# THE CONSTRUCTION OF MIRROR SYMMETRY † ⋆

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## ABSTRACT

The construction of mirror symmetry in the heterotic string is reviewed in the context of Calabi–Yau and Landau–Ginzburg compactifications. This framework has the virtue of providing a large subspace of the configuration space of the heterotic string, probing its structure far beyond the present reaches of solvable models. The construction proceeds in two stages: First all singularities/catastrophes which lead to ground states of the heterotic string are found. It is then shown that not all ground states described in this way are independent but that certain classes of these Landau–Ginzburg/Calabi–Yau string vacua can be related to other, simpler, theories via a process involving fractional transformations of the order parameters as well as orbifolding. This construction has far reaching consequences. Firstly it allows for a systematic identification of mirror pairs that appear abundantly in this class of string vacua, thereby showing that the emerging mirror symmetry is not accidental. This is important because models with mirror flipped spectra are a priori independent theories, described by distinct Calabi–Yau/Landau–Ginzburg models. It also shows that mirror symmetry is not restricted to the space of string vacua described by theories based on Fermat potentials (corresponding to minimal tensor models). Furthermore it shows the need for a better set of coordinates of the configuration space or else the structure of this space will remain obscure. While the space of Landau–Ginzburg vacua is *not* completely mirror symmetric, results described in the last part suggest that the space of Landau–Ginburg *orbifolds* possesses this symmetry.

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†Dedicated to the memory of B.Warr and T.I.A.Hu–Man.

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Even though the general structure of the configuration space of the heterotic string remains to a large extent terra incognita, some of its important properties have been uncovered. Perhaps the most interesting of these is the recently discovered mirror symmetry of the space of (2,2)–supersymmetric vacua [8][24] . Ideally questions about the space of ground states should be analyzed starting from first principles, given an appropriate parametrization of this manifold. Not too much progress however has been made along this avenue. Instead one proceeds somewhat indirectly. The symmetry principles of string theory are used to formulate a set of consistency conditions which are solved explicitly. Unfortunately this introduces some uncertainty as to whether the part of the space of vacua that has been uncovered via these constructions represents a typical slice of the whole space. Properties that are generic in specific constructions may not at all be features of the total space one is interested in but instead could merely be artefacts of the techniques employed.

An example of such an artefact is furnished by the class of heterotic string vacua described by **c**omplete **i**ntersection **C**alabi–**Y**au manifolds embedded in products of projective spaces (CICYs). In this class the number of generations and antigenerations of the models are parametrized by the only two independent Hodge numbers $(h^{(1,1)}, h^{(2,1)})$ that exist on such manifolds and the number of light generations of these theories is measured by the Euler number $\chi = 2(h^{(1,1)} - h^{(2,1)})$. The results for the latter turn out to lie in the range $-200 \leq \chi \leq 0$ [4]. In Fig.1 the Euler number of all CICY vacua is plotted versus the sum of the two independent Hodge numbers [23].

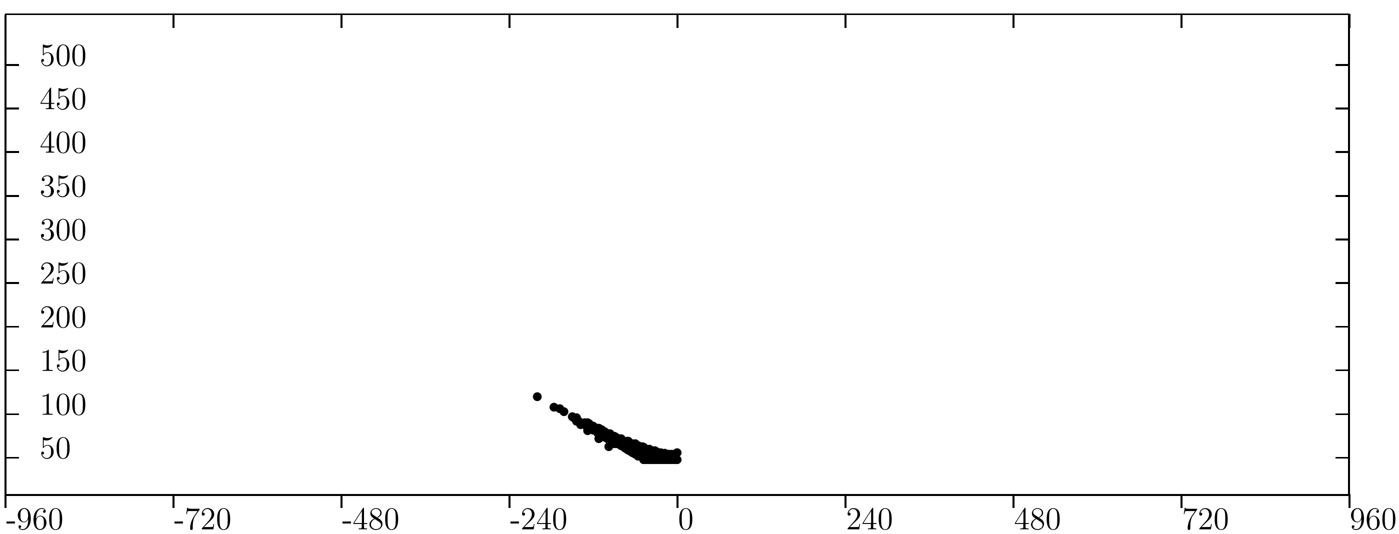



**Fig. 1** *A plot of the Euler number versus* $(h^{(1,1)} + h^{(2,1)})$ *for the spectra of the 7868 CICYs.*

At the time when the class of CICYs was constructed only very few manifolds with positive Euler number were known and hence one might naively have concluded that the vast majority of the complete construction [44, 17] of the set of heterotic vacua based on tensor products of minimal $N = 2$ superconformal field theories [21]. The resulting space is again rather asymmetric even though *some* mirror pairs appear in this construction. Fig.2 contains again a plot of the Euler numbers versus the sum of generations and antigenerations for those theories.

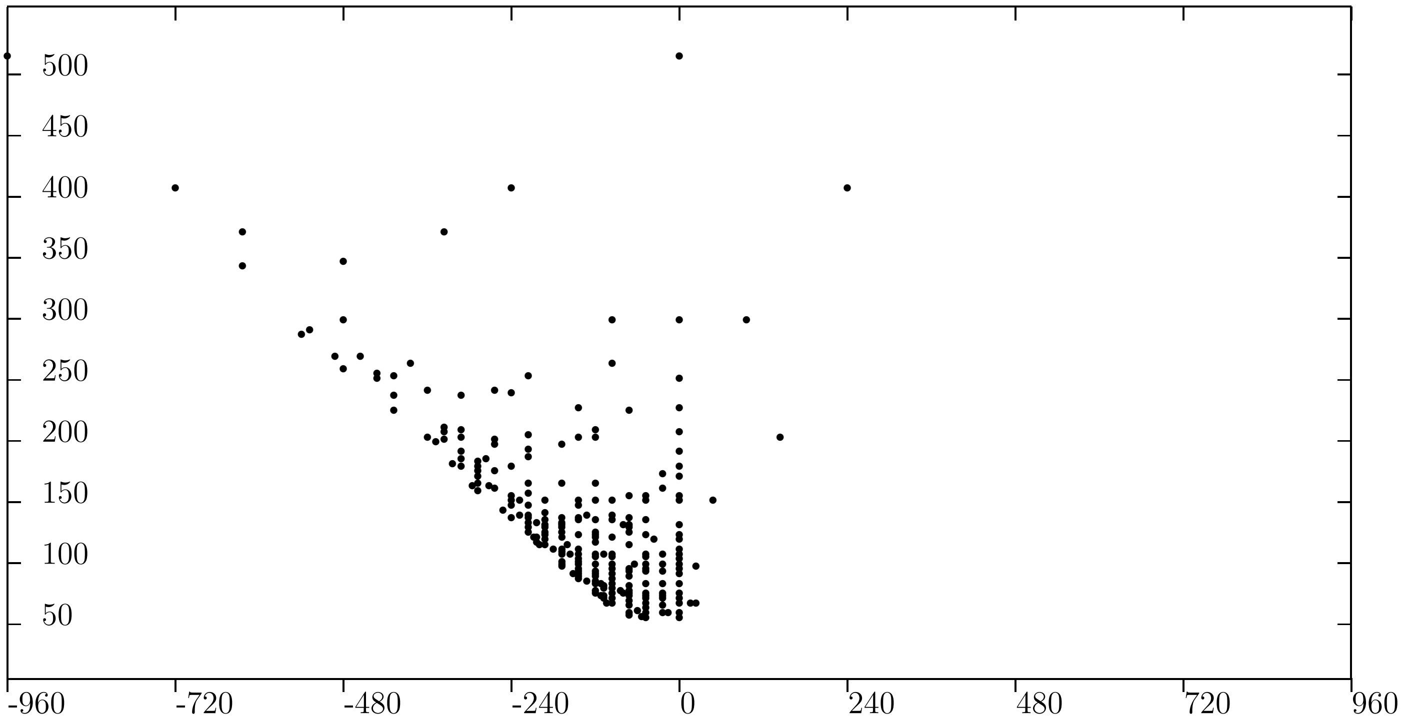



**Fig. 2** *A plot of the Euler number versus* $(h^{(1,1)} + h^{(2,1)})$ *for all ADE tensor models.*

As it turns out, however, that the idea of an asymmetric space of ground states is not correct. The purpose of this review is to first describe in Part I a class of string vacua whose spectra are almost symmetrically distributed in a range of positive and negative Euler numbers, secondly to show in Part II that this is not an accident and thirdly to present evidence, in Part III, that mirror symmetry is a 'robust' property in that it is not an artefact of any one construction.

Sections 2 and 3 are devoted to the construction of a class of Calabi–Yau manifolds which may be realized by polynomials in weighted $\mathbb{P}_4'$*s*. The result of this investigation [8] are some 6,500 examples [1]. This class of vacua is of considerable interest because it interpolates between the previously studied class the CICYs [4] mentioned above, which have negative Euler numbers and the orbifolds of tori which have positive Euler number.

More recently the construction of all Calabi–Yau manifolds embedded in weighted $\mathbb{P}_4$ and,

[1]It was shown in [45] and will be discussed in later sections that not all these spaces are distinct.

in [34][38]. The total class consists of some 10,000 Landau–Ginzburg configurations, the resulting spectra of which have been plotted in Fig.3.

The most remarkable feature of this class of manifolds is immediately apparent from Fig 3. It is evident that the manifolds are very evenly divided between positive and negative Euler numbers the distribution exhibitting an approximate but compelling symmetry under $\chi \to -\chi$. This resonates with the observation made with regard to conformal field theories that the distinction between particles and antiparticles is purely one of convention and the suggestion that for every Calabi–Yau manifold with Euler number $\chi$ there should be one with Euler number $-\chi$.

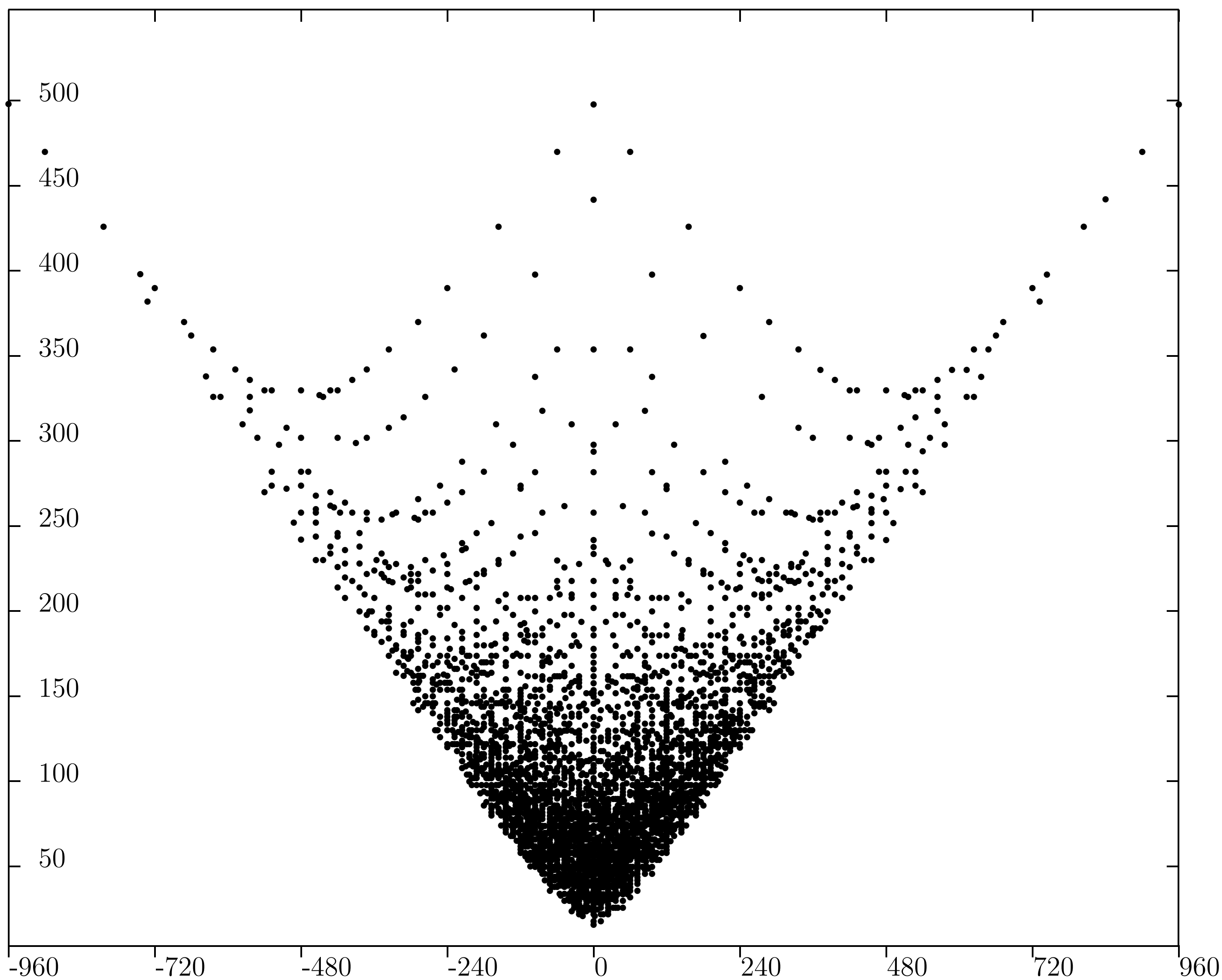



**Fig. 3** *A plot of Euler numbers against $\bar{n}_g + n_g$ for the 2997 spectra of all the LG potentials.*

The richness of the list can be appreciated from the fact that the number of distinct spectra in metry emerges only when a sufficiently large subspace of the configuration space of the heterotic string is probed. Furthermore this class is also of potential phenomenological interest. Whereas it had previously proved very difficult to find Calabi–Yau manifolds with $\chi = \pm 6$ the construction of weighted CICYs leads to some tens such manifolds. These spaces are not of immediate phenomenological interest since they are all simply connected. Hence the best place to seek interesting models may well be among manifolds with Euler number $\pm 6k$, with $k > 1$, although it may also be possible to construct interesting (2,0)–models [10] from the manifolds of Euler number $\pm 6$ which have reduced gauge groups due to an enlarged background gauge group. This however is a question for future work.

With benefit of hindsight it seems odd that no systematic construction of such manifolds was previously attempted. Hypersurfaces in weighted projective spaces were first discussed in the physics literature by Yau [60] and Strominger and Witten[57] who constructed a number of examples with large negative Euler number $-200 > \chi > -300$. Subsequently Kim, Koh and Yoon [33] constructed a few further examples with Euler numbers in a similar range. A complicating feature of these examples was the assumed need to avoid the singular sets of the embedding $\mathbb{P}_4$, coupled with the large values of $|\chi|$ achieved, this led to the impression that the construction was contrived and the resulting manifolds very complicated. To those elucidating the connections between exactly soluble conformal theories and Calabi–Yau manifolds [47][25] however it was apparent that avoiding the singular sets of the embedding space was unnecessary since the singularities on the resulting hypersurface can be resolved while maintaining the condition $c_1 = 0$. This greatly increases the number of manifolds that can be constructed in this way. In fact of the 10,000 odd examples that have been constructed in refs. [8][34] only a small subset consisting of spaces described by polynomials of Fermat type do not intersect the singular sets. Resolution of singularities has the effect of raising the Euler number and in this way the many Euler numbers plotted in Fig. 3 are achieved including many values of moderate size that may be of interest for model building.

Even though the high degree of symmetry of this space of groundstates under the flip of the sign of the Euler numbers strongly suggests a relation between dual pairs there is a priori no reason as far as the different Landau–Ginzburg potentials are concerned why this should be the case. Looking more closely at the Hodge pairs of [8] seems to confuse the issue since generically one finds for a given dual pair of Hodge numbers not two manifolds but *many* and it is unclear which of the possible combinations (if any) are in fact related.

It is therefore of interest to ask whether a systematic procedure can be established which relates consists of a combination of an orbifolding and a nonlinear transformation with somewhat unusual properties in that it involves fractional powers. Even though this procedure can be formulated in the manifold picture it is most easily motivated in Landau–Ginzburg language. By applying this construction to a certain class of mirror candidates it is possible to establish a close connection between dual vacua. The description of this construction comprises Part II of this review.

The starting point is the well known equivalence of the affine D–invariant in the N=2 superconformal minimal series to the $\mathbb{Z}_2$–orbifold of the diagonal invariant at the same level. In section 4 this equivalence will be lifted to the full vacuum described by a tensor product of $N=2$ minimal theories. It is necessary to reformulate this equivalence in terms of the Landau–Ginzburg (LG) potential and its projectivization as a Calabi–Yau (CY) manifold since for most of our models no exactly solvable theory is known. The computations in this framework then either involve orbifoldizing LG–potentials [58] or modding out discrete groups of CY–manifolds and resolving singularities [51][56]. Once this procedure has been formulated in the simple framework of the tensor series of $N=2$ minimal models it will be clear how to proceed in general. Sections 5 and 6 extend the discussion to more general potentials and illustrate to what extent our LG–potentials can be viewed as orbifolds. As already mentioned a general technique to find mirror pairs will emerge. Another, more mathematical, aspect which lends itself to an analysis via fractional transformation is the strange duality of Arnold.

Finally, in Part III, the emphasis is shifted from Landau–Ginzburg theories to their orbifolds. Even though it is possible to map via the fractional transformations of Part II many classes of orbifolds of the models described in Part I to complete intersections it will become clear that not all orbifolds can be understood this way, at least with the techniques presently available. Hence it is useful to consider orbifolds in their own right firstly as a potential pool of new models and also in order to pursue the question raised at the beginning of the introduction regarding the 'robustness' the mirror property against a change of technique [2]. In ref. [36] we have constructed the complete set of Landau–Ginzburg potentials with an arbitrary number of fields involving only couplings between at most two fields (i.e. arbitrary combinations of Fermat, 1–Tadpole and 1–Loop polynomials) and implemented some 40 odd actions of phase symmetries. It turns out that for this class of orbifolds the mirror property improves from about eighty percent for the LG–potentials to about 94%. Figure 4 contains the plot of the resulting spectra.

[2] An orbifold analysis of minimal exactly solvable models has been performed in ref. [24]. The interesting result is that the set of all orbifolds emerging from a given diagonal theory is selfdual (see also ref. [3])

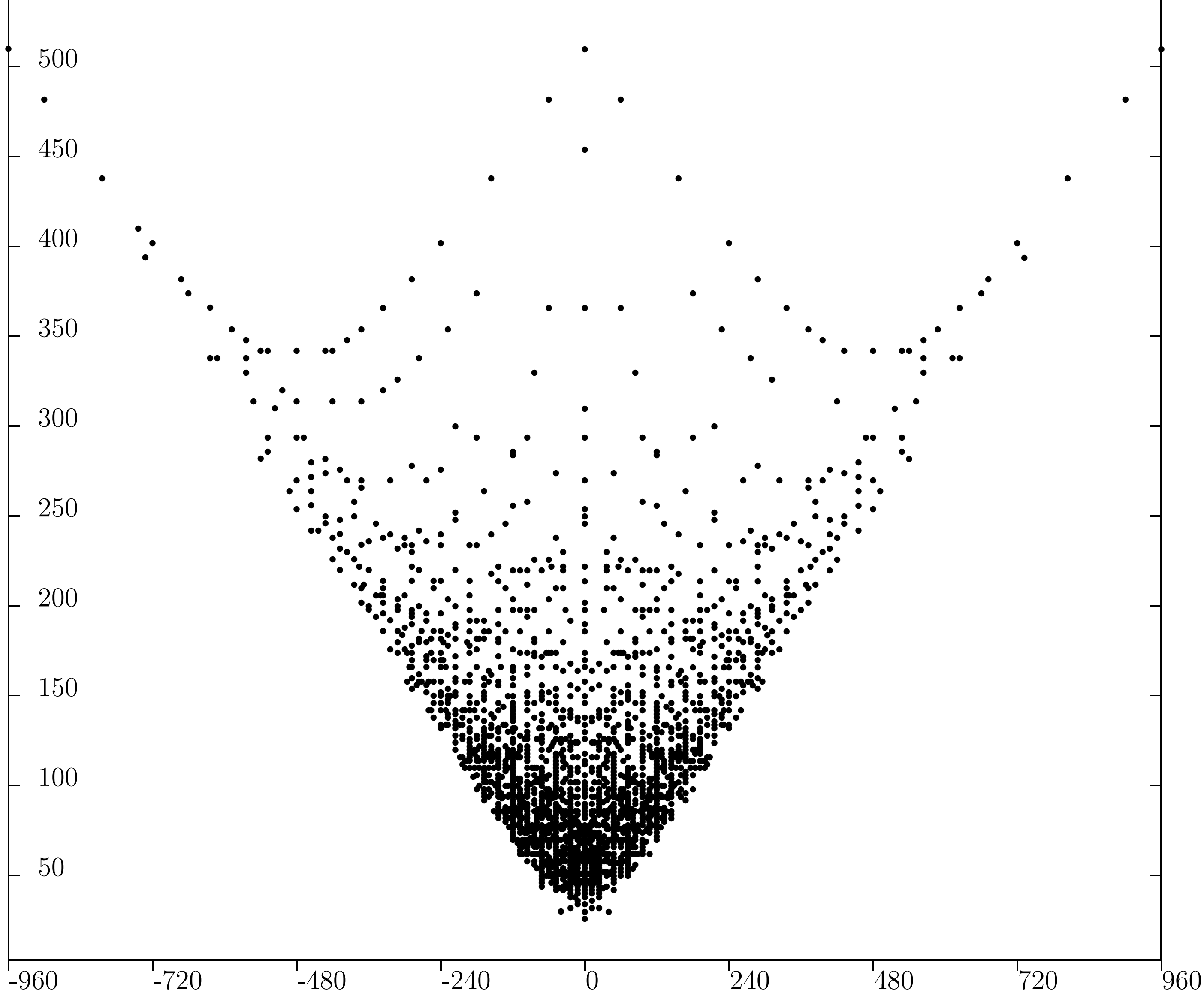



**Fig. 4** *A plot of Euler numbers against $\bar{n}_g + n_g$ for the 1900 odd spectra of all the LG potentials and phase orbifolds constructed.*

It is obvious from Fig. 4 that the upper boundary of the distribution of spectra is the same for the orbifolds as for the complete intersection manifolds. Very likely this boundary is in fact a property of the total moduli space of all three–dimensional Calabi–Yau manifolds. It would be interesting to see whether all Calabi–Yau Hodge numbers fall into the limits defined by the Figures 3 & 4. As expected the lower part of the plot has shifted since new theories with smaller numbers for the total number of fields have appeared. It is to be expected that, as the construction becomes more complete, the structure of the lower part will change again. In fact there are well known manifolds that lie below the models presented here; these however involve permutation groups.

# Construction of Landau–Ginzburg Theories and Weighted CICYs.

Even though there is some overlap between the sets of string vacua described by Landau–Ginzburg vacua on the one hand and Calabi–Yau manifolds on the other it is not, at present, clear whether the former is contained in the latter. It is appropriate justified to separate the discussion of these two classes somewhat. In Sections 2 and 3 the emphasis will be on the explicit construction of a set of CY manifolds embedded in weighted $\mathbb{P}_4$ whereas the remaining sections of Part I will be concerned with the complete class of LG vacua.

## 2 Complete Intersection Manifolds in Weighted $\mathbb{P}_4$.

This section contains some elements of the theory of hypersurfaces defined by polynomials in weighted projective spaces. An extensive discussion of these spaces can be found in [13].

A weighted $\mathbb{P}_4$ with weights $(k_1, k_2, k_3, k_4, k_5)$, which will be denoted by $\mathbb{P}_{(k_1,k_2,k_3,k_4,k_5)}$, is most easily described in terms of 5 complex 'homogeneous coordinates' $(z_1, z_2, z_3, z_4, z_5)$, not all zero, which are subject to the identification

$$(z_1, ..., z_5) \simeq (\lambda^{k_1} z_1, ..., \lambda^{k_5} z_5) \tag{1}$$

for all nonzero $\lambda \in \mathbb{C}$. Thus a weighted projective space is a generalization of ordinary projective space and $\mathbb{P}_4 = \mathbb{P}_{(1,1,1,1,1)}$ in this notation. In the following, when referring to a generic weighted $\mathbb{P}_4$, we shall frequently consider the weights to be understood and write $\mathbb{P}_4$ for $\mathbb{P}_{(k_1,k_2,...,k_5)}$.

The first point to note concerning these spaces is the fact that weighted projective spaces have orbifold singularities owing to the identification (1), except for the case that the weights are all unity. This is most easily seen by setting $z_j = (\zeta_j)^{k_j}$ so that (1) becomes $(\zeta_1, ..., \zeta_5) \simeq \lambda(\zeta_1, ..., \zeta_5)$. However in virtue of the definition of $\zeta_i$ we must also identify $\zeta_j \simeq e^{2\pi i/k_j}\zeta_j$. So we see that

$$\mathbb{P}_{(k_1,k_2,k_3,k_4,k_5)} = \frac{\mathbb{P}_4}{\mathbb{Z}_{k_1} \times \cdots \times \mathbb{Z}_{k_5}} \tag{2}$$

These identifications lead to singular sets. A simple example is $\mathbb{P}_{(1,1,1,2,5)}$ which has weights that are mutually prime. If we take in turn $\lambda = -1$ and $\lambda = \alpha$ with $\alpha$ a fifth root of unity in (1) then

$$
\begin{aligned}
(z_1, z_2, z_3, z_4, z_5) &\simeq (-z_1, -z_2, -z_3, z_4, -z_5) \\
&\simeq (\alpha z_1, \alpha z_2, \alpha z_3, \alpha^2 z_4, z_5).
\end{aligned} \tag{3}
$$

Consider now a neighborhood of the point $(0,0,0,1,0)$. We can take coordinates on the neighborhood by setting $\lambda = z_4^{-1/2}$ and writing $u_j = z_j/z_4^{1/2}$ for $j = 1,2,3$ and $u_5 = z_5/z_4^{5/2}$. We can therefore think of points in the neighborhood as corresponding to $(u_1, u_2, u_3, 1, u_5)$. However from the first of identifications (3) it follows that

$$
(u_1, u_2, u_3, 1, u_5) \simeq (-u_1, -u_2, -u_3, 1, -u_5) \tag{4}
$$

so that there is a $\mathbb{Z}_2$ identification on the space and hence on a neighborhood of the point and the action fixes $(0,0,0,1,0)$. In the same way there is a $\mathbb{Z}_5$ action which fixes $(0,0,0,0,1)$. In this case the singular set consists of points owing to the fact that the weights are mutually prime. Consider now $\mathbb{P}_{(1,1,1,2,4)}$ then the identification is

$$
(z_1, z_2, z_3, z_4, z_5) \simeq (\lambda z_1, \lambda z_2, \lambda z_3, \lambda^2 z_4, \lambda^4 z_5) \tag{5}
$$

If we set $\lambda = z_5^{-1/4}$ and choose coordinates $w_j = z_j/z_5^{1/4}$, $w_4 = z_4/z_5^{1/2}$ then due to the freedom to set $\lambda = -1$ in (5) we have

$$
(w_1, w_2, w_3, w_4, 1) \simeq (-w_1, -w_2, -w_3, w_4, 1) \tag{6}
$$

and we see that we have a $\mathbb{Z}_2$ action which fixes the curve $(0,0,0,z_4,z_5)$. There is in addition a $\mathbb{Z}_4$ action with fixed point $(0,0,0,0,1)$ that lies within this curve. In general there is a fixed point for each weight that is greater than unity, a fixed curve for every pair of weights $k_i, k_j$ whose greatest common factor, which we denote by $(k_i, k_j)$, is greater than unity, a fixed surface for each triple with $(k_i, k_j, k_l) > 1$ and so on.

We wish to study Calabi–Yau hypersurfaces defined by polynomials in the homogeneous coordinates. We require the polynomials to be transverse, that is $p = 0$ and $dp = 0$ have no common solution. Given the weights of the ambient space the requirement of a vanishing first Chern class fixes the degree of the polynomial as in the case of CICYs. To derive the explicit condition we digress briefly on the Chern classes of the submanifold $\mathcal{M}$ defined by the equation $p = 0$. We denote by $\mathcal{T}_{\mathbb{P}_4}$ and $\mathcal{T}_{\mathcal{M}}$ the tangent spaces to the $\mathbb{P}_4$ and $\mathcal{M}$ and by $\mathcal{N}$ the normal bundle of $\mathcal{M}$ in $\mathbb{P}_4$. Proceeding in a standard manner we have

$$
\mathcal{T}_{\mathbb{P}_4} = \mathcal{T}_{\mathcal{M}} \oplus \mathcal{N} \tag{7}
$$

tangent space of $\mathbb{P}_4$ may be thought of as the set of vectors

$$\mathcal{V} = \mathcal{V}^j \frac{\partial}{\partial z^j} \tag{8}$$

which act on bona fide functions of the homogeneous coordinates, i.e. functions of degree zero. From Euler's theorem for homogeneous functions of degree $m$ we have that

$$\sum_j k_j z^j \frac{\partial}{\partial z^j} f = mf. \tag{9}$$

So when acting on functions of degree zero we see that we may regard the $\mathcal{V}^j$ as independent apart from the identification $\mathcal{V}^j \simeq \mathcal{V}^j + k_j z^j$. This reduces the dimension of the space of $\mathcal{V}^j$ to 4 as is necessary. It follows that

$$\mathcal{T}_{\mathbf{P}_4} = \mathcal{O}(k_1) \oplus \cdots \oplus \mathcal{O}(k_5)/\mathcal{O} \tag{10}$$

where $\mathcal{O}(k)$ is a line bundle with $c_1 = kJ$ and $J$ is the Kähler class. It is also the line bundle whose fibre coordinates transform like a polynomial of degree $k$. $\mathcal{O}$ is the trivial bundle. Since $\mathcal{O}(k)$ is one dimensional we have $c(\mathcal{O}(k)) = 1 + kJ$ and hence

$$c(\mathcal{T}_{\mathbf{P}_4}) = \prod_{j=1}^{5} (1 + k_j J)\,. \tag{11}$$

The defining polynomial $p$ can be regarded as a fibre coordinate on $\mathcal{N}$ so if $p$ is of degree $d$ we have $\mathcal{N} = \mathcal{O}(d)$ and $c(\mathcal{N}) = 1 + dJ$. Hence

$$c(\mathcal{T}_{\mathcal{M}}) = \frac{\prod_{j=1}^{5} (1 + k_j J)}{(1 + dJ)} \tag{12}$$

It follows that $c_1 = 0$ is the condition

$$d = \sum_j k_j\ . \tag{13}$$

We record here also an expression for $c_3$, obtained by expanding (12) to third order, which is useful for the computation of the Euler number of these spaces

$$c_3 = -\frac{1}{3}\left(d^3 - \sum_{j=1}^{5} k_j^3\right) J^3\ . \tag{14}$$

There are, roughly speaking, two sorts of singularities that can arise for a hypersurface defined by the vanishing of a polynomial $p$. The first is that the locus $p = 0$ intersects the $\mathbb{Z}_n$–singularities of the ambient space. This does not pose a difficulty however as these can in general be resolved.

from being transverse. This is in contradistinction to CICYs where every configuration admits a smooth representative (in fact almost all representatives are smooth [22]). To see this let

$$d_j := d - k_j \ , \ j = 1, \ldots, 5 \tag{15}$$

and expand $p$ in powers of $z_1$, say,

$$p = \sum_{r=0}^{a_1} C_r(z_m)\, z_1^r \ , \quad m \neq 1 \ . \tag{16}$$

It follows from Euler's Theorem (9) that $p = 0$ and $dp = 0$ if and only if there is a simultaneous solution of the equations

$$\frac{\partial p}{\partial z_j} = 0 \ , \ j = 1, \ldots, 5 \ . \tag{17}$$

Applying this to (16) we have

$$\begin{aligned} \frac{\partial p}{\partial z_1} &= \sum_{r=1}^{a_1} r C_r(z_m)\, z_1^{r-1} \\ \frac{\partial p}{\partial z_m} &= \sum_{r=0}^{a_1} \frac{\partial C_r}{\partial z_m}\, z_1^r \end{aligned} \tag{18}$$

Now the degrees of the $C_r$ are fixed by (16)

$$\begin{aligned} \deg\left(C_r\right) &= d - rk_1 = d_1 - (r-1)k_1 \ , \ \text{for } r \geq 1 \ , \\ \deg\left(\frac{\partial C_r}{\partial z^m}\right) &= d_m - rk_1 \ . \end{aligned} \tag{19}$$

and a polynomial of negative degree is understood to vanish identically. Unless at least one of the coefficients in (18) has degree precisely zero then equations (17) will be satisfied for $z_1 = 1$ and $z_m = 0$ and $p$ will not be transverse. This leads to the following neccessary condition on the weights if $p$ is to be transverse:

*For each $i$ there must exist a $j$ such that $k_i | d_j$ ($k_i$ divides $d_j$).*

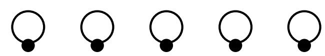
This condition is quite restrictive. For example it is immediate that the only manifolds of the form $\mathbb{P}_{(1,1,1,1,k)}[k+4]$ that it allows are those for the four values $k = 1, 2, 3, 4$, and the assiduous reader may care to check that apart from these there only eleven other allowed cases of the form $\mathbb{P}_{(1,1,1,k,l)}[k+l+3]$, where $k+l+3$ indicates the degree of the polynomial.

This criterion however is not a sufficient condition. A counterexample being furnished by the configuration $\mathbb{P}_{(1,2,2,2,2)}[9]$ whose most general polynomial is of the form

$$p(z_1, z_2, z_3, z_4, z_5) = z_1 \tilde{p}(z_1, z_2, z_3, z_4, z_5). \tag{20}$$

a neighborhood $U_5$ say, on which we can take $z_5 = 1$. Taking the differential gives $dp = \tilde{p}dz_1 + z_1 d\tilde{p}$ which has a zero for $z_1 = 0$ and $\tilde{p} = 0$. These equations give two constraints in 4 variables and therefore always have a solution. Since the polynomial constraint is identically satisfied it follows that this configuration cannot admit a transverse realization. Thus there are further criteria which must be satisfied in order for a polynomial to be transverse. In refs. [8] a list of transverse polynomials was constructed. A little thought shows that polynomials of Fermat type for which $k_i|d_i$ for each $i$ *are* transverse. These are of the form

$$z_1^a + z_2^b + z_3^c + z_4^d + z_5^e \qquad (21)$$

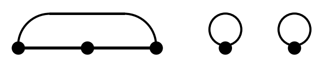


to which we have appended a diagrammatic shorthand.

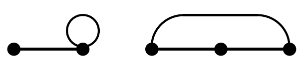
There are other types which are also always transverse such as

$$z_1^a z_2 + z_2^b + z_3^c + z_4^d + z_5^e \qquad (22)$$

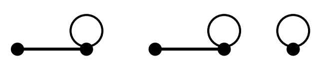
$$z_1^a z_2 + z_2^b z_3 + z_3^c z_1 + z_4^d + z_5^e \qquad (23)$$



Parts of these expressions corresponding to connected subdiagrams can also be combined together to produce yet other transverse polynomials such as



or



(24)

What is needed then is a classification of nondegenerate weighted homogeneous polynomials in five variables. Unfortunately, there exists as yet no such classification and indeed its formulation seems to be a hard problem.

There *does* exist a classification of smooth polynomials in three variables [2] and what has been done in [8] is to extend this to a partial classification of polynomials in five variables. These constructions do not describe all possible polynomials but they do respresent a minimal extension of Arnold's classification to the case of five variables. Table 1 contains the polynomial types implemented in [8] to construct nondegenerate polynomials in five variables. By combining the nineteen types listed below in the way described above one obtains thirty different five dimensional catastrophes.

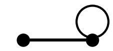
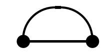
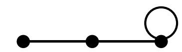
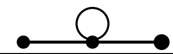
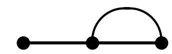
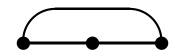
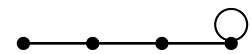
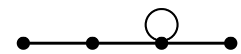
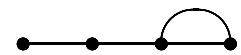
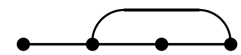
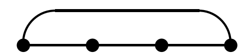
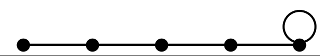
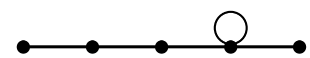
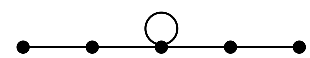
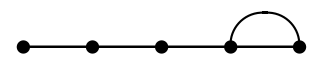
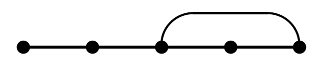
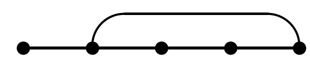
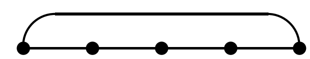
| # | Polynomial Type | Diagram |
|---|---|---|
| 1 | $z_1^a$ |  |
| 2 | $z_1^a z_2 + z_2^b$ |  |
| 3 | $z_1^a z_2 + z_2^b z_1$ |  |
| 4 | $z_1^a z_2 + z_2^b z_3 + z_3^c$ |  |
| 5 | $z_1^a z_2 + z_2^b + z_3^c z_2 + z_1^p z_3^q$ |  |
| 6 | $z_1^a z_2 + z_2^b z_3 + z_3^c z_2 + z_1^p z_3^q$ |  |
| 7 | $z_1^a z_2 + z_2^b z_1 + z_3^c z_1$ |  |
| 8 | $z_1^a z_2 + z_2^b z_3 + z_3^c z_4 + z_4^d$ |  |
| 9 | $z_1^a z_2 + z_2^b z_3 + z_3^c + z_4^d z_3 + z_2^p z_4^q$ |  |
| 10 | $z_1^a z_2 + z_2^b z_3 + z_3^c z_4 + z_4^d z_3 + z_2^p z_4^q$ |  |
| 11 | $z_1^a z_2 + z_2^b z_3 + z_3^c z_4 + z_4^d z_2 + z_1^p z_4^q$ |  |
| 12 | $z_1^a z_2 + z_2^b z_3 + z_3^c z_4 + z_4^d z_1$ |  |
| 13 | $z_1^a z_2 + z_2^b z_3 + z_3^c z_4 + z_4^d z_5 + z_5^e$ |  |
| 14 | $z_1^a z_2 + z_2^b z_3 + z_3^c z_4 + z_4^d + z_5^e z_4 + z_3^p z_5^q$ |  |
| 15 | $z_1^a z_2 + z_2^b z_3 + z_3^c + z_4^d z_3 + z_5^e z_4 + z_2^p z_4^q$ |  |
| 16 | $z_1^a z_2 + z_2^b z_3 + z_3^c z_4 + z_4^d z_5 + z_5^e z_4 + z_3^p z_5^q$ |  |
| 17 | $z_1^a z_2 + z_2^b z_3 + z_3^c z_4 + z_4^d z_5 + z_5^e z_3 + z_2^p z_5^q$ |  |
| 18 | $z_1^a z_2 + z_2^b z_3 + z_3^c z_4 + z_4^d z_5 + z_5^e z_2 + z_1^p z_5^q$ |  |
| 19 | $z_1^a z_2 + z_2^b z_3 + z_3^c z_4 + z_4^d z_5 + z_5^e z_1$ |  |

**Table 1.** *The polynomial types that have been implemented.*

The last problem that confronts the construction of all Calabi–Yau manifolds in weighted projective space is the question whether the list is finite. Again we compare the new situation with the one in the case of CICYs. There the condition of vanishing first Chern class leads to just one configuration that can be found in the case of one projective space and one polynomial, namely the quintic $\mathbb{P}_4[5]$. For a weighted $\mathbb{P}_4$ however this condition leads to the equation (13). It seems at first that this equation has an infinite number of solutions. In fact a little thought shows that for each of the thirty polynomial types there are restrictions on the range of possible weights. For the types of conformal field theories which have all been constructed [43][44][42][17]. Consider a Fermat polynomial

$$p = z_1^{a_1} + z_2^{a_2} + z_3^{a_3} + z_4^{a_4} + z_5^{a_5} \tag{25}$$

with weights $(k_1, ...., k_5)$ and degree

$$d = k_1 a_1 = k_2 a_2 = k_3 a_3 = k_4 a_4 = k_4 a_5 \ . \tag{26}$$

The condition of vanishing first Chern class becomes

$$1 = \sum_{i=1}^{5} \frac{1}{a_i} \tag{27}$$

It is possible to iteratively bound the $a_i$, which we take to be ordered such that $a_i \leq a_{i+1}$, in virtue of the fact that $\sum_i \frac{1}{a_i}$ is a decreasing function of all its arguments. The smallest possible value for $a_1$ is 2. The next step consists in finding the smallest possible value for $a_2$. Using the lower bound on $a_1$ condition (27) becomes

$$\frac{1}{2} = \sum_{i=2}^{5} \frac{1}{a_i}, \tag{28}$$

from which it follows that the smallest possible value for $a_2$ is 3. Proceeding in this manner we end up with the condition

$$\frac{1}{42} - \frac{1}{43} = \frac{1}{a_5} \tag{29}$$

from which we find $a_5 = 1806$ which turns out to be the highest power that arises in our polynomials.

As a second example consider polynomials of type 2,

$$p = z_1^{a_1} + z_2^{a_2} + z_3^{a_3} + z_4^{a_4} + z_5^{a_5} z_4. \tag{30}$$

In this case we have

$$d = k_1 a_1 = k_2 a_2 = k_3 a_3 = k_4 a_4 = k_4 + k_5 a_5. \tag{31}$$

and the condition of vanishing first Chern class is now

$$1 = \sum_{i=1}^{5} \frac{1}{a_i} - \frac{1}{a_4 a_5} \tag{32}$$

Taking again $a_1 = 2$ we proceed as above. Using the lower bound on $a_1$ condition (32) becomes

$$\frac{1}{2} = \sum_{i=2}^{5} \frac{1}{a_i} - \frac{1}{a_4 a_5}. \tag{33}$$

$a_3 = 7$ and then $a_4 = 43$. Finally we find the condition

$$\frac{1}{42 \cdot 43} = \frac{1}{a_5} - \frac{1}{43 \cdot a_5} \tag{34}$$

whence $a_5 = 42^2 = 1764$. In a similar way we find constraints on all other types of polynomials.

# 3 Computation of the Spectrum

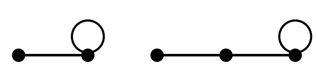
Having constructed these 6,500 odd spaces one wants, of course, to know about the spectrum, especially about the number of light generations. There are several methods available to compute these numbers. First there is of course the geometrical analysis that can be used to compute the independent Hodge numbers of a Calabi–Yau manifold. Another method that is more useful for the class of spaces at hand are techniques for computing the spectrum in the framework of Landau–Ginzburg theories. In those cases for which an exactly solvable theory corresponding to the model is known, techniques from conformal field theory are available as well. Unfortunately for most of the theories constructed in the previous section no exactly solvable model is known and hence the tools from conformal field theory, even though most powerful, are not available here.

The very first step in a systematic analysis is, of course, the determination of the number of light generations, i.e. the computation of the Euler number. This can be done by computing the integral of the third Chern class using the fact that $\int J^3 = 1/\prod k_j$ and taking into account the contributions from the singularities [60]

$$\chi = -\frac{\frac{1}{3}\left(d^3 - \sum k_j^3\right) d}{\prod k_j} - \sum_i \frac{\chi(S_i)}{n_i} + \sum_i n_i \chi(S_i) \tag{35}$$

where $\chi(S_i)$ is the Euler number of the singular set $S_i$ and $n_i$ is the order of the cyclic symmetry group $\mathbb{Z}_{n_i}$.

This can be illustrated with an example. Consider the manifold

$\mathbb{P}_{(4,4,5,5,7)}[25]$  . (36)

In this example we have three types of singular sets; first there is a $\mathbb{Z}_4$–curve $C$ with Euler number $\chi(C) = 2$. Next there are five $\mathbb{Z}_5$–points and finally the $\mathbb{Z}_7$ leads to one additional fixed point. Put together with $\chi_s = -44\frac{5}{14}$ for the Euler number of the singular space this gives $\chi = -6$ for this space.

atically as it entails a detailed analysis of the singular sets $S_i$ for each manifold. These not only depend on the divisibility property of the weights but also on the type of the individual catastrophe involved and therefore need to be determined on a case by case basis. This is not easily automated.

It is much easier to use a result of Vafa [58] on the Euler number of $c = 9$, $N = 2$ Landau–Ginzburg models. His formula specialized to the case at hand yields

$$\chi = \frac{1}{d} \sum_{l,r=0}^{d-1} (-1)^{r+l+d} \prod_{lq_i, rq_i \in \mathbf{Z}} \frac{d-k_i}{k_i}, \tag{37}$$

where $q_i = k_i/d$. Computing the Euler number for all 10,839 odd spaces leads to the results of Figure 3. As already mentioned the Euler number $-960 \leq \chi \leq 960$. Among these spaces are many that lead to 2, 3 and 4 light generations.

The next question then is how the Euler number actually splits up into generations and antigenerations. Again it is possible to use both, manifold techniques and Landau–Ginzburg type methods. As we have already easy methods to compute the Euler number $\chi = 2(h^{(1,1)} - h^{(2,1)})$ we only need to compute either the number of generations of the number of antigenerations to have the complete Hodge diamond.

We consider first the geometrical techniques and compute the number of antigenerations. In a projective space there would be nothing to compute since in this case the dimensions of this cohomology group is always one, its only contribution coming from the Kähler form. In the case of weighted projective spaces the Kähler form of course is not the only contribution because the blow–ups of the singular sets introduce new (1,1)–forms. The singular sets consist of points and/or curves. The techniques for blowing up points have been discussed in [51] and the contributions coming from blowing up curves have been discussed in [56]. These are in fact the only types of singularity that can arise if $p$ is transverse. In other words that singular subsets of $M$ cannot have dimension 2 or 3. First we show that the embedding $\mathbb{P}_4$ cannot have singular sets of dimension greater than or equal to 3. Recall that the $\mathbb{P}_4$ has a singular point for each $k_i > 1$, a singular curve for each pair with $(k_i, k_j) > 1$, a singular subset of dimension 2 for each triple with $(k_i, k_j, k_l) > 1$ *etc*. In the definition of $\mathbb{P}_4$ we have assumed that the weights have no common factor so there are no singular sets of dimension 4. Consider the possibility of a singular set of dimension 3. Such a set would correspond to weights such that

$$(k_2, k_3, k_4, k_5) = m > 1 \;\; \text{but} \;\; m \not| k_1 \ .$$

Since $m$ does not divide $k_1$ but does divide the other $k$'s it cannot divide the degree $d = \sum_{j=1}^{5} k_j$. Every monomial of degree $d$ must therefore contain at least one factor of $z_1$. Thus $p(z) = z_1 \tilde{p}(z)$

generically intersect $M$ in subsets of dimension 1. It remains to show that a singular subset of dimension 2 cannot lie within $M$. To this end suppose there is a fixed point set of dimension 2 for the identification (1) and that this subset lies within $M$. We may choose coordinates $(x, y, z)$ such that, locally, the fixed point set is $(x, y, 0)$. Suppose the identification is represented by a matrix $A$. Since $A$ fixes $(x, y, 0)$ it has the form

$$A = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & c \end{pmatrix} .$$

The three–form $dx \wedge dy \wedge dz$ transforms as

$$dx \wedge dy \wedge dz \longrightarrow \det A \, dx \wedge dy \wedge dz$$

we must have det $A = 1$ and hence $c = 1$. Since $A$ is contained in a finite group $A^n = 1$ for some $n$, however

$$A^n = \begin{pmatrix} 1 & 0 & na \\ 0 & 1 & nb \\ 0 & 0 & 1 \end{pmatrix}$$

so $a$ and $b$ must, in fact, vanish. Thus the only identification that fixes a two–dimensional subset is the identity.

When resolving the singularities it needs to be checked that the blown up manifolds are still Calabi–Yau manifolds. For the case of singular points this has been discussed in ref. [51] whereas the case of singular curves was first analyzed in ref. [56].

In order to resolve curves consider the action of a $\mathbb{Z}_n$ on a weighted CICY leaving a curve invariant. In the three–dimensional Calabi–Yau manifold the normal bundle of this curve has fibres $\mathbb{C}_2$ and therefore this discrete group induces an action on the fibres described by the matrix

$$\begin{pmatrix} \alpha^{mq} & 0 \\ 0 & \alpha^m \end{pmatrix} \tag{38}$$

where $\alpha$ is an n$^{th}$ root of unity, $0 \leq m \leq n$, and $q, n$ have no common divisor. This action has an isolated singularity which needs to be resolved. The essential point is that the singularity of $\mathbb{C}_2/\mathbb{Z}_n$ can be described as the singular set of the surface

$$S: \quad z_3^n = z_1 z_2^{n-q} \tag{39}$$

in $\mathbb{C}_3$ and therefore can be resolved by a construction that is completely determined by the type $(n, q)$ of the action through the method of continued fractions. The expansion of $\frac{n}{q}$ in a continued

$$\begin{aligned} \frac{n}{q} &= [b_1, ..., b_s] \\ &= b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \frac{1}{b_s}}} \end{aligned} \tag{40}$$

determines the numbers $b_i$ which specify uniquely the plumbing process which replaces the singularity. Furthermore, it also determines the additional generators of the cohomology, since the number of $\mathbb{P}_1$'s necessary to resolve the singularity is precisely the number of steps needed in the evaluation of $\frac{n}{q} = [b_1, .., b_s]$. The reason for this is that the singularity is replaced by a bundle which is constructed of $s+1$ patches with $s$ transition functions that are specified by the $b_i$'s. Each of these glueing steps introduces a sphere which in turn supports a (1,1)–form. A standard shorthand notation for the geometry of the blow–up is through what is called a Hirzebruch–Jung tree [28] which in the case of the blow–up of a $\mathbb{Z}_n$ action is just an SU$(s+1)$ Dynkin diagram with the negative values of the $b_i's$ attached to the nodes. Each node in the diagram corresponds to a sphere and the diagram shows which spheres intersect each other (in the case of the $\mathbb{Z}_n$–action only the neighboring spheres intersect) whereas the $b_i$ determine the intersection numbers.

In order to show that these blow–up procedures can be applied, we have to show that the determinant (38) is always 1. This can be done by checking the invariance of the holomorphic threeform $\Omega$ under $\mathbb{Z}_n$.

Consider first manifolds of Fermat type. A singular curve in such a manifold is signalled by three weights of the ambient space that are not coprime

$$(k_1, k_2, k_3) = n > 1 \tag{41}$$

($n$ does not divide $k_4, k_5$). The integer $n$ defines a $\mathbb{Z}_n$ discrete group. The action of $\mathbb{Z}_n$ is given by

$$(z_1, z_2, z_3, z_4, z_5) \mapsto (z_1, z_2, z_3, \alpha^{k_4} z_4, \alpha^{k_5} z_5), \tag{42}$$

where $\alpha$ is an $n$th root of unity. In order to show that the blown–up manifold is still of Calabi–Yau type one needs to show that $\Omega$ is invariant. In this case the action of $\mathbb{Z}_n$ on $\Omega$ is

$$\Omega \mapsto \alpha^{k_4+k_5} \Omega. \tag{43}$$

The condition for $\Omega$ to be invariant is, therefore, that

$$\alpha^{k_4+k_5} = 1, \tag{44}$$

$$1 = \alpha^d = \alpha^{\sum k_i} = \alpha^{k_4+k_5}. \tag{45}$$

Now suppose the curve is embedded in a non–Fermat Calabi–Yau manifold. Then the possibility exists that there are only two weights for which

$$(k_1, k_2) = n > 1 \tag{46}$$

($n$ does not divide $k_3, k_4, k_5$), i.e. $k_1, k_2$ are not constrained by the polynomial. In this case the coordinates parametrizing the curve occur as

$$z_1^{l_1} z_p, \quad z_2^{l_2} z_q. \tag{47}$$

The action of $\mathbb{Z}_n$ is given by

$$(z_1, z_2, z_3, z_4, z_5) \mapsto (z_1, z_2, \alpha^{k_2} z_3, \alpha^{k_4} z_4, \alpha^{k_5} z_5), \tag{48}$$

and the condition for $\Omega$ to be invariant becomes

$$\alpha^{k_3+k_4+k_5} = \alpha^{k_p} = \alpha^{k_q}. \tag{49}$$

As before

$$d = k_1 l_1 + k_p = k_2 l_2 + k_q = nr_1 l_1 + k+_p = nr_2 l_2 + k_q \tag{50}$$

hence

$$\alpha^d = \alpha^{k_3+k_4+k_5} = \alpha^{k_p} = \alpha^{k_q} \tag{51}$$

which was to be shown.

Returning now to the discussion of the example $\mathbb{P}_{(4,4,5,5,7)}[25]$, we find from the results of ref. [51] that a $\mathbb{Z}_5$–point blow–up of this manifold contributes two (1,1)–forms, whereas a $\mathbb{Z}_7$–point blow–up leads to three additional generators of the second cohomology. Since the $\mathbb{Z}_5$ singular set $\mathbb{P}_1[5]$ consists of five points and the $\mathbb{Z}_7$ singular set consists just of one point all point blow–ups together contribute a total of thirteen (1,1)–forms. To find out how many (1,1)–forms the blow–up of the $\mathbb{Z}_4$–curve contributes we need to check the induced action of this discrete group action on the normal bundle of the curve [56].

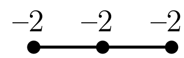
In our example (36) this induced action

$$\mathbb{C}_2 \ni (z_1, z_2) \longmapsto (\alpha z_1, \alpha^3 z_2) \tag{52}$$

the surface

$$z_3^4 = z_1 z_2. \tag{53}$$

The resolution is specified by the Hirzebruch–Jung tree




which in turn is determined completely by the continued fraction

$$\frac{4}{3} = 2 - \frac{1}{2 - \frac{1}{2}}. \tag{54}$$

Therefore the blow–up of the curve contributes three more (1,1)–forms. Taking into account the Kähler form of the ambient space then leads to a total of 17 (1,1)–forms for this manifold.

As is evident however the manifold technique is awkward to implement in a systematic way and is better used as an independent check of the Landau–Ginzburg type formulation of this problem by Vafa who constructs a Poincaré–type polynomial for the $l^{\rm th}$ twisted sector

$$Tr_l((t\bar{t})^{dJ_0}) = t^{d\left(Q_l + \frac{1}{6}c_T\right)} \bar{t}^{d\left(-Q_l + \frac{1}{6}c_T\right)} \prod_{lq_i \in \mathbf{Z}} \left( \frac{1 - (t\bar{t})^{d-k_i}}{1 - (t\bar{t})^{k_i}} \right) \tag{55}$$

with

$$\begin{aligned} Q_l &= \sum_{lq_i \notin \mathbf{Z}} \left( lq_i - [lq_i] - \frac{1}{2} \right) , \\ \frac{1}{6} c_T &= \sum_{lq_i \notin \mathbf{Z}} \left( \frac{1}{2} - q_i \right) \end{aligned} \tag{56}$$

where $t$ and $\bar{t}$ are formal variables, $d$ is the degree of the Landau–Ginzburg potential, the $q_i = k_i/d$ are the normalized weights of the fields and $[lq_i]$ is the integer part of $lq_i$. Expanding this polynomial in powers in $t$ and $\bar{t}$ it is possible to read off the contributions to the various cohomology groups from the different sectors of the twisted LG–theory. The (2,1) forms for example are given by the number of fields with charge (1,1), i.e. the coefficient of $(t\bar{t})^d$. In general, the number of $(p,q)$ forms are given by the coefficients of $t^{(3-p)d}\bar{t}^{qd}$ in the Poincaré polynomials summed over all sectors $l = 0, 1, \ldots, d-1$.

# 4 Landau–Ginzburg Vacua

In this section we briefly review the construction of all Landau–Ginzburg vacua based on superpotentials with an isolated singularity at the origin. Consider a string ground state based on a Landau–Ginzburg theory which we assume to be $N = 2$ supersymmetric since we demand $N = 1$ spacetime the action takes the form

$$\mathcal{A} = \int d^2z d^4\theta \; K(\Phi_i, \bar{\Phi}_i) + \int d^2z d^2\theta^- \; W(\Phi_i) + \int d^2z d^2\theta^+ \; W(\bar{\Phi}_i) \tag{57}$$

where $K$ is the Kähler potential and the superpotential $W$ is a holomorphic function of the chiral superfields $\Phi_i$. Since the ground states of the bosonic potential are the critical points of the superpotential of the LG theory we demand their existence. The type of critical points we need is determined by the fact that we wish to keep the fermions in the theory massless; hence we assume that the critical points are completely degenerate. Furthermore we require that all critical points be isolated, since we wish to relate the finite dimensional ring of monomials associated to such a singularity with the chiral ring of physical states in the Landau–Ginzburg theory, in order to construct the spectrum of the corresponding string vacuum. Finally we demand that the theory is conformally invariant; from this follows, relying on some assumptions regarding the renormalization properties of the theory, that the Landau–Ginzburg potential is quasihomogeneous. In other words, we require to be able to assign, to each field $\Phi_i$, a weight $q_i$ such that for any non–zero complex number $\lambda \in \mathbb{C}^\star$

$$W(\lambda^{q_1}\Phi_1, \ldots, \lambda^{q_n}\Phi_n) = \lambda W(\Phi_1, \ldots, \Phi_n). \tag{58}$$

Thus we have formulated the class of potentials that we need to consider: quasihomogeneous polynomials that have an isolated, completely degenerate singularity (which we can always shift to the origin).

Associated to each of the superpotentials, $W(\Phi_i)$ is a so–called catastrophe which is obtained by first truncating the superfield $\Phi_i$ to its lowest bosonic component $\phi_i(z, \bar{z})$, and then going to the field theoretic limit of the string by assuming $\phi_i$ to be constant $\phi_i = z_i$. Writing the weights as $q_i = k_i/d$, we will denote by

$$\mathbb{C}_{(k_1,k_2,\ldots,k_n)}[d] \tag{59}$$

the set of all catastrophes described by the zero locus of polynomials of degree $d$ in variables $z_i$ of weight $k_i$.

The affine varieties described by these polynomials are not compact and hence it is necessary to implement a projection in order to compactify these spaces. In Landau–Ginzburg language, this amounts to an orbifolding of the theory with respect to a discrete group $\mathbb{Z}_d$ the order of which is the degree of the LG potential [58]. The spectrum of the orbifold theory will contain twisted states which, together with the monomial ring of the potential, describe the complete spectrum of the

$$\mathbb{C}^\star_{(k_1,k_2,...,k_n)}[d] \tag{60}$$

and call it a configuration.

In manifold speak the projection should lead to a three–dimensional Kähler manifold, with vanishing first Chern class. For a general Landau–Ginzburg theory no unambiguous universal prescription for doing so has been found and, as we will describe in Section 4, none can exist. One way to compactify amounts to simply imposing projective equivalence

$$(z_1, ...., z_n) \equiv (\lambda^{k_1} z_1, ....., \lambda^{k_n} z_n) \tag{61}$$

which embeds the hypersurface described by the zero locus of the polynomial into a weighted projective space $\mathbb{P}_{(k_1,k_2,...,k_n)}$ with weights $k_i$. The set of hypersurfaces of degree $d$ embedded in weighted projective space will be denoted by

$$\mathbb{P}_{(k_1,k_2,...,k_n)}[d]. \tag{62}$$

For a potential with five scaling variables this construction is completely sufficient in order to pass from the Landau–Ginzburg theory to a string vacuum [47][25] provided $d = \sum_{i=1}^5 k_i$, which is the condition that these hypersurfaces have vanishing first Chern class. For more than five variables, however, this type of compactification does not lead to a string vacuum.

Even though the precise relation between LG theories and CY manifolds is not known for the most general case certain facts are known. Since LG theories with five variables describe a CY manifold embedded in a 4 complex dimensional weighted projective space one might expect e.g. that LG potentials with 6, 7, etc., variables describe manifolds embedded in 5, 6, etc., dimensional weighted projective spaces. This is not correct.

In fact none of the models with more than five variables is related to manifolds embedded in one weighted projective space. Instead they describe Calabi–Yau manifolds embedded in products of weighted projective space. A simple example is furnished by the LG potential in six variables

$$W = \Phi_1 \Psi_1^2 + \Phi_2 \Psi_2^2 + \sum_{i=1}^{3} \Phi_i^{12} + \Phi_4^3 \tag{63}$$

which corresponds to the exactly solvable model described by the tensor product of $N = 2$ minimal theories at the levels

$$(22^2 \cdot 10 \cdot 1)_{D^2 \cdot A^2}, \tag{64}$$

belongs to the LG configuration

$$\mathbb{C}^{\star}_{(2,11,2,11,2,8)}[24]^{(3,243)}_{-480} \tag{65}$$

and is equivalent to the weighted complete intersection Calabi–Yau (CICY) manifold in the configuration

$$\begin{matrix} \mathbb{P}_{(1,1,1,4,6)} \\ \mathbb{P}_{(1,1)} \end{matrix} \begin{bmatrix} 1 & 12 \\ 2 & 0 \end{bmatrix}^{(3,243)}_{-480} \tag{66}$$

described by the intersection of the zero locus of the two potentials

$$\begin{aligned} p_1 &= x_1^2 y_1 + x_2^2 y_2 \\ p_2 &= y_1^{12} + y_2^{12} + y_3^{12} + y_4^3 + y_5^2. \end{aligned} \tag{67}$$

Here we have added a trivial factor $\Phi_5^2$ to the potential and taken the field theory limit via $\phi_i(z,\bar{z}) = y_i$, where $\phi_i$ is the lowest component of the chiral superfield $\Phi_i$. The first column in the degree matrix (66) indicates that the first polynomial is of bidegree (2,1) in the coordinates $(x_i, y_j)$ of the product of the projective line $\mathbb{P}_1$ and the weighted projective space $\mathbb{P}_{(1,1,1,4,6)}$ respectively, whereas the second column shows that the second polynomial is independent of the projective line and of degree 12 in the coordinates of the weighted $\mathbb{P}_4$. The superscripts in (65) and (66) describe the dimensions $(h^{(1,1)}, h^{(2,1)})$ of the fields corresponding to the cohomology groups $(H^{(1,1)}, H^{(2,1)})$, whereas the subscript is the Euler number of the configuration.

It should be noted however that Landau–Ginzburg potentials in six variables do not describe the most general complete intersection in products of weighted spaces, simply because not all of these manifolds involve trivial factors, or put differently, quadratic monomials. A simple example is the manifold that corresponds to the Landau–Ginzburg theory $(1 \cdot 16^3)_{A \cdot E_7^3}$ with LG potential

$$W = \sum_{i=1}^{3} \left( \Phi_i^3 + \Phi_i \Psi_i^3 \right) + \Phi_4^3. \tag{68}$$

This theory describes a codimension–2 Calabi–Yau manifold embedded in

$$\begin{matrix} \mathbb{P}_2 \\ \mathbb{P}_3 \end{matrix} \begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}. \tag{69}$$

This space has 8 (1,1)–forms and 35 (2,1)–forms which correspond to the possible complex deformations in the two polynomials $p_1, p_2$ [54].

Associated to this Calabi–Yau manifold in a product of ordinary projective spaces is an auxiliary algebraic manifold in a weighted six–dimensional projective space

$$\mathbb{P}_{(2,2,2,3,3,3,3)}[9]. \tag{70}$$

$\Psi_i$ and the last four come from the $\Phi_i$. We want to compute the number of complex deformations of this manifold, i.e. we want to compute the number of monomials of charge 1.

The most general monomial is of the form $\prod_i \Phi_i \prod \Psi_j$. It is easy to show explicitly that there are precisely 35 monomials by writing them down but the following remarks may suffice. There are four different types of possible monomials, depending on whether they contain the fields $\Phi_i$ not at all (I), linearly (II), quadratically (III) or cubically (IV). First note that monomials of type (I) do not contribute to the marginal operators. Monomials of type (II) have to contain cubic monomials in the $\Psi$. Since there are three fields $\Psi_i$ available, we obtain a total of 40 marginal operators of this type. Because of the equation of motion nine of these are in fact in the ideal and we are left with 31 complex deformations of this type. Monomials quadratic in the $\Phi_i's$ again do not contribute, whereas those cubic in the $\Phi_i$ fields contribute the remaining 4. Indeed, there are 20 cubic monomials in terms of the four $\Phi_i$ but using the equations of motion one finds that 16 of those are in the ideal.

In other words, for the total of $60 = 40 + 20$ monomials of degree 9 (or charge 1) it is possible to fix the coefficients of 9 of the 40 by allowing linear field redefinitions of the first three coordinates and also to fix the coefficients of 16 of the 20 via linear field redefinitions of the last four coordinates. Hence even though the ambient space is singular and the manifold hits these singularities in a $\mathbb{P}_2$ and in a cubic surface $\mathbb{P}_3[3]$ the resolution does not contribute any complex deformations because these surfaces are simply connected.

It should be emphasized that this manifold is not the physical internal part of a string ground state, but that it plays an auxiliary role, which allows to discuss just one particular sector of the string vacuum, namely the complex deformations.

Before turning to the problem of constructing LG configurations satisfying the constraints described above, we wish to make some remarks regarding the validity of the requirements formulated in the previous paragraph.

Even though the assumptions formulated in refs. [47][59] and reviewed above seem rather reasonable, and previous work shows that the set of such Landau–Ginzburg theories certainly is an interesting and quite extensive class of models, it is clear that it is not the most general class of (2,2) vacua. Although it provides a rather large set of different models [2], which contains many

[2] The rather extravagant values that have been mentioned in the literature as the number of possible (2,2) vacua are based on extrapolations that do not take into account the problem of overcounting that is generic to all of these different constructions.

framework.

Perhaps the simplest example that does not fit into the classification above is that of the Calabi–Yau manifolds in

$$\mathbb{P}_5[4\ \ 2], \tag{71}$$

described by the intersection of two hypersurfaces defined by a quartic and a quadratic polynomial in a five–dimensional projective space $\mathbb{P}_5$ because of the purely quadratic polynomial that appears as one of the constraints defining the hypersurface. The requirement that the singularity be completely degenerate seems, in fact, to exclude a great many of the CICY manifolds, the complete class of which was constructed in ref. [4]. An important set of manifolds in that class that does not fit into the present framework either is defined by polynomials of bidegree (1,4) and (1,1)

$$\begin{matrix}\mathbb{P}_1\\ \mathbb{P}_4\end{matrix}\begin{bmatrix}1 & 1\\ 1 & 4\end{bmatrix}. \tag{72}$$

The superpotential $W = p_1 + p_2$ is not quasihomogeneous, nor does it have an isolated singularity [4]. Thus it appears that there ought to be a generalization of the framework described above, which allows a modified LG description of these and other string vacua. This however we leave for future work.

# 5 Transversality of Catastrophes

The most explicit way of constructing a Landau–Ginzburg vacuum is, of course, to exhibit a specific potential that satisfies all the conditions imposed by the requirement that it ought to describe a consistent ground state of the string. Even though much effort has gone into the classification of singularities of the type described in the previous section, such polynomials have not been classified yet. As already mentioned above the mathematicians have classified polynomials with at most three variables [2], which is two short of the lowest number of variables that is needed in order to construct a vacuum that allows a formulation of a four–dimensional low–energy effective theory [5].

In Sections 1–3 a set of potentials in five variables was described which represents an obvious generalization of the polynomials that appear in two–dimensional catastrophes. After imposing the

[3]Such as vacua constructed tensor models based on the ADE minimal models [47][25][43][44][17][53] and level–1 Kazama–Suzuki models [39][15][52], as well as G/H LG theories [46] related to Kazama–Suzuki models of higher levels.

[4]These manifolds are important in the context of possible phase transitions between Calabi–Yau string vacua [6] via the process of splitting and contraction introduced in [4].

[5]The complete list of K3 representations embedded in $dim_{\mathbb{C}} = 3$ weighted projective space $\mathbb{P}_{(k_1,k_2,k_3,k_4)}$ has been obtained in [50].

of LG theories survive and all these solutions were constructed. It is clear that this classification of singularities is not complete, even after restricting to five variables. It is indeed easy to construct polynomials that are not contained in the classification of [8], a simple one being furnished by the example [49]

$$\mathbb{P}_{(15,10,6,1)}[45] \ni \left\{z_1(z_1^2 + z_2^3 + z_3^5) + z_4^{45} + z_2^2 z_3^4 z_4 = 0\right\}. \tag{73}$$

This polynomial is not of any of the types analysed discussed above but it is nevertheless transverse.

Knowledge of the explicit form of the potential of a LG theory is very useful information when it comes to the detailed analysis of such a model. It is however not necessary if only limited knowledge, such as the computation of the spectrum of the theory, is required. In fact the only ingredients necessary for the computation of the spectrum of a LG vacuum [58] are the anomalous dimensions of the scaling fields as well as the fact that in a configuration of weights there exists a polynomial of appropriate degree with an isolated singularity. However, it is much easier to check whether there exists such a polynomial in a configuration than to actually construct such a potential. The reason is a theorem by Bertini [26], which asserts that if a polynomial does have an isolated singularity on the base locus then, even though this potential may have worse singularities away from the base locus, there exists a deformation of the original polynomial that only admits an isolated singularity anywhere. Hence we only have to find criteria that guarantee at worst an isolated singularity on the base locus. It is precisely this problem that was addressed in the mathematics literature [14] at the same time as the explicit construction of LG vacua was started in ref. [8]. The main point of the argument in [14] is the following.

Suppose we wish to check whether a polynomial in $n$ variables $z_i$ with weights $k_i$ has an isolated singularity, i.e. whether the condition

$$dp = \sum_i \frac{\partial p}{\partial z_i} dz_i = 0 \tag{74}$$

can be solved at the origin $z_1 = \cdots = z_n = 0$. According to Bertini's theorem, the singularities of a general element in $\mathbb{C}_{(k_1,...,k_n)}[d]$ will lie on the base locus, i.e., the intersection of the hypersurface and all the components of the base locus, described by coordinate planes of dimension $k = 1, ..., n$. Let $\mathcal{P}_k$ such a $k$–plane, which we may assume to be described by setting to zero the coordinates $z_{k+1} = \cdots = z_n$. Expand the polynomials around the non–vanishing coordinates $z_1, ..., z_k$

$$p(z_1, ..., z_n) = q_0(z_1, ..., z_k) \ + \sum_{j=k+1}^{n} q_j(z_1, ..., z_k) z_j + h.o.t. \tag{75}$$

Clearly, if $q_0 \neq 0$ then $\mathcal{P}_k$ is not part of the base locus and hence the hypersurface is transverse. If on the other hand $q_j = 0$, then $\mathcal{P}_k$ is part of the base locus and singular points can occur on intersection to be empty, then the potential is smooth on the base locus.

Thus we have found that the conditions for transversality in any number of variables is the existence for any index set $\mathcal{I}_k = \{1, ..., k\}$ of

1. either a monomial $z_1^{a_1} \cdots z_k^{a_k}$ of degree $d$
2. or of at least $k$ monomials $z_1^{a_1} \cdots z_k^{a_k} z_{e_i}$ with distinct $e_i$.

Assume on the other hand that neither of these conditions holds for all index sets and let $\mathcal{I}_k$ be the subset for which they fail. Then the potential has the form

$$p(z_1, ..., z_n) = \sum_{j=k+1}^{n} q_j(z_1, ..., z_k) z_j \; + \; \cdots \tag{76}$$

with at most $k-1$ non–vanishing $q_j$. In this case the intersection of the hypersurfaces $\mathcal{H}_j$ will be positive and hence the polynomial $p$ will not be transverse.

As an example for the considerable ease with which one can check whether a given configuration allows for the existence of a potential with an isolated singularity, consider the polynomial of Orlik and Randall

$$p = z_1^3 + z_1 z_2^3 + z_1 z_3^5 + z_4^{45} + z_2^2 z_3^4 z_4. \tag{77}$$

Condition (74) is equivalent to the system of equations

$$\begin{array}{rclrcl} 0 & = & 3z_1^2 + z_2^3 + z_3^5, & 0 & = & 3z_1 z_2^2 + 2z_2 z_3^4 z_4 \\ 0 & = & 5z_1 z_3^4 + 4z_2^2 z_3^3 z_4, & 0 & = & z_2^2 z_3^4 + 45 z_4^{44}. \end{array} \tag{78}$$

which, on the base locus, collapses to the trivial pair of equations $z_2 z_3 = 0 = z_2^3 + z_5^5$. Hence this configuration allows for such a polynomial. To check the system away from the base locus clearly is much more complicated.

By adding a fifth variable $z_5$ of weight 13 it is possible to define a Calabi–Yau deformation class $\mathbb{P}_{(1,6,10,13,15)}[45]^{(17,53)}_{-72}$, a configuration not considered in [8].

# 6 Finiteness Considerations

The problem of finiteness has two parts: first one has to put a constraint on the number of scaling fields that can appear in the LG theory and then one has to determine limits on the exponents with the central charge of a Landau–Ginzburg theory with fields of charge $q_i$

$$c = 3\sum(1-2q_i) =: \sum c_i \tag{79}$$

has to be $c = 9$ in order to describe a string vacuum.

It should be clear that without any additional input the number of Landau–Ginzburg vacuum configurations that can be exhibited is infinite. This is to be expected simply because it is known from the construction of CICYs [8] that it is often possible to rewrite a manifold in an infinite number of ways and we ought to encounter similar things in the LG framework. A trivial way to do this is to simply add mass terms that do not contribute to the central charge. Even though trivial such mass terms are important and necessary for LG theories, not only in orbifold constructions [36] but also in order to relate them to CY manifolds. Consider e.g. the codimension–four Calabi–Yau manifold

$$\begin{matrix} \mathbb{P}_1 \\ \mathbb{P}_2 \\ \mathbb{P}_2 \\ \mathbb{P}_2 \end{matrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \tag{80}$$

with the defining polynomials

$$\begin{array}{rclrcl} p_1 &=& \sum_{i=1}^2 u_i^2 v_i, & p_2 &=& \sum_{i=1}^3 v_i^2 w_i \\ p_3 &=& \sum_{i=1}^3 w_i^2 x_i, & p_4 &=& \sum_{i=1}^3 x_i^2 \end{array} \tag{81}$$

the superpotential $W = \sum p_i$ of which lives in

$$\mathbb{C}^\star_{(5,5,6,6,6,4,4,4,8,8,8)}[16]^{10}_{-32} \tag{82}$$

and has an isolated singularity at the origin. All eleven variables are coupled and hence this example appears to involve three fields with zero central charge in a nontrivial way.

It turns out, however, that the manifold (80) is equivalent to a manifold with nine variables. One way to see this is by making use of some topological identities introduced in [4]. First consider the well–known isomorphism

$$\mathbb{P}_2[2] \equiv \mathbb{P}_1, \tag{83}$$

which allows us to rewrite the manifold above as

$$\begin{matrix} \mathbb{P}_1 \\ \mathbb{P}_2 \\ \mathbb{P}_2 \\ \mathbb{P}_1 \end{matrix} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}. \tag{84}$$

Using the surface identity [4]

$$\begin{matrix} \mathbb{P}_1 \\ \mathbb{P}_2 \end{matrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{matrix} \mathbb{P}_1 \\ \mathbb{P}_1 \end{matrix} \tag{85}$$

$$\begin{matrix} \mathbb{P}_1 \\ \mathbb{P}_2 \\ X \end{matrix} \begin{bmatrix} 2 & 0 \\ 1 & a \\ 0 & M \end{bmatrix} = \begin{matrix} \mathbb{P}_1 \\ \mathbb{P}_1 \\ X \end{matrix} \begin{bmatrix} a \\ a \\ M \end{bmatrix} \tag{86}$$

shows that this space is in turn equivalent to

$$\begin{matrix} \mathbb{P}_1 \\ \mathbb{P}_1 \\ \mathbb{P}_1 \\ \mathbb{P}_2 \end{matrix} \begin{bmatrix} 2 & 0 \\ 2 & 0 \\ 0 & 2 \\ 1 & 2 \end{bmatrix}, \tag{87}$$

a manifold with only nine homogeneous coordinates. It should be noted that the LG potential of (87), defined by the sum of the two polynomials, certainly does not have an isolated singularity. Furthermore it is not possible to even assign weights to the fields such that the central charge comes out to be nine! It is thus possible, by applying topological identities, to extend the applicability of Landau–Ginzburg theories to types of Calabi–Yau manifolds that were hitherto completely inaccessible by the standard formulation.

Further insight into the problem of redundancy in the construction of LG potentials can be gained by an LG theoretic analysis of this example. From the weights of the scaling variables in the LG configuration above it is clear that the spectrum of this LG configuration remains the same if the last three coordinates are set to zero. In the potential

$$W = \sum_{i=1}^{2} \left( u_i^2 v_i + v_i^2 w_i + w_i^2 x_i + x_i^2 \right) + (v_3^2 w_3 + w_3^2 x_3 + x_3^2) \tag{88}$$

described by the CICY polynomials (81), these variables cannot be set to zero because they are coupled to other fields; hence it seems impossible to reduce the number of fields. Consider however the following change of variables

$$y_i = x_i + \frac{1}{2} w_i^2, \quad i = 1, 2, 3. \tag{89}$$

It follows from these transformations that the potential defined by by (88) is equivalent to

$$W = \sum_{i=1}^{2} \left( u_i^2 v_i + v_i^2 w_i - \frac{1}{4} w_i^4 \right) + v_3^2 w_3 - \frac{1}{4} w_3^4. \tag{90}$$

Adding a trivial factor and splitting this potential into three separate polynomials

$$p_1 = u_1^2 v_1 + u_2^2 v_2, \quad p_2 = v_1^2 w_1 + v_2^2 w_2 + v_3^2 w_3, \quad p_3 = -\frac{1}{4} \left( w_1^4 + w_2^4 + w_3^4 \right) + x^2 \tag{91}$$

we see that the original model is equivalent to a weighted configuration

$$\begin{matrix} \mathbb{P}_1 \\ \mathbb{P}_2 \\ \mathbb{P}_{(1,1,1,2)} \end{matrix} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 4 \end{bmatrix}, \tag{92}$$

equivalent to two different (weighted) CICY representations

$$\begin{matrix}\mathbb{P}_1\\ \mathbb{P}_1\\ \mathbb{P}_1\\ \mathbb{P}_2\end{matrix}\begin{bmatrix}2&0\\2&0\\0&2\\1&2\end{bmatrix} \equiv \begin{matrix}\mathbb{P}_1\\ \mathbb{P}_2\\ \mathbb{P}_2\\ \mathbb{P}_2\end{matrix}\begin{bmatrix}2&0&0&0\\1&2&0&0\\0&1&2&0\\0&0&1&2\end{bmatrix} \equiv \begin{matrix}\mathbb{P}_1\\ \mathbb{P}_2\\ \mathbb{P}_{(1,1,1,2)}\end{matrix}\begin{bmatrix}2&0&0\\1&2&0\\0&1&4\end{bmatrix}. \tag{93}$$

To summarize the last few paragraphs, we have shown two things: first that by adding trivial factors and coupling them to the remaining fields we can give a Landau–Ginzburg description of a larger class of Calabi–Yau manifolds than previously thought possible. Furthermore we can use topological identities to obtain an LG formulation of CY manifolds, which do not admit a canonical LG potential at all. Incidentally we have also shown that it is possible to relate complete intersection manifolds embedded in products of projective spaces to weighted complete intersection manifolds embedded in products of weighted projective space.

Similarly the number of fields can grow without bound if we not only allow fields that do not contribute to the central charge but also fields with a negative contribution. Again such fields provide redundant descriptions of simpler LG theories; they are nevertheless important for the LG/CY relation and occur in the constructions of splitting and contraction introduced in ref. [4]. Even though these constructions were discussed in [4] only in the context of Calabi–Yau manifolds embedded in products of ordinary projective spaces they readily generalize to the more general framework of weighted projective spaces.

In special circumstances, the splitting or contraction process does not change the spectrum of the theory and hence it provides another tool to relate LG potentials with at most nine variables manifolds with more than nine homogeneous coordinates. Consider e.g. the manifold embedded in

$$\begin{matrix}\mathbb{P}_{(1,1)}\\ \mathbb{P}_{(1,1,1,1,3)}\end{matrix}\begin{bmatrix}2&0\\1&6\end{bmatrix}^{(2,128)}_{-252} \tag{94}$$

which is described by the zero locus of the two polynomials

$$\begin{aligned}p_1 &= x_1^2 y_1 + x_2^2 y_2\\ p_2 &= y_1^6 + y_2^6 + y_3^6 + y_4^6 + y_5^2\end{aligned} \tag{95}$$

that can be described by a superpotential $W = p_1 + p_2$ defining a Landau–Ginzburg theory in

$$\mathbb{C}^{\star}_{(5,5,2,2,2,2,6)}[12]^{(2,128)}_{-252}. \tag{96}$$

This manifold can be rewritten via an ineffective split in an infinite sequence of different represen-

$$\begin{matrix} \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(1,1,1,1,3)} \end{matrix} \begin{bmatrix} 2 & 0 \\ 1 & 6 \end{bmatrix} \longrightarrow \begin{matrix} \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(1,1,1,1,3)} \end{matrix} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 6 \end{bmatrix} \longrightarrow \begin{matrix} \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(1,1,1,1,3)} \end{matrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 6 \end{bmatrix} \longrightarrow \cdots \tag{97}$$

which are described by LG potentials in the alternating classes

$$\mathbb{C}^{\star 7}_{(5,5,2,2,2,2,6)}[12] \longrightarrow \mathbb{C}^{\star 9}_{(1,1,10,10,2,2,2,2,6)}[12] \longrightarrow \mathbb{C}^{\star 11}_{(5,5,2,2,10,10,2,2,2,2,6)}[12] \longrightarrow \cdots \tag{98}$$

i.e. the infinite sequence belongs to the configurations

$$\mathbb{C}^{\star 7+4k}_{(5,5,2,2,10,10,2,2,10,10,...,2,2,10,10,2,2,6)}[12] \tag{99}$$

where the part $(2,2,10,10)$ occurs $k$ times, and

$$\mathbb{C}^{\star 5+4k}_{(1,1,10,10,2,2,10,10,2,2,...10,10,2,2,2,2,6)}[12]. \tag{100}$$

where $(10,10,2,2)$ occurs $k$ times.

The construction above easily generalizes to a number of examples which all belong to a class of spaces discussed in ref. [56]. Consider manifolds embedded in

$$\begin{matrix} \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(k_1,k_1,k_3,k_4,k_5)} \end{matrix} \begin{bmatrix} 2 & 0 \\ k_1 & k \end{bmatrix}, \tag{101}$$

where $k = k_1 + k_3 + k_4 + k_5$. These spaces can be split into the infinite sequences

$$\begin{matrix} \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(k_1,k_1,k_3,k_4,k_5)} \end{matrix} \begin{bmatrix} 2 & 0 \\ k_1 & k \end{bmatrix} \longrightarrow \begin{matrix} \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(k_1,k_1,k_3,k_4,k_5)} \end{matrix} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & k_1 & k \end{bmatrix} \longrightarrow \begin{matrix} \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(k_1,k_1,k_3,k_4,k_5)} \end{matrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & k_1 & k \end{bmatrix} \longrightarrow \cdots \tag{102}$$

If the weights are such that $k/k_i$ is an integer, then it is easy to write down the tensor model that corresponds to it (but this is not essential). In such models the levels $l_i$ of the tensor model $l_1^2 \cdot l_3 \cdot l_4 \cdot l_5$ in terms of the weights are given by

$$l_1 = l_2 = \frac{2k}{k_1} - 2, \quad l_i = \frac{k}{k_i} - 2, \; i = 3,4,5 \tag{103}$$

and the corresponding LG potentials live in

$$\mathbb{C}^{\star 7}_{(k-k_1,k-k_1,2k_1,2k_1,2k_3,2k_4,2k_5)}[2k] \longrightarrow \mathbb{C}^{\star 9}_{(k_1,k_1,2(k-k_1),2(k-k_1),2k_1,2k_1,2k_3,2k_4,2k_5)}[2k] \longrightarrow \cdots \tag{104}$$

i.e. they belong to the sequences

$$\mathbb{C}^{\star 7+4p}_{((k-k_1),(k-k_1),2k_1,2k_1,2(k-k_1),2(k-k_1),....,2k_1,2k_1,2k_3,2k_4,2k_5)}[2k] \tag{105}$$

$$\mathbb{C}^{\star 9+4k}_{(k_1,k_1,2(k-k_1),2(k-k_1),2k_1,2k_1,\ldots,2k_3,2k_4,2k_5)}[2k]. \tag{106}$$

where $(2(k-k_1), 2(k-k_1), 2k_1, 2k_1)$ occurs $p$ times.

All these models are constructed in such a way that they have central charge nine, but in contrast to the example discussed previously, there now appear fields with negative central charge. In the case at hand, however, these dangerous fields only occur in a *coupled* subpart of the theory; the smallest subsystem which involves these fields and which one can isolate is in fact a theory with positive central charge. In the series of splits just described e.g., the fields with negative central charge that occur in the first split always appear in the subsystem described by the configurations

$$\mathbb{C}^{\star}_{(k_1,2(k-k_1),2k_1)}[2k] \tag{107}$$

with potentials of the form

$$x^2y + yz + z^{2k/k_1}. \tag{108}$$

Thus the contribution to the central charge of this sector becomes

$$c = 3\left[\left(1-\frac{k_1}{k}\right) + \left(1-\frac{2(k-k_1)}{k}\right) + \left(1-\frac{2k_1}{k}\right)\right], \tag{109}$$

which is always positive. This formula suggests that it ought to be possible to dispense with the variables $y, z$ altogether, as their total contribution to the central charge adds up to zero, and that this theory is equivalent to that of a single monomial of degree $k/k_1$.

More generally one may consider the Landau–Ginzburg theory defined by the potential

$$x_1^a x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_n^b. \tag{110}$$

From the central charge

$$c = \left\{ \begin{array}{ll} 6\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right), & n \text{ even} \\ 3\left(1-\frac{2}{ab}\right), & n \text{ odd} \end{array} \right\}, \tag{111}$$

as well as from the dimension of the chiral ring

$$\dim R_n = \prod\left(\frac{1}{q_i}-1\right) = \left\{ \begin{array}{ll} ab-b-1, & n \text{ even} \\ ab-1, & n \text{ odd} \end{array} \right\}, \tag{112}$$

we expect this theory to be equivalent to

$$x_1^a x_2 + x_2x_3 + \cdots x_{n-1}x_n + x_n^b = \left\{ \begin{array}{ll} x_1^a x_n + x_n^b, & n \text{ even} \\ x_1^{ab}, & n \text{ odd} \end{array} \right\}. \tag{113}$$

$$\mathbb{C}^\star_{(1,1,1,6,6,6,3,3,3,3,3,3)}[9] \equiv \mathbb{C}^\star_{(1,1,1,3,3)}[9] \qquad (114)$$

as well as many other identifications which are rather nontrivial in the context of the associated manifolds.

It follows from the above considerations that we have to assume, in order to avoid redundant reconstructions of LG theories, that the central charge of all scaling fields of the potential should be positive. In order to relate the potentials to manifolds, we may then add one or several trivial factors or more complicated theories with zero central charge.

Using the above results we derive in the following more detailed finiteness conditions. Observe first that from eq. (79) written as

$$\sum_{i=1}^{r} q_i = \left(\frac{r}{2} - \frac{c}{6}\right) := \hat{c} \qquad (115)$$

we obtain $r > c/3$.

Now let $p$ be a polynomial of degree $d$ in $r$ variables. For the index set $\mathcal{I}_1$ the conditions for transversality imply the existence of $n_i \in \mathbb{N}^+$, $i = 1, \ldots, r$ and of a map $j : \mathcal{I}_r \rightarrow \mathcal{I}_r$ such that for all $i$ there exists $j(i)$ such that

$$q_i = \frac{1 - q_{j(i)}}{n_i}. \qquad (116)$$

Let us first see how many non-trivial fields can occur at most. Fields which have charge $q_i \leq 1/3$ contribute $c_i \geq 1$ to the conformal anomaly. Now consider fields with larger charge. Since we assume $c_i > 0$, they are in the range $1/3 < q_i < 1/2$. Among these fields the transversality condition (1.) cannot hold, because two of them are not enough and three of them are too many fields in order to form a monomial of charge one. Transversality condition (2.) implies that each of them has to occur together with a partner field $z_{j(i)}$. These pairs contribute to the conformal anomaly according to (116,79) $c_i + c_{j(i)} > 2$, so we can conclude that $r \leq c$.

In order to construct all transversal LG potentials for a given $c$, we choose a specific $r$ in the range obtained above and consider all possible maps $j$ of which there are $r^r$. Without restricting on the generality, we may *then* assume the $n_i$ to be ordered $n_1 \leq \cdots \leq n_r$. Starting with (115) we obtain via eq. (116) and the positivity of the charges a bound $n_1 < r/c$. Now we choose $n_1$ in the allowed range and use (116) in order to eliminate the $q_1$, if necessary in favour of the $q_2, \ldots, q_r$. This yields an equation of the general form ($p = 2$)

$$\sum_{i=p}^{r} w_i^{(p)} q_i = \hat{c}^{(p)}. \qquad (117)$$

a finite bound for $n_p$. Assume $\hat{c}^{(p)} > 0$ and let $\mathcal{I}_+$ be the indices of the positive $w_i^{(p)}$; then one has $n_p < (\sum_{i\in\mathcal{I}_+} w_i^{(p)})/\hat{c}^{(p)}$; likewise if $\hat{c}^{(p)} < 0$ we have $n_p < (\sum_{i\in\mathcal{I}_-} w_i^{(p)})/\hat{c}^{(p)}$.

Consider the case $\hat{c}^{(p)} = 0$. If the $w_i^{(p)}$ are indefinite we get no bound from (117). However we will show that the existence of certain monomials, which are required by the transversality conditions, implies a bound for $n_p$. Let $\mathcal{I}_a$ denote the indices of the already bounded $n_i$ and $\mathcal{I}_b$ the others. The charge of the field $z_a$ with $a \in \mathcal{I}_a$ will depend on the unknown charge of a field $z_{b(a)}$ with $b \in \mathcal{I}_b$ if there is a chain of indices $a_0 = a, a_1 = j(a), \ldots, a_l = j(\ldots j(a) \ldots) =: b(a)$ linked by the map $j$. The charge of $z_a$ is given by

$$q_a = \frac{1}{n_a} - \frac{1}{n_a n_{a_1}} + \cdots - \frac{(-1)^l}{n_a \ldots n_{l-1}} + \frac{(-1)^l q_{b(a)}}{n_a \ldots n_{l-1}}. \tag{118}$$

Indefiniteness of the $w_i^{(p)}$ can only occur if there are fields $z_a$, $a \in \mathcal{I}_a$, with odd $l$, i.e. the last term in (118) $s_a q_{b(a)} := (-1)^l/(n_a \ldots n_{a_{l-1}} q_{b(a)})$ is negative. Call the index set of these fields $\mathcal{I}_a^-$. Assume first that the transversality condition (1.) holds. This implies the existence of $m_i \in \mathbb{N}^+$ ($m_i < 2n_i$) such that $\sum_{i\in\mathcal{I}_a^-} m_i q_i = 1$. For the unknown $q_i$, $i \in \mathcal{I}_b$, we get an equation of the form $\sum w_i q_i = \epsilon$, which yields a bound for the lowest $n_i$, $i \in \mathcal{I}_b$, since $w_i > 0$. The lowest possible value for $\epsilon > 0$ can be readily calculated from the denominators occurring in (118). If transversality condition (2.) applies, we have $|\mathcal{I}_a^-|$ equations of the form $\sum_{i\in\mathcal{I}_a^-} m_i^{(j)} q_i = 1 - q_{e_j}$ which can be rewritten as $\sum_{i\in\mathcal{I}_b} w_i^{(j)} q_i = \epsilon^{(j)}$. Only if all $w_i^{(j)}$ happen to be indefinite and all $\epsilon^{(j)}$ are zero we get *no bound* from this condition. Assuming this to be true we have

$$\sum_{i\in\mathcal{I}_b} m^{(j)} s_i q_i - s_{e_j} q_{b(e_j)} = 0, \tag{119}$$

where $s_{e_j} := 1$ and $b(e_j) := b$ if $e_j \in \mathcal{I}_b$. Note that $\sum_i m_i^{(j)} \geq 2$ in order to avoid quadratic mass terms. Now we can rewrite (117) in the form

$$\sum s_i q_{b(i)} = 0. \tag{120}$$

If one uses now (119) and $\sum_i m_i^{(j)} \geq 2$ in order to eliminate the negative $s_i$, one finds $\sum_i w_i q_i \leq 0$ with $w_i > 0$, which is in contradiction with the positivity of the charges, hence we get a bound in any case.

This procedure of restricting the bound for $n_p$, given $n_i, \ldots, n_{p-1}$, was implemented in a computer program. It allows all configurations to be found without testing unnecessarily many combinations of the $n_i$. The actual upper bounds for the $(n_i, \ldots, n_r)$ in the four–variable case are $(7, 17, 83, 1805)$, and we have found 2390 configurations which allow for transversal polynomials. In trivial mass term $z_5^2$ in the four–variable case, the configurations mentioned so far lead to three–dimensional Calabi–Yau manifolds described by a one polynomial constraint in a four–dimensional weighted projective space.

The same figures for the six–variable case and seven–variable case are $(3,3,5,11,41,482)$, 2567 and $(2,2,3,4,8,26,242)$, 669 respectively, leading to a total of 3236 combinations. Likewise for eight– and nine–variable potentials the bounds become $(2,2,2,2,2,2,3,3,5,14)$ with 47 examples and $(2,2,2,2,2,2,2,2,2)$ with 1 example respectively. The lists with all models can be found in [34].

# 7 Results and Comparisons

We have constructed 10,839 Landau–Ginzburg theories with 2997 different spectra, i.e. pairs of generations and antigenerations. The massless spectrum is very rough information about a theory and it is likely that the degeneracy is lifted to a large degree when additional information, such as the number of singlets and/or the Yukawa couplings, becomes available. We expect the situation to be very similar to the class of CICYs [4], which only leads to some 250 different spectra [23], but for which a detailed analysis of the Yukawa couplings [7] shows that it contains several thousand distinct theories.

It is clear however that there is in fact some redundancy in this class of Landau–Ginzburg theories even beyond the one discussed in the previous sections. In the list there appear, for instance, two theories with spectrum $(h^{(1,1)}, h^{(2,1)}, \chi) = (2, 122, -240)$ involving five variables

$$\mathbb{P}_{(2,2,2,1,7)}[14] \ni \{z_1^7 + z_2^7 + z_3^7 + z_4^{14} + z_5^2 = 0\} \tag{121}$$

and

$$\mathbb{P}_{(1,1,1,1,3)}[7] \ni \{y_1^7 + y_2^7 + y_3^7 + y_4^7 + y_4 y_5^2 = 0\}. \tag{122}$$

Using the fractional transformations introduced in ref. [45] it is easy to show that these two models are equivalent, even though this is not obvious by just looking at the potentials. To see this consider first the orbifold

$$\mathbb{P}_{(2,2,2,1,7)}[14]/\mathbb{Z}_2 : [\ 0\ 0\ 0\ 1\ 1] \tag{123}$$

of the first model where the action indicated by $\mathbb{Z}_2 :\ [0\,0\,0\,1\,1]$ means that the first three coordinates remain invariant, whereas the latter two variables are to be multiplied with the generator of the cyclic $\mathbb{Z}_2$, $\alpha = -1$. Since the action of this $\mathbb{Z}_2$ on the weighted projective space is part of the hand the fractional transformation

$$z_i = y_i,\ i = 1, 2, 3; \quad z_4 = y_4^{1/2}, \quad z_5 = y_4^{1/2} y_5 \tag{124}$$

associated with this symmetry [45] defines a 1–1 coordinate transformation on the orbifold, which transforms the first theory into the second; these are therefore equivalent as well.

Similarly the equivalences

$$\begin{aligned} \mathbb{P}_{(2,2,2,3,9)}[18]^{(4,112)}_{-216} &= \mathbb{P}_{(1,1,1,3,3)}[9] \\ \mathbb{P}_{(2,6,6,7,21)}[42]^{(17,65)}_{-96} &= \mathbb{P}_{(1,3,3,7,7)}[21] \\ \mathbb{P}_{(2,5,14,14,35)}[70]^{(27,59)}_{-64} &= \mathbb{P}_{(1,5,7,7,15)}[35] \end{aligned} \tag{125}$$

can be shown, as well as a number of others.

It should be noted that even though we now have constructed Landau–Ginzburg potentials with an arbitrary number of scaling fields, the basic range of the spectrum has not changed as compared with the results of [8] where it was found that the spectra of all 6000 odd theories constructed there lead to Euler numbers which fall in the range [6]

$$-960 \leq \chi \leq 960. \tag{126}$$

In fact not only do all the LG spectra fall into this range, all known Calabi–Yau spectra and all the spectra from exactly solvable tensor models are contained in this range as well! This suggests that perhaps the spectra of all string vacua based on $c = 9$ will be found within this range. To put it differently, we conjecture that the Euler numbers of all Calabi–Yau manifolds are contained in the range $-960 \leq \chi \leq 960$.

Similarly to the results in [8], the Hodge pairs do not pair up completely. In fact the mirror symmetry of the space of Landau–Ginzburg vacua is just about 77%. It thus appears that orbifolding is an essential ingredient in the construction of a mirror–symmetric slice of the configuration space of the heterotic string. It is in fact easy to produce examples of orbifolds whose spectrum does not appear in our list of LG vacua. An example of a mirror pair is furnished by the orbifold

$$\mathbb{P}_{(1,1,1,1,2)}[6]^{(1,103)}_{-204} / \mathbb{Z}_6 : [\,3 \quad 2 \quad 1 \quad 0 \quad 0\,] \tag{127}$$

which has the spectrum $(11, 23, -24)$ and the space

$$\mathbb{P}_{(1,1,1,1,2)}[6]^{(1,103)}_{-204} / \mathbb{Z}_6 \times \mathbb{Z}_3 : \begin{bmatrix} 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \end{bmatrix} \tag{128}$$

[6] This should be compared with the result for the complete intersection Calabi–Yau manifolds where $-200 \leq \chi \leq 0$ [4]. See Figure 1.

structed so far [36] is indeed much closer to being mirror–symmetric than the space of LG theories itself. Even though the construction in [36] is incomplete, about 94% of the Hodge numbers pair up.

By now there exist many different prescriptions to construct string vacua and we would like to put the LG framework into the context of other left–right–symmetric constructions. Among the more prominent ones other than Calabi–Yau manifolds, which are obviously closely related to LG theories, there are constructions which have traditionally been called, somewhat misleadingly, orbifolds (what is meant are orbifolds of tori) [11], free–fermion constructions [31], lattice constructions [40], and interacting exactly solvable models [21][32].

None of these classes are known completely, even though much effort has gone into the exploration of some of them. Because powerful computational tools are available, toroidal orbifolds have been analysed in some detail [12][27] and much attention has focused on the explicit construction of exactly solvable theories in the context of tensor models via $N = 2$ superconformal minimal theories [43][44][17] and Kazama–Suzuki models [15][39][46][52][19] [1] as well as their in depth analysis [39][55][20][9]. In the context of (2,2) vacua, orbifolds lead only to some tens of distinct models, whereas the known classes of exactly solvable theories lead only to a few hundred models with distinct spectra. Similar results have been obtained so far via the covariant–lattice approach [41] and hence it is obvious that these constructions do not exhaust the configuration space of heterotic string by far. The class of Landau–Ginzburg string vacua thus appears as a rather extensive source of (2,2)–symmetric models.

Finally we should remark that the (2,2) Landau–Ginzburg theories we have constructed here can be used to build a probably much larger class of (2,0) models along the lines described in [10], using an appropriate adaptation of the work of [5][48] [35][16] in order to determine the instantons on which the existence of a certain split of a vector bundle has to be checked.

# Mirror Pairs via Fractional Transformations.

Even though the emerging mirror symmetry of the construction discussed in Part I is clearly very suggestive there is a priori no reason why models with mirror flipped spectra should be related at all. After all, Hodge numbers do not classify manifolds topologically and knowing the spectrum of a physical theory is only a very first step in the analysis of its properties. It turns out that the detailed analysis of the models rests both mathematically and physically on the computation of the Yukawa couplings. This, in general, is a difficult business. It is therefore gratifying that it is possible to proceed via a different line of thought to relate different models, namely via a direct map between different theories. In particular cases this rather general map turns out to be the mirror map.

## 8 From A–Models to D–Models via Orbit Construction

It is well known from the ADE–classification of partition functions of ($N=2$)minimal models that the D–type models are equivalent to A–type models modded out by a $\mathbb{Z}_2$ discrete symmetry. As a consequence subsets of vacua of the Heterotic String involving the $N=2$ minimal series are also related via orbifold constructions [3]. In terms of the corresponding Landau-Ginzburg potentials of the exact models this can be seen as follows. The potential corresponding to a model at level $k$ with a diagonal affine invariant is [30]

$$\mathrm{A}_{k+1} \quad \sim \quad \Phi_1^{k+2} + \Phi_2^2 \tag{129}$$

where, with hindsight, a trivial factor has been added. The LG potential corresponding to the exact model at the same level [4] but with the D–invariant is described by [47][59]

$$\mathrm{D}_{\frac{k}{2}+2} \quad \sim \quad \tilde{\Phi}_1^{(k+2)/2} + \tilde{\Phi}_1\tilde{\Phi}_2^2. \tag{130}$$

By defining the transformation of the scaling variables

$$\Phi_1 = \tilde{\Phi}_1^{1/2} \quad \text{and} \quad \Phi_2 = \tilde{\Phi}_1^{1/2}\tilde{\Phi}_2 \tag{131}$$

[3]In this article orbifold construction will be understood to include twisted modes in the conformal field theory or Landau–Ginzburg construction and blow–up modes in the geometric formulation.

[4]The D–models only exist for even level $k=2n$.

transformation has a constant Jacobian and therefore one might naively expect that they are in fact equivalent descriptions of a given theory since their path integrals are equivalent. It is clear however that this is not the case as the two theories have a different spectrum: the local ring of the diagonal theory consists of states

$$\mathcal{R}_{\mathrm{A}_{k+1}} = \{1, \Phi_1, \Phi_1^2, \cdots, \Phi_1^k\} \tag{132}$$

i.e. the theory has $k+1$ states. The D–theory however has only $\frac{k}{2}+2$ states in its spectrum

$$\mathcal{R}_{\mathrm{D}_{\frac{k}{2}+2}} = \{1, \tilde{\Phi}_2, \tilde{\Phi}_1, \cdots, \tilde{\Phi}_1^{\frac{k}{2}}\}. \tag{133}$$

Hence these two theories are not the same. The resolution of this puzzle comes from the fact that the transformation of (131) is not a coordinate transformation since it is not a bijection but is 2–1 as it stands. To make it 1–1 we should identify

$$\Phi_i \sim -\Phi_i, \quad i = 1, 2. \tag{134}$$

Of the $k+1$ states of $\mathrm{A}_{k+1}$–models $\frac{k}{2}+1$ are invariant with respect to the action of this $\mathbb{Z}_2$. By including the one twisted state we find precisely the states of the non–diagonal D–theory.

This construction can immediately be applied to string compactification proper. It may seem unlikely at first that the simple modding of a $\mathbb{Z}_2$ should account for the different behaviour that the various tensor models exhibit under the exchange of a diagonal invariant by a D–invariant. A quick look at the results of refs. [44][17] shows that this exchange generically changes the spectrum in some arbitrary way without any obvious systematics. However in some special cases the spectrum is flipped, exchanging generations and anti–generations while in other models the spectrum does not change at all. The reason for this erratic behaviour can be traced to the fact that the other coordinates involved in the construction determine the fixed point structure of the discrete group in an essential way.

Consider the following example where the exchange of the affine invariant does not change the spectrum [5]

$$\mathbb{P}_{(1,7,2,2,2)}[14] \quad \ni \quad (12\cdot 5^3)_{A_{13}\otimes A_6^3} \quad \longrightarrow \quad (12\cdot 5^3)_{D_8\otimes A_6^3} \quad \in \quad \mathbb{P}_{(1,3,1,1,1)}[7]. \tag{135}$$

where we have added one trivial factor [6]. The reason that the spectrum does not change after the replacement of the diagonal affine invariant by the D–invariant is rather obscure from the point

[5] The notation of ref. [44] is used.

[6] An explanation for the necessity of this trivial factor and when it is to be added can be found in [44].

invariants but can be understood easily from the orbifold point of view. In the manifold picture it simply follows from the fact that the $\mathbb{Z}_2$–action $[1,1,0,0,0]$, generated by

$$(\Phi_1,\Phi_2,\Phi_3,\Phi_4,\Phi_5) \longrightarrow (\alpha\Phi_1,\alpha\Phi_2,\Phi_3,\Phi_4,\Phi_5), \tag{136}$$

(where $\alpha^2=1, \alpha\neq 1$) is not an action at all on the Calabi–Yau manifold embedded in the ambient weighted projective space but it is part of the projective equivalence transformation because we have

$$\mathbb{P}_{(1,7,2,2,2)} = \mathbb{C}_5/\sim \tag{137}$$

with $(\Phi_1,\Phi_2,\Phi_3,\Phi_4,\Phi_5) \sim (\lambda\Phi_1,\lambda^7\Phi_2,\lambda^2\Phi_3,\lambda^2\Phi_4,\lambda^2\Phi_5)$, where $\lambda\in\mathbb{C}^*$. Therefore the $\mathbb{Z}_2$ does not transform the projective coordinates at all but is trivial. In the conformal field theory picture this translates into the fact that the action is actually part of the modding that has to be done to implement the GSO projection. This example shows that not all our models are independent but that some models appear in different representations.

A more complicated example is furnished by the pair of minimal models

$$\mathbb{P}_{(1,5,1,1,2)}[10]^1_{-288} \quad = \quad (8^3\cdot 3)_{A_9^3\otimes A_4} \longrightarrow (8^3\cdot 3)_{D_6\otimes A_9^2\otimes A_4} \quad = \quad \mathbb{P}_{(2,4,1,1,2)}[10]^3_{-192}. \tag{138}$$

The superscript denotes the dimension of the second cohomology group $h^{(1,1)} = \dim \mathrm{H}^{(1,1)}(\mathcal{M})$ whereas the subscript denotes the Euler number. It is again not obvious from the point of view of the affine invariants involved why the spectrum changes the way it does but the result is clear from the orbifold construction. Defining the $\mathbb{Z}_2$–action as $[1,1,0,0,0]$ it follows that in this case the map is nontrivial. Its fixed point set consists of two curves $C_1=\mathbb{P}_{(1,1,2)}[10]$ with Euler number $\chi_{C_1}=-30$ and $C_2=\mathbb{P}_{(1,5,2)}[10]$ with $\chi_{C_2}=-2$. The Euler number of the resolved orbit manifold is therefore

$$\chi = -\frac{288}{2} - \frac{(-30-2)}{2} + 2(-30-2) = -192. \tag{139}$$

The cohomology can be computed as well. Given the Euler number we only need to compute either the second cohomology group or the number of (2,1)–forms. Using the results of [56] it is clear that each of the curves contributes one additional generator to the second cohomology, i.e. for the resolved manifold we have $h^{(1,1)}=3$ and therefore the result of (138).

These simple identifications of a class of Landau-Ginzburg potentials with orbifolds of other potentials via fractional transformations can be considered as a generalization of the strange duality known from the exceptional singularities of modality one [2]. One of the 'strangely dual singularity at the origin is described by the two polynomials

$$K_{14} \quad : \quad \mathbb{P}_{(3,12,8)}[24] \ \ni \ \{\Phi_1^8 + \Phi_2^2 + \Phi_3^3 = 0\} \tag{140}$$

$$Q_{10} \quad : \quad \mathbb{P}_{(6,9,8)}[24] \ \ni \ \{\tilde{\Phi}_1^4 + \tilde{\Phi}_1\tilde{\Phi}_2^2 + \tilde{\Phi}_3^3 = 0\}. \tag{141}$$

The dimension of the chiral ring of these singularities is 10 and 14, respectively, as indicated by the subscript of $Q$ and $K$. Each of these catastrophes is characterized by two triplets of numbers, the Dolgachev numbers and the Gabrielov numbers. For the above singularity $Q_{10}$ the corresponding pair of triplets is $\mathcal{D}(Q_{10}) = (2,3,9)$ and $\mathcal{G}(Q_{10}) = (3,3,4)$ for the Dolgachev and Gabrielov numbers respectively whereas the singularity $K_{14}$ leads to the triplets $(3,3,4)$ and $(2,3,9)$ For the above pair $\mathcal{D}(Q_{10}) = \mathcal{G}(K_{14})$ and $\mathcal{G}(Q_{10}) = \mathcal{D}(K_{14})$. It is in this sense that the two polynomials are called dual [7].

From our previous results it is clear that an alternative way to relate these two singularities flows from the description of affine D–invariants as orbifolds of diagonal invariants. The above polynomials can be viewed as the Landau–Ginzburg potentials of the tensor product $(1 \cdot 6)$

$$Q_{10} \sim (1 \cdot 6)_{A_2 \otimes D_5} \quad \text{and} \quad K_{14} \sim (1 \cdot 6)_{A_2 \otimes A_7} \tag{142}$$

where we have added a trivial factor for the $K_{14}$ singularity. Therefore it follows that

$$Q_{10} = K_{14}/\mathbb{Z}_2. \tag{143}$$

More explicitly consider the chiral ring of $K_{14}$

$$\mathcal{R}_K = \{1, \Phi_1, \Phi_1^2, ..., \Phi_1^6, \Phi_3, \Phi_1\Phi_3, ..., \Phi_1^6\Phi_3\}. \tag{144}$$

The action of the $\mathbb{Z}_2$ is defined as $[1,0,1]$ on the fields $(\Phi_1, \Phi_2, \Phi_3)$. The two sectors of the orbifold consist of the eight invariant states of $\mathcal{R}_K$ and of two twisted states giving the total of ten states necessary for $Q$.

This strangely dual pair can again be used as building block of string compactifications. In the models constructed in [44][17] this involves vacua which have $(1 \cdot 6)_{A_1 \otimes A_7}$ or $(1 \cdot 6)_{A_1 \otimes D_5}$ as subfactors.

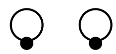
The ADE models make up only a very small part of the string compactifications constructed in [8] and it is natural to ask whether it is possible to relate spaces in this set to one another in a similar way and if so whether it is possible to construct *all* of these models as orbifolds of some basic

[7] The precise nature of these numbers are not of concern here. More details can be found in ref. [2]

in the following section.

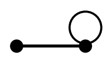
## 9 Fractional Transformations

In the notation of [8] the transition from the diagonal affine invariant to the non–diagonal D–invariant can be formulated as the transition from a Fermat type polynomial with diagram



$$\Phi_1^a + \Phi_2^b \tag{145}$$

to particular polynomials of the type of a tadpole



$$\tilde{\Phi}_1^c \tilde{\Phi}_2 + \tilde{\Phi}_2^d \tag{146}$$

Equations (129,130) clearly describe only a very small subset of these polynomials and one can ask whether transformations generalizing (131) are possible. Indeed they are.

Consider the transformation

$$\Phi_1 = \tilde{\Phi}_1^{c/a} \tilde{\Phi}_2^{1/a}, \qquad \Phi_2 = \tilde{\Phi}_2^{d/b} \tag{147}$$

which transforms the polynomial in (145) into the one of eq. (146). The next step is to find the constraints on the weights of the fields that make the Jacobian of this transformation constant. They are given by

$$c = a \quad \text{and} \quad d = b\left[1 - \frac{1}{a}\right]. \tag{148}$$

As in the case of the previous transformation however this transformation is not well defined yet and we have to find the discrete group which makes the transformation of variables well defined. Suppose that we have a map

$$\Phi_1 \longrightarrow \alpha\Phi_1 \quad \text{and} \quad \Phi_2 \longrightarrow \beta\Phi_2 \tag{149}$$

where $\alpha, \beta$ are roots of unity. The condition that the change of variables in eq. (147) is invariant determines $\alpha$ as the generator of $\mathbb{Z}_a$ and $\beta = \alpha^{a-1}$. Similarly observation regarding the inverse transformation leads to the group that one has to mod out by in the nondiagonal theory. Only after modding out these cyclic groups does transformation (147) become well defined.

The isomorphism can be summarized concisely with the following diagram

$$\mathbb{C}_{\left(\frac{b}{g_{ab}},\frac{a}{g_{ab}}\right)}\left[\frac{ab}{g_{ab}}\right] \ni \left\{z_1^a + z_2^b = 0\right\} \Big/ \mathbb{Z}_b : [\,(b-1) \quad 1\,]$$

$$\sim \mathbb{C}_{\left(\frac{b^2}{h_{ab}},\frac{a(b-1)-b}{h_{ab}}\right)}\left[\frac{ab(b-1)}{h_{ab}}\right] \ni \left\{y_1^{a(b-1)/b} + y_1 y_2^b = 0\right\} \Big/ \mathbb{Z}_{b-1} : [\,1 \quad (b-2)\,]\,. \tag{150}$$

Here $g_{ab}$ is the greatest common divisor of $a$ and $b$ and $h_{ab}$ is the greatest common divisor of $b^2$ and $(ab - a - b)$. The action of a cyclic group $\mathbb{Z}_b$ of order $b$ denoted by $[m \quad n]$ indicates that the symmetry acts like $(z_1, z_2) \mapsto (\alpha^m z_1, \alpha^n z_2)$ where $\alpha$ is the $b^{th}$ root of unity.

It is easy to generalize this analysis to general transformations from Fermat–type polynomials to tadpole–type polynomials with $N$ points attached. Consider the transformation

$$\sum_{i=1}^{N} \Phi_i^{l_i} \longrightarrow \left(\sum_{j=1}^{N-1} \tilde{\Phi}_j^{n_j}\tilde{\Phi}_{j+1}\right) + \tilde{\Phi}_N^{n_N} \tag{151}$$

defined by

$$\begin{aligned}\Phi_i &= \tilde{\Phi}_i^{n_i/l_i}\tilde{\Phi}_{i+1}^{1/l_i} \qquad\qquad i = 1, .., N-1;\\ \Phi_N &= \tilde{\Phi}_N^{n_N/l_n}.\end{aligned} \tag{152}$$

The condition that the Jacobian of this transformation is constant again leads to constraints on the exponents:

$$n_1 = l_1; \tag{153}$$

$$n_i = l_i\left[1 - \frac{1}{l_{i-1}}\right]\,,\ i = 2, ..., N. \tag{154}$$

Transformation (152) then becomes

$$\Phi_1 = \tilde{\Phi}_1\tilde{\Phi}_2^{1/l_1}; \tag{155}$$

$$\Phi_i = \tilde{\Phi}_i^{(l_{i-1}-1)/l_{i-1}}\tilde{\Phi}_{i+1}^{1/l_i} \qquad\qquad i = 2, .., N-1; \tag{156}$$

$$\Phi_N = \tilde{\Phi}_N^{(l_{N-1}-1)/l_{N-1}}. \tag{157}$$

As before, this transformation is not well defined, but one can find a discrete group under which it is invariant. Let

$$\Phi_i \longrightarrow \alpha_i\Phi_i; \tag{158}$$

it then follows that under this transformation

$$\tilde{\Phi}_N \longrightarrow \alpha_N^{l_{N-1}/(l_{N-1}-1)}\tilde{\Phi}_N, \tag{159}$$

$$\alpha_N^{l_{N-1}/(l_{N-1}-1)} = 1 \tag{160}$$

for the generator $\alpha_N$. Plugging the solutions of this constraint iteratively into eqs. (157) one finds that all generators $\alpha_i$ have to satisfy $\alpha_i^{l_i} = 1$ , $i = 1, .., N-1$ for the transformation of variables to be invariant under the identification (158). The group by which one has to mod out is $\prod_{i=1}^{N-1} \mathbb{Z}_{l_i}$. Again one has to similarly analyze the inverse map. It is obvious that this generalization just amounts to repeated application of the original isomorphism (150).

Consider e.g. the manifold

$$\mathbb{P}_{(1,3,3,3,5)}[15]^3_{-144} = \{\Phi_1^{15} + \Phi_2^5 + \Phi_3^5 + \Phi_4^5 + \Phi_5^3 = 0\}. \tag{161}$$

We will show now that the orbifold of this model with respect to two cyclic groups $\mathbb{Z}_5$, the first generated by $[0,4,1,0,0]$ the second one by $\mathbb{Z}_5 : [0,0,4,1,0]$, is isomorphic to the manifold

$$\mathbb{P}_{(16,60,45,39,80)}[240]^{75}_{144} \;\; = \{\tilde{\Phi}_1^{15} + \tilde{\Phi}_2^4 + \tilde{\Phi}_2\tilde{\Phi}_3^4 + \tilde{\Phi}_3\tilde{\Phi}_4^5 + \tilde{\Phi}_5^3 = 0\}. \tag{162}$$

Consider the first $\mathbb{Z}_5$. The fixed points of this action are determined by the requirement that

$$(\Phi_1, \alpha^4\Phi_2, \alpha\Phi_3, \Phi_4, \Phi_5) = c(\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5) \tag{163}$$

where $c \in \mathbb{C}^*$. The fixed point set consists of one curve $\mathbb{P}_{(1,3,5)}[15]$ with Euler number $\chi = -6$ and one further fixed point $\mathbb{P}_{(3,5)}[15]$. The Euler number of the resolved orbifold therefore is $\chi = \frac{-144}{5} - \frac{1}{5}(-6+1) + 5(-6+1) - \frac{1}{5} + 5 = -48$. The corresponding weighted CY–manifold of this space is described by

$$\mathbb{P}_{(4,15,9,12,20)}[60] \;\; = \;\; \{\phi_1^{15} + \phi_2^4 + \phi_2\phi_3^5 + \phi_4^5 + \phi_5^3 = 0\}. \tag{164}$$

and the coordinate transformation is defined as follows

$$(\Phi_1, \Phi_2, ..., \Phi_5) = (\phi_1, \phi_2^{4/5}, \phi_2^{1/5}\phi_3, \phi_4, \phi_5). \tag{165}$$

Modding out by the other $\mathbb{Z}_5$ leads to a fixed point set of three curves

$$\mathbb{P}_{(4,15,20)}[60]_{15}, \quad \mathbb{P}_{(15,9,20)}[60]_{16}, \quad \mathbb{P}_{(15,12,20)}[60]_{11}. \tag{166}$$

These three curves intersect in the point $\mathbb{P}_{(15,20)}[60] = \mathbb{P}_{(3,4)}[12]$ and the Euler number of the resolved manifold is given by

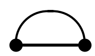
$$\chi = -\frac{48}{5} - \frac{1}{5}\,(15 + 11 + 16 - 2 \times 5) + 5\,(15 + 11 + 16 - 2 \times 5) = 144. \tag{167}$$

when in fact we only should have done so once.

We have thus established that the discrete groups of the LG–CY theories play a role comparable to the discrete symmetries of the fusion rules in conformal field theories even though for most of the models at hand the corresponding exact theory is not known. We have seen that in general there are constraints on the weights of the potentials that can be identified with with orbifolds of other models. Nevertheless, in the types of potentials discussed above there was enough freedom to have nontrivial identifications. This is not true in general as we will show in the next section.

# 10 Loop Potentials

Consider potentials of the type



$$\tilde{\Phi}_1^c \tilde{\Phi}_2 + \tilde{\Phi}_2^d \tilde{\Phi}_1 \tag{168}$$

and more general polynomials of this type with an arbitrary number of fields. For these polynomials it is possible to find a 1–1 coordinate transformation from Fermat–type polynomials, but the condition that the Jacobian is constant leads to the conclusion that only deformations of the original space can be obtained in this way.

The transformation of a Fermat type polynomial into a loop–type polynomial

$$\sum_{i=1}^{N} \Phi_i^{l_i} \longrightarrow \left( \sum_{i=1}^{N-1} \tilde{\Phi}_i^{n_i} \tilde{\Phi}_{i+1} \right) + \tilde{\Phi}_N^{n_N} \tilde{\Phi}_1. \tag{169}$$

leads to the transformation of variables

$$\begin{aligned} \Phi_i &= \tilde{\Phi}_i^{n_i/l_i} \tilde{\Phi}_{i+1}^{1/l_i} \qquad i = 1, .., N-1; \\ \Phi_N &= \tilde{\Phi}_N^{n_N/l_N} \tilde{\Phi}_1^{1/l_N}. \end{aligned} \tag{170}$$

The condition that the Jacobian be constant leads to the constraints

$$n_1 = l_1 \left[ 1 - \frac{1}{l_N} \right]; \tag{171}$$

$$n_i = l_i \left[ 1 - \frac{1}{l_{i-1}} \right], \qquad i = 2, .., N; \tag{172}$$

so that the coordinate transformation (170) becomes

$$\Phi_1 = \tilde{\Phi}_1^{(l_N - 1)/l_N} \tilde{\Phi}_2^{1/l_1} \tag{173}$$

$$\Phi_i = \tilde{\Phi}_i^{(l_{i-1} - 1)/l_{i-1}} \tilde{\Phi}_{i+1}^{1/l_i}, \qquad i = 2, .., N. \tag{174}$$

the Fermat type polynomial, i.e. the loop–type model is a trivial deformation of the original one.

# Landau–Ginzburg Orbifolds.

Even though the class of Heterotic Vacua described in Part I is the largest constructed so far it clearly does not encompass all spectra that are known. Consider e.g. the orbifold of the Fermat quintic in $\mathbb{P}_4[5]$ with respect to the action

$$\mathbb{Z}_5 : [\,3 \quad 1 \quad 1 \quad 0 \quad 0\,] \tag{175}$$

which leads to a space with spectrum $(h^{(1,1)}, h^{(2,1)}, \chi) = (21, 17, 8)$. No complete intersection model with such Hodge numbers appears among the results of [34]. An obvious question of course is whether one could use the fractional transformation discussed in Part II to *construct* the complete intersection representation of this orbifold. Unfortunately this yields a singular space. Thus it is unclear at this point whether a complete intersection of this orbifold exist. There are many examples of this type, some of which will be mentioned below.

Orbifolding is important for the construction of mirrors as well because in many examples the weighted CICY representation of a mirror is not known whereas it is easy to construct the mirror as an orbifold. A simple example is furnished by the manifold

$$\begin{matrix} \mathbb{P}_2 \\ \mathbb{P}_3 \end{matrix} \begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}. \tag{176}$$

the mirror of which can easily be constructed as the orbifold with respect to the action

$$\mathbb{Z}_9 \times \mathbb{Z}_3 \;:\; \begin{bmatrix} 6 & 1 & 3 & 2 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 \end{bmatrix} \tag{177}$$

when viewed as a symmetry on the corresponding LG theory embedded in $\mathbb{C}_{(3,2,3,2,3,2,3)}[9]$. It is unclear however what the CICY representation of this mirror is.

A further motivation for the construction of orbifolds has been mentioned already in the introduction. Namely in order to get some idea of how 'robust' the mirror property of the configuration space is it is useful to implement different types of constructions. Until the proper framework for mirror symmetry and its explicit form has been found this appears to be one feasible avenue for collecting support for the existence [8].

This last part of the review contains a brief description of the work done in [36] in which some 40odd types of actions have been considered on all Landau–Ginzburg potentials that can be build

[8] There are of course also other reasons to look for new models but this is another story.

combinations thereof.

## 11 Actions of Symmetries: General Considerations

It is useful to first discuss some general aspects that are important for group actions on Landau–Ginzburg theories that have been orbifolded with respect to the U(1)–symmetry in order to describe string vacua with $N = 1$ spacetime supersymmetry.

An obvious question when considering orbifolds is whether there is any a priori insight into what spectra are possible for the orbifolds of a given model with respect to a particular set of symmetries. This question is of particular interest if the goal is to produce orbifolds with presribed spectra, say models with a small number of fields where the difference between the number of generations and antigenerations is three.

Even though it is possible to formulate constraints on the orbifold spectrum for particular types of actions, we know of no constraints that hold in full generality, or even for arbitrary cyclic actions. One very simple class of symmetries are those without fixed points. For such actions there are no twisted sectors and hence there exists a simple formula expressing the Euler number $\chi_{orb}$ of the orbifold in terms of the Euler characteristic $\chi$ of the covering space and the order $|G|$ of the group

$$\chi_{orb} = \frac{\chi}{|G|}. \tag{178}$$

The vast majority of actions however do have fixed points and hence the result above does not apply very often.

For orbifolds with respect to cyclic groups of prime order there exists a generalization of this result. For such group actions it was shown in the first reference in [12] that

$$\bar{n}^g_{orb} - n^g_{orb} = (|G|+1)\left(\bar{n}^g_{inv} - n^g_{inv}\right) - \left(\bar{n}^g - n^g\right), \tag{179}$$

where $n^g_{orb}$, $n^g_{inv}$, $n^g$ are the numbers of generations of the orbifold theory, the invariant sector and the original LG theory, respectively.

Consider then the problem of constructing an orbifold with a prescribed Euler number $\chi_{orb}$ from a given theory. Only for fixed point free actions will the order of the group be completely specified as $|G| = \chi/\chi_{orb}$. It is important to realize that in general the order of the group by which a theory is orbifolded does *not* determine its spectrum – the precise form of the action of the symmetry is important.

Even though we don't know a priori what the invariant sector of the orbifold will be we do know that its associated Euler number must be an integer

$$\chi_{inv} = \frac{\chi + \chi_{orbi}}{|G| + 1} \quad \in \mathbb{N}. \tag{180}$$

This simple condition does lead to restrictions for the order of the group. Suppose, e.g., that we wish to check whether the quintic threefold admits a three–generation orbifold: For the deformation class of the quintic

$$\mathbb{C}_{(1,1,1,1,1)}[5]: \ \chi = -200 \tag{181}$$

the order of the discrete group in question must satisfy the constraint $-206/(|G|+1) \in \mathbb{Z}$, implying $|G| = 102$. Hence there exists no three–generation orbifold of the quintic with respect to a discrete group with prime order. A counterexample for nonprime orders is furnished by the following theory

$$\mathbb{C}_{(2,2,2,3,3,3,3)}[9]: \ (\bar{n}^g, n^g, \chi) = (8, 35, -54), \tag{182}$$

which corresponds to a CY theory embedded in a product of two projective spaces by two polynomials of bidegree $(0, 3)$ and $(3, 1)$ [54][55] i.e. the Calabi–Yau manifold of this model is embedded in an ambient space consisting of a product of two projective spaces

$$\begin{matrix} \mathbb{P}_2 \\ \mathbb{P}_3 \end{matrix} \begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}. \tag{183}$$

Suppose we are searching for three–generation orbifolds of this space with $\chi_{orb} = \pm 6$ . If $\chi = -6$ the constraint is not very restrictive and allows a number of possible groups $|G| \in \{2, 3, 5, 11, 19, 29\}$. Even though it is not known whether any of these groups lead to a three–generation model it *is* known that at a particular point in the configuration space of (182) described by the superpotential

$$W = \sum_{i=1}^{3} (\Phi_i^3 + \Phi_i \Psi_i^3) + \Phi_4^3 \tag{184}$$

a symmetry of order nine exists that leads to a three–generation model [54].

Our interest however is not restricted to models with particular spectra for reasons explicated in the introduction. Hence we wish to implement general types of actions regardless of their fixed point structure and order. A general analysis of symmetries for an arbitrary Landau–Ginzburg potential is beyond the scope of this paper; instead we restrict our attention to the types of potentials that we have constructed explicitly. Before we discuss these types we should remark upon a number of aspects concerning actions on string vacua defined by LG–theories.

It is important to note that depending on the weights (or charges) of the original LG theory it can and does happen that actions that take rather different forms when considered as actions on of the U(1) projection. It is easiest to explain this with an example. Consider the superpotential

$$W = \Phi_1^{18} + \Phi_2^{18} + \Phi_3^3 + \Phi_4^3 + \Phi_4\Phi_5^3 \tag{185}$$

which belongs to the configuration $\mathbb{C}_{(1,1,6,6,4)}[18]^9_{-204}$ (here the superscript denotes the number of antigenerations and the subscript denotes the Euler number of the configuration). At this particular point in moduli space we can, e.g., consider the orbifolds with respect to the actions

$$\begin{aligned} \mathbb{Z}_3 &: [0 \;\; 0 \;\; 1 \;\; 0 \;\; 2], \quad (13, 79, -132) \\ \mathbb{Z}_3 &: [1 \;\; 1 \;\; 1 \;\; 0 \;\; 0], \quad (13, 79, -132) \\ \mathbb{Z}_3 &: [1 \;\; 0 \;\; 1 \;\; 0 \;\; 1], \quad (14, 44, -60), \end{aligned} \tag{186}$$

where the notation $\mathbb{Z}_a : [p_1 \;\ldots\; p_n]$ indicates that the fields $\Phi_i$ transform with phases $(2\pi i p_i/a)$ under the generator of the $\mathbb{Z}_a$ symmetry. It is clear from the last action in (186) that the order of a group is, in general, not suffient to determine the resulting orbifold spectrum but that the specific form of the way the symmetry acts is essential.

Since the first two actions lead to the same spectrum we are led to ask whether the two resulting orbifolds are equivalent. Theories with the same number of light fields need, of course, not be equivalent and to show whether they are is, in general a rather involved analysis, entailing the transformation behaviour of the fields and the computation of the Yukawa couplings.

In the case at hand it is, however, very easy to check this question. The first two actions only differ by the 6$^{\text{th}}$ power of the canonical $\mathbb{Z}_{18}$ which is given by $\mathbb{Z}_3 : [1\;1\;0\;0\;1]$. Since the orbifolding with respect to this group is always present in the construction of a LG vacuum the fist two orbifolds in *eq.* (186) are trivially equivalent.

Another important point is the role of trivial factors in the LG theories. Given a superpotential $W_0$ with the correct central charge to define a Heterotic String vacuum we always have the freedom to add trivial factors to it

$$W = W_0 + \sum_i \Phi_i^2, \tag{187}$$

since neither the central charge nor the chiral ring are changed by this operation. As we restrict our attention to symmetries with unit determinant, we gain, however, the possibility to cancel a negative sign of the determinant by giving some $\Phi_i$ a nontrivial transformation property under a $\mathbb{Z}_{2n}$. Adding a trivial factor hence changes the symmetry properties of the LG–potential with regards to this class of symmetries. [9] If we wish to relate the vacuum described by the potential

[9] In LG theories the determinant restriction is necessary for modular invariance and can be avoided by introducing discrete torsion [29].

LG–potential

$$W_0 = \Phi_1^{12} + \Phi_2^{12} + \Phi_3^6 + \Phi_4^6 \tag{188}$$

which has $c = 9$ and charges $(\frac{1}{12}, \frac{1}{12}, \frac{1}{6}, \frac{1}{6})$ and hence is a member of the configuration $\mathbb{C}_{(1,1,2,2)}[12]$. Only after adding the necessary trivial factor this theory can be orbifolded with an action defined by $\mathbb{Z}_2 : [\ 1\ 0\ 0\ 0\ 1\ ]$ acting on the Fermat polynomial in $\mathbb{C}_{(1,1,2,2,6)}[12]$; this action leads to the orbifold spectrum $(4, 94, -180)$ and is not equivalent to any symmetry that acts only on the first four variables with determinant 1. Neglecting the addition of the quadratic term to the LG potential $W_0$ would have meant missing the above spectrum as one of the possible orbifold results.

Finally it should be noted that obviously we have to make *some* choice about which points in moduli space we wish to consider. Different members of a moduli space have, in general, drastically different symmetry properties. An example is the well known quintic theory which we already mentioned. The most symmetric point in the 101 dimensional space of complex deformations of the quintic is described by the Fermat polynomial

$$W = \sum_i \Phi_i^5, \tag{189}$$

which has a discrete symmetry group of order $5! \cdot 5^4$. Any deformation breaks most of these symmetries but in some cases new symmetries appear which turn out to lead to new spectra. An example is furnished by the quintic described by a combination of two 1–Tadpole polynomials and one Fermat monomial

$$W = \Phi_1^5 + \Phi_2^5 + \Phi_2\Phi_3^4 + \Phi_4^5 + \Phi_4\Phi_5^4 \tag{190}$$

which, when orbifolded with respect to the symmetry $\mathbb{Z}_2 : [0\ 0\ 1\ 0\ 1]$ leads to a model with spectrum $(3, 59, -112)$. This spectrum cannot be obtained via orbifolding the Fermat quintic by any of its supersymmetry preserving symmetries.

# 12 Phase Actions: Implementation and Results

Consider then a potential $W$ with $n$ order parameters normalized such that the degree $d$ takes the lowest value such that all order parameters have integer weight. In the following we discuss potentials of the type

$$W = \sum_i \Phi_i^{a_i} + \sum_j \left(\Phi_j^{e_j} + \Phi_j \Psi_j^{f_j}\right) + \sum_k \left(\Phi_k^{e_k}\Psi_k + \Phi_k \Psi_k^{f_k}\right) \tag{191}$$

**FERMAT POTENTIALS**: Clearly the potential $W = \sum_{i=1}^n \Phi_i^{a_i}$ is invariant under $\prod_i \mathbb{Z}_{a_i}$, i.e. the phases of the individual fields, acting like

$$\Phi_i \longrightarrow e^{2\pi i \frac{m_i}{a_i}} \Phi_i. \tag{192}$$

For some divisor $a$ of $\mathrm{lcm}(a_1, \ldots, a_n)$ and $\frac{m_i}{a_i} = \frac{p_i}{a}$ we denote such an action by

$$\mathbb{Z}_a : \ [\, p_1 \quad p_2 \quad \cdots \quad p_n \,], \quad 0 \leq \ p_i \leq \ a - 1. \tag{193}$$

and require that $a$ divides $\sum p_i$ in order to have determinant 1.

We have implemented such symmetries in the form

$$\mathbb{Z}_a : \ [\, (a - \sum_l i_l) \quad i_1 \quad \cdots \quad i_p \quad (a - \sum_m j_m) \quad j_1 \quad \cdots \quad j_q \quad \cdots ] \tag{194}$$

with the obvious divisibility conditions. For small $p$ and $q$ these symmetries can act on a large number of spaces and therefore lead to many different orbifolds, but as $p, q$ get larger the number of resulting orbifolds decreases rapidly. We have stopped implementation of more complicated actions when the number of results for the different orbifold Hodge pairs was of the order of a few tens. As already mentioned above, the precise form of the action is very important when considering symmetries with fixed points since the order itself is not sufficient to determine the orbifold spectrum.

More complicated symmetries can be constructed via multiple actions by multiplying single actions of the type described above

$$\prod_c \mathbb{Z}_{a_c} : \quad [\, (a_c - \sum_l i_{c,l}) \quad i_{c,1} \quad \cdots \quad i_{c,p} \quad (a_c - \sum_m j_{c,m}) \quad j_{c,1} \quad \cdots \quad j_{c,q} \quad \cdots ]\,. \tag{195}$$

We have considered (an incomplete set of) actions of this type with up to six twists (i.e. six $\mathbb{Z}_a$ factors). Again the precise form of the action is rather important.

**TADPOLE AND LOOP POLYNOMIALS**: The action of the generator of the maximal phase symmetry within a tadpole or loop sector is

$$\mathbb{Z}_{\mathcal{O}} : [\, -f \quad 1\,], \tag{196}$$

where $\mathcal{O} = ef$ or $ef - 1$, respectively. If we want unit determinant within one sector, we must take our generator to the $n^{\mathrm{th}}$ power with some $n$ fulfilling $n(f-1)/\mathcal{O} \in \mathbb{Z}$. With $\omega = gcd(f-1, \mathcal{O})$ the action of the resulting subgroup can be chosen to be

$$\mathbb{Z}_\omega : \quad [\, (\omega - 1) \quad 1\,]. \tag{197}$$

and tadpole/loop parts involve phases acting both on the tadpole/loop part as well as on a number of Fermat monomials. As was the case with pure Fermat polynomials we have also implemented multiple actions of the type considered above.

We have implemented some forty different actions of the types described in the previous paragraphs. These symmetries lead to a large number of orbifolds not all of which are distinct however for reasons explained in the previous section. Our computations have concentrated on the number of generations and anti–generations of these models and we have found some 1900 distinct Hodge pairs. This set of spectra shows a mirror symmetry that is even higher than the one exhibited by the complete intersection vacua: whereas about 80spectra have mirror partners!

It is obvious from this plot that there is a large overlap between the results of [8] and the orbifolds constructed here. This might indicate that the relation established in [45] between orbifolds of Landau–Ginzburg theories and other Landau–Ginzburg theories is a general phenomenon and not restricted to the particular classes of actions which were analysed in [45].

Models with a low number of fields are clearly of particular interest. There are two aspects to this question, as mentioned in the introduction – low numbers for the *difference* of generations and anti–generations (more precisely one wants the number 3 here) and low values for the *total* number of generations and anti–generations. As far as the latter are concerned the following 'low–points' are the 'highlights' among the results for phase symmetry orbifolds.

The lowest models have $\chi = 0$, more precisely the spectra (9,9,0) and (11,11,0). These spectra appear many times in different orbifolds of Fermat type; an example for the first one being

$$\mathbb{C}_{(1,\ldots,1)}[9]/\mathbb{Z}_3^2 : \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \qquad (198)$$

or, even simpler,

$$\mathbb{C}_{(4,4,4,4,4,4,3,3)}[12]/\mathbb{Z}_3 : [1\ 1\ 1\ 0\ 0\ 0\ 0\ 0]. \qquad (199)$$

The second one can be constructed e.g. as

$$\mathbb{C}_{(4,3,3,3,3,2)}[12]/\mathbb{Z}_4^2 : \begin{bmatrix} 0 & 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & 0 \end{bmatrix}. \qquad (200)$$

Other examples with a total of 22 generations and anti–generations are the following orbifolds of the Fermat quintic:

$$\mathbb{Z}_5 : [\,0 \quad 1 \quad 2 \quad 3 \quad 4\,], \qquad (1, 21, -40) \qquad (201)$$

and

$$\mathbb{Z}_5^2 : \begin{bmatrix} 3 & 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 1 & 0 \end{bmatrix}, \qquad (21, 1, 40) \qquad (202)$$

no new such models aside from the known three–generation models [34].

Via orbifolding a number of such models can be found, which all, however, have a fairly large number of generations and antigenerations. We list those in Table 2.

| # | Configuration | Potential | Action | Spectrum |
|---|---|---|---|---|
| 1 | $\mathbb{C}_{(9,2,5,9,2)}[27]^{16}_{-66}$ | $\Phi_1^3+\Phi_2^{11}\Phi_3+\Phi_2\Phi_3^5+\Phi_4^3+\Phi_4\Phi_5^9$ | $\mathbb{Z}_3$ : [1 0 0 0 2] | $(18,21,-6)$ |
| 2 | $\mathbb{C}_{(9,2,5,3,8)}[27]^{10}_{-54}$ | $\Phi_1^3+\Phi_2^{11}\Phi_3+\Phi_2\Phi_3^5+\Phi_4^9+\Phi_4\Phi_5^3$ | $\mathbb{Z}_2$ : [0 1 1 0 2] | $(21,18,6)$ |
| 3 | $\mathbb{C}_{(17,6,9,3,16)}[51]^{15}_{-102}$ | $\Phi_1^3+\Phi_2^7\Phi_3+\Phi_2\Phi_3^5+\Phi_4^{17}+\Phi_4\Phi_5^3$ | $\mathbb{Z}_2$ : [0 1 1 0 0] | $(31,34,-6)$ |
| 4 | $\mathbb{C}_{(15,15,2,9,4)}[45]^{23}_{-30}$ | $\Phi_1^3+\Phi_2^3+\Phi_2\Phi_3^{15}+\Phi_4^5+\Phi_4\Phi_5^9$ | $\mathbb{Z}_3$ : [1 0 2 0 0] | $(23,20,6)$ |
| 5 | $\mathbb{C}_{(15,15,10,3,2)}[45]^{22}_{-54}$ | $\Phi_1^3+\Phi_2^3+\Phi_2\Phi_3^3+\Phi_4^{15}+\Phi_4\Phi_5^{21}$ | $\mathbb{Z}_3$ : [1 0 2 0 0] | $(35,32,6)$ |

**Table 2.** *Three–generation orbifold models; models which are equivalent up to the U(1) projection are not listed separately.*

By using the relation established in [45] between LG/CY–theories via fractional transformations it can be shown that the orbifold #1 in Table 2,

$$\mathbb{C}_{(2,5,9,2,9)}[27]^{16}_{-66}/\mathbb{Z}_3 : [\ 0\ 0\ 0\ 2\ 1], \tag{203}$$

for which the covering model is described by the polynomial

$$W=\Phi_1^{11}\Phi_2+\Phi_1\Phi_2^5+\Phi_3^3+\Phi_3\Phi_4^9+\Phi_5^3, \tag{204}$$

is isomorphic to the orbifold

$$\mathbb{C}_{(2,5,9,3,8)}[27]^{10}_{-54}/\mathbb{Z}_2 : [\ 0\ 0\ 0\ 1\ 1] \tag{205}$$

where the covering theory is described by the polynomial

$$W=\Phi_1^{11}\Phi_2+\Phi_1\Phi_2^5+\Phi_3^3+\Phi_3\Phi_4^6+\Phi_4\Phi_5^3. \tag{206}$$

The latter is a theory involving a subtheory with couplings among three scaling fields and hence goes beyond the types of potentials we have implemented. This example indicates that more complicated examples than the ones investigated here are likely to yield more (perhaps more realistic) three generation models.

The covering spaces of all the three generation models are described by either tadpole or loop type polynomials, and with our actions none of the Fermat type polynomials leads to a three generation model. It should be noted that these orbifolds exist only at particular points in moduli space.

It is clear that the structure of the configuration space of the Heterotic String is not particularly well understood and that much remains to be done; it is apparent from the results described above that what has been achieved so far at best is little more than scratching the surface. It might be hoped that once a completely mirror symmetric part of the moduli space has been constructed a representative part of the complete space has been uncovered. Pursueing work along the lines described above is certainly promising in this regard; very likely it is possible to generate a mirror symmetric subspace by all orbifolds of the Landau–Ginzburg theories listed in [34] or, more generally, to construct all weighted complete intersection Calabi–Yau manifolds and their orbifolds.

Having done that it still remains to show that all potential mirror partners in this symmmetric subspace of ground states are in fact related. Even though many types of LG–potentials constructed in [8] admit, via fractional transformations, an interpretation as orbifolds not every mirror potential can be constructed in this way at present. It is therefore clear that a generalization of this type of mirror map is necessary.

Aside from the question of mirror symmetry the orbifold technique is extremely useful to get insight into both, the detailed structure of the vacua constructed via such a classification of LG–potentials and the relation between these vacua. Properties of the ground states that are obscure from the point of view of the superpotentials alone or from the point of view of the partition function of the underlying conformal field theory (if known) become rather obvious in the orbifold picture.

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