{f_j\}$ can be constructed inductively once we know how to get f_1 from the starting map f_0. Since f_0 is (g_1, g_2)-short, both $\delta_1 = g_1 - f_0^*h_1$ and $\delta_2 = g_2 - f_0^*h_2$ are positive definite Riemannian metrics on V such that $a^2\delta_1 < \delta_2 < b^2\delta_1$. Therefore we can apply Lemma 3.1 for the pair (δ_1, δ_2) and obtain

$$\delta_1/2 = \sum_{i=1}^{\infty} \phi_i^2 d\psi_i^2 \quad \text{and} \quad \delta_2/2 = \sum_{i=1}^{\infty} c_i^2\phi_i^2 d\psi_i^2,$$

where ϕ_i and ψ_i are smooth functions on V as described in the lemma, and c_i's are constants satisfying $a < c_i < b$ for all i.

The map f_1 will be obtained through a sequence of smooth maps $\{\bar{f}_i\}$ as $\lim_i \bar{f}_i = f_1$, where $\bar{f}_0 = f_0$, and

$$\begin{array}{c}\bar{f}_i^*(h_1) = \bar{f}_{i-1}^*(h_1) + \phi_i^2 d\psi_i^2 + E_i \\ \bar{f}_i^*(h_2) = \bar{f}_{i-1}^*(h_2) + c_i^2\phi_i^2 d\psi_i^2 + E_i'\end{array} .$$

The error terms E_i and E_i' can be controlled simultaneously according to our requirement. The procedure of obtaining $\bar{f}_i$ from $\bar{f}_{i-1}$ is the same for each i and so we only describe how to get $\bar{f}_1$ from $\bar{f}_0$.

Suppose $supp\,\phi_1 \subset U_p$, where $U_p \in \mathcal{U}, \mathcal{U}$ being the covering of V described in Section 3. Since f_0 is regular and $a < c_1 < b$, where a, b satisfy the hypothesis of Theorem 1.1, we can get an extension $\tilde{f}_0 : U_p \times \mathbb{R}^2 \longrightarrow \mathbb{R}^q$ of the map $\bar{f}_0$ such that

$$\tilde{f}_0^*(h_1) = f_0^*(h_1) \oplus dt^2 \oplus ds^2 \text{ and } \tilde{f}_0^*(h_2) = f_0^*(h_2) \oplus c_1^2 dt^2 \oplus c_1^2 ds^2$$

on $U_p \times \{0\}$ (see Section 6). Moreover, the difference $\tilde{f}_0^*(h) - f_0^*(h) \oplus dt^2 \oplus ds^2$ on $U_p \times D_\varepsilon^2$ is of the order of ε, where D_ε^2 denotes the ε-ball in $\mathbb{R}^2$ centred at the origin.

Now, proceeding exactly as in the proof of Lemma 2.2 we get a smooth map $\bar{f}_1 : V \longrightarrow \mathbb{R}^q$ such that

$$\begin{array}{c}\bar{f}_1^*(h_1) = f_0^*(h_1) + \phi_1^2 d\psi_1^2 + \varepsilon_1^2 d\phi_1^2 \\ \bar{f}_1^*(h_2) = f_0^*(h_2) + c_1^2\phi_1^2 d\psi_1^2 + c_1^2\varepsilon_1^2 d\phi_1^2.\end{array}$$

The map $\bar{f}_1$ obtained above may not be regular. However, a small perturbation near $supp\,\phi_2$ can make it regular there (as regularity is a generic property). Hence we can repeat the above steps on $\bar{f}_1$ to get $\bar{f}_2$ and so on in succession. The sequence $\{\bar{f}_i\}$ then converges uniformly to a C^∞-map f_1 since only finitely many ϕ_i's are non-vanishing on any open set U_p. Now observe that

$$\begin{array}{c}g_1 - f_1^*(h_1) = \frac{1}{2}(g_1 - f_0^*(h_1)) - \sum_i O(\varepsilon_i) \\ g_2 - f_1^*(h_2) = \frac{1}{2}(g_2 - f_0^*(h_2)) - \sum_i c_i^2 O(\varepsilon_i).\end{array}$$

We can choose ε_i at each stage in such a way that

(a) $g_1 - f_1^*(h_1)$ and $g_2 - f_1^*(h_2)$ are positive definite;

(b) the error terms satisfy the inequalities:

$$\textstyle\sum_i O(\varepsilon_i) < \frac{1}{6}\|g_1 - f_0^*(h_1)\|_0$$

Then inequalities (2) follow from the discussion made in Section 2. Further, if we have

$$\begin{array}{c}\sum_i(b^2 - c_i^2)O(\varepsilon_i) < \frac{1}{2}[b^2(g_1 - f_0^*h_1) - (g_2 - f_0^*h_2)] \\ \sum_i(c_i^2 - a^2)O(\varepsilon_i) < \frac{1}{2}[(g_2 - f_0^*h_2) - a^2(g_1 - f_0^*h_1)]\end{array}$$

then f_1 also satisfies inequality (1). Now, by Remark 5.5 we can assume f_1 to be (h_1, h_2)-regular. This completes the proof of Theorem 1.1 □

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