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CRITICAL SOLUTION FOR A HILL'S TYPE PROBLEM

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Abstract

We studied the problem of two satellites attracted by a center of force. Assuming the motion of the center of mass of the two satellites describes a keplerian circular motion around the center of force we regularized the collision between them using the Levi-Civita procedure. The existence of a constant of motion in the extended phase space allowed us to study the stability of the solution where the two satellites are tied together in their circular motion around the center of force. We call this solution the critical solution. A theorem of M Kummer is applied to prove, in specific conditions, the existence of two one-parametric families of almost periodic orbits for the satellites motion that bifurcates from the critical solution.

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1. Introduction

Suppose a natural satellite is in a circular motion around a center of force (a planet or a star for example) and suddenly it breaks into two pieces. What is the dynamics of the pieces? In this paper we studied this problem and proved the existence of two (orbitally) stable almost periodic orbits that bifurcate from the original motion. The approach is the same as used in [CC] and We describe it succinctly. Consider the planar problem of three bodies of masses m_0,m_1 and m_2 in the case where m_1 and m_2 are much smaller than m_0. The mutual atraction of the two small bodies can be usually neglected and the problem reduces in a fair approximation to two independent two-body problems. However if the distance between the two small bodies is small their mutual attraction can no longer be ignorated. This is known as Hill's problem [H]. We take a different approach to Hill's problem. First, instead of taking the limits $m_1, m_2 \to 0$ we fix the body of mass m_0 at the origin and assume $m_0 = 1$. We take $m_1, m_2 << 1$ but no hierarchy for the masses m_1 and m_2 will be assumed. Second, we assume that the center of mass of the two-satellite system is on a circular orbit around the center of force. This situation is similar to the circular Hill problem [I]. The collision between the satellites is them regularized using the canonical form of the Levi-Civita regularization [SS] and it is found that the solution where the two satelites are tied together in a circular motion around the center of force is a relative equilibria for the system. We will call this solution the critical solution. In the extended phase space we found an S^1 action that generalizes the usual angular momentum (if the two satelite system is in the infinity with respect to the center of force this constant of motion reduces to the usual angular momentum). This action is free and proper and we can use the standards procedures of symplectic reduction in a trivial way to reduce the dimension of the problem. The reduced phase space has dimension 4 and We apply Normal Form theory results to study the stability of the critical point that represents the critical solution. In the reduced space we can prove that (see section (5.3) for the definitions) if $\sigma > 0$ the critical solution is unstable and if $\sigma < 0$ the critical solution is Lyapounov stable. Moreover, in the case $\sigma < 0$ and under mild conditions on the parameters, a theorem of M. Kummer is them aplied to prove the existence of two one-parametric families of stable periodic orbits in the reduced space, the parameter being the energy. Those periodic orbits corresponds to almost periodic orbits in the full phase space. The organization of the paper is as follows:

In section 2 We review the Levi-Civita transformation from a Hamiltonian point of view. We also introduce without proofs the theorems used in the construction of our model.

In section 3 We construct the model. The model is regularized and contrary to the usual study of Hill's problem We do not use a rotating system of coordinates (synodical) and do not truncate the series expansions.

In section 4 We prove the existence of a constant of motion in the extended phase space and use it to reduce the dimension of the system. An interesting feature of the the constant of motion is that it isvalid for the full Hamiltonian and is preserved no matter the order of the truncation of the series. This makes the system well suitable for numerical investigations of the cracking-sattelite problem.

In section 5 we do the stability analysis of the critical point that represents the critical solution of the satelites and expand the Hamiltonian around this point in Normal Form. Then We apply Kummer's theorem to obtain our main result (Theorem (5.2)).

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2. Hamiltonian Regularization of the Kepler Problem

We review the Levi-Civita regularization under the eyes of Geometric Mechanics (for a general treatment of Geometric Mechanics see e.g. ([AM],[MR])). The theorems introduced in this section will be used to build our model. The mathematical theorems will be stated without proofs. Let M be a symplectic manifold and w its symplectic two-form. In Celestial Mechanics problems M is usually the cotangent bundle of a configuration space C, i.e., $M = T^*C$ and w is the standard symplectic two-form. The following theorem states that any transformation of coordinates in C generates a canonical transformation (symplectomorphism) of T^*C.

Theorem 2.1. *Let $\phi : C \to C$ be a diffeomorphism. Then $\Phi : T^*C \to T^*C$, $\Phi = (d\phi^T)^{-1}$ is a canonical transformation. Here $d\phi^T$ is the adjoint of the derivative of ϕ.*

Example: The Levi-Civita Transformation For the planar Kepler problem the configuration space is given by $C = \mathbb{R}^2 - \{(0,0)\}$ and the phase space is given by T^*C. Since we will be considering time reparametrizations it is convenient to work in the extended phase space. This is nothing more then the usual phase space direct product with $\mathbb{R}^2$. Physically this means that energy and time are included as canonically conjugated variables, i.e., our configuration space will be given by $\tilde{C} = C \times \mathbb{R}$ and parametrized by (q_1, q_2, t). The phase space will be given by $T^*\tilde{C}$ and parametrized by $(q_u, q_v, t, p_u, p_v, E)$. The symplectic two form is the canonical one: $w = dq_u \wedge dp_u + dq_v \wedge dp_v + dt \wedge dE$. The Levi-Civita transformation $\phi : C \to C$ is given by $\phi(q_1, q_2) = \left(u^2 - v^2, 2\,u\,v\right)$. Looking C as imbeeded in $\tilde{C}$ this induces the transformation $\tilde{\phi}(q_1, q_2, t) = \left(u^2 - v^2, 2\,u\,v, t\right)$. Therefore

$$d\tilde{\phi} = \begin{pmatrix} 2\,u & -2\,v & 0 \\ 2\,v & 2\,u & \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

and

$$(d\tilde{\phi}^T)^{-1} = \frac{1}{4\,(u^2+v^2)} \begin{pmatrix} u & -v & 0 \\ v & u & 0 \\ 0 & 0 & 4\,(u^2+v^2) \end{pmatrix}. \tag{2}$$

The canonical transformation $\tilde{\Phi}$ is given by

$$\tilde{\Phi}(q_1, q_2, t, p_1, p_2, E) = (u^2 - v^2, 2\,u\,v, t, \frac{u\,p_u - v\,p_u}{4\xi^2}, \frac{v\,p_u + u\,p_v}{4\xi^2}, E)$$

where $\xi^2 = u^2 + v^2$. The Hamiltonian of the Kepler problem is given by

$$H = \frac{p_1^2}{2} + \frac{p_2^2}{2} - \frac{\kappa}{r}$$

where $r = \sqrt{q_1^2 + q_2^2}$ and κ is a constant. Observe that under the transformation $\tilde{\Phi}$ we have

$$\bar{H}(u, v, t, p_u, p_v, E) = H(\tilde{\Phi}(q_1, q_2, t, p_1, p_2, E)) = \frac{p_u^2 + p_v^2}{8\,\xi^2} - \frac{\kappa}{\xi^2}.$$

The important fact here is that the Hamiltonian becomes homogeneous of degree -2 in $|\xi|$. We will call $\xi = (u, v)$ and $p_\xi = (p_u, p_v)$.

Remark We use the term $dt \wedge dE$ istead of the common term $dE \wedge dt$. This is motivated by the symmetry of our model (see Definition (4.1)). The two different choices corresponds to different parametrizations of the Hamiltonian solutions (doing $t \to -t$ We change the form $dt \wedge dE$ in $dE \wedge dt$). This will become clear in the next example.

Theorem 2.2. *Let H, F be two Hamiltonians such that $\{H = E_h\} = \{F = E_f\}$ as sets, where $E_h, E_f \in \mathbb{R}$. Thus the Hamiltonian flow generated by H at the level E_h is equal to the Hamiltonian flow of F at the level E_f up to reparametrization.*

Example: The Regularized Kepler Problem Consider the Hamiltonians $\bar{H}$ and $\mathcal{H} = \xi^2\,(\bar{H} - E)$. We have that as sets $\{\bar{H} = E\} = \{\mathcal{H} = 0\}$. Observe also, that $|\xi| = 0$ is a removable singularity for $\mathcal{H}$. By the theorem, the induced flows at the respective levels are equal up to reparametrization. In fact writing Hamilton's equations for $\mathcal{H}$ we obtain that

$$\begin{cases} \frac{d\xi}{ds} = \xi^2\,\frac{\partial \bar{H}}{\partial p_\xi}, \\ \frac{dp_\xi}{ds} = -2\xi\,\left(\bar{H} - E\right) - \xi^2\,\frac{\partial \bar{H}}{\partial \xi} \\ \frac{dt}{ds} = -\xi^2 \\ \frac{dE}{ds} = 0 \end{cases} \tag{3}$$

At $\mathcal{H} = 0$, i.e., at $\bar{H} = E$ it follows that the equations write as

$$\begin{cases} \frac{d\xi}{ds} = \xi^2\,\frac{\partial \bar{H}}{\partial p_\xi}, \\ \frac{dp_\xi}{ds} = -\xi^2\,\frac{\partial \bar{H}}{\partial \xi} \\ \frac{dt}{ds} = -\xi^2 \\ \frac{dE}{ds} = 0 \end{cases} \tag{4}$$

The fifth equations is just a time reparametrization (the Sundman reparametrization). Doing $t \to -t$ (see the remark above) we rewrite those equations as

$$\begin{cases} \frac{d\xi}{dt} = \frac{\partial \bar{H}}{\partial p_\xi}, \\ \frac{dp_\xi}{dt} = -\frac{\partial \bar{H}}{\partial \xi}; \end{cases} \tag{5}$$

but those are Hamilton's equations for H at the level set E. Observe that equations (4) are not singular at $|\xi| = 0$.

3. The Circular Hill's Problem

The Hamiltonian $\bar{H}$ of the planar problem of two bodies of mass m_1 and m_2 attracted by a center of force of mass m_0 at the origin is

$$\bar{H} = \frac{\bar{p}_1^2}{2m_1} + \frac{\bar{p}_2^2}{2m_2} - G\frac{m_0 m_1}{|\bar{q}_1|} - G\frac{m_0 m_2}{|\bar{q}_2|} - G\frac{m_1 m_2}{|\bar{q}_1 - \bar{q}_2|}$$

where $\bar{q}_1$ and $\bar{q}_2$ are the coordinates of the bodies of masses m_1 and m_2 respectively, $\bar{p}_1$ and $\bar{p}_2$ their conjugate momenta and G is the gravitational constant. We choose units such that $G = 1$ and set $m_0 = 1$. This Hamiltonian represents a Hill problem when $m_1, m_2 << 1$. We introduce a proportionality factor $\lambda \in (0, \infty)$ such that $m_2 = \lambda m_1$. The Hamiltonian becomes

$$\bar{H} = \frac{\bar{p}_1^2}{2m_1} + \frac{\bar{p}_2^2}{2\lambda m_1} - \frac{m_1}{|\bar{q}_1|} - \frac{\lambda m_1}{|\bar{q}_2|} - \frac{\lambda\, m_1^2}{|\bar{q}_1 - \bar{q}_2|}.$$

Let $\bar{w} = d\bar{q}_1 \wedge d\bar{p}_1 + d\bar{q}_2 \wedge d\bar{p}_2$ denote the standard symplectic 2-form. Let $X_{\bar{H}}$ be the Hamiltonian vector field generated by $\bar{H}$. Consider the fiber scaling given by

$$\Phi(\bar{q}_1, \bar{q}_2, \bar{p}_1, \bar{p}_2) = (q_1, q_2, m_1 p_1, m_1 p_2).$$

Under this scaling we have

$$\bar{H} = m_1 \left\{ \frac{p_1^2}{2} + \frac{p_2^2}{2\lambda} - \frac{1}{|q_1|} - \frac{\lambda}{|q_2|} - \frac{\lambda m_1}{|q_1 - q_2|} \right\};$$

and

$$\bar{w} = m_1 \left(dq_1 \wedge dp_1 + dq_2 \wedge dp_2 \right).$$

Dividing Hamilton's equations $i_{X_{\bar{H}}} \bar{w} = d\bar{H}$ by m_1 we see that it suffices to study the Hamiltonian flow given by the Hamiltonian

$$H = \frac{p_1^2}{2} + \frac{p_2^2}{2\lambda} - \frac{1}{|q_1|} - \frac{\lambda}{|q_2|} - \frac{\lambda\, m_1}{|q_1 - q_2|} \tag{6}$$

with standard symplectic 2-form $w = dq_1 \wedge dp_1 + dq_2 \wedge dp_2$.

We introduce Jacobi variables ρ and r by

$$\begin{cases} q_1 = \rho - \frac{\lambda}{1+\lambda} r, \\ \\ q_2 = \rho + \frac{1}{1+\lambda} r. \end{cases}$$

Here ρ represents the position of the center of mass of the two satellites and r represents their relative position vector. Assuming $\frac{|r|}{|\rho|} < \frac{1+\lambda}{\lambda}$ we can expand the terms

$$\frac{1}{|\rho - \frac{\lambda}{1+\lambda} r|} \quad \text{and} \quad \frac{1}{|\rho + \frac{1}{1+\lambda} r|}$$

in power series ([Br]). The Hamiltonian (6) becomes

$$H = \left(\frac{p_\rho^2}{2\,\bar{\lambda}} - \frac{\bar{\lambda}}{|\rho|} \right) + \left(\frac{p_r^2}{2\,\Gamma} - \frac{\lambda\, m_1}{|r|} \right) - \frac{1}{|\rho|} \sum_{n=1}^{\infty} P_n(\cos\alpha) \left(\frac{|r|}{|\rho|} \right)^n \Lambda_n. \tag{7}$$

where α is the positively oriented angle between ρ and r,

$$\Lambda_n = \Gamma^n \left(1 + (-1)^n \lambda\right),$$

$\bar{\lambda} = 1 + \lambda$ and $\Gamma = \lambda\, \bar{\lambda}^{-1}$.

3.1. **The Circular Motion Hypothesis:** At this point we make the principal assumption of this work, namely, we assume that $\rho = (\rho_x, \rho_y)$, the vector representing the position of the center of mass of the two satellites, describes a circular keplerian orbit of radius $|\rho_0|$ around the center of force, i.e. ρ is a circular solution of $\ddot{\rho} = -\frac{1}{|\rho|^3}\rho$ yielding

$$\rho = |\rho_0| \left(\cos(\omega t), \sin(\omega t)\right),$$

where $\omega = |\rho_0|^{-\frac{3}{2}}$. By the second law of Kepler the energy of the center of mass is given by $E_{cm} = -\frac{\bar{\lambda}}{|\rho_0|}$, and Hamiltonian (7) becomes

$$H = -\frac{\bar{\lambda}}{|\rho_0|} + \left(\frac{p_r^2}{2\Gamma} - \frac{\lambda m_1}{|r|}\right) - \frac{1}{|\rho_0|}\sum_{n=1}^{\infty} P_n(\cos\alpha)\left(\frac{|r|}{|\rho_0|}\right)^n \Lambda_n. \tag{8}$$

We remark that this Hamiltonian is time dependent since the angle θ depends explicitely on time. Since energy of system (8) is not preserved we extend phase space from $\mathbb{R}^4$ to $\mathbb{R}^6$ by including the canonically conjugated pair (E, t). Our new Hamiltonian system is given by

$$\begin{cases} \bar{\mathcal{H}} = -E - \frac{\bar{\lambda}}{|\rho_0|} + \left(\frac{p_r^2}{2\,\Gamma} - \frac{\lambda m_1}{|r|}\right) - \frac{\bar{\lambda}}{|\rho_0|}\sum_{n=1}^{\infty} P_n(\cos\alpha)\left(\frac{|r|}{|\rho_0|}\right)^n \Lambda_n, \\ w = du \wedge dp_u + dv \wedge dp_v + dt \wedge dE, \end{cases} \tag{9}$$

where we must restrict our attention to the level set $\bar{\mathcal{H}} = 0$. Denoting the new time by f it follows from Hamilton's equation $\frac{dt}{df} = 1$. By choice we identify f and t.

3.2. **Levi-Civita Regularization:** We regularize the collision between the two satellites. Observe that on the regularized system the collision state $r = 0$ is an equilibrium point.

Theorem 3.1. *The flow of system (9) is up to a reparametrization equal to the flow of system*

$$\begin{cases} \mathcal{H} = \frac{p_\xi^2}{2\,\Gamma} - \frac{1}{4}\left(\bar{\lambda}\,\epsilon + E\right)|\xi|^2 - \epsilon^2 \sum_{n=1}^{\infty} \epsilon^{n-1} P_n\left(\cos(\alpha/2)\right)\left(\frac{|\xi|^2}{4}\right)^{n+1} \Lambda_n. \\ w = du \wedge dp_u + dv \wedge dp_v + dt \wedge dE. \end{cases} \tag{10}$$

Proof. Writing $\rho = (\rho_x, \rho_y)$ and $r = (r_x, r_y)$ we write the Levi-Civita transformation [SS]

$$\begin{cases} r_x = u^2 - v^2, \\ r_y = 2\,u\,v, \\ \rho_x = w^2 - z^2, \\ \rho_y = 2\,w\,z. \end{cases} \tag{11}$$

By theorem (2.1), transformation (11) lifts to the cotangent bundle as a canonical transformation. Under this lift Hamiltonian (9) becomes

$$\bar{\mathcal{H}} = -E - \frac{\bar{\lambda}}{|\gamma_0|^2} + \frac{1}{|\xi|^2}\left(\frac{p_\xi^2}{8\,\Gamma} - \lambda\, m_1\right) - \frac{1}{|\gamma_0|^2}\sum_{n=1}^{\infty} P_n(\cos(\alpha/2))\left(\frac{|\xi|}{|\gamma_0|}\right)^{2n} \Lambda_n. \tag{12}$$

Now consider the Hamiltonian

$$\mathcal{H} = |\xi|^2 \bar{\mathcal{H}} \tag{13}$$

in extended phase space with symplectic 2-form given by

(14) $$w = du \wedge dp_u + dv \wedge dp_v + dt \wedge dE.$$

Since the hypersurfaces $\{\bar{\mathcal{H}} = 0\}$ and $\{\mathcal{H} = 0\}$ are equal it follows by theorem (2.2) that the Hamiltonian flow of (12) at the level set $\bar{\mathcal{H}} = 0$ is a reparametrization of the Hamiltonian flow of (13) at the level set $\mathcal{H} = 0$. (12) and (13) yields

$$\mathcal{H} = -\lambda\, m_1 + \frac{p_\xi^2}{8\,\Gamma} - \left(\frac{\bar{\lambda}}{|\gamma_0|^2} + E\right)|\xi|^2 - \frac{|\xi|^2}{|\gamma_0|^2}\sum_{n=1}^{\infty} P_n\left(\cos(\alpha/2)\right)\left(\frac{|\xi|^{2n}}{|\gamma_0|^{2n}}\right)\Lambda_n.$$

We are interested on the flow of $\mathcal{H}$ at the level 0. We can eliminate the constant $-\lambda\, m_1$ of the Hamiltonian by considering the level $\lambda\, m_1$ instead. Writing $\epsilon = \frac{1}{|\gamma_0|^2}$ and doing the symplectic scaling $p_\xi \to 2\, p_\xi$, $\xi \to \xi/2$ we have

(15) $$\mathcal{H} = \frac{p_\xi^2}{2\,\Gamma} - \frac{1}{4}\left(\bar{\lambda}\,\epsilon + E\right)|\xi|^2 - \epsilon^2\sum_{n=1}^{\infty}\epsilon^{n-1}\, P_n\left(\cos(\alpha/2)\right)\left(\frac{|\xi|^2}{4}\right)^{n+1}\Lambda_n;$$

proving the theorem. □

4. Symmetry and Reduction

The increase in the dimension of the phase space is the price to be paid in doing the regularization. For time dependent systems this is quite natural since extending the phase space is the way to recover the aparently lost Hamiltonian formalism. For Hamiltonian system (10) We discovered an action of S^1 in the extended configuration space of the system for which the Hamiltonian is invariant. Geometrically this action is quite simple to describe:

Let $\vec{r} = (u, v, t) \in \mathbb{R}^3$ and $\vec{p} = (p_u, p_v, E) \in \mathbb{R}^{3\,*}$.

Definition 4.1. For $\theta \in S^1$ define the action $S^1 \times \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\theta \cdot \vec{r} = \left(\cos(\theta)\, u - \sin(\theta)\, v\,,\ \sin(\theta)\, u + \cos(\theta)\, v\,,\ t + \frac{4}{w}\theta\right).$$

The S^1-action just defined is a rotation when restricted to the plane (u, v) and a translation when restricted to the u axis. The rotation factor θ and the translation factor $\frac{4}{w}\theta$ are such that the action acts as a rigid rotation on the triangle made up of the two-satelites and the center of force.

Lemma 4.2. *The momentum map $\mathcal{J}(r,p) : T^*\left(T\mathbb{R}^3\right) \to s^*(1)$ of the action (4.1) is given by*

$$\mathcal{J}(r,p) = -u\, p_v + v\, p_u + \frac{4}{w}E.$$

Proof. This action lifts to the cotangent bundle $T^*\mathbb{R}^3$ as

(16) $$\theta \cdot (\vec{r}, \vec{p}) = (\theta \cdot \vec{r}, \theta \cdot \vec{p})$$

where

$$\theta \cdot \vec{p} = \left(\cos(\theta)\, p_u - \sin(\theta) p_v, \sin(\theta) p_u + \cos(\theta) p_v, E\right).$$

We identify the Lie algebra of S^1 with $\mathbb{R}$. Let $\nu \in s(1)$. Denote by $\Xi : T^*\mathbb{R}^3 \to T(T^*\mathbb{R}^3)$ the infinitesimal generator of the action. Then

$$\Xi(\vec{r},\vec{p}) = \left(-\nu\, v, \nu\, u, \nu\, \frac{4}{w}, -\nu\, p_v, \nu\, p_u, 0\right).$$

The infinitesimal generator is an Hamiltonian vector field. In fact $w(\Xi, .) = dJ_\nu$ where

$$J_\nu = -\nu\, u\, p_v + \nu\, v\, p_u + \nu\, \frac{4}{w}\, E.$$

But $J_\nu(r,p) =< \nu, \mathcal{J}(r,p) >$ where $\mathcal{J}(r,p) : T^*(T\mathbb{R}^3) \to s(1)^*$ is the momentum map for the action and $<,>$ represents the pairing between the algebra and its dual what gives the result. □

Lemma 4.3. *Hamiltonian (15) is invariant under the action (4.1).*

Proof. It suffices to prove that

$$\cos(\alpha/2) = \frac{\left(u^2 - v^2\right)\, \cos(wt/2) + 2\, u\, v\, \sin(wt/2)}{|\xi|^2},$$

is invariant. In fact, we only need to prove that the numerator is invariant. Denoting by $(\bar{u}, \bar{v}, \bar{t}) = \theta \cdot (u, v, t)$ we have that $\bar{u} = \cos(\theta)u - \sin(\theta)v$, $\bar{v} = \sin(\theta)u + \cos\theta(v)$, $\bar{t} = t + \frac{4}{w}\theta$. A straightforward computation gives the result. □

Remark 1 Since $w = \epsilon^{-\frac{3}{2}}$ it follows that in the limit $\epsilon = 0$, i.e., when the distance between the center of mass of the satelites and of the center of force is infinite, the momentum map reduces analytically to the usual angular momentum of the two-satelite system. In this paper ϵ will <u>not</u> be treated as pertubation parameter and will be a small constant.

Remark 2 The invariance of $\cos(\alpha/2)$ implies that the momentum map is preserved for any truncation of the Hamiltonian (15) regardles the order of the truncation.

Since the momentum map is conserved along the flow we can reduce the dimensionality of the system. Fixing a level set of $\mathcal{J}(r,p) = c$, for $c \in R$ and writing

$$E = \frac{c + u\, p_v - v\, p_u}{4\, \epsilon^{\frac{3}{2}}},$$

(10) becomes

$$\mathcal{H} = \frac{p_\xi^2}{2\,\Gamma} - \frac{1}{4}\left(\bar{\lambda}\,\epsilon + \frac{c + u\, p_v - v\, p_u}{4\,\epsilon^{\frac{3}{2}}}\right)|\xi|^2 - \epsilon^2 \sum_{n=1}^{\infty} \epsilon^{n-1}\, P_n\left(\cos(\alpha/2)\right)\left(\frac{|\xi|^2}{4}\right)^{n+1} \Lambda_n,$$

that we write as

$$\mathcal{H} = \frac{1}{2\,\Gamma}\left(p_u^2 + p_v^2\right) - \frac{\sigma}{2}\left(u^2 + v^2\right) + H_4 + H_6 + ... \tag{17}$$

where

$$\sigma = \frac{1}{2}\left(\bar{\lambda}\epsilon + \frac{c}{4\,\epsilon^{\frac{3}{2}}}\right),$$

and H_n is a homogeneous polynomial of degree n.

Since E is now a ciclic variable (what implies that $\frac{dt}{ds} = 0$) the Hamiltonian (15) is an Hamiltonian in the reduced 4-dimensional phase space (u, v, p_u, p_v) parametrized by c and $t(0)$.

Observe that if $\sigma < 0$ the degree 2 term of the hamiltonian represents a ressonant harmonic oscillator.

5. Normal Form and Stability

The origin in the reduced phase space is a critical point for the Hamiltonian equations of (17). Physically, in the non-reduced space, this critical point represents the solution in which the two-satellites are 'glued' together and revolving in a circular orbit around the center of force.

5.1. **Critical Point Analysis:**

Theorem 5.1. *For the reduced Hamiltonian (17) we have that the origin is an unstable critical point if $\sigma > 0$ and it is a Lyapounov stable critical point if $\sigma < 0$.*

Proof. Let $\vec{x} = (u, v, p_u, p_v)$. Then the linearized system given by (17) at the critical point $\vec{x}_0$ is given by $\dot{\vec{x}} = JD^2H(\vec{x}_0)\,\vec{x}$ where J is the canonical symplectic matrix and D^2 denotes the Hessian. At the origin $\vec{0} = (0,0,0,0)$ we have that

$$JD^2H(\vec{0}) = \begin{pmatrix} 0 & 0 & \frac{1}{\Gamma} & 0 \\ 0 & 0 & 0 & \frac{1}{\Gamma} \\ -\frac{\sigma}{2} & 0 & 0 & 0 \\ 0 & -\frac{\sigma}{2} & 0 & 0 \end{pmatrix}.$$

The eigenvalues of $JD^2H(\vec{0})$ are given by

$$e_1 = \frac{\sqrt{\Gamma\,\sigma}}{2} \text{ and } e_2 = -\frac{\sqrt{\Gamma\,\sigma}}{2},$$

each with multiplicity two. Therefore, *if $\sigma > 0$ the critical point is unstable.* If $\sigma < 0$ then the quadratic part of (17) is positive definite and *the critical point is Lyapounov stable.* □

In the case $\sigma < 0$ we can prove the existence of two one-parameter families of stable periodic orbits parametrized by the energy.

Theorem 5.2. *If $\frac{\epsilon^4\,\Lambda_1}{16\,\sigma^3\Gamma} \neq 1$ then the Hamiltonian flow induced by (17) has two stable one-parametric families of periodic solutions, the family parameter being the energy.*

We prepare the notation to write a Normal Form expansion for the Hamiltonian (17). First we do the symplectic scaling

$$u \to \frac{u}{(|\sigma|\,\Gamma)^{\frac{1}{4}}} \quad \text{and} \quad p_u \to (|\sigma|\,\Gamma)^{\frac{1}{4}}\,p_u$$

for the u, p_u pair and then do the same scaling for the v, p_v pair. Dividing the Hamiltonian by $\frac{|\sigma|}{\Gamma}^{\frac{1}{2}}$ (what amounts to change the energy level) we obtain

$$\mathcal{H} = \frac{p_u^2}{2} + \frac{u^2}{2} + \frac{p_v^2}{2} + \frac{v^2}{2} + H_4 + H_6 + ..., \tag{18}$$

where

$$H_4 = (u^2 + v^2)\,\left\{\alpha_1\,(u^2 - v^2)\,\cos(w\,t_0) + 2\,\alpha_2\,u\,v\,\sin(w\,t_0) + \beta\,(u\,p_v - v\,p_u)\right\},$$

$$\alpha_1 = -\frac{\epsilon^2\,\Lambda_1}{4\,|\sigma|^{\frac{3}{2}}\,\Gamma^{\frac{1}{2}}}\,\cos(w\,t_0),$$

$$\alpha_2 = -\frac{\epsilon^2\,\Lambda_1}{4\,|\sigma|^{\frac{3}{2}}\,\Gamma^{\frac{1}{2}}}\,\sin(w\,t_0),$$

and

$$\beta = \frac{1}{16\,\epsilon|\sigma|}.$$

This is a hamiltonian where the quadratic part is in $1:1$ ressonance. We define the vector $\delta = (1,1)$ that represents the ressonance. The symplectic transformations that leave the quadratic term of (18) invariant constitute the group $U(2)$, and correspondingly the Gustavson normal form of our Hamiltonian is a function over the Lie Algebra $u(2)$. We write

$$z_1 = \frac{1}{\sqrt{2}}\,(u + ip_u),\;\; \bar{z_1} = \frac{1}{\sqrt{2}}\,(u - ip_u);$$

$$z_2 = \frac{1}{\sqrt{2}}\,(v + ip_v),\;\; \bar{z_2} = \frac{1}{\sqrt{2}}\,(v - ip_v).$$

We also define

$$z_j = N_j^{\frac{1}{2}}\,i\,e^{\theta_j} \text{ for j=1,2.}$$

In those coordinates the symplectic 2-form write as

$$w = i\sum_{j=1}^{2} dz_j \wedge d\bar{z_j} = dN_j \wedge d\theta_j,$$

and the Poisson bracket write as

$$\{f,g\} = i\sum_{j=1}^{2}\left(\frac{\partial f}{\partial z_j}\frac{\partial g}{\partial \bar{z_j}} - \frac{\partial g}{\partial z_j}\frac{\partial f}{\partial \bar{z_j}}\right).$$

Let $\bar{z} = (\bar{z}_1, \bar{z}_2)$ and $z = (z_1, z_2)$. We define the following quadratic forms

$$M_i = \frac{1}{2}\bar{z}^t\,\mathbf{s_i}z \tag{19}$$

for $i = 0,1,2,3$ and where

$$\mathbf{s_0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\; \mathbf{s_1} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},\; \mathbf{s_2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\; \mathbf{s_3} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

are the Pauli matrices. Denote $J = M_0 = \frac{1}{2}\,(z_1\bar{z_1} + z_2\bar{z_2})$. From the definitions it follows that

$$J^2 = M_1^2 + M_2^2 + M_3^2. \tag{20}$$

Lemma 5.3. *(Normal Form Theorem) There exists a formal change of coordinates $z_k \to \hat{z}_k$ such that in the new variables $\hat{z}_k$ the Hamiltonian (18) has the form $2J + K$ where*

$$K = \sum_{m=2}^{\infty} K_m(J, M_1, M_2, M_3)$$

is a formal sum of homogeneous polinomials of degre m and $\{K_m, J\} = 0$ far all m.

Proof. See ([Mos]). □

We will put the Hamiltonian (18) in normal form up to the fourth order. Denote by N the hamiltonian in normal form. Let $(\xi, p_\xi) \to (\psi, \eta)$ be the formal change of coordinates that will bring $\mathcal{H}$ to N. We write this change of coordinates with the help of a generating function $W = W^{(2)} + W^{(3)} + W^{(4)} + \dots$. Since the quadratic part of $\mathcal{H}$ is already in normal form and observing that the Hamiltonian has no term of order 3 we write

$$W(\xi,\eta) = u\,\eta_1 + v\,\eta_2 + W^{(4)} + \dots.$$

This gives

$$\left\{ \begin{array}{l} \psi = \xi + \frac{\partial W^{(4)}}{\partial \eta} + \dots\,, \\ \\ p_\xi = \eta + \frac{\partial W^{(4)}}{\partial \xi} + \dots\,. \end{array} \right. \tag{21}$$

Therefore we can write

$$H(\xi, \frac{\partial W}{\partial \xi}) = N(\frac{\partial W}{\partial \eta}, \eta). \tag{22}$$

We define two integer vectors $\vec{k} = (k_1, k_2)$ and $\vec{l} = (l_1, l_2)$ where $k_i, l_i \in \mathbb{Z}$ for $i = 1, 2$. We also define the variables

$$\zeta_1 = u + i\eta_1\ ,\ \bar{\zeta_1} = u - i\eta_1,$$

and

$$\zeta_2 = v + i\eta_2\ ,\ \bar{\zeta_2} = v - i\eta_2.$$

We also write

$$\zeta^k\,\zeta^l = \prod_{\nu=1}^{n} \zeta_\nu^{k_\nu}\,\bar{\zeta_\nu}^{l_\nu}.$$

N will be in normal form if its expansion in the variables ζ_i and $\bar{\zeta_i}$ contains only terms $\zeta^k\,\zeta^l$ with

$$(k - l, \delta) = 0. \tag{23}$$

Expanding (22) and choosing $W^{(4)}$ such that (22) is satisfied order by order we obtain after some algebra that

$$\begin{aligned} N = \zeta_1\,\bar{\zeta_1} + \zeta_2\,\bar{\zeta_2} \quad &+ \quad \frac{3\,\alpha_1}{8}\left\{\zeta_1^2\,\bar{\zeta}_1^2 - \zeta_2^2\,\bar{\zeta_2}^2\right\} \\ &+ \quad \frac{3\,\alpha_2}{8}\left\{\zeta_1^2\,\bar{\zeta_1}\,\bar{\zeta_2} + \bar{\zeta_1}^2\,\zeta_1\,\zeta_2 + \zeta_1\,\zeta_2\,\bar{\zeta_2}^2 + \bar{\zeta_1}\,\bar{\zeta_2}\,\zeta_2^2\right\} + O(6), \end{aligned}$$

that can be factored as

$$\begin{aligned} N = \zeta_1\,\bar{\zeta_1} + \zeta_2\,\bar{\zeta_2} \quad &+ \quad \frac{3\,\alpha_1}{2}\left(\frac{\zeta_1\,\bar{\zeta_1} + \zeta_2\,\bar{\zeta_2}}{2}\right)\left(\frac{\zeta_1\,\bar{\zeta_1} - \zeta_2\,\bar{\zeta_2}}{2}\right) \\ &+ \quad \frac{3\,\alpha_2}{2}\left(\frac{\zeta_1\,\bar{\zeta_1} + \zeta_2\,\bar{\zeta_2}}{2}\right)\left(\frac{\zeta_1\,\bar{\zeta_2} + \bar{\zeta_1}\,\zeta_2}{2}\right) + \mathcal{O}(6) \end{aligned}$$

Doing a relabeling of the variables and using (19) We finally have

$$N = 2\,J + \frac{3}{2}J\ (\alpha_1\,M_3 + \alpha_2 M_2) + \mathcal{O}(6)$$

At this point we define the unit vector (see (20))

$$\vec{s} = (s_1, s_2, s_3) = \frac{1}{J}(M_1, M_2, M_3)$$

and write

$$N = 2\,J + \frac{3}{2}\,J^2\,(\alpha_1\,s_3 + \alpha_2\,s_2) + \mathcal{O}(6).$$

We call $\tilde{N}$ the fourth-order truncation of N, i.e.

$$\tilde{N} = 2\,J + \frac{3}{2}\,J^2\,(\alpha_1\,s_3 + \alpha_2\,s_2)$$

For future use we call

$$K = \frac{3}{2}\,(\alpha_1\,s_3 + \alpha_2\,s_2)\,.$$

The Hamiltonian $\tilde{N}$ induces a flow on the unit sphere

$$S^2 = \left\{(s_1, s_2, s_3) \in \mathbb{R}^3 \mid s_1^2 + s_2^2 + s_3^2 = 1\right\}$$

in the three-dimensional space. It turns out that this induced flow determines the flow of the hamiltonian $\tilde{N}$ in $\mathbb{R}^4$:

Theorem 5.4. *(Kummer, M **) The flow induced by $\tilde{N}$ on S^2 determines the flow induced by $\tilde{N}$ in $\mathbb{R}^4$ and moreover, to each critical point of $\tilde{N}$ on S^2 corresponds a periodic orbit of $\tilde{N}$ on $\mathbb{R}^4$.*

Proof. We sketch Kummer's result. To relate the induced flow on S^2 with the flow of $\mathbb{R}^4$ introduce the variables (J, A_J, M_3, A_m) where

$$A_J = \theta_1 + \theta_2 \;\text{ and }\; A_m = \theta_1 - \theta_2.$$

The canonical 2-form has the representation

$$w = dJ \wedge dA_J + dM_3 \wedge dA_m.$$

Now we can compute that

$$M_1 = \sqrt{J^2 - M_3^2}\,\sin(A_m),$$

$$M_2 = \sqrt{J^2 - M_3^2}\,\cos(A_m).$$

Over S^2 We have that

$$s_1 = \sqrt{1 - {s_3}^2}\,\sin(A_m),$$

$$s_2 = \sqrt{1 - {s_3}^2}\,\cos(A_m).$$

Introducing A_z by $s_3 = \cos(A_z)$ we have that A_m, A_z are polar coordinates on S^2. In terms of coordinates J, A_m, A_z, A_J the symplectic form writes as

$$w = dJ \wedge dA_J + (\cos(A_z)dJ - J\sin(A_z)dA_z) \wedge dA_m.$$

Hamilton's equations for the Hamiltonian $\tilde{N}$ with respect to those variables will write as

$$\begin{cases} \dot{J} = 0, \\ \dot{A_J} = 2 + J\left\{2\,F + cotg(A_z)\frac{\partial F}{\partial A_z}(s)\right\}, \\ \dot{A_m} = -\frac{J}{\sin(A_z)}\,\frac{\partial F}{\partial A_z}(s), \\ \dot{A_z} = \frac{J}{\sin(A_z)}\,\frac{\partial F}{\partial A_m}(s) \end{cases} \tag{24}$$

Therefore the induced flow in S^2 determines the flow in $\mathbb{R}^4$. Also, the critical points of the function $\tilde{N}$ in S^2 are the critical points of the flow it induces on S^2. From the equations we have that to each critical point s_0 of $\tilde{N}$ in S^2 corresponds a periodic orbit in $\mathbb{R}^4$, in fact at s_0 we will have that $\frac{\partial F}{\partial A_z}(s_0) = \frac{\partial F}{\partial A_m}(s_0) = 0$. □

The next result of M. Kummer shows that the periodic orbits of $\tilde{N}$ predicted by the last theorem persist when we consider the full Hamiltonian. We state the theorem: Let s_0 be a critical point of $\tilde{N}$ over S^2. Without lost of generality we can assume that $s_0 = (0, 0, -1) = e_z$. Define

$$A_{11} = \left(\frac{\partial^2 K}{\partial x^2}\right),\ A_{22} = \left(\frac{\partial^2 K}{\partial y^2}\right),\ A_{12} = \left(\frac{\partial^2 K}{\partial x \partial y}\right),$$

$$A_{33} = \left(\frac{\partial^2 K}{\partial z^2}\right),\ A_{23} = \left(\frac{\partial^2 K}{\partial y \partial z}\right),\ A_{13} = \left(\frac{\partial^2 K}{\partial x \partial z}\right);$$

all of them computed at e_z. Let

$$\lambda = -\left(\frac{\partial K}{\partial z}\right)$$

also computed at e_z. Also let

$$A = \lambda - A_{11},\ B = \lambda - A_{22},\ C = \lambda - A_{33},$$

and

$$D = \begin{vmatrix} A & 0 & -A_{13} \\ 0 & B & -A_{23} \\ -A_{13} & -A_{23} & C \end{vmatrix}$$

Theorem 5.5. *(i) To each unstable critical point of the flow that $\tilde{N}$ induces on S^2 there corresponds an unstable one-parametric family of periodic solutions of the equations associated with N, the family parameter being the energy.*

(ii) An analogous statement holds if the critical point is stable provided the following expression is non-zero:

$$12(A^3\,A_{23}^2 + B^3\,A_{13}^2) + 4\,(AB)^2\,(A + B + C) - 3D(A + B)^2, \tag{25}$$

where $AB = \Delta(\lambda) > 0$.

Proof. See ref (**) □

The proof of theorem (5.2) is a direct consequence Kummer's result.

Proof. The critical points of $\tilde{N}$ on S^2 are the critical points of K on S^2. Using Lagrange multipliers we have that s_0 is a critical point of K on S^2 if and only if there is a real number λ such that

$$\nabla K_{s_0} = \lambda\, s_0.$$

But

$$\nabla K_{s_0} = (0, \alpha_2, \alpha_1)$$

therefore we can take the solutions to be

$$\lambda = \pm\sqrt{\alpha_1^2 + \alpha_2^2} \text{ and } s_0 = \pm\,(0, \frac{\alpha_2}{\lambda}, \frac{\alpha_1}{\lambda}).$$

Whithout loss of generality we assume only the 'plus' solution. Making the rotation

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{\alpha_1}{\lambda} & \frac{\alpha_2}{\lambda} \\ 0 & -\frac{\alpha_2}{\lambda} & -\frac{\alpha_1}{\lambda} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

we have that s_0 gets mapped into e_z and that K in the new system of coordinates writes as

$$K = -\frac{3}{2}\sqrt{{\alpha_1}^2 + {\alpha_2}^2}\, z$$

The critical point is clearly stable. We also compute trivially that

$$A_{ij} = 0 \text{ for } i, j = 1, 2, 3.$$

Therefore the expression (25) reduces to

$$12(\lambda^3 - \lambda^5).$$

This is zero only if $|\lambda| = 1$, but this possibility is eliminated by the hypothesis that

$$\frac{\epsilon^4\,\Lambda_1}{16\,\sigma^3\,\Gamma} \neq 1$$

$\square$

Remark This theorem is valid no matter the order of the truncation of N.

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