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FINITENESS OF THE DISCRETE SPECTRUM OF THE THREE-PARTICLE SCHRÖDINGER OPERATOR

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Abstract

We analyse the spectrum of the three-particle Schrödinger operator with pair contact and three-particle interactions on the neighboring nodes on a three-dimensional lattice. We show that the essential spectrum of this operator is the union of two segments, one of which coincides with the spectrum of an unperturbed operator and the other called two-particle branch. We will prove finiteness of the discrete spectrum of the Schrödinger operator at all parameter values of the problem.

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1. INTRODUCTION

Finiteness of the discrete spectrum of the SO (Schrödinger operator) H of a system of three arbitrary rapidly decreasing interactions was established by Yafaev in [1] and by Zhislin in [2]. In [3,4,5], it was established that the discrete spectrum of the SO can also be infinite.

The so-called DSO's (discrete SO) which are the lattice analogues of SO in the continuous space appear in the models of solid state physics [6] and in the lattice field theory [7]. For these operators, it is interesting to study finiteness or infiniteness of the discrete spectrum. The works [8,9] were devoted to this problem.

In [10], the DSO of a system of three identical particles (bosons) interacting via pairwise contact attractive potentials was considered. It is proved that the discrete spectrum of three-particle DSO in the case when the operators described by the subsystems of two particles have no virtual levels, is finite.

Note that in the continuous (Euclidean space) case, the energy of the motion of the center-of-mass can be separated from the total Hamiltonian so that the essential spectrum and "bound states" are eigenvectors of the energy operator with total momentum separated (and this operator does not, in fact, depend on the values of the total momentum).

In lattice terms the "center-of-mass separation" corresponds to a realization of the Hamiltonians as a "fibered operator", i.e., as the "direct integral of a family of operators" $H_{\mu,\lambda}(K)$ depending on the values of the total quasi-momentum $K\in T^3$ (T^3 being the three dimensional torus, $\mu > 0$ and $\lambda > 0$ are energies of the interactions of two and three-particles respectively). In this case a "bound state" is an eigenvector of the operator $H_{\mu,\lambda}(K)$ for some $K\in T^3$. Typically, this eigenvector depends continuously on K.

In this paper we consider the difference of DSO $H_{\mu,\lambda}(K)$ with pair contact and three-particle interactions on the neighboring nodes.

We prove finiteness of the discrete spectrum of the operator $H_{\mu,\lambda}(K)$ for K in the some set $\prod \subset T^3$ and all $\mu > 0, \quad \lambda > 0$.

2. Description of difference three-particle DSO

Let Z^3 be a three dimensional lattice, $\ell_2((Z^3)^3)$ the Hilbert space of square summable functions which are defined on $(Z^3)^3$, and $\ell_2^s((Z^3)^3) \subset \ell_2((Z^3)^3)$ a subspace consisting of functions $\psi(n_1, n_2, n_3)$ which are symmetric with respect to the permutation of any two arguments.

In the coordinate representation, the DSO of a system of three bosons with pair contact and three-particle interactions on the neighboring nodes acts on the space $\ell_2^s((Z^3)^3)$ and has the form

$$(\tilde{H}_{\mu,\lambda}\varphi)(n_1, n_2, n_3) = \frac{1}{2}\sum_{|s|=1}[3\varphi(n_1, n_2, n_3) - \varphi(n_1 + s, n_2, n_3) - \varphi(n_1, n_2 + s, n_3) -$$

$$-\varphi(n_1, n_2, n_3 + s)] - \mu(\delta_{n_1n_2} + \delta_{n_2n_3} + \delta_{n_3n_1})\varphi(n_1, n_2, n_3) - \frac{\lambda}{2}[\delta_{n_1n_2}(\delta_{|n_1-n_3|1}+$$

$$+\delta_{|n_2-n_3|1}) + \delta_{n_2 n_3}(\delta_{|n_1-n_3|1} + \delta_{|n_2-n_1|1}) + \delta_{n_3 n_1}(\delta_{|n_2-n_1|1} + \delta_{|n_2-n_3|1})]\varphi(n_1,n_2,n_3).$$

Here $\mu > 0$ and $\lambda > 0$ are the interaction energies of two and three particles respectively; δ_{nm} is the Kronecker delta; $s = (s^{(1)}, s^{(2)}, s^{(3)}) \in Z^3$, $\mid s \mid = \mid s^{(1)} \mid + \mid s^{(2)} \mid + \mid s^{(3)} \mid$.

The operator $\tilde{H}_{\mu,\lambda}$ commutes with the group of operators $\{\tilde{U}_s, s \in Z^3\}$ acting on the Hilbert space $\ell_2^s((Z^3)^3)$ by the formula

$$\tilde{U}_s\varphi(n_1,n_2,n_3) = \varphi(n_1+s, n_2+s, n_3+s).$$

Let T^3 be the three-dimensional torus, $L_2^s((T^3)^3)$ the space of square integrable symmetric functions defined on $(T^3)^3$.

In the momentum representation the DSO described above acts in $L_2^s((T^3)^3)$ to

$$(H_{\mu,\lambda}f)(k_1,k_2,k_3) = (\varepsilon(k_1)+\varepsilon(k_2)+\varepsilon(k_3))f(k_1,k_2,k_3) -$$

$$-\mu\sum_{\alpha=1}^{3}\int \delta(k_\alpha - k'_\alpha)\delta(k_\beta + k_\gamma - k'_\beta - k'_\gamma)f(k'_1,k'_2,k'_3)dk'_1dk'_2dk'_3 -$$

$$-\lambda\int\sum_{i=1}^{3}[3-\varepsilon(k_i-k'_i)]\delta(k_1+k_2+k_3-k'_1-k'_2-k'_3)f(k'_1,k'_2,k'_3)dk'_1dk'_2dk'_3.$$

Here $\alpha \neq \beta \neq \gamma \neq \alpha$; $\varepsilon(p) = \sum_{i=1}^{3}(1-cos p_i)$, $p = (p_1,p_2,p_3) \in T^3$, $\delta(k)$ is the three-particle Dirac delta function. Here we choose a unit measure on the torus T^3, i.e. $\int_{T^3} dk = 1$. Throughout, an integral without limits denotes integration over the whole range of variation of the variables of integration.

Taking the Fourier transform the group of translations $\{\tilde{U}_s, \ s \in Z^3\}$, can be transformed to the group of operators $\{U_s, s \in Z^3\}$ so that

$$(U_s f)(k_1,k_2,k_3) = exp\{-i(s, k_1+k_2+k_3)\}f(k_1,k_2,k_3),$$

where $(k,s) = \sum_{i=1}^{3} k_i s_i$ is a scalar product of vectors $k \in T^3$, $s \in Z^3$.

Let $K = k_1 + k_2 + k_3$ be the total quasimomentum of the three particles system and $F_K = \{(k_1,k_2,k_3) \in (T^3)^3 : K = k_1+k_2+k_3\}$ be a six-dimensional manifold. We denote by $L(K)$ the Hilbert space of all square integrable functions defined on F_K and satisfy the conditions $f(k_1,k_2) = f(k_2,k_1) = f(k_1,k_3)$, $k_3 = K - k_1 - k_2$.

The operator $H_{\mu,\lambda}$ commutes with the group $\{U_s, \quad s \in Z^3\}$. Therefore $H_{\mu,\lambda}$ is represented as a direct operator integral (see [11]) of the family $\{H_{\mu,\lambda}(K), K \in T^3\}$, acting on the Hilbert space of $L(K)$ as follows

$$H_{\mu,\lambda}(K)f(p,q) = H_0(K)f(p,q) - \mu\sum_{\alpha=1}^{3}(V_\alpha f)(p,q) - \lambda\sum_{\alpha=1}^{3}(\tilde{V}_\alpha V_\alpha f)(p,q). \qquad (1)$$

Here

$$H_0(K)f(p,q) = \varepsilon_K(p,q)f(p,q), \quad \varepsilon_K(p,q) = \varepsilon(p)+\varepsilon(q)+\varepsilon(K-p-q)$$

and

$$(V_\alpha f)(k_\alpha,k_\beta)=\int f(k_\alpha,s)ds,\quad (\tilde V_\alpha f)(k_\alpha,k_\beta)=\int [3-\varepsilon(k_\alpha-t)]f(t,k_\beta)dt.$$

Theorem 1. *For all $K\in T^3,\mu>0,\ \lambda>0$ the operator $H_{\mu,\lambda}(K)$ has no eigenvalue lying to the right of the essential spectrum.*

Let $h_\mu(k)$ be a two-particle DSO acting on the space $\tilde L(k)$ having the form

$$(h_\mu(k)f)(p)=\varepsilon_k(p)f(p)-\mu\int f(s)ds, \tag{2}$$

where $\varepsilon_k(p)=\varepsilon(k-p)+\varepsilon(p)$, $\tilde L(k)$ is the Hilbert space of the functions $f\in L_2(T^3)$, satisfying for each $k\in T^3$ the following condition $f(p)=f(k-p)$.

We recall some known facts (see [10]). Let $m(k)$ and $M(k)$ be the minimal and maximal values of the function $\varepsilon_k(p)$ respectively. For any $z\in C\setminus[m(k),M(k)]$ define

$$\Delta(\mu,k,z)=1-\mu D(k,z),\quad D(k,z)=\int(\varepsilon_k(q)-z)^{-1}dq.$$

For each $\mu>0$ denote by $G_\mu\subset T^3$ the set $\{k\in T^3: D(k,m(k))>1/\mu\}$. Then for any $k\in G_\mu$ the operator $h_\mu(k)$ has a unique simple eigenvalue $z_\mu(k)<m(k)$ and for the spectrum $\sigma(h_\mu(k))$ of the operator $h_\mu(k)$ the equality

$$\sigma(h_\mu(k))=\{z_\mu(k)\}\cup[m(k),M(k)] \tag{3}$$

holds. Let $\mu_0=\max(D(k,m(k)))^{-1}$ then for any $\mu\in(0,\mu_0]$ the function $z_\mu(k)$ is defined on G_μ (and $G_\mu=T^3$ for $\mu>\mu_0$). The eigenvalue $z_\mu(k)\equiv z(\mu,k)$ is a solution of the equation $D(k,z)=1/\mu$ and continuously depends on parameters $\mu>0$ and $k\in G_\mu$ and strictly decreases on $\mu\in(0,\infty)$.

Let $\prod$ be a set of points $K\in T^3\setminus\{0\}$, at which minimums $E_{\mu K}=z_\mu(p)+\varepsilon(K-p)$ are nondegenerate. The set $\prod$ is not empty and contains punctured neighborhood of zero, since $E_{\mu 0}(p)$ has a unique nondegenerate minimum at the point $p=0$.

Theorem 2. *For any $K\in\prod$, $\mu>0$ and $\lambda>0$ the operator $H_{\mu,\lambda}(K)$ has only a finite discrete spectrum.*

3. Structure of the essential spectrum of the DSO

Let $L_\alpha\subset L_2((T^3)^2)$ be the subspace consisting of functions $f\in L_2((T^3)^2)$ satisfying the condition $f(k_\alpha,k_\beta)=f(k_\alpha,k_\gamma)$, where k_α, k_β, k_γ are connected with relation $k_\alpha+k_\beta+k_\gamma=K$.

Let $h_{\alpha\mu}(K)=H_0(K)-\mu V_\alpha\ (\alpha=1,2,3)$ be the "channel operator" acting on the space L_α according to

$$h_{\alpha\mu}(K)f(k_\alpha,k_\beta)=\varepsilon_K(k_\alpha,k_\beta)f(k_\alpha,k_\beta)-\mu\int f(k_\alpha,s)ds.$$

Since the system consists of three identical particles then the spectra of channel operators $h_{\alpha\mu}(K),\alpha=1,2,3$, coincide, that is $\sigma(h_{\alpha\mu}(K))=\sigma(h_{\beta\mu}(K))$.

It is easy to check, that the channel operator $h_{\alpha\mu}(K)$ commutes with the group of operators $\{u_s, s \in Z^3\}$ of the form $(u_s f)(k_\alpha, k_\beta) = exp(i(k_\alpha, s))f(k_\alpha, k_\beta)$, therefore $h_{\alpha\mu}(K)$ decomposes into the direct operator integrals

$$h_{\alpha\mu}(K) = \int_{T^3} \oplus[h_\mu(k) + \varepsilon(k_\alpha)I]dk,$$

where $h_\mu(k)$ is the two-particle DSO defined by formula (2).

Let

$$\varepsilon_{min}(K) = \min_{p,q\in T^3} \varepsilon_K(p,q); \qquad \varepsilon_{max}(K) = \max_{p,q\in T^3} \varepsilon_K(p,q).$$

From the Theorem of the spectrum of decomposable operators and the representation (3) it follows that the spectrum $\sigma(h_{\alpha\mu}(K))$ of the channel operator $h_{\alpha\mu}(K)$ coincides with the set $Im\varepsilon_K \cup ImE_{\mu K}$, i.e.

$$\sigma(h_{\alpha\mu}(K)) = Im\varepsilon_K \cup ImE_{\mu K},$$

and the discrete spectrum is absent.

The operator V_α commutes with $\tilde{V}_\alpha$, i.e $\tilde{V}_\alpha V_\alpha = V_\alpha \tilde{V}_\alpha$ and denote by $W_\alpha = \tilde{V}_\alpha V_\alpha$. The operator $H_{\mu,\lambda}(K)$ acting on the Hilbert space $L(K)$ by formula (1) is represented as a sum of operators

$$H_{\mu,\lambda}(K) = H_\mu(K) - \lambda W,$$

where

$$H_\mu(K) = H_0(K) - \mu V, \quad V = V_1 + V_2 + V_3 \;\; W = W_1 + W_2 + W_3.$$

Note, that $\lambda W = H_\mu(K) - H_{\mu,\lambda}(K)$ is the integral operator. Consequently it is compact. Therefore according to the Weyl's Theorem the essential spectrum of these operators coincides, i.e.

$$\sigma_{ess}(H_{\mu,\lambda}(K)) = \sigma_{ess}(H_\mu(K)).$$

Lemma 1. The essential spectrum of $H_\mu(K)$ coincides with the spectrum of the channel operator $h_{\alpha\mu}(K)$, i.e. (see [10])

$$\sigma_{ess}(H_{\mu,\lambda}(K)) = \sigma_{ess}(H_\mu(K)) = \sigma(h_{\alpha,\mu}(K)) = Im\varepsilon_K \cup ImE_{\mu K}.$$

The first part $Im\varepsilon_K = [\varepsilon_{min}(K), \varepsilon_{max}(K)]$ of the spectrum of $h_{\alpha\mu}(K)$ does not depend on μ and coincides with the spectrum of the unperturbed three-particle operator $H_0(K)$ and it is called the three-particle branch of the essential spectrum of $H_\mu(K)$. We denote it by $\sigma_{three}(H_\mu(K))$. The set $ImE_{\mu K}$ coincides with the range of values of the function $E_{\mu K}(p) = z_\mu(K-p) + \varepsilon(p)$. It is called the two-particle branch of the essential spectrum of $H_\mu(K)$ and we denote it by $\sigma_{two}(H_\mu(K))$ (see [10]).

According to the definition, the three-particle branch $[\varepsilon_{min}(K), \varepsilon_{max}(K)]$ of the essential spectrum of the operator $H_\mu(K)$ does not depend on $\mu > 0$ and the two-particle branch $\cup_k\{E_{\mu K}(k)\}$ displaces to the left with increasing $\mu > 0$. That is $E_{min}(\mu, K) = \min_k E_{\mu K}(k)$

monotonically decreases with increasing μ. Therefore at some threshold value $\mu = \mu(K)$ the left edge $E_{min}(\mu, K)$ of the two-particle branch and the left edge $\varepsilon_{min}(K)$ of the three-particle branch of the essential spectrum of $H_\mu(K)$ coincide. The value of $\mu(K)$ is defined by the following equality:

$$\mu(K) = \min_{p\in T^3}\Big(D(k-p, \varepsilon_{min}(K) - \varepsilon(p)\Big)^{-1}, \ \tilde{\mu}(K) = \max_{p\in T^3}\Big(D(K-p, \varepsilon_{min}(K) - \varepsilon(p)\Big)^{-1}.$$

Lemma 2. a) If $\mu \le \mu(K)$, then $\sigma_{ess}(H_\mu(K)) = [\varepsilon_{min}(K), \varepsilon_{max}(K)]$;
b) if $\mu \in (\mu(K), \tilde{\mu}(K)]$, then $\sigma_{ess}(H_\mu(K)) = [E_{min}(\mu, K), \varepsilon_{max}(K)]$, where $E_{min}(\mu, K)$ is lower bound of the set $\sigma_{two}(H_\mu(K))$ and $E_{min}(\mu, K) < \varepsilon_{min}(K)$;
c) if $\mu > \tilde{\mu}(K)$, $\sigma_{ess}(H_\mu(K)) = [E_{min}(\mu, K), E_{\max}(\mu, K)] \cup [\varepsilon_{min}(K), \varepsilon_{max}(K)]$, where $E_{max}(\mu, K)$ is an upper bound of the set $\sigma_{two}(H_\mu(K))$ and $E_{max}(\mu, K) < \varepsilon_{min}(K)$.

Remark. In the case $K = 0$ we have

$$\frac{1}{\mu(0)} = \int \frac{dq}{2\varepsilon(q)}, \qquad \tilde{\mu}(0) = 12,$$

$$\sigma_{three}(H_\mu(0)) = [0, \frac{27}{2}], \qquad \sigma_{two}(H_\mu(0)) = [z_\mu(0), 12 - \mu].$$

If $\mu = \mu(0)$, then the two-particle branch $\sigma_{two}(H_\mu(0))$ coincides with the segment $[0, 12 - \mu(0)]$ in this case the left edges of two-particle and three-particle branches coincide. For the case $\mu = 12$ the right edge of two-particle branch and the left edge of three-particle branch coincide. If $\mu > 12$ then these branches are mutually disjoint.

The operator $V = V_1 + V_2 + V_3$ is positive (see [10]). Similarly we can show that the operator $\tilde{V}_\alpha, \alpha = 1, 2, 3$ is positive. The operators V_α and $\tilde{V}_\alpha$ commute, therefore $W_\alpha = \tilde{V}_\alpha V_\alpha \ge 0$. Consequently $W = W_1 + W_2 + W_3$ is positive.

We denote by $R_0(z)$ the resolvent of $H_0(K)$. For any $z \in C \setminus \sigma_{ess}(H_\mu(K))$ we define the operator $\hat{T}_{\mu,\lambda}(K, z)$, acting on the space $L_2(T^3)$ by the form

$$\hat{T}_{\mu,\lambda}(K, z) = (I - \mu V_\alpha R_0(z))^{-1} V_\alpha R_0(z)[2\mu I + \lambda \sum_{\beta=1}^{3} \tilde{V}_\beta]. \tag{4}$$

The following lemma establishes a connection between the eigenvalues of operators $H_{\mu,\lambda}(K)$ and $\hat{T}_{\mu,\lambda}(K, z)$.

Lemma 3. The number $z \in C \setminus \sigma_{ess}(H_\mu(K))$ is an eigenvalue of $H_{\mu,\lambda}(K)$ iff the number 1 is an eigenvalue of $\hat{T}_{\mu,\lambda}(K, z)$.

Proof. Necessity. Let $z \in C \setminus \sigma_{ess}(H_\mu(K))$ be an eigenvalue of the operator $H_{\mu,\lambda}(K)$, i.e. let the equation

$$(H_0(K) - zI)f = \mu \sum_{\alpha=1}^{3} V_\alpha f + \lambda \sum_{\alpha=1}^{3} \tilde{V}_\alpha V_\alpha f \tag{5}$$

have a nonzero solution $f \in L(K)$. On introducing the notation

$$g_\alpha = g(k_\alpha) = (V_\alpha f)(k_\alpha) = \int f(k_\alpha, t)dt, \tag{6}$$

we derive the following relation from (5)

$$f = \mu R_0(z)\sum_{i=1}^{3} g_\alpha + \lambda R_0(z)\sum_{\alpha=1}^{3}\tilde V_\alpha g_\alpha. \tag{7}$$

Substitution of (7) for f in (6) leads to the conclusion that the equation

$$(I-\mu V_\alpha R_0(z))g_\alpha = \mu V_\alpha R_0(z)\sum_{\beta\neq\alpha}^{3} g_\beta + \lambda V_\alpha R_0(z)\sum_{\beta=1}^{3}\tilde V_\beta g_\beta \tag{8}$$

has a nonzero solution. For each $z\in C\setminus\sigma_{ess}(H_\mu(K))$ the operator $I-\mu V_\alpha R_0(z)$ is invertible, therefore if we multiply (8) from the left by the operator $(I-\mu V_\alpha R_0(z))^{-1}$ we get

$$g_\alpha = (I-\mu V_\alpha R_0(z))^{-1} V_\alpha R_0(z)[\mu\sum_{\beta\neq\alpha}^{3} g_\beta + \lambda\sum_{\beta=1}^{3}\tilde V_\beta g_\beta]. \tag{9}$$

Because of the identity of particles the functions g_α, $\alpha=1,2,3$, represent the same function $g\in L_2(T^3)$, and the operator on the right-hand side of (9) does not depend on α. Therefore we do not get a system of equations, but we get the equation $g=\hat T_{\mu,\lambda}(K,z)g$. Here $\hat T_{\mu,\alpha}(K,z)$ is defined by (4). It follows from, that $g\in L_2(T^3)$ is an eigenfunction of $\hat T_{\mu,\lambda}(K,z)$ corresponding to the eigenvalue 1. The operator $\hat T_{\mu,\lambda}(K,z)$ is called the Faddeev type operator. This proves the necessity of the Lemma.

Sufficiency. Let the number 1, for some $z\in C\setminus\sigma_{ess}(H_\mu(K))$ be an eigenvalue of the operator $\hat T_{\mu,\lambda}(K,z)$ and let $g\in L_2(T^3)$ be the corresponding eigenfunction. Then $g_i=g(k_i)$ satisfies equation (9) and the function f is defined by relation (7), belongs to $L(K)$, satisfies $H_{\mu,\lambda}(K)f=zf$ and $f\not\equiv 0$. It follows from the inclusion $g_i\in L_2(T^3)$ and boundedness of $(\varepsilon_K(k_1,k_2)-z)^{-1}$ that $f\in L(K)$. Summing the equality (9) all over indices α for the function f defined by equality (7) we obtain the relation $H_{\mu,\lambda}(K)f=zf$. To prove that $f\not\equiv 0$, we will check that it is possible to restore φ_i by f according to the formula $g_i=V_i f\not\equiv 0$. This proves sufficiency and hence Lemma 3 is proved completely.

4. Finiteness of the discrete spectrum of the DSO

Lemma 4. For any $K=(K_1,K_2,K_3)$, $K_i\neq\pi, i=1,2,3$ the function $\varepsilon_K(p,q)$ has a unique nondegenerate minimum at the point $(K/3,K/3)$. If at some $i\in\{1,2,3\}$, $K_i=\pi$, then the function ε_K has several coincident nondegenerate minimums.

Proof of Lemma 4. Since the function ε_K consists of three identical terms $\varepsilon_{K_i}(p_i,q_i)=3-\cos(K_i-p_i-q_i)-\cos p_i-\cos q_i, i=1,2,3$, each of which depends only on two real-valued arguments $p_i,q_i\in[-\pi,\pi], i=1,2,3$, then it is enough to study minimums of functions ε_{K_i}. The function ε_{K_i} is real-valued analytic with period 2π with respect to each argument p_i and q_i, therefore it is enough to find all critical points of ε_{K_i} and compare the values of the function ε_{K_i} at these points. Calculating partial derivatives of the function ε_{K_i} and solving a system of trigonometric equations with respect to unknowns p_i, q_i, we find all critical points of the function ε_{K_i}. They have the forms

$$(\frac{K_i}{3},\frac{K_i}{3});\quad (\frac{K_i+2\pi}{3},\frac{K_i+2\pi}{3});\quad (\frac{K_i-2\pi}{3},\frac{K_i-2\pi}{3});$$

$$(K_i+\pi,K_i+\pi);\quad (K_i+\pi,-K_i);\quad (-K_i,K_i+\pi).$$

For any $K_i\in(-\pi,\pi)$ the minimum of ε_{K_i} is reached at a unique point $(K_i/3,K_i/3)$ which is nondegenerate. Only in the case $K_i=\pi$, the minimum of ε_{K_i} is reached at two points $(\pi/3,\pi/3)$ and $(-\pi/3,-\pi/3)$ and both points are nondegenerate. Nondegeneracy is checked directly and $\varepsilon_{min}(K)=3\varepsilon(K/3)$, for all $K\in T^3$.

Lemma 5. Let $K\neq 0$, *then*

$$\max_q D(K-q,\varepsilon_{min}(K)-\varepsilon(q)) > D(K-q,\varepsilon_{min}(K)-\varepsilon(q))|_{q=K/3}.$$

Moreover for any $q\in T^3$ *the operator* $h_{\alpha\mu}(K,q)$ *has no virtual level at the left edge of the essential spectrum of the operator* $H_\mu(K)$.

Proof of Lemma 5. We can show that the maximal value of the function $D(K-q,z_0-\varepsilon(q))$ does not reach at the point $q=K/3$, that is at the point $q=K/3$ the necessary condition of extremum is not fulfilled

$$\frac{\partial D(K-q,\varepsilon_{min}(K)-\varepsilon(q))}{\partial q_i}|_{q=K/3} = sin(K_i/3)\int\frac{(1-cosp_i)dp}{(\varepsilon_K(p,K/3)-\varepsilon_{min}(K))^2}\neq 0.$$

The unique operator which can have a virtual level is the operator $h_{\alpha\mu}(K,K/3)$, since the left bounds of the essential spectrums of $h_{\alpha\mu}(K,q)$ and $H_\mu(K)$ coincide at the unique value $q=K/3$. For $q\neq K/3$ these bounds do not coincide. Therefore the left edge $\varepsilon_{min}(K)$ of the spectrum of $H_\mu(K)$ does not have a virtual level for the operators $h_{\alpha\mu}(K,q)$ at $q\neq K/3$.

Let $\mu=\mu(K)$, $z_0=\varepsilon_{min}(K)$, $\Delta(p,z)=1-\mu(K)D(K-p,z-\varepsilon(p))$. It follows from Lemma 5 that $\min_p\Delta(p,z_0)=\Delta(Q(K),z_0)=0$ and $Q(K)\neq K/3$. For any $K\in T^3$ there exist $\delta>0, C_1, C_2$ such that the following inequalities are fulfilled (see [10])

$$a) C_1(|\,p-K/3\,|^2+|\,q-K/3\,|^2)\le\varepsilon_K(p,q)-z_0\le$$

$$\le C_2(|\,p-K/3\,|^2+|\,q-K/3\,|^2)\quad\text{for}\quad(p,q)\in(U_\delta(K/3))^2. \qquad (10)$$

$$b)\varepsilon_K(p,q)-z\ge\varepsilon_K(p,q)-z_0\ge C_1\quad\text{for}\quad(p,q)\notin(U_\delta(K/3))^2.$$

Lemma 6. The conjugate operator $\hat{T}^*_{\mu(K),\lambda}(K,z)\equiv\hat{T}^*_\lambda(z)$ *is compact for any* $z\le z_0$ *in the Banach space* $C(T^3)$. *There exist* $C>0$ *and* $\delta>0$ *such that the following inequality*

$$\|\hat{T}^*_\lambda(z_0)-\hat{T}^*_\lambda(z)\|\le C\sqrt{z_0-z}\quad \textit{for}\quad z\in(z_0-\delta,z_0)$$

is valid.

Proof of Lemma 6. The kernel $\hat{T}^*_\lambda(z)$ has a form:

$$\hat{T}^*_\lambda(p,q;z)=2\mu(K)\frac{(\varepsilon_K(p,q)-z)^{-1}}{\Delta(q;z)}+\lambda\int\frac{[9-\varepsilon_{K-3p}(q-p,s-p)]ds}{(\varepsilon_K(q,s)-z)\Delta(q;z)}.$$

For any $z<z_0$ the compactness of $\hat{T}^*_\lambda(z)$ follows from the continuity of the kernel of $\hat{T}^*_\lambda(p,q;z)$. We will prove the estimate $\|\hat{T}^*_\lambda(z_0)-\hat{T}^*_\lambda(z)\|\le C\sqrt{z_0-z}$. For any $g\in C(T^3)$, $\|g\|_\infty=\max_p\mid g(p)\mid\le 1$ we have

$$\|\hat{T}^*_\lambda(z_0)g-\hat{T}^*_\lambda(z)g\|_\infty\le$$

$$\le\|g\|_\infty\max_p[2\mu(K)\int|\frac{\Delta(q;z)(\varepsilon_K(p,q)-z)-\Delta(q;z_0)(\varepsilon_K(p,q)-z_0)}{\Delta(q;z)(\varepsilon_K(p,q)-z)\Delta(q;z_0)(\varepsilon_K(p,q)-z_0)}|dq+$$

$$+9\lambda\int|\frac{\Delta(q;z)(\varepsilon_K(q,s)-z)-\Delta(q;z_0)(\varepsilon_K(q,s)-z_0)}{\Delta(q;z)(\varepsilon_K(q,s)-z)\Delta(q;z_0)(\varepsilon_K(q,s)-z_0)}|dqds].$$

Using (10) and the estimate

$$\mid\Delta(q;z)(\varepsilon_K(p,q)-z)-\Delta(q;z_0)(\varepsilon_K(p,q)-z_0)\mid\le C(z_0-z),$$

$$C_1[(p-Q(K))^2+z_0-z]\le\Delta(p;z)\le C_2[(p-Q(K))^2+z_0-z],$$

we have:

$$\|\hat{T}^*_\lambda(z_0)g-\hat{T}^*_\lambda(z)g\|_\infty\le C\|g\|_\infty(z_0-z)$$

$$\max_{p\in T^3}[\int\frac{[(p-\frac{K}{3})^2+(q-\frac{K}{3})^2+z_0-z]^{-1}dq}{(q-Q(K))^2((q-Q(K))^2+z_0-z)[(p-\frac{K}{3})^2+(q-\frac{K}{3})^2]}+$$

$$+\int\frac{[(q-\frac{K}{3})^2+(s-\frac{K}{3})^2+z_0-z]^{-1}dqds}{(q-Q(K))^2((q-Q(K))^2+z_0-z)[(q-\frac{K}{3})^2+(s-\frac{K}{3})^2]}]. \qquad (11)$$

Consider the first term in the square brackets on the right-hand side of the inequality (11). The integral over the set $T^3\setminus(U_\delta(\frac{K}{3})\cup U_\delta(Q(K)))$ is uniformly bounded in $p\in T^3$. We estimate the integral over the domain $U_\delta(\frac{K}{3})\cup U_\delta(Q(K))$. Passing on to the spherical coordinate system in the integrals over the sets $U_\delta(\frac{K}{3})$, $U_\delta(Q(K))$ and substituting $p=\frac{K}{3}$ we have:

$$(C_1+C_2\int_0^\delta\frac{r^2dr}{r^2(r^2+z_0-z)})\le C_1+\frac{C_2}{\sqrt{z_0-z}}.$$

By analogous reasonings we can show that the second term in the square bracket on the right-side of (18) is bounded by the same value, therefore

$$\|\hat{T}^*_\lambda(z_0)g-T^*_\lambda(z)g\|_\infty\le(z_0-z)(C_1+\frac{C_2}{\sqrt{z_0-z}})\|g\|_\infty.$$

It follows from here that

$$\|\hat{T}^*_\lambda(z_0)-T^*_\lambda(z)\|\le C\sqrt{z_0-z}. \qquad (12)$$

Compactness of the limit operator $\hat{T}^*_\lambda(z_0)$ follows from the compactness of $\hat{T}^*_\lambda(z), z<z_0$, and the inequality (12). Thus Lemma 6 is proved.

Along with the Faddeev type operator $\hat{T}_{\mu,\lambda}(K,z)$ we will consider the operator $T_{\mu,\lambda}(K,z) = (I-\mu V_\alpha R_0(z))^{1/2}\hat{T}_{\mu,\lambda}(K,z)(I-\mu V_\alpha R_0(z))^{-1/2}$, i.e.

$$T_{\mu,\lambda}(K,z) = (I-\mu V_\alpha R_0(z))^{-1/2} V_\alpha R_0(z)[2\mu + \lambda \sum_{\alpha=1}^{3} \tilde{V}_\alpha](I-\mu V_\alpha R_0(z))^{-1/2}. \qquad (13)$$

Note, that first term on the right side of (13) is a self-adjoint operator and the equations $T_{\mu,\lambda}(K,z)g = g$ and $\hat{T}_{\mu,\lambda}(K,z)\varphi = \varphi$ are equivalent.

Lemma 7. Let $K \in \prod$ then the operator $T_{\mu(K),\lambda}(K,z) \equiv T_\lambda(z)$ is a Hilbert-Schmidt operator for any $z \leq z_0$ and uniformly continuous in $z = z_0$.

Proof of Lemma 7 immediately follows, if we use the nondegeneracy minimums of ε_K and Δ.

Proof of Theorem 1. It follows from the above, that the right edge of the essential spectrum of $H_{\mu,\lambda}(K)$ is equal to $\varepsilon_{max}(K)$. Since the operators $V = V_1 + V_2 + V_3$ and $W = W_1 + W_2 + W_3$ are positive the following inequality

$$\sup_{||f||=1} (H_{\mu,\lambda}(K)f,f) = \sup_{||f||=1} [(H_0(K)f,f) - \mu(Vf,f) - \lambda(Wf,f)] \leq$$
$$\leq \sup_{||f||=1} (H_0(K)f,f) = \varepsilon_{max}(K)$$

holds. It means, that $\sigma(H_{\mu,\lambda}(K)) \cap (\varepsilon_{max}(K), \infty) = \emptyset$, i.e. the operator $H_{\mu,\lambda}(K)$ does not have any points of spectrum on the semi-axes $(\varepsilon_{max}(K), \infty)$.

Proof of Theorem 2. We will assume the contrary, that is the operator $H_{\mu,\lambda}(K)$ has infinitely many eigenvalues z_i such that $\lim_{i\to\infty} z_i = z_0 = \varepsilon_{min}(K)$. Let $f_i \in L(K)$ be the normed eigenfunction of $H_{\mu,\lambda}(K)$, corresponding to the eigenvalue z_i. From the self-adjointness of $H_{\mu,\lambda}(K)$ it follows that the system $\{f_i\}$ is orthogonal, that is $(f_i, f_j) = \delta_{ij}$. Since any orthogonal system weakly converges to zero, an orthogonal system of eigenfunctions $\{f_i\}$ weakly converges to zero too.

It follows from Lemmas 6-7.

Lemma 8. Let $K \in \prod$ and $\{f_i\}$ be the orthonormal system of eigenfunctions of the operator $H_{\mu,\lambda}(K)$ and $g_i(p) = \int f_i(p,q)dq$. Then:
a) the sequence $\{g_i\}$ strongly converges to zero.
b) there exists a constant $C > 0$ not depending on i and $\delta > 0$ such that

$$|\, g_i(p) \,| \leq C \quad for \quad p \in U_\delta(K/3). \qquad (14)$$

Continuation of the proof of Theorem 2. The function $g_i(p) = \int f_i(p,q)dq$ satisfies the Faddeev equation at $z = z_i$. It follows from Lemma 8 that the sequence $\{g_i\}$ strongly converges to zero and is uniformly bounded in the neighborhood of the point $p = K/3$. Using the representation (see (7))

$$f_i(p,q) = \mu \frac{g_i(p) + g_i(q) + g_i(K-p-q)}{\varepsilon_K(p,q) - z_i} + \lambda \int \frac{[3 - \varepsilon_{K-3t}(p-t, q-t)]g_i(t)dt}{\varepsilon_K(p,q) - z_i}$$

we will show that the sequence $\{f_i\}$ strongly converges to zero. This convergence contradicts the condition of the normalization of f_i and completes the proof of Theorem 2. In fact, according to the properties of the norm we have

$$\|f_i\| \leq 3\mu(\int |\frac{g_i(p)}{\varepsilon_K(p,q)-z_i}|^2 dpdq)^{1/2} + 9\lambda(\int |g_i(p)|^2 dp \int \frac{dpdq}{(\varepsilon_K(p,q)-z_i)^2})^{1/2}. \quad (15)$$

Since $\varepsilon_K(p,q) - z_i \geq C > 0$ for $(p,q) \not\in U_\delta^2(K/3)$, then the first integral on the right-hand side of (15) over the set $(T^3)^2 \setminus U_\delta^2(K/3)$ converges to zero because of the convergence of the sequence $\{g_i\}$. Estimate the integral over the domain $U_\delta^2(K/3)$. According to (14) we have

$$| \frac{g_i(p)}{\varepsilon_K(p,q)-z_i} | \leq \frac{C}{\varepsilon_K(p,q)-z_0}, \quad \text{for} \quad (p,q) \in U_\delta^2(K/3).$$

Since the minimum of ε_K is nondegenerate it follows from here that the majorizing function belongs to $L_2((T^3)^2)$. By virtue of absolute continuation of the Lebesgue integral, the integral over the set $U_\delta^2(K/3)$ can be estimated by a small quantity. Because of the strong convergence of the sequence $\{g_i\}$, it follows that the second term on the right-hand side of inequality (15) converges to zero at $i \to \infty$. It means that $\|f_i\| \to 0$ as $i \to \infty$.

Proof of the Theorem in the case $\mu \neq \mu(K)$ is similar to that of the Theorem in [10] . Theorem 2 is proved.

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