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Dirac spinors for Doubly Special Relativity

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ABSTRACT

We construct a Dirac equation that is consistent with one of the recently-proposed schemes for relativistic transformations with two observer-independent scales (a velocity scale, still naturally identified with the speed-of-light constant, and a length/momentum scale, possibly given by the Planck length/momentum). We exploit the fact that in the energy-momentum sector the transformation laws are governed by a nonlinear realization of the Lorentz group. We find that the nonlinearity, which is due to the introduction of the second observer-independent scale, only induces a mild deformation of the structure of Dirac spinors.

After more than 70 years of study [1, 2] the "quantum-gravity problem", the problem of reconciling/unifying gravity and quantum mechanics, is still unsolved. Even the best developed quantum-gravity theories [3, 4] still lack any observational support [5, 6, 7] and are still affected by serious deficiencies in addressing some of the "conceptual issues" that arise at the interplay between gravity and quantum mechanics[1]. One can conjecture that the lack of observational support might be due to the difficulties of the relevant phenomenology [5, 6, 7] and that the conceptual issues might be eventually settled, but it is also legitimate to take as working assumption that all quantum-gravity theories so far considered are incorrect. At present it is even conceivable that the empasse in the study of the quantum-gravity problem might be due to the inadequacy of some of the key (and apparently most natural) common assumptions of quantum-gravity approaches. One of us recently proposed [9] an alternative path toward quantum gravity based on the possibility that Lorentz symmetry, usually assumed to be unaffected by the interplay between gravity and quantum mechanics, is deformed by the presence of the Planck length L_p ($L_p \sim 10^{-33} cm$): "special relativity" would be replaced by a "doubly special relativity", in which, in addition to the familiar[2] velocity scale c, also a second scale, a length scale λ (momentum scale $1/\lambda$), is introduced as observer-independent feature of the laws of transformation between inertial observers. λ can be naturally (though not necessarily) identified with the Planck length.

The fact that in some doubly-special-relativity scenarios the scale $1/\lambda$ turns out to set the maximum value of momentum [9, 10, 11] and/or energy [12, 13, 14, 15, 16] attainable by fundamental particles might be a useful tool for quantum-gravity research. In particular, it appears likely that [9, 14] the idea of a doubly special relativity may find applications in the study of certain noncommutative spacetimes. Moreover, while the deformation is soft enough to be consistent with all presently-available data, some of the predictions of doubly-special-relativity scenarios are testable [9, 13, 17] with forthcoming experiments [18], and therefore these theories may prove useful also in the wider picture of quantum-gravity research, as a training camp for the general challenge of setting up experiments capable of reaching sensitivity to very small (Planck-length suppressed) quantum-spacetime effects.

Some of these testable predictions, which concern spin-half particles, have been obtained at a rather heuristic level of analysis, since, so far, no DSR formulation[3] of spinors had been presented. We provide here this missing element of DSR theories. We focus on the specific DSR scheme used as illustrative example in the studies [9] that proposed the DSR idea, but our approach appears to be applicable to a wider class[4] of DSR schemes, including the one recently proposed by Maguejio and Smolin in Ref. [12] and the wider class of DSR schemes even more recently considered in Refs. [14, 15]. In fact, in all these DSR schemes the introduction of the second observer-independent scale relies on a nonlinear realization of the Lorentz group: the generators that govern the rules of transformation between inertial observers still

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[1] Examples of these conceptual issues are the so-called "problem of time" and "background-independence problem" [8].

[2] In presence of an observer-independent length scale the fact that our observations, on photons which inevitably have wavelengths that are much larger than the Planck length, are all consistent with a wavelength-independent speed of photons must be analyzed more cautiously [9]: it is only possible to identify the speed-of-light constant c as the speed of long-wavelength photons.

[3] From here onward "DSR" stands for "Doubly Special Relativity".

[4] Indeed, in concurrent work by Ahluwalia and Kirchbach [19] and in work in progress by Magueijo and Smolin [20], completely analogous results are being found concerning Dirac spinors in other DSR schemes.

satisfy the Lorentz algebra, but their action on energy-momentum space is modified. This does not appear to be a necessary feature of DSR theories, but it does characterize all DSR schemes so far considered and it plays a central role in the structure of our proposal. Another key aspect of our analysis is the fact that it is fully formulated in energy-momentum space, where the DSR schemes so far considered appear to admit a rather intuitive physical interpretation. The spacetime sector of these DSR schemes requires more caution, especially in light of the possible emergence of noncommutative geometry (whose operative understanding is still under investigation [9, 21]) and the subtleties that DSR introduces [14] in the structure of the duality between spacetime and energy-momentum space.

In preparation for our analysis of a DSR formulation of the Dirac equation it is useful to briefly review the structure of the ordinary Dirac equation. We revisit and describe the ordinary Dirac equation in a way that will provide a useful starting point for our DSR deformation. The approach we adopt is based on the one of Ref. [22]. We start by introducing operators $\overrightarrow{A}$ and $\overrightarrow{B}$ that are related to the generators of rotations, $\overrightarrow{J}$, and boosts, $\overrightarrow{K}$, through

$$\overrightarrow{A} = \frac{1}{2}(\overrightarrow{J} + i\overrightarrow{K}) \tag{1}$$

$$\overrightarrow{B} = \frac{1}{2}(\overrightarrow{J} - i\overrightarrow{K}) \tag{2}$$

The usefulness of these generators $\overrightarrow{A}$ and $\overrightarrow{B}$ reflects the familiar relation between the Lorentz algebra and the algebra $SU(2) \otimes SU(2)$. In fact, from the Lorentz-algebra relations for $\overrightarrow{J}$ and $\overrightarrow{K}$ it follows that

$$[A_l, A_m] = i\epsilon_{lmn}A_n \ , \tag{3}$$

$$[B_l, B_m] = i\epsilon_{lmn}B_n \ , \tag{4}$$

$$[A_l, B_m] = 0 \ . \tag{5}$$

Spinors can be labeled with a pair of numbers (j, j') characteristic of the eigenvalues of $\overrightarrow{A}^2$ and $\overrightarrow{B}^2$. In particular, "left-handed" and "right handed" spinors correspond to the cases $\overrightarrow{A}^2 = 0$ and $\overrightarrow{B}^2 = 0$ respectively. Left-handed spinors are labeled by $(\frac{1}{2}, 0)$ and their transformation rules for generic Lorentz-boost "angle" (rapidity) $\overrightarrow{\xi}$ and rotation angle $\overrightarrow{\theta}$ are

$$\psi_L \rightarrow \exp\left(i\frac{\overrightarrow{\sigma}}{2}\cdot\overrightarrow{\theta} - \frac{\overrightarrow{\sigma}}{2}\cdot\overrightarrow{\xi}\right)\psi_L \ , \tag{6}$$

where $\overrightarrow{\sigma}$ denotes the familiar 2×2 Pauli matrices. Analogously, right-handed spinors are labeled by $(0, \frac{1}{2})$ and transform according to

$$\psi_R \rightarrow \exp\left(i\frac{\overrightarrow{\sigma}}{2}\cdot\overrightarrow{\theta} + \frac{\overrightarrow{\sigma}}{2}\cdot\overrightarrow{\xi}\right)\psi_R \tag{7}$$

In particular, under a pure Lorentz boost from the rest frame to an inertial frame in which the particle has spatial momentum $\overrightarrow{p}$

$$\psi_R(\overrightarrow{p}) = e^{\frac{1}{2}\overrightarrow{\sigma}\cdot\overrightarrow{\xi}}\psi_R(0) = \left(\cosh\left(\frac{\xi}{2}\right) + \overrightarrow{\sigma}\cdot\overrightarrow{n}\sinh\left(\frac{\xi}{2}\right)\right)\psi_R(0) \ , \tag{8}$$

and

$$\psi_L(\overrightarrow{p}) = e^{-\frac{1}{2}\overrightarrow{\sigma}\cdot\overrightarrow{\xi}}\,\psi_L(0) = \left(\cosh\left(\frac{\xi}{2}\right) - \overrightarrow{\sigma}\cdot\overrightarrow{n}\,\sinh\left(\frac{\xi}{2}\right)\right)\psi_L(0)\ , \tag{9}$$

where $\overrightarrow{n}$ is the unit vector in the direction of the boost (and therefore characterizes the direction of the space momentum of the particle) and on the right-hand sides of Eqs. (8) and (9) the dependence on momentum is also present implicitly through the special-relativistic relations[5] between the boost parameter ξ and energy E,

$$\cosh\xi = \frac{E}{m}\ , \tag{10}$$

and (the "dispersion relation") between energy and spatial momentum

$$E^2 = \overrightarrow{p}^2 + m^2\ . \tag{11}$$

for given mass m of the particle.

One must then codify the fact that left-handed and right-handed spinors cannot be distinguished at rest. One way to do this[6] relies on the condition $\psi_R(0) = \psi_L(0)$, from which it follows that

$$\begin{pmatrix} -I & F^+(\xi) \\ F^-(\xi) & -I \end{pmatrix}\begin{pmatrix} \psi_R(\overrightarrow{p}) \\ \psi_L(\overrightarrow{p}) \end{pmatrix} = 0\ , \tag{12}$$

where

$$F^{\pm}(\xi) = 2\left(\cosh^2\left(\frac{\xi}{2}\right) - \frac{1}{2}\pm\overrightarrow{\sigma}\cdot\overrightarrow{n}\,\sinh\left(\frac{\xi}{2}\right)\cosh\left(\frac{\xi}{2}\right)\right)\ . \tag{13}$$

Using Eqs. (10) and (11) it is easy to explicitate the dependence on the particle's energy-momentum which is coded in the ξ-dependence of Eq. (12). This leads to the ordinary Dirac equation formulated in energy-momentum space[7]

$$(\gamma^\mu p_\mu - m)\,\psi(\overrightarrow{p}) = 0\ , \tag{14}$$

where γ^μ are the familiar "γ matrices" and

$$\psi(\overrightarrow{p}) \equiv \begin{pmatrix} \psi_R(\overrightarrow{p}) \\ \psi_L(\overrightarrow{p}) \end{pmatrix}\ . \tag{15}$$

The path we followed in reviewing the derivation of the ordinary special-relativistic Dirac equation provides a natural starting point for our announced deformation within the DSR framework. In fact, we relied exclusively on the algebric properties of the generators of boosts and rotations (the properties of the Lorentz algebra,

[5]In order to render some of our equations more compact we adopt conventions with $c \to 1$. This should not create any confusion since in DSR the speed-of-light constant preserves its role as observer-independent scale (but in DSR it is accompanied by a second observer-independent scale λ) and the careful reader can easily reinstate $c \neq 1$ by elementary dimensional-analysis considerations.

[6]Since we are here only concerned with the basics of the DSR deformation of Dirac spinors, we take the liberty to set aside the possible phase difference between $\psi_R(0)$ and $\psi_L(0)$.

[7]The space-time formulation of the Dirac equation is then obtained straightforwardly through Fourier transform: $(i\gamma^\mu\partial_\mu - m)\psi(x) = 0$.

without making use of the specific representation of the generators of boosts and rotations as differential operators on energy-momentum space that is adopted in special relativity) and on Eqs. (10) and (11), the ordinary special-relativistic relations between energy and rapidity (boost parameter connecting to the rest frame) and between energy and momentum. The algebraic properties of the generators of boosts and rotations remain unmodified in the DSR scheme considered in Refs. [9] (and in the other DSR schemes considered in Refs. [12, 14, 15]). In fact, the nonlinearity needed in order to introduce the second observer-independent scale is implemented by adopting a deformed representation as differential operators on energy-momentum space of the generators of boosts and rotations, but these deformed generators still satisfy the Lorentz algebra. Therefore in the derivation of the Dirac equation the only changes are introduced by the DSR deformations of the relations between energy and rapidity and between energy and momentum.

In the DSR scheme considered in Refs. [9], on which we focus here, the relation between energy and momentum (the dispersion relation) is[8]

$$2\lambda^{-2}\cosh(\lambda E) - \overrightarrow{p}^2 e^{\lambda E} = 2\lambda^{-2}\cosh(\lambda m) \ . \tag{16}$$

The relation between rapidity and energy that holds in the DSR scheme considered in Refs. [9], can be deduced from the structure of the corresponding DSR-deformed boost transformations, which have been studied in Refs. [9, 11]. Focusing again on a pure Lorentz boost from the rest frame to an inertial frame in which the particle has spatial momentum $\overrightarrow{p} \equiv |\overrightarrow{p}|\overrightarrow{n}$ one easily finds [11]

$$E(\xi) = m + \lambda^{-1}\ln\Big(1 - \sinh(\lambda m)\, e^{-\lambda m}(1-\cosh\xi)\Big) \ . \tag{17}$$

Therefore the boost parameter ξ can be expressed as a function of the energy using

$$\cosh\xi = \frac{e^{\lambda E} - \cosh(\lambda m)}{\sinh(\lambda m)} \ . \tag{18}$$

In the DSR derivation of the Dirac equation the Eqs. (16) and (18) must replace the Eqs. (10) and (11) of the ordinary special-relativistic case. All the steps of the derivation that used the algebra properties of the boost generators apply also to the DSR context (since, as emphasized above, the Lorentz-algebra relations remain undeformed in the DSR scheme considered in Refs. [9], and in the other DSR schemes considered in Refs. [12, 14, 15]).

We are thefore ready[9] to write down the DSR-deformed Dirac equation:

$$\begin{pmatrix} -I & F^+_\lambda(E,m) \\ F^-_\lambda(E,m) & -I \end{pmatrix} \begin{pmatrix} \psi_R(\overrightarrow{p}) \\ \psi_L(\overrightarrow{p}) \end{pmatrix} = 0 \tag{19}$$

[8]The dispersion relation (16) adopted in the DSR scheme considered in Refs. [9] might deserve special interest since it had already appeared in the mathematical-physics literature on deformation of the Poincaré algebra [23, 24], where it corresponds to the so-called "deformed mass casimir", and in work on a quantum-gravity approach based on noncritical string theory [25].

[9]We obtain here the DSR-deformed Dirac equation for the four-component spinor $\psi(\overrightarrow{p})$. We take some liberty in denoting with $\psi_R(\overrightarrow{p})$ two of the components of $\psi(\overrightarrow{p})$ and with $\psi_L(\overrightarrow{p})$ the remaining two components. In fact, especially if, as suggested in Refs. [9, 14], the DSR deformation should rely on a noncommutative spacetime sector the action of "space-Parity" transformations on energy-momentum space and on our spinors might involve some subtle issues [21]. The labels "R" and "L" on our DSR spinors are therefore at present only used for bookkeeping (they are reminders of the role that these components of the DSR Dirac spinor play in the $\lambda \to 0$ limit).

where

$$F_\lambda^\pm(E,m) = \frac{e^{\lambda E} - \cosh(\lambda m) \pm \overrightarrow{\sigma}\cdot\overrightarrow{n}\left(2e^{\lambda E}\left(\cosh(\lambda E) - \cosh(\lambda m)\right)\right)^{\frac{1}{2}}}{\sinh(\lambda m)} \ . \tag{20}$$

Introducing

$$D_0^\lambda(E,m) \equiv \frac{e^{\lambda E} - \cosh(\lambda m)}{\sinh(\lambda m)} \tag{21}$$

and

$$D_i^\lambda(E,m) \equiv \frac{n_i\left(2e^{\lambda E}\left(\cosh(\lambda E) - \cosh(\lambda m)\right)\right)^{\frac{1}{2}}}{\sinh(\lambda m)} \tag{22}$$

the DSR-deformed Dirac equation can be rewritten as

$$\left(\gamma^\mu D_\mu^\lambda(E,m) - I\right)\psi(\overrightarrow{p}) = 0 \tag{23}$$

where again the γ^μ are the familiar "γ matrices".

The nature of this DSR deformation of the Dirac equation becomes more transparent by rewriting (22) taking into account the DSR dispersion relation (16):

$$D_i^\lambda(\overrightarrow{p},m) = \frac{e^{\lambda E}}{\lambda^{-1}\sinh(\lambda m)} p_i \ . \tag{24}$$

In particular, as one should expect, in the limit $\lambda \to 0$ one finds

$$D_i^\lambda(E,m) \to \frac{E}{m} \ , \tag{25}$$

$$D_i^\lambda(\overrightarrow{p},m) \to \frac{p_i}{m} \ , \tag{26}$$

and the familiar special-relativistic Dirac equation is indeed obtained in the $\lambda \to 0$ limit.

It is also easy to verify that the determinant of the matrix $(\gamma^\mu D_\mu^\lambda(E,m) - I)$ vanishes, as necessary. In fact,

$$\det\left(\gamma^\mu D_\mu^\lambda(E,m) - I\right) = \left(\sinh^2(\lambda m) - \left(e^{\lambda E} - \cosh(\lambda m)\right)^2 + \frac{e^{2\lambda E}}{\lambda^{-2}}\overrightarrow{p}^2\right)^2 =$$

$$= \left(\frac{e^{\lambda E}}{\lambda^{-2}}\left(-2\lambda^{-2}\cosh(\lambda E) + \overrightarrow{p}^2 e^{\lambda E} + 2\lambda^{-2}\cosh(\lambda m)\right)\right)^2 = 0 \ , \tag{27}$$

where the last equality on the right-hand side follows from the DSR dispersion relation.

Our DSR-deformed Dirac equation of course leads to the DSR-deformed Weyl equation in the case of massless particles. In terms of the "DSR helicity" of our massless spinors one finds:

$$(\overrightarrow{\sigma}\cdot\hat{p})\,\psi_{R,L}(\overrightarrow{p}) = \pm\psi_{R,L}(\overrightarrow{p}) \ , \tag{28}$$

where $\hat{p} \equiv \overrightarrow{p}/|\overrightarrow{p}|$. The operator $\overrightarrow{\sigma}\cdot\hat{p}$ still has eigenvalues ± 1 as in the ordinary special-relativistic case.

In summary the DSR description of spinors appears to require only a relatively mild deformation of the familiar special-relativistic formulas. Our DSR-deformed Dirac equation differs from the ordinary Dirac equation only through the dependence on energy-momentum of the coefficients of the γ^μ matrices. The difference between the DSR coefficients, $[D_0^\lambda(E,m), D_i^\lambda(\overrightarrow{p},m)]$, and the ordinary ones, $[E/m, \overrightarrow{p}/m]$, is very small ($\lambda$-suppressed, Planck-length suppressed) for low-energy particles, and in particular the difference vanishes in the zero-momentum limit. Still it is plausible that the new effects might be investigated experimentally in spite of their smallness, following the strategy outlined in the recent literature [5, 6, 7] on the search of Planck-length suppressed effects. In particular, the sensitive context of neutrino oscillations should be considered from this perspective.

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References

[1] J. Stachel, "Early History of Quantum Gravity", in "Black Holes, Gravitational Radiation and the Universe", B.R. Iyer and B. Bhawal eds. (Kluwer Academic Publisher, Netherlands, 1999).

[2] C. Rovelli, gr-qc/0006061 (in Procedings of the 9th Marcel Grossmann Meeting on Recent Developments in Theoretical and Experimental General Relativity, Gravitation and Relativistic Field Theories, Rome, Italy, 2-9 Jul 2000).

[3] M.B. Green, J.H. Schwarz and E. Witten, "Superstring theory" (Cambridge Univ. Press, Cambridge, 1987); J. Polchinski, "Superstring Theory and Beyond", (Cambridge University Press, Cambridge, 1998).

[4] C. Rovelli, Living Rev. Rel. 1 (1998) 1; T. Thiemann, gr-qc/0110034; A. Ashtekar, gr-qc/0112038; L. Smolin, "Three Roads to Quantum Gravity" (Weidenfeld and Nicolson, London, 2000).

[5] G. Amelino-Camelia, gr-qc/9910089, Lect. Notes Phys. 541 (2000) 1; gr-qc/0204051.

[6] S. Sarkar, gr-qc/0204092

[7] D.V. Ahluwalia, gr-qc/0205121.

[8] A. Ashtekar and J.J. Stachel (eds.), "Conceptual Problems in Quantum Gravity" (Birkhauser, Boston, 1991).

[9] G. Amelino-Camelia, Int. J. Mod. Phys. D11 (2002) 35 [arXiv:gr-qc/0012051]; Phys. Lett. B510 (2001) 255 [arXiv:hep-th/0012238].

[10] J. Kowalski-Glikman, Phys. Lett. A286 (2001) 391 [arXiv:hep-th/0102098].

[11] N. R. Bruno, G. Amelino-Camelia and J. Kowalski-Glikman, Phys. Lett. B522 (2001) 133 [arXiv:hep-th/0107039].

[12] J. Magueijo and L. Smolin, Phys. Rev. Lett. 88 (2002) 190403. [arXiv:hep-th/0112090].

[13] G. Amelino-Camelia, D. Benedetti and F. D'Andrea, hep-th/0201245.

[14] J. Kowalski-Glikman and S. Nowak, hep-th/0203040.

[15] J. Lukierski and A. Nowicki, hep-th/0203065.

[16] S. Judes and M. Visser, hep-th/0205067.

[17] G. Amelino-Camelia and T. Piran, Phys. Rev. D64 (2001) 036005. [arXiv:astro-ph/0008107].

[18] J.P. Norris, J.T. Bonnell, G.F. Marani, J.D. Scargle, astro-ph/9912136; A. de Angelis, astro-ph/0009271.

[19] D.V. Ahluwalia and M. Kirchbach, in preparation.

[20] L. Smolin, private communications.

[21] G. Amelino-Camelia and S. Majid, Int. J. Mod. Phys. A15 (2000) 4301 [arXiv:hep-th/9907110].

[22] D.V. Ahluwalia and M. Kirchbach, Int. J. Mod. Phys. D10 (2001) 811.

[23] S. Majid and H. Ruegg, Phys. Lett. B334 (1994) 348.

[24] J. Lukierski, H. Ruegg and W.J. Zakrzewski Ann. Phys. 243 (1995) 90.

[25] G. Amelino-Camelia, J. Ellis, N.E. Mavromatos and D.V. Nanopoulos, Int. J. Mod. Phys. A12 (1997) 607.