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VARIANTS OF MOREAU'S SWEEPING PROCESS

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Abstract

In this paper we prove the existence and uniqueness of two variants of Moreau's sweeping process $-u'(t) \in N_{C(t)}(u(t))$, where in one variant we replace $u(t)$ by $u'(t)$ in the right-hand side of the inclusion and in the second variant $u'(t)$ and $u(t)$ are respectively replaced by $u''(t)$ and $u'(t)$.

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1 Introduction

In the seventies Moreau [14] introduced and studied the evolution problem

$$-u'(t) \in N_{C(t)}(u(t)) \quad \text{a.e. in } [0,T], u(0) = u_0 \in C(0) \tag{1.1}$$

which describes the motion of a ball inside a ring. Here $u(t)$ is the position of the ball at time t and $C(t)$ is the ring at time t. $N_{C(t)}(u(t))$ denotes the outward normal cone to the set $C(t)$ at the position $u(t)$. Thus (1.1) tells us that the velocity $u'(t)$ of the ball has to point inwards to the ring at almost every time $t \in [0,T]$. The initial condition $u(0) \in C(0)$ states that the ball is initially contained in the ring. (1.1) is known as the Moreau's sweeping process. This includes evolution variational inequality as a special case.
Find $u(t) \in K$ a.e. such that

$$\langle u'(t), v-u\rangle \geq \langle f, v-u\rangle \tag{1.2}$$

for all $v \in K$, K is a subset of a Hilbert space H, $u : [0,T] \to H, f \in L_2(0,T;H^*)$.

Several extensions and applications of the Moreau sweeping process in diverse fields [7]-[15], [20]-[22] have been studied. For a lucid introduction of this process along with numerical aspects and applications we, particularly, refer to Moreau [15]. While studying the heat control problem one encounters the following evolution variational inequality.
Find $u = u(x,t)$ such that $u'(t) = \partial u(\cdot,t)/\partial t \in H^1(\Omega)$ and

$$\langle u'(t), v-u'(t)\rangle + a(u(t), v-u'(t)) + j(v) - j(u'(t)) \leq \langle f(t), v-u'(t)\rangle \tag{1.3}$$

where $j(\cdot)$ is convex and lower semicontinuous with values in $(-\infty,+\infty)$ but not identically $+\infty$ (for details see [4, 80-94] and [5, 454-476]). In particular we may consider variational inequality [1-6, 16-19] of the type
Find $u = u(x,t)$ such that $u'(t) \in H^1(\Omega)$

$$\langle u'(t), v-u'(t)\rangle \geq 0 \tag{1.4}$$

and look for existence and uniqueness of solution of a variant of Moreau process, namely
Find $u = u(x,t) \in C(t)$ such that $u'(t) \in C(t)$ and

$$-u'(t) \in N_{C(t)}(u'(t)) \tag{1.5}$$

which includes (1.4) as a special case.

The variational inequality of the type (1.6) is the formulation of the dynamic analogue of the Signorini problem (see [4, 154-162] and [5, 476-487]).
Find $u'(t) \in C(t)$ for all t such that

$$\left.\begin{array}{rl} \langle u''(t), v-u'(t)\rangle & + \quad a(u(t), v-u'(t)) + j(v) - j(u'(t)) \\ & \geq \quad \langle f(t), v-u'(t)\rangle \end{array}\right\} \tag{1.6}$$

for all $v \in C(t)$ and with the initial conditions $u(0) = u_0,\ u'(0) = u_1$. A natural question is whether the following sweeping process has a unique solution:
Find $u(t) \in C(t)$ such that $u'(t) \in C(t)$ a.e. t and

$$-u''(t) \in N_{C(t)}(u'(t)), \quad u(0) = u_0,\ u'(0) = u_1\,. \tag{1.7}$$

The main goal of this paper is to study existence and uniqueness of sweeping processes described by (1.5) and (1.7).

2 Notation and Preliminaries

Let H be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$. For a closed convex subset C of H the set

$$N_C(x) = \{y \in H \mid \langle y, v - x \rangle \leq 0, \forall v \in C\}, x \in C,$$

denotes the normal cone to C at x. Let $d_H(A, B)$ denote the Hausdorff distance between two subsets A and B of H and it is defined as follows

$$d_H(A, B) = \max\{\sup_{x \in B} d(x, A), \sup_{x \in A} d(x, B)\} \tag{2.1}$$

where $d(x, A) = \inf\{\|x - y\| \mid y \in A\}$.
For any Banach space X, we denote by $C^m([0, T]; X)$ the space of continuous functions $u : [0, T] \to X$ that have continuous derivatives up to and including those of order m on $[0, T]$ with the norm

$$\|u\|_{C^m([0,T];X)} = \sum_{i=0}^{m} \max_{0 \leq t \leq T} \|u^{(i)}(t)\|_X \tag{2.2}$$

and by $L_p(0, T; X)$ for $1 \leq p < \infty$ the space of all measurable functions $u : (0, T) \to X$ for which

$$\|u\|_{L_p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \right)^{1/p} < \infty\,. \tag{2.3}$$

The space of measurable functions $u : (0, T) \to X$ which is essentially bounded and denoted by $L_\infty(0, T; X)$ and this space is endowed with norm

$$\|u\| = \text{ess} \sup_{0 \leq t \leq T} \|u(t)\|_X \tag{2.4}$$

Some properties of those spaces are listed in Theorem 2.1 [23].

Theorem 2.1 *Let m be a nonnegative integer and $1 \leq p \leq \infty$. Let X be a Banach space.*

- *a) $C^m([0, T]; X)$ with the norm (2.2) is a Banach space.*

- *b) $L_p(0, T; X)$ is a Banach space if we identify functions that are equal almost everywhere in $(0, T)$.*

c) If X is a Hilbert space with inner product $\langle \cdot, \cdot \rangle_X$ then $L_2(0,T;X)$ is also a Hilbert space with the inner product

$$\langle u, v \rangle_{L_2(0,T;X)} = \int_0^T \langle u(t), v(t) \rangle_X \, dt \,. \tag{2.5}$$

The topological dual of a Banach space X is defined by X^* and the operation of an element $u^* \in H^*$ on an element $u \in X$ is represented by (u^*, u). If X is separable the $L_1(0,T;X^*)$ is separable and $(L_1(0,T;X))^* = L_\infty(0,T,X^*)$. If X is a Hilbert space then $(L_2(0,T;X))^* = L_2(0,T,X^*)$. For a Hilbert space H, we define by $W^{1,2}(0,T;H)$ the space of functions $u \in L_2(0,T;H)$ such that $u' \in L_2(0,T;H)$, equipped with the norm

$$\|u\|^2_{W^{1,2}(0,T;H)} = \|u\|^2_{L_2(0,T;H)} + \|u'\|^2_{L_2(0,T;H)} \tag{2.6}$$

where u' denotes the generalized derivative of f on $(0,T)$. A function $w = u^{(n)}$ is the generalized derivative of the function u on $(0,T)$ if and only if

$$\int_0^T \phi^{(n)}(t) u(t) \, dt = (-1)^n \int_0^T \phi(t) w(t) \, dt \tag{2.7}$$

for all $\phi \in C_0^\infty(0,T)$ - the space of infinitely differentiable functions having compact support. The integrals in (2.7) exist if $u, w \in L_1(0,T;H)$. The generalized derivative is unique, if the function $u : [0,T] \to H$ is continuous and the derivative

$$u'(t) = \lim_{h \to 0} \frac{u(t+h) - u(t)}{h} \tag{2.8}$$

exist for all $t \in [0,T]$ as a limiting value in H; and $u' : [0,T] \to H$ is also continuous then u' is the generalized derivative of u on $(0,T)$. Moreover, if $u \in L_2(0,T;H)$ then $u' \in L_2(0,T;H^*)$. The following results [23] are needed in our subsequent discussion.

Theorem 2.2 ([23], p.421) *Let H be a Hilbert space and let $u : [0,T] \to H$ be Lipschitz continuous, that is*

$$\|u(t) - u(s)\| \le L|t - s| \quad \text{for all } t, s \in [0,T] \tag{2.9}$$

and fixed $L \ge 0$. Then

a) For almost all $t \in [0,T]$, the function u has a derivative,

$$u'(t) = \lim_{h \to 0} \frac{u(t+h) - u(t)}{h}$$

and

$$u(t) = u(0) + \int_0^t u'(s) \, ds \quad \text{for all } t \in [0,T]$$

b) For almost all $t \in [0,T]$

$$\|u'(t)\| \le L$$

and u' is the generalized derivative of u on $(0,T)$.

An operator $A : H \to H^*$ is called monotone if

$$\langle Au - Av, u - v\rangle \geq 0 \quad \text{for all } u, v \in H \tag{2.10}$$

A is called strongly monotone if there is a constant $\beta > 0$ such that

$$\langle Au - Av, u - v\rangle \geq \beta\|u - v\| \quad \text{for all } u, v \in H \tag{2.11}$$

A is maximal monotone if and only if

$$R(A + I) = H\,.$$

For this characterization which was also proposed by Minty and other related results see [23, chapter 32].

It may be observed that if $X = R$ then u' for $u : X = R \to X^* = R$ is strongly monotone if u is C^2 and $u''(t) > c$ for all $t \in R$ and fixed $c > 0$. u' is strongly monotone if u is C^1 and satisfies $u'(t) - u'(s) \geq c(t - s)$ for all $t \geq s \in R$ and $c > 0$.

Let

$$\begin{aligned} D_1(u) = u' \quad &, \quad \mathrm{dom}(D_1) = \{u \in W^{1,2}(0, T; H) \mid u(0) = 0\} \subseteq H \\ D_2(u) = u' \quad &, \quad \mathrm{dom}(D_2) = \{u \in W^{1,2}(0, T; H) \mid u(0) = u(T)\} \subseteq H \end{aligned}$$

Then $D_1 : \mathrm{dom}(D_1) \to H^*$ and $D_2 : \mathrm{dom}(D_2) \to H^*$ are maximal monotone operators.

A moving set valued map $t \to C(t)$ is called Lipschitz continuous if

$$d_H(C(t), C(s)) \leq L|t - s|,\ t, s \in [0, T] \tag{2.12}$$

for some constant $L > 0$. Our aim is to prove that for a Lipschitz continuous moving set $C(t)$ there exists a unique solution to (1.5). By a solution of (1.5) we mean a function $u : [0, T] \to H$ such that

a) $u(0) = u_0$

b) $u(t) \in C(t)$ for almost every $t \in [0, T]$

c) $u'(t) \in C(t)$ for almost every $t \in (0, T)$

d) $-u'(t) \in N_{C(t)}(u'(t))$ for almost every $t \in [0, T]$

The following discretization process is needed for the proof of the solution of the sweeping process. We fix $n \in N$ and choose a time discretization

$$0 = t_0^n < t_1^n < \ldots < t_{m-1}^n < t_m^n = T \tag{2.13}$$

with $t_{i+1}^n - t_i^n \leq \frac{1}{n}$, $0 \leq i \leq m-1$. We may set $t_i^n = \frac{i}{n}$, but we need not fix the discretization explicitly. The value of m will depend on n and $m \to \infty$ for $n \to \infty$. We define the step approximation $u^n : [0,T] \to H$ as follows. Let

$$u_0^n = u_0,\ u_{i+1}^n = u_i^n + \operatorname{proj}(0, C(t_{i+1}^n)) \in C(t_{i+1}^n), \tag{2.14}$$

$0 \leq i \leq m-1$. The u_n are defined via linear interpolation

$$u_n(t) = u_i^n + \frac{t - t_i^n}{t_{i+1}^n - t_i^n}(u_{i+1}^n - u_i^n),\ t \in [t_i^n, t_{i+1}^n]\,. \tag{2.15}$$

For $x \in H$ an element y of C is called the projection of x on $C \subset H$ (C is closed and convex) written as

$$y = \operatorname{proj}(x, C) \text{ if } \|x - y\| = d(x, C) = \inf_{z \in C} \|x - z\|. \tag{2.16}$$

Equivalently $y = \operatorname{proj}(x, C)$ if

$$\langle y - x, y - z \rangle \leq 0 \text{ for all } z \in C\,. \tag{2.17}$$

For our discussion we assume that $0 \in C(t)$ and $C(t)$ is a cone and $u'(t) \in C(t)$ whenever $u'(t)$ exists and $u(t) \in C(t)$.

3 Existence Results and related Lemmas

Theorem 3.1 *Let $t \to C(t)$ be Lipschitz continuous, that is, satisfy* (2.12) *and $C(t) \subset H$ be nonempty, closed and convex for every $t \in [0,T]$. Let $u_0 = u(0), u_0^1 = u'(0)$ belong to $C(0)$. Then there exists a unique solution $u : [0,T] \to H$ of* (1.5) *which is Lipschitz continuous. In particular, $u \in L_\infty(0,T;H)$ and $u' \in L_\infty(0,T;H)$.*

Theorem 3.2 *Let $t \to C(t)$ be Lipschitz continuous, that is, satisfy* (2.12) *and $C(t) \subset H$ be nonempty, closed and convex for every $t \in [0,T]$. Let $u_0^1 = u(0), u_0^2 = u''(0)$ belong to $C(0)$. Then there exists a unique solution $u : [0,T] \to H$ of* (1.7) *which is Lipschitz continuous. In particular, $u \in L_\infty(0,T;H)$, $u' \in L_\infty(0,T;H)$ and $u'' \in L_2(0,T;H)$.*

Lemma 3.1 ([13], p.10) *Let H be a Hilbert space and $\{u_n\}$ be a sequence of functions $u_n : [0.T] \to H$ that is bounded uniformly in norm and variation, that is,*

$$\begin{aligned} &\|u_n(t)\| \leq M_1, \quad n \in N, \quad t \in [0,T] \text{ and} \\ &var(u_n) \leq M_2, \quad n \in N \end{aligned} \tag{3.1}$$

for some constants $M_1, M_2 > 0$ independently of $n \in N$ and $t \in [0,T]$. Then there exists a subsequence $\{u_{n_k}\}$ and a function $u : [0,T] \to H$ such that $var(u) \leq M_2$ and $u_{n_k}(t) \to u(t)$ weakly in H for all $t \in [0,T]$, that is,

$$\langle u_{n_k}(t), z \rangle \to \langle u(t), z \rangle \quad \text{for all } z \in H \tag{3.2}$$

as $k \to \infty$.

Lemma 3.2 ([10] or [23], p.258) *a) Let $u_n \to u$ weakly in H. The*

$$\|u\| \leq \lim_{n\to\infty}\inf \|u_n\| \tag{3.3}$$

holds.

b) If $u_n \in C + \bar{B}_{\epsilon_n}(0)$ for some closed convex $C \subset H$ and some sequence $\epsilon_n \to 0$, then $u \in C$.

Lemma 3.3 (Rockafellar, R.T. see [10]) *Let $\{v_n\}$ be a sequence of functions $v_n : [0,T] \to H$ such that $v_n \to v_*$ in the weak* topology of $L_\infty([0,T];H)$, that is,*

$$\int_0^T \langle v_n(t), \phi(t)\rangle \, dt \to \int_0^T \langle v_*(t), \phi(t)\rangle \, dt \quad \text{as } n \to \infty \tag{3.4}$$

for all $\phi \in L_1([0,T];H)$. Suppose that for each $t \in [0,T]$ the set $C(t) \subset H$ is nonempty, closed and convex such that (2.12) is satisfied. Let

$$\Phi(v) = \int_0^T \delta^*(v(t), c(t))\, dt \tag{3.5}$$

for $v \in L_\infty(0,T;H)$, where $\delta^*(x,C) = \sup\{\langle x, c\rangle \mid c \in C\}$ for $x \in H$. Then Φ is lower semi-continuous, that is,

$$\Phi(v_*) \leq \lim_{n\to\infty} \inf \Phi(v_n)\,.$$

Lemma 3.4 ([10]) *Let $u : [0,T] \to H$ be a continuous function that is differentiable at almost every point $t \in (0,T)$. Then*

a) $\int\limits_0^T \langle u'(t), u(t)\rangle \, dt = \frac{1}{2}|u(T)|^2 - \frac{1}{2}|u(0)|^2$

b) $\frac{1}{2}(\frac{d}{dt}|u'(t)|^2) = \langle u'(t), u'(t)\rangle = \|u'(t)\|^2\,.$

4 Proof of Theorem 3.1

Step 1. First of all we show that if u is a weak limit of u_n given by (2.15) then $u \in L_\infty(0,T;H)$, that is, $|u(t)| \leq M$ for almost every $t \in [0,T]$. It can be seen that

$$\|u^n_{i+1} - u^n_i\| \leq d_H(C(t^n_i), C(t^n_{i+1})) \leq L|t^n_i - t^n_{i+1}| \tag{4.1}$$

where we have used discretization in Section 2, (2.14) and (2.12). If u_n is defined by (2.15) then

$$\begin{aligned}
\operatorname{var}(u_n) &= \sum_{i=1}^{m-1} \|u_n(t^n_{i+1}) - u_n(t^n_i)\| = \sum_{i=1}^{m-1} \|u^n_{i+1} - u^n_i\| \\
&\leq L \sum_{i=1}^{m-1} (t^n_{i+1} - t^n_i) = LT = M_2\,, \\
\|u^n_{i+1}\| &\leq \|u^n_i\| + L(t^n_{i+1} - t^n_i)
\end{aligned}$$

and

$$\begin{aligned}\|u_n(t)\| &\leq \|u_i^n\| + L\|u_{i+1}^n - u_i^n\| \\ &\leq \|u_0\| + L(t_{i+1}^n - t_i^n) \\ &\leq |u_0| + LT = M_1 \end{aligned} \tag{4.2}$$

for $t \in [t_i^n, t_{i+1}^n]$. Consequently the desired result $\|u_n(t)\| \leq M_1$ holds for $t \in [0, T]$ as the above relation is true for all $n \in N$ and $t \in [0, T]$. Since $\{u_n(t)\}$ is a bounded sequence in Hilbert space H we can extract a subsequence still denoted by $u_n(t)$ which converges weakly in H say $u_n(t) \to u(t)$ weakly for all $t \in [0, T]$ (Lemma 3.1).

Step 2. $t \to u(t)$ is Lipschitz continuous.

Let $u_n(0) = u_0$ then weak limit of $u_n(0) = u(0) = u_0$. For $t \in [t_j^n, t_{j+1}^n]$ and $s \in [t_i^n, t_{i+1}^n]$ for some $0 \leq j, i \leq m-1$ (without loss of generality we can assume $i \leq j$, that is, $t_i^n \leq t_j^n$) from (2.12) and (4.1) we obtain

$$\begin{aligned}\|u_n(t) - u_n(s)\| &\leq \|u_n(t) - u_n(t_j^n)\| + \sum_{k=i+1}^{j-1} \|u_n(t_{k+1}^n) - u_n(t_k^n)\| \\ &\qquad + \|u_n(t_{i+1}^n) - u_n(s)\| \\ &\leq \frac{t - t_j^n}{t_{j+1}^n - t_j^n}\|u_{j+1}^n - u_j^n\| + \sum_{k=i+1}^{j-1} \|u_{k+1}^n - u_k^n\| \\ &\qquad + \frac{t_{i+1}^n - s}{t_{i+1}^n - t_i^n}\|u_{i+1}^n - u_i^n\| \\ &\leq \sum_{k=i}^{j} \|u_{k+1}^n - u_k^n\| \leq L \sum_{k=i}^{j} \|t_{k+1}^n - t_k^n\| = L(t_{j+1}^n - t_i^n)\end{aligned}$$

or

$$\|u_n(t) - u_n(s)\| \leq L|t - s| + |s - t_i^n| + |t_{j+1}^n - t| \leq L\left(|t - s| + \frac{2}{n}\right). \tag{4.3}$$

By (4.3) and Lemma 3.2(a) we get

$$\|u(t) - u(s)\| \leq \lim_{n\to\infty} \inf \|u_n(t) - u_n(s)\| \leq L|t - s|$$

as weak limit of $(u_n(t) - u_n(s)) = u(t) - u(s)$. Therefore u is Lipschitz continuous and by Theorem 2.2(a), $u'(t)$ exists for almost every t and $\|u'(t)\| \leq L$ by Theorem 2.2(b). Hence $u' \in L_\infty(0, T; H)$. Clearly $u'(0) = u_0'$.

Step 3. To show that $u(t) \in C(t)$.

By (2.13) and (2.15) we have

$$\frac{t - t_i^n}{t_{i+1}^n - t_j^n}\|u_{j+1}^n - u_j^n\| \leq L\|t - t_i^n\| \leq L(t_{i+1}^n - t_i^n) \leq \frac{L}{n}$$

for $t \in [t_i^n, t_{i+1}^n]$. Hence by (2.15), (2.16) and (2.12), for $t \in [t_i^n, t_{i+1}^n]$,

$$\begin{aligned} u_n(t) &\in C(t_i^n) + \bar{B}_{L/n}(0) \\ &\subset C(t) + \bar{B}_{L(t-t_i^n)}(0) + \bar{B}_{L/n}(0) \\ &\subset C(t) + \bar{B}_{2L/n}(0)\,. \end{aligned} \tag{4.4}$$

(2.13) has been used in the last step. It is clear that (4.4) holds for all $n \in N$ and $t \in [0,T]$, and so Lemma 3.2 yields $u(t) \in C(t)$ for all $t \in [0,T]$.

Step 4. To show that u is a solution of (1.5).

By (2.14) and (2.17) we have

$$\langle u_{i+1}^n - u_i^n, u_{i+1}^n - u_i^n - v\rangle \le 0, \quad v \in C(t_{i+1}^n)\,. \tag{4.5}$$

From (2.15), (4.1) and (2.13) we obtain

$$\|u_n(t) - u_{i+1}^n\| = \frac{t_{i+1}^n - t}{t_{i+1}^n - t_i^n}\|u_{i+1}^n - u_i^n\| \le L(t_{i+1}^n - t) \le \frac{L}{n}\,, \tag{4.6}$$

$t \in [t_i^n, t_{i+1}^n]$. Since by (2.12),

$$\begin{aligned} C(t) &\subset C(t_{i+1}^n) + \bar{B}_{L(t_{i+1}^n - t)}(0) \\ &\subset C(t_{i+1}^n) + \bar{B}_{L/n}(0) \end{aligned}$$

for $t \in [t_i^n, t_{i+1}^n]$, we find from (4.5) and (4.1) that

$$\begin{aligned} \langle u_{i+1}^n - u_i^n, u_n'(t) - v\rangle &= \langle u_{i+1}^n - u_i^n, u_{i+1}^n - w\rangle + \\ &\quad \langle u_{i+1}^n - u_i^n, [u_n'(t) - (u_{i+1}^n - u_i^n)] + [w - v]\rangle \\ &\le \|u_{i+1}^n - u_i^n\|\left(\frac{L}{n} + \frac{L}{n}\right) \le \frac{2L}{n}(t_{i+1}^n - t_i^n) \end{aligned} \tag{4.7}$$

for $t \in [t_i^n, t_{i+1}^n]$ and $c \in C(t)$. In the interior (t_i^n, t_{i+1}^n), u_n is differentiable with derivative $u_n'(t) = (t_{i+1}^n - t_i^n)^{-1}(u_{i+1}^n - u_i^n)$ and hence by (4.7) we get

$$\langle u_n'(t), u_n'(t) - v\rangle \le \frac{M}{n}\,, \quad t \in (t_i^n, t_{i+1}^n) \tag{4.8}$$

$v \in C(t)$. The estimate (4.1) also shows that

$$\|u_n'(t)\| \le L\,, \quad t \ne t_i^n\,,$$

hence

$$\|u_n'(t)\|_{L_\infty(0,T;H)} \le L\,, \quad n \in N\,.$$

Since $L_\infty(0,T;H)$ is the dual space of $L_1(0,T;H)$, it is a consequence of the Banach-Alaoglu theorem that we may extract a further subsequence, again indexed by n (see for example [10] or

[23], p. 260), such that $u_n' \to v_*$ for some $v_* \in L_\infty(0,T;H)$ the consequence being in the weak star topology on $L_\infty(0,T;H)$. This means that for all $\phi \in L_1(0,T;H)$

$$\int_0^T \langle u_n'(t), \phi(t)\rangle \, dt \to \int_0^T \langle v_*(t), \phi(t)\rangle \, dt \text{ as } n \to \infty\,.$$

According to the differentiability of u_n,

$$u_n(t) = u_0 + \int_0^t u_n'(s)\, ds\,, \quad t \in [0,T]\,.$$

It can be seen that

$$u_n(t) = u_0 + \int_0^t v_*(s)\, ds\,, \quad t \in [0,T]\,.$$

This again shows $u : [0,T] \to H$ is differentiable for almost every point $t \in (0,T)$, and moreover $u'(t) = v_*(t)$ for almost every $t \in (0,T)$. In particular $-u_n' \to -u'$ in the weak star topology on $L_\infty(0,T;H)$. By lemma 3.3 this gives

$$\int_0^T \delta^*(-u'(t), C(t))\, dt \le \lim_{n\to\infty} \inf \int_0^T \delta^*(-u_n'(t), C(t))\, dt \tag{4.9}$$

(for a definition of $\delta^*(\cdot,\cdot)$ see Lemma 3.3). It is clear that

$$\int_0^T \langle u'(t), u'(t)\rangle \, dt \le \lim_{n\to\infty} \inf \int_0^T \langle u_n'(t), u_n'(t)\rangle \, dt \tag{4.10}$$

Taking supremum w.r.t. v in (4.8) and integrating over $[0,T]$ we find that

$$\int_0^T \left[\delta^*(-u_n'(t), C(t)) + \langle u_n'(t), u_n'(t)\rangle\right]\, dt \le \frac{MT}{n} \tag{4.11}$$

for $n \in N$. Using a well known property of the limit inferior of a sequence (4.9), (4.10) and (4.11) we get

$$\int_0^T \left[\delta^*(-u'(t), C(t)) + \langle u'(t), u'(t)\rangle\right]\, dt \le 0\,. \tag{4.12}$$

We have shown in Step 3 that $u(t) \in C(t), t \in [0,T]$, and so $u'(t) \in C(t)$. By the definition of $\delta^*(\cdot,\cdot)$ we get

$$\delta^*(-u'(t), C(t)) + \langle u'(t), u'(t)\rangle = 0$$

for almost every $t \in (0,T)$. Thus for any $v \in C(t)$

$$\begin{aligned} \langle -u'(t), u(t)\rangle &= \delta^*(-u'(t), C(t)) \\ &\ge \langle -u'(t), v\rangle \end{aligned}$$

or

$$\langle -u'(t), v - u'(t)\rangle \le 0\,.$$

Hence $u(t)$ is a solution of (1.5)

Step 5. Uniqueness of solution.

Let u_1 and u_2 be two solutions of (1.5) then

$$\langle -u_1'(t), v - u_1'(t)\rangle \leq 0 \tag{4.13}$$

and

$$\langle -u_2'(t), v - u_2'(t)\rangle \leq 0\,. \tag{4.14}$$

Put $v = u_2'(t)$ and $v = u_1'(t)$ respectively in (4.13) and (4.14) then we get

$$\langle -u_1'(t), u_2'(t) - u_1'(t)\rangle \leq 0 \tag{4.15}$$

$$\langle -u_2'(t), u_1'(t) - u_2'(t)\rangle \leq 0\,. \tag{4.16}$$

From (4.15) and (4.16) we get

$$\langle u_1'(t) - u_2'(t), u_1'(t) - u_2'(t)\rangle \leq 0$$

or

$$\|w'(t)\|^2 = 0 \tag{4.17}$$

where $w(t) = u_1'(t) - u_2'(t)$. From (4.17) we get

$$\int_0^t |w'(\xi)|^2\, d\xi = 0 \quad \text{as } w(0) = 0\,.$$

Be Lemma 3.4

$$\int_0^t \frac{1}{2}\frac{d}{d\xi}(|u'(\xi)|^2) = 0$$

or $u(\xi) = 0, \forall \xi \in (0,T)$ or $u_1(\xi) = u_2(\xi), \forall \xi \in (0,T)$.

5 Proof of Theorem 3.2

Let $u'(t) = \phi(t)$ and $u''(t) = \phi'(t)$. Then $\phi'(t) \in N_{C(t)}(\phi(t))$ for almost every $t \in (0,T)$ holds by Theorem 2 [10] (Theorem 2.1 [13, p. 141] or Moreau [14]) provided $\phi(t) \in C(t)$. $t \to \phi(t)$ is Lipschitz continuous with constant L. In particular $|\phi'(t)| = |u''(t)| \leq L$ for almost every $t \in (0,T)$ and so $u''(t) \in L_\infty(0,T;H)$. Let $\phi_1(t)$ and $\phi_1(t)$ be two solutions of (1.7) then

$$\langle -\phi_1'(t), v - \phi_1(t)\rangle \leq 0 \tag{5.1}$$

$$\langle -\phi_2'(t), v - \phi_2(t)\rangle \leq 0\,. \tag{5.2}$$

By (5.1) and (5.2) we get

$$\langle \phi_1'(t) - \phi_2'(t), \phi_1(t) - \phi_2(t) \rangle \le 0\,. \tag{5.3}$$

By Lemma 3.4 we have

$$\frac{1}{2}\frac{d}{dt}(|\phi_1(t) - \phi_2(t)|^2) = |\langle \phi_1'(t) - \phi_2'(t), \phi_1(t) - \phi_2(t) \rangle| \le 0$$

almost everywhere in $(0, T)$. Integration yields

$$\|\phi_1(t) - \phi_2(t)\|^2 \le \|\phi_1(0) - \phi_2(0)\|^2 = \|\phi_1^0 - \phi_2^0\|$$

$t \in [0, T]$. In particular, if $\phi_1^0 = \phi_2^0$ then the solution is unique.

6 Relationship with Degenerate Sweeping Processes

Kunze and Monteiro Marques [8] have proved the following theorems.

Theorem 6.1 *Let $A : dom(A) \to 2^H$ be a maximal and strongly monotone operator and for any $t \in [0, T]$, $C(t) \neq \phi \subset H$ be closed and convex set and $t \to C(t)$ be Lipschitz continuous. If in addition the following conditions are satisfied*

a) $C(0)$ is bounded or there exists a function $M : [0, \infty) \to [0, \infty)$ which maps bounded sets such that

$$\|Ax\| = \sup\{|y| : y \in Ax\} \le M(|x|) \quad \textit{for } x \in dom(A)\,,$$

b) $dom(A) \cap \bar{B}_r(0)$ is relatively compact for every $r > 0$ or $C(t) \cap \bar{B}_r(0)$ is compact for every $t \in [0, T]$ and $r > 0$.

Then there exists a Lipschitz continuous function $u : [0, T] \to H$, $u(t) \in dom(A)$ a.e., such that for every $u_0 \in dom(A)$ with $Au_0 \cap C(0) \neq \phi$

$$v(t) \in Au(t) \cap C(t) \quad \textit{a.e.}$$

and

$$-u'(t) \in N_{C(t)}(v(t)) \quad \textit{a.e. in } [0, T] \tag{6.1}$$

Theorem 6.2 *Let $A : H \to H$ be linear, bounded and self adjoint such that $\langle Ax, x \rangle \ge \beta\|x\|^2$ for $x \in H$. If $t \to C(t)$ is Lipschitz continuous where $t \in [0, T]$. $C(t) \subset H$ is closed and convex and $Au_0 \in C(0)$, then* (6.1) *has a unique solution which is Lipschitz continuous.*

It may be observed that in some special cases Theorem 3.1 and Theorem 3.2 can be derived from Theorem 6.1 and Theorem 6.2. For example, if A is as D_1 or D_2 defined in Section 2, $H = R$ and u is of C_2 class with $u''(t) > c$ for all $t \in R$ and fixed c then Theorem 6.1 reduces to Theorem 3.1 provided $C(0)$ is bounded.
If we choose $A = u'$ in Theorem 6.2 then u' satisfies the condition $\langle u', u \rangle \ge \beta\|u\|^2$, is linear and self adjoint. However u' is bounded only almost everywhere and so Theorem 3.1 cannot be obtained as a special case of Theorem 6.2.

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