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TRANSIENT ANALYSIS OF A DISCOURAGED ARRIVALS QUEUE

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Abstract

The transient solution is obtained analytically using continued fractions for a state-dependent birth-death queue in which potential customers are discouraged by the queue length. This queueing system is then compared with the well-known infinite server queueing system which has the same steady state solution as the model under consideration, whereas their transient solutions are different. A natural measure of speed of convergence of the mean number in the system to its stationarity is also computed. We also determine the distributions in discrete time of the number of customers in line and of the busy period in closed form.

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1 Introduction

In the study of queueing systems the emphasis has been on obtaining steady state solution as it is simple to derive and straightforward techniques can be employed. But in many potential applications steady state measures of system performance simply do not make sense when the practitioner needs to know how the system will operate up to some specified time [24]. Time-dependent analysis helps us to understand the behaviour of a system when the parameters involved are perturbed and it can contribute to the costs and benefits of operating a system. In addition, such transient analysis is useful in obtaining optimal solutions which lead to the control of the system. There has been a resurgence of interest in the time-dependent analysis of birth-death queueing models (see, for example, [14, 20]).

The exact time-dependent analysis of the state-dependent queueing systems is usually difficult and often impossible. Even in the simple $M/M/1$ queue which is a birth-death process with constant birth and death rates, analytical solution involves an infinite series of Bessel functions and their integrals (see, for example, [19, 21]). In real world problems the underlying birth and death rates are complex and the difficulty is compounded in the transient analysis of such models.

In this work, the transient solution to a state-dependent birth-death queueing model in which potential customers are discouraged by queue length is obtained using continued fractions, both in continuous time and in discrete time. In continuous time, the Laplace transforms of the density function for the length of the busy period and the mean busy period are also deduced. This solution is then compared with the well-known infinite server queueing model to illustrate that these two models having different transient behaviours lead to the same steady state solution. This is also depicted through graphs. A measure of speed of convergence towards stationarity is computed in terms of the parameters of the model. In discrete time, we obtain the generating function of the busy period distribution and the mean busy period.

The model under consideration is the birth-death queueing system with the birth and death rates as given below:

$$\lambda_n = \frac{\lambda}{n+1}, \quad n = 0, 1, 2, \ldots \quad \text{and } \mu_n = \mu, \quad n = 1, 2, 3, \ldots . \tag{1.1}$$

This *discouraged arrivals* single server queueing system is useful to model a computing facility that is solely dedicated to batch-job processing [[18], p.105]. Job submissions are discouraged when the facility is heavily used and can be modelled as a Poisson process with the state-dependent arrival rate. The time taken to process each job is exponentially distributed with a constant service rate regardless of the number of jobs in the system.

The well-known infinite server queue, denoted as $M/M/\infty$ queue, is often used to analyse manufacturing processes and to model phenomena in telecommunication networks. In the context of broadband integrated services digital networks based upon the asynchronous transfer mode (ATM), this system has been pointed out to be of interest when studying open loop statistical multiplexing of data connections on an ATM network [9].

The discouraged arrivals queue has been studied in the past by Natvig [17], van Doorn [8] and Chihara [6] and here the arrivals are geared (or could be controlled) in accordance with the availability of service. However, the transient solution has not been obtained sofar explicitly in closed form. In this work, we have obtained the transient solution analytically in closed form by employing a new and effective continued fraction methodology. In this study the underlying forward Kolmogorov differential-difference equations are first transformed into a set of linear algebraic equations by employing Laplace transforms. This transform is then represented as a continued fraction and the inversion is carried out analytically.

2 Continued Fractions

Continued fraction approximations often provide good representations for transcendental functions, much more generally useful than the classical representation by power series. In addition, a number of problems have been found for which algorithms involving continued fractions lend themselves to high-speed computer operations. A systematic study of the theory of continued fractions with stress on computation can be found in Jones and Thron [11]. Its application to the study of birth-death processes, a special Markov process, was initiated by Murphy and O'Donohoe [16]. On account of their algorithmic nature, they are used extensively in applied areas like numerical analysis, computer science, the theory of automata, electronic communications. This importance has grown further with the advent of fast computing facilities.

A continued fraction is denoted by

$$\cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ldots}}}$$

or equivalently by

$$\frac{a_1}{b_1+}\ \frac{a_2}{b_2+}\ \frac{a_3}{b_3+}\ \ \cdots$$

where the a_n's and b_n's are real or complex numbers. The value obtained by retaining the first n terms and omitting the remaining terms is called the n-th convergent. For any continued fraction the exact value of the fraction lies between two neighbouring convergents. All even numbered convergents lie to the left of the exact value, that is they give an approximation to the exact value by defect. All odd numbered convergents lie to the right of the exact value, that is they give an approximation to the exact value by excess.

Conolly and Langaris [7], and Parthasarathy and Lenin [20] have applied continued fraction methodology, which was till then used only to obtain numerical solutions, to obtain the transient solution of birth and death processes analytically. We now apply this technique to obtain analytically the transient system size probabilities of our models.

Some of the identities which are used in the following sections will be now presented.

The *confluent hypergeometric function*, also referred to as Kummer function, is denoted by ${}_1F_1(a;c;z)$ and is defined by

$$
\begin{aligned}
{}_1F_1(a;c;z) &= 1+\frac{a}{c}\frac{z}{1!}+\frac{a(a+1)}{c(c+1)}\frac{z^2}{2!}+\cdots &(2.1)\\
&= \sum_{k=0}^{\infty}\frac{(a)_k}{(c)_k}\frac{z^k}{k!} &(2.2)
\end{aligned}
$$

for $z \in \mathbb{C}$, parameters $a, c \in \mathbb{C}$ (c not a negative interger), with $(\alpha)_n$, known as Pochhammer symbol, defined as

$$
(\alpha)_n = \begin{cases} 1, & n=0 \\ \alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-1), & n \geq 1. \end{cases} \qquad (2.3)
$$

Observe that ${}_1F_1(0;c;z)=1$ and ${}_1F_1(1;2;-z)=\dfrac{1-e^{-z}}{z}$. Also, this confluent hypergeometric function satisfies the following recurrence relation [[1], (13.4.7), p.507]

$$
c(c-1)\ {}_1F_1(a-1,c-1,z) - az\ {}_1F_1(a+1,c+1,z) = c(c-1-z)\ {}_1F_1(a,c,z). \qquad (2.4)
$$

The following identity is from Lorentzen and Waadeland [[15], (4.1.5), p.573].

$$
\frac{{}_1F_1(a+1;c+1;z)}{{}_1F_1(a;c;z)} = \frac{c}{c-z+}\ \frac{(a+1)z}{c-z+1+}\ \frac{(a+2)z}{c-z+2+}\ \cdots.
$$

which can be rewritten as

$$
c\frac{{}_1F_1(a;c;z)}{{}_1F_1(a+1;c+1;z)} - (c-z) = \frac{(a+1)z}{c-z+1+}\ \frac{(a+2)z}{c-z+2+}\ \cdots. \qquad (2.5)
$$

The following identities are from Andrews [2].

$$
\begin{aligned}
(c-a)\ {}_1F_1(a;c+1,z) + a\ {}_1F_1(a+1;c+1;z) &= c\ {}_1F_1(a;c;z), &(2.6)\\
c\ {}_1F_1(a+1;c;z) - c\ {}_1F_1(a,c,z) &= z\ {}_1F_1(a+1;c+1,z), &(2.7)\\
(c-a)\ {}_1F_1(a-1,c,z) + (2a-c+z)\ {}_1F_1(a,c,z) &= a\ {}_1F_1(a+1,c,z), &(2.8)\\
\sum_{k=0}^{\infty}\frac{(a)_k}{(c)_k}\frac{y^k}{k!}\ {}_1F_1(a+k;c+k;x) &= {}_1F_1(a;c;x+y). &(2.9)
\end{aligned}
$$

In the sequel, for any function $f(t)$, let

$$
\hat{f}(s) = \int_0^{\infty} e^{-st} f(t)\,dt
$$

denotes its Laplace transform.

In the next section, we derive the transient solution in continuous time for the model under consideration by employing continued fractions.

3 Discouraged Arrivals Queue

Let $P_n(t), n = 0, 1, 2, \ldots$ be the probability that there are n customers in the system at time t. Then, the forward Kolmogorov equations for the Model A are

$$\begin{aligned} \frac{dP_0(t)}{dt} &= \mu P_1(t) - \lambda P_0(t), \\ \frac{dP_n(t)}{dt} &= \tfrac{\lambda}{n} P_{n-1}(t) + \mu P_{n+1}(t) - (\tfrac{\lambda}{n+1} + \mu) P_n(t), \quad n \geq 1. \end{aligned} \tag{3.1}$$

Assume that initially the system is empty. By taking Laplace transforms, (3.1) is reduced to a system of simultaneous equations given by

$$\begin{aligned} (s+\lambda)\hat{P}_0(s) - 1 &= \mu \hat{P}_1(s), \\ \left(s + \frac{\lambda}{n+1} + \mu\right) \hat{P}_n(s) &= \frac{\lambda}{n} \hat{P}_{n-1}(s) + \mu \hat{P}_{n+1}(s), \quad n \geq 1. \end{aligned} \tag{3.2}$$

From the first equation of (3.2), we obtain

$$\hat{P}_0(s) = \frac{1}{s + \lambda - \mu \frac{\hat{P}_1(s)}{\hat{P}_0(s)}} \tag{3.3}$$

and the second equation of (3.2) can be written as, for $n \geq 1$,

$$\frac{\hat{P}_n(s)}{\hat{P}_{n-1}(s)} = \frac{\frac{\lambda}{n}}{s + \frac{\lambda}{n+1} + \mu - \mu \frac{\hat{P}_{n+1}(s)}{\hat{P}_n(s)}}. \tag{3.4}$$

Iterating this equation, we get

$$\frac{\hat{P}_n(s)}{\hat{P}_{n-1}(s)} = \frac{\frac{\lambda}{n}}{s + \frac{\lambda}{n+1} + \mu -} \;\; \frac{\frac{\lambda}{(n+1)}\mu}{s + \frac{\lambda}{n+2} + \mu -} \;\; \cdots . \tag{3.5}$$

By substituting (3.5) in (3.3) we get a continued fraction expression for $\hat{P}_0(s)$ as

$$\hat{P}_0(s) = \frac{1}{s + \lambda -} \;\; \frac{\lambda\mu}{s + \frac{\lambda}{2} + \mu -} \;\; \frac{\frac{\lambda}{2}\mu}{s + \frac{\lambda}{3} + \mu -} \;\; \cdots . \tag{3.6}$$

By making use of (2.5) the above equation simplifies to

$$\begin{aligned} \hat{P}_0(s) &= \left\{ s + \lambda + (s+\mu) \left[\left(\frac{s\lambda}{(s+\mu)^2} + 1 \right) \frac{{}_1F_1\left(1; \frac{s\lambda}{(s+\mu)^2} + 1; \frac{-\lambda\mu}{(s+\mu)^2}\right)}{{}_1F_1\left(2; \frac{s\lambda}{(s+\mu)^2} + 2; \frac{-\lambda\mu}{(s+\mu)^2}\right)} \right.\right. \\ &\qquad \left.\left. - \left(\frac{\lambda}{s+\mu} + 1 \right) \right] \right\}^{-1} \\ &= \left\{ (s+\mu) \left(\frac{s\lambda}{(s+\mu)^2} + 1 \right) \frac{{}_1F_1\left(1; \frac{s\lambda}{(s+\mu)^2} + 1; \frac{-\lambda\mu}{(s+\mu)^2}\right)}{{}_1F_1\left(2; \frac{s\lambda}{(s+\mu)^2} + 2; \frac{-\lambda\mu}{(s+\mu)^2}\right)} - \mu \right\}^{-1} . \end{aligned}$$

Use of (2.6) yields

$$\begin{aligned} \hat{P}_0(s) &= \frac{1}{s} \left\{ \left(\frac{s\lambda}{(s+\mu)^2} + 1 \right) \frac{{}_1F_1\left(1; \frac{s\lambda}{(s+\mu)^2} + 1; \frac{-\lambda\mu}{(s+\mu)^2}\right)}{{}_1F_1\left(2; \frac{s\lambda}{(s+\mu)^2} + 2; \frac{-\lambda\mu}{(s+\mu)^2}\right)} \right. \\ &\qquad \left. + \frac{\lambda\mu}{(s+\mu)^2} \frac{{}_1F_1\left(1; \frac{s\lambda}{(s+\mu)^2} + 2; \frac{-\lambda\mu}{(s+\mu)^2}\right)}{{}_1F_1\left(2; \frac{s\lambda}{(s+\mu)^2} + 2; \frac{-\lambda\mu}{(s+\mu)^2}\right)} \right\}^{-1} . \end{aligned}$$

Again, making use of (2.7) and using the fact that ${}_1F_1\,(0;c;x) = 1$, we obtain

$$\hat{P}_0(s) \;=\; \frac{1}{s\left(\frac{s\lambda}{(s+\mu)^2}+1\right)}\;{}_1F_1\left(2;\frac{s\lambda}{(s+\mu)^2}+2;\frac{-\lambda\mu}{(s+\mu)^2}\right). \tag{3.7}$$

The above expression is equivalent to the one given by Natvig [17]. In a similar way, we will find $\hat{P}_n(s)$. Making use of (2.6), (2.7) and (2.8), we can write (3.5) as, for $n \geq 1$,

$$\frac{\hat{P}_n(s)}{\hat{P}_{n-1}(s)} \;=\; \frac{1}{n}\,\frac{\frac{(n+1)\lambda}{s+\mu}}{\left(\frac{s\lambda}{(s+\mu)^2}+n+1\right)}\,\frac{{}_1F_1\left(n+2;\frac{s\lambda}{(s+\mu)^2}+n+2;\frac{-\lambda\mu}{(s+\mu)^2}\right)}{{}_1F_1\left(n+1;\frac{s\lambda}{(s+\mu)^2}+n+1;\frac{-\lambda\mu}{(s+\mu)^2}\right)}.$$

Iterating this equation and using (3.7), we get

$$\begin{aligned}\hat{P}_n(s) \;&=\; \hat{P}_0(s)\prod_{i=1}^{n}\frac{\hat{P}_i(s)}{\hat{P}_{i-1}(s)}\\ &=\; \frac{\frac{n+1}{s}\left(\frac{\lambda}{s+\mu}\right)^n}{\left(\frac{s\lambda}{(s+\mu)^2}+1\right)\left(\frac{s\lambda}{(s+\mu)^2}+2\right)\ldots\left(\frac{s\lambda}{(s+\mu)^2}+n+1\right)}\\ &\qquad\times\,{}_1F_1\left(n+2;\frac{s\lambda}{(s+\mu)^2}+n+2;\frac{-\lambda\mu}{(s+\mu)^2}\right).\end{aligned} \tag{3.8}$$

Now, we will invert (3.8) by expanding the function as given below.

$$\begin{aligned}\hat{P}_n(s) \;&=\; \frac{(n+1)\lambda^{n+1}}{(s+\mu)^n}\,\frac{(s+\mu)^{2(n+1)}}{s\lambda[s\lambda+(s+\mu)^2]\cdots[s\lambda+(n+1)(s+\mu)^2]}\\ &\qquad\times\sum_{k=0}^{\infty}\frac{(n+2)_k}{\left(\frac{s\lambda}{(s+\mu)^2}+n+2\right)_k}\,\frac{\left(\frac{-\lambda\mu}{(s+\mu)^2}\right)^k}{k!}\\ &=\; \sum_{k=0}^{\infty}\frac{\lambda^{n+k+1}(-\mu)^k(n+k+1)!}{k!\;n!}\\ &\qquad\times\frac{1}{(s+\mu)^{2k+n}}\;\frac{(s+\mu)^{2(n+k+1)}}{\prod\limits_{i=0}^{n+k+1}[s\lambda+i(s+\mu)^2]}\\ &=\; \sum_{k=0}^{\infty}\frac{\lambda^{n+k+1}(-\mu)^k}{k!\;n!}\,\frac{1}{(s+\mu)^{2k+n}}\sum_{i=0}^{n+k+1}\binom{n+k+1}{i}(-1)^i\frac{1}{s\lambda+i(s+\mu)^2}\\ &=\; \sum_{k=0}^{\infty}\frac{\lambda^{n+k+1}(-\mu)^k}{k!\;n!}\sum_{i=0}^{n+k+1}\binom{n+k+1}{i}(-1)^i\\ &\qquad\qquad\times\frac{1}{(s+\mu)^{2k+n}}\,\frac{1}{s\lambda+i(s+\mu)^2}\end{aligned} \tag{3.9}$$

which on inversion becomes

$$P_n(t) \;=\; \sum_{k=0}^{\infty}\frac{\lambda^{n+k+1}(-\mu)^k}{k!\;n!}\sum_{i=0}^{n+k+1}\binom{n+k+1}{i}(-1)^i h_{n+2k,\,i}(t), \qquad n\geq 0 \tag{3.10}$$

where

$$
\begin{aligned}
h_{0,\ 0}(t) &= \frac{1}{\lambda}, && (3.11)\\
h_{0,1}(t) &= \frac{1}{2\sqrt{\frac{\lambda^2}{4}+\lambda\mu}}\left\{e^{-\left(\frac{\lambda}{2}+\mu-\sqrt{\frac{\lambda^2}{4}+\lambda\mu}\right)t}-e^{-\left(\frac{\lambda}{2}+\mu+\sqrt{\frac{\lambda^2}{4}+\lambda\mu}\right)t}\right\}, && (3.12)\\
h_{n+2k,\ 0}(t) &= \frac{1}{\lambda(n+2k-1)!}\int_0^t e^{-\mu y}y^{n+2k-1}dy, \quad \text{for } n+2k>0 && (3.13)
\end{aligned}
$$

and, for $n+2k,\ i\geq 1$,

$$
\begin{aligned}
h_{n+2k,\ i}(t) &= \frac{1}{(n+2k-1)!\,2i\sqrt{\frac{\lambda^2}{4i^2}+\frac{\lambda\mu}{i}}}\\
&\times\left\{e^{-\left(\frac{\lambda}{2i}+\mu-\sqrt{\frac{\lambda^2}{4i^2}+\frac{\lambda\mu}{i}}\right)t}\int_0^t e^{\left(\frac{\lambda}{2i}-\sqrt{\frac{\lambda^2}{4i^2}+\frac{\lambda\mu}{i}}\right)y}y^{n+2k-1}dy\right.\\
&\left.-e^{-\left(\frac{\lambda}{2i}+\mu+\sqrt{\frac{\lambda^2}{4i^2}+\frac{\lambda\mu}{i}}\right)t}\int_0^t e^{\left(\frac{\lambda}{2i}+\sqrt{\frac{\lambda^2}{4i^2}+\frac{\lambda\mu}{i}}\right)y}y^{n+2k-1}dy\right\}. && (3.14)
\end{aligned}
$$

Thus (3.10) gives explicit time-dependent system size probabilities for the discouraged arrivals queueing system.

We observe that $h_{n+2k,\ i}(t)\to 0$, for $i\geq 1$, as $t\to\infty$ and

$$
\begin{aligned}
\lim_{t\to\infty} h_{n+2k,\ 0}(t) &= \frac{1}{\lambda(n+2k-1)!}\int_0^\infty e^{-\mu y}y^{n+2k-1}dy\\
&= \frac{1}{\lambda}\frac{1}{\mu^{n+2k}}
\end{aligned}
$$

and hence the steady state solutions are given by

$$
\begin{aligned}
p_n &= \lim_{t\to\infty} P_n(t)\\
&= \sum_{k=0}^{\infty}\frac{\lambda^{n+k+1}(-\mu)^k}{k!\ n!}\frac{1}{\lambda}\frac{1}{\mu^{n+2k}}\\
&= exp\left\{\frac{-\lambda}{\mu}\right\}\frac{\left(\frac{\lambda}{\mu}\right)^n}{n!}, \quad n=0,1,2,\ldots. && (3.15)
\end{aligned}
$$

We have obtained above the transient solution for the system size probabilities with the assumption that there are zero customers in the system initially. One gets a complicated expression for $P_n(t)$ with the initial number m in the system by defining certain polynomials in s; incorporating with $\hat{P}_n(s)$ and then inverting it [16].

Aliter:

If

$$
g_n(s) = \frac{\left(\frac{s\lambda}{(s+\mu)^2}+1\right)\left(\frac{s\lambda}{(s+\mu)^2}+2\right)\ldots\left(\frac{s\lambda}{(s+\mu)^2}+n+1\right)}{\frac{n+1}{s}\left(\frac{\lambda}{s+\mu}\right)^n}\hat{P}_n(s),
$$

then the second equation in (3.2) reduces to

$$\left(\frac{s\lambda}{(s+\mu)^2}+n+2\right)\left(\frac{s\lambda}{(s+\mu)^2}+n+1\right)g_{n-1}(s)-(n+2)\left(\frac{-\lambda\mu}{(s+\mu)^2}\right)g_{n+1}(s)$$
$$=\left(\frac{s\lambda}{(s+\mu)^2}+n+2\right)\left(\frac{\lambda}{s+\mu}+n+1\right)g_n(s)$$

We identify this equation with the recurrence relation (2.4) with $a=n+2, c=\frac{s\lambda}{(s+\mu)^2}+2$, $z=\frac{-\lambda\mu}{(s+\mu)^2}$. It is seen that this also holds for the first equation of (3.2). Thus, we will have

$$g_n(s) \;=\; {}_1F_1\left(n+2;\frac{s\lambda}{(s+\mu)^2}+n+2;\frac{-\lambda\mu}{(s+\mu)^2}\right)$$

and hence

$$\hat{P}_n(s) \;=\; \frac{\frac{n+1}{s}\left(\frac{\lambda}{s+\mu}\right)^n}{\left(\frac{s\lambda}{(s+\mu)^2}+1\right)\dots\left(\frac{s\lambda}{(s+\mu)^2}+n+1\right)}\;{}_1F_1\left(n+2;\frac{s\lambda}{(s+\mu)^2}+n+2;\frac{-\lambda\mu}{(s+\mu)^2}\right)$$

which is the same as (3.8) obtained earlier.

Busy Period:

Let the length of the busy period, a random variable, be denoted by T and $b_A(t)$ be its probability density function. Let the arrival of a customer start the busy period and now the system is at state '1'. Now, from (3.9), we finally deduce the Laplace transform $\hat{b}_A(s)$ of the density function $b_A(t)$ of the busy period as

$$\hat{b}_A(s) \;=\; \frac{\frac{2\mu}{s+\mu}}{\frac{s\lambda}{(s+\mu)^2}+2}\;\frac{{}_1F_1\left(3;\frac{s\lambda}{(s+\mu)^2}+3;\frac{-\lambda\mu}{(s+\mu)^2}\right)}{{}_1F_1\left(2;\frac{s\lambda}{(s+\mu)^2}+2;\frac{-\lambda\mu}{(s+\mu)^2}\right)}. \tag{3.16}$$

The mean busy period is given by

$$\begin{aligned}
E(T) &= \lim_{s\to 0}\left(\frac{1-\hat{b}_A(s)}{s}\right)\\
&= \lim_{s\to 0}\frac{1}{s+\mu}\,\frac{{}_1F_1\left(1;\frac{s\lambda}{(s+\mu)^2}+2;\frac{-\lambda\mu}{(s+\mu)^2}\right)}{{}_1F_1\left(2;\frac{s\lambda}{(s+\mu)^2}+2;\frac{-\lambda\mu}{(s+\mu)^2}\right)}\\
&= \frac{1}{\mu}\,e^{\frac{\lambda}{\mu}}\;{}_1F_1\left(1;2;\frac{-\lambda}{\mu}\right)\\
&= \frac{e^{\frac{\lambda}{\mu}}-1}{\lambda}
\end{aligned}$$

In the next section, we obtain the transient solution for the infinite server queue by our methodology and compare it with the model under consideration, *i.e.,* the discouraged arrivals queue.

4 Infinite Server Queue

Let $R_n(t), n = 0, 1, 2, \ldots$ be the probability that there are n customers in the system at time t. Then, the forward Kolmogorov equations for this system are

$$\begin{aligned}
\frac{dR_0(t)}{dt} &= \mu R_1(t) - \lambda R_0(t), \\
\frac{dR_n(t)}{dt} &= \lambda R_{n-1}(t) + (n+1)\mu R_{n+1}(t) - (\lambda + n\mu)R_n(t), \quad n \geq 1.
\end{aligned} \tag{4.1}$$

Assume that initially the system is empty. The above system of equations can easily be solved using generating functions. However, we give here an alternate approach in tune with the analysis given in the previous section.

Taking Laplace transforms (4.1) becomes

$$\begin{aligned}
(s+\lambda)\hat{R}_0(s) - 1 &= \mu \hat{R}_1(s), \\
(s+\lambda+n\mu)\hat{R}_n(s) &= \lambda \hat{R}_{n-1}(s) + (n+1)\mu \hat{R}_{n+1}(s), \quad n \geq 1.
\end{aligned} \tag{4.2}$$

By repeated substitution of the second equation into the first equation of (4.2) we get a continued fraction expression for $\hat{R}_0(s)$ as

$$\hat{R}_0(s) = \frac{1}{s+\lambda-} \; \frac{\lambda\mu}{s+\lambda+\mu-} \; \frac{2\lambda\mu}{s+\lambda+2\mu-} \cdots . \tag{4.3}$$

Using the identity (2.5),

$$\hat{R}_0(s) = \frac{1}{s} \, {}_1F_1\left(1; \frac{s}{\mu}+1; \frac{-\lambda}{\mu}\right). \tag{4.4}$$

By a similar procedure used for the discouraged arrivals queue, we obtain

$$\begin{aligned}
\hat{R}_n(s) &= \frac{\lambda^n}{s(s+\mu)(s+2\mu)\cdots(s+n\mu)} \, {}_1F_1\left(n+1; \frac{s}{\mu}+n+1; \frac{-\lambda}{\mu}\right). \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k (n+k)!}{k!\, n!} \left(\frac{\lambda}{\mu}\right)^{n+k} \frac{\mu^{n+k}}{\prod_{i=0}^{n+k}(s+i\mu)}.
\end{aligned}$$

Using partial fraction expansion the above equation becomes

$$\hat{R}_n(s) = \sum_{k=0}^{\infty} \frac{(-1)^k (n+k)!}{k!\, n!} \left(\frac{\lambda}{\mu}\right)^{n+k} \mu^{n+k} \sum_{i=0}^{\infty} \frac{(-1)^i}{\mu^{n+k} i!(n+k-i)!} \frac{1}{s+i\mu}$$

which on inversion becomes

$$\begin{aligned}
R_n(t) &= \sum_{k=0}^{\infty} \frac{(-1)^k (n+k)!}{k!\, n!} \left(\frac{\lambda}{\mu}\right)^{n+k} \left(1-e^{-\mu t}\right)^{n+k} \\
&= \frac{\left[\frac{\lambda}{\mu}\left(1-e^{-\mu t}\right)\right]^n}{n!} exp\left\{-\frac{\lambda}{\mu}\left(1-e^{-\mu t}\right)\right\}, \quad n = 0, 1, 2, \ldots .
\end{aligned} \tag{4.5}$$

We also see that the steady state solutions are given by

$$\begin{aligned}
r_n &= \lim_{t\to\infty} R_n(t) \\
&= exp\left\{\frac{-\lambda}{\mu}\right\} \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!}, \quad n = 0, 1, 2, \ldots .
\end{aligned} \tag{4.6}$$

Aliter:

If

$$g_n(s) \;=\; \frac{s(s+\mu)\ \ldots\ (s+n\mu)}{\lambda^n}\hat{R}_n(s),$$

then the second equation in (4.2) reduces to

$$\begin{aligned}\left(\frac{s}{\mu}+n+1\right)\left(\frac{s}{\mu}+n\right)g_{n-1}(s)-(n+1)\left(\frac{-\lambda}{\mu}\right)g_{n+1}(s)\\ =\left(\frac{s+\lambda}{\mu}+n\right)\left(\frac{s}{\mu}+n+1\right)g_n(s)\end{aligned}$$

We identify this equation with the recurrence relation (2.4) with $a = n+1, c = \frac{s}{\mu}+n+1$, $z = \frac{-\lambda}{\mu}$. It is seen that this also holds for the first equation of (4.2). Thus, we will have

$$g_n(s) \;=\; {}_1F_1\left(n+1; \frac{s}{\mu}+n+1; \frac{-\lambda}{\mu}\right)$$

and hence

$$\hat{R}_n(s) \;=\; \frac{\lambda^n}{s(s+\mu)(s+2\mu)\cdots(s+n\mu)}\ {}_1F_1\left(n+1; \frac{s}{\mu}+n+1; \frac{-\lambda}{\mu}\right)$$

which is the same as (4.5) obtained earlier.

Busy Period:

Proceeding on similar lines as we have done for the discouraged arrivals queue, we obtain the Laplace transform $\hat{b}_B(s)$ of the density function $b_B(t)$ of the busy period as

$$\begin{aligned}\hat{b}_B(s) &= \frac{s+\lambda}{\lambda}-\frac{s}{\lambda}\frac{1}{{}_1F_1\left(1;\frac{s}{\mu}+1;\frac{-\lambda}{\mu}\right)}\\ &= \frac{\mu}{s+\mu}\ \frac{{}_1F_1\left(2;\frac{s}{\mu}+2;\frac{-\lambda}{\mu}\right)}{{}_1F_1\left(1;\frac{s}{\mu}+1;\frac{-\lambda}{\mu}\right)}. \end{aligned}\tag{4.7}$$

For different values of λ with $\mu = 1$, Karlin [12] has given the first three zeros of the function ${}_1F_1\left(1;\frac{s}{\mu}+1;\frac{-\lambda}{\mu}\right)$ which, for many purposes, are sufficient to approximate the probability density function of the duration of the busy period for large values of t. The mean busy period in this case can be easily deduced to be $\dfrac{e^{\frac{\lambda}{\mu}}-1}{\lambda}$, same as the one given by Takács [23].

We observe that the transient solutions for the two models under consideration are not the same (see (3.10) and (4.5)) whereas the steady state solutions are (see (3.15) and (4.6)). This underlines the importance of the transient analysis of systems under study. It is also observed that both the models under consideration have the same mean busy period.

For the purpose of illustration of our observations, we plot the graphs of system size probabilities for the two models by assuming certain values for the parameters λ and μ with the assumption that initially the system is empty.

In Figure 1, some of the system size probabilities, $P_0(t), P_1(t), P_2(t), P_3(t)$ and $P_5(t)$, are plotted for the two models with the parameter values $\lambda = 4.8, \mu = 1.3$. It can be observed from the figure that while the discouraged arrivals queue attains the equilibrium distribution around 15 time units, the infinite server queue reaches it more rapidly around 5 time units.

5 Convergence to Stationarity of Mean

In the performance evaluation of queueing systems the approximation of the underlying stochastic processes by their stationary versions is of considerable importance. As observed in the previous section, the two models under consideration have the same steady state solution but different transient solutions. One would naturally be interested to study the speed of convergence of the underlying process towards its stationarity. Recently, Stadje and Parthasarathy [22] showed how to compute measures of speed of convergence for the $M/M/c$ queue. For the models under consideration, we compute a measure of speed of convergence of the mean number in the system to its stationary value in terms of the model parameters λ and μ.

Let $\hat{G}_A(z,s)$ and $\hat{G}_B(z,s)$ be the Laplace transform of the generating functions for $P_n(t)$ for the discouraged arrivals queue and the infinite server queue respectively. In the sequel, we consider only the discouraged arrivals queue and the analysis for the infinite server queue follows in similar lines. Now,

$$\hat{G}_A(z,s) = \sum_{n=0}^{\infty} \hat{P}_n(s) z^n$$

where $\hat{P}_n(s)$ is given by (3.8). By making use of (2.9) we can deduce

$$\begin{aligned}
\hat{G}_A(z,s) &= \frac{1}{s\left(\frac{s\lambda}{(s+\mu)^2}+1\right)} \sum_{n=0}^{\infty} \frac{(2)_n}{\left(\frac{s\lambda}{(s+\mu)^2}+2\right)_n} \frac{\left(\frac{\lambda z}{s+\mu}\right)^n}{n!} \\
&\qquad \times {}_1F_1\left(n+2; \frac{s\lambda}{(s+\mu)^2}+n+2; \frac{-\lambda\mu}{(s+\mu)^2}\right) \\
&= \frac{1}{s\left(\frac{s\lambda}{(s+\mu)^2}+1\right)}\, {}_1F_1\left(2; \frac{s\lambda}{(s+\mu)^2}+2; \frac{\lambda z}{s+\mu} - \frac{\lambda\mu}{(s+\mu)^2}\right). \qquad (5.1)
\end{aligned}$$

Observe that

$$\lim_{s\to 0} s\hat{G}_A(z,s) = e^{\frac{\lambda}{\mu}(z-1)}$$

as expected. By differentiating n times $\hat{G}_A(z,s)$ and putting $z = 1$ we can find the Laplace transform of the factorial moments. Now, let $M_n(t)$ be the n^{th} factorial moment. Then,

$$\begin{aligned}
\hat{M}_n(s) &= \left.\frac{\partial^n}{\partial z^n}\hat{G}_A(z,s)\right|_{z=1} \\
&= \frac{(n+1)!\left(\frac{\lambda}{s+\mu}\right)^n}{s\left(\frac{s\lambda}{(s+\mu)^2}+1\right)\cdots\left(\frac{s\lambda}{(s+\mu)^2}+n+1\right)} \\
&\qquad \times {}_1F_1\left(n+2; \frac{s\lambda}{(s+\mu)^2}+n+2; \frac{s\lambda}{(s+\mu)^2}\right). \qquad (5.2)
\end{aligned}$$

By expanding and simplifying we get

$$\begin{aligned}\hat{M}_n(s) &= \sum_{k=0}^{\infty} \frac{\lambda^{n+k+1}}{k!} \left(\frac{s}{(s+\mu)^2} \right)^k \\ &\qquad \times \sum_{i=0}^{n+k+1} \binom{n+k+1}{i} (-1)^i \frac{1}{(s+\mu)^n} \frac{1}{s\lambda + i(s+\mu)^2}\end{aligned}$$

which on inversion becomes

$$M_n(t) = \sum_{k=0}^{\infty} \frac{\lambda^{n+k+1}}{k!} g^{*k}(t) * \sum_{i=0}^{n+k+1} \binom{n+k+1}{i} (-1)^i h_{n,\,i}(t) \tag{5.3}$$

where $'*'$ denotes the convolution and $g^{*k}(t)$ is the k-fold convolution of the function $g(t)$ given by

$$g(t) = e^{-\mu t}(1 - \mu t)$$

and the functions $h_{n,\,i}(t)$ are as defined in (3.11), (3.12), (3.13) and (3.14).

In particular, the Laplace transform of the mean number $M_1(t)$ in the system at time t is

$$\hat{M}_1(s) = \frac{2\frac{\lambda}{s+\mu}}{s\left(\frac{s\lambda}{(s+\mu)^2}+1\right)\left(\frac{s\lambda}{(s+\mu)^2}+2\right)} \, {}_1F_1\left(3; \frac{s\lambda}{(s+\mu)^2} + 3; \frac{s\lambda}{(s+\mu)^2}\right) \tag{5.4}$$

$$= \frac{\frac{\lambda}{s+\mu}}{s\left(\frac{s\lambda}{(s+\mu)^2}+1\right)} \, {}_1F_1\left(1; \frac{s\lambda}{(s+\mu)^2} + 2; \frac{s\lambda}{(s+\mu)^2}\right). \tag{5.5}$$

which on inversion yields

$$M_1(t) = \sum_{k=0}^{\infty} \frac{\lambda^{k+2}}{k!} g^{*k}(t) * \sum_{i=0}^{k+2} \binom{k+2}{i} (-1)^i h_{1,i}(t) \tag{5.6}$$

where $g^{*k}(t)$ and $h_{1,i}$ are as given in (3.13) and (3.14). In a similar way, the variance can also be obtained.

Let $X(t)$ be the number of customers in the system at time t and suppose that $X(0) = 0$. Then it is well-known that $X(t)$ is stochastically increasing to a random variable X distributed according to the stationary distribution (see [13]). Consequently, $E(X(t)) \uparrow E(X)$ as $t \uparrow \infty$. Now we will find the measure, say I_A, of the speed of convergence to the stationarity in term of the parameters λ and μ. Thus,

$$\begin{aligned}I_A &= \int_0^{\infty} (E(X) - E(X(t)))\, dt \\ &= \lim_{s\to 0} \int_0^{\infty} e^{-st} (E(X) - E(X(t)))\, dt \\ &= \lim_{s\to 0} \frac{1}{s} \left(\frac{\lambda}{\mu} - s\hat{M}_1(s) \right).\end{aligned}$$

Using (5.5), we obtain

$$I_A = \lim_{s\to 0} \frac{1}{s} \left(\frac{\lambda}{\mu} - \frac{\frac{\lambda}{s+\mu}}{\left(\frac{s\lambda}{(s+\mu)^2}+1\right)} \, {}_1F_1\left(1; \frac{s\lambda}{(s+\mu)^2} + 2; \frac{s\lambda}{(s+\mu)^2}\right) \right).$$

On simplification, we finally deduce

$$I_A = \frac{\lambda}{\mu^2}\left(1+\frac{\lambda}{2\mu}\right). \tag{5.7}$$

Observe that I_A gives the area between the two graphs, the graphs of time-dependent mean number in the system and its stationary counterpart. The smaller the number I_A the faster the convergence.

For the infinite server queue, we have

$$\hat{G}_B(z,s) = \frac{1}{s}\,{}_1F_1\left(1;\frac{s}{\mu}+1;\frac{\lambda}{\mu}(z-1)\right)$$

which on inversion reduces to

$$G_B(z,t) = \exp\left\{-\frac{\lambda}{\mu}\left(1-e^{-\mu t}\right)(1-z)\right\}$$

with mean $\frac{\lambda}{\mu}\left(1-e^{-\mu t}\right)$. Thus, we deduce

$$I_B = \frac{\lambda}{\mu^2} \tag{5.8}$$

which is, as one would expect, less than that of the one for the discouraged arrivals queue and hence the infinite server queue attains equilibrium faster than the discouraged arrivals queue.

As an illustration, in Figure 2, the expected system sizes for the two models are drawn for $\lambda = 3.8, \mu = 1.3$. The discouraged arrivals queue reaches the steady state value around 15 time units whereas the infinite server queue attains it around 5 time units. It can also be observed that the area between the graph of mean system size of the discouraged arrivals queue and its steady state counterpart, *i.e.,* a straight line parallel to the time axis, is more than that of the infinite server queue which confirms our observations.

In the succeeding sections, we analyse the discouraged arrivals queue in discrete time.

6 Discrete Time Solution of the Discouraged Arrivals Queue

With the rapid growth and technological innovation in computer communications, there has been considerable interest in discrete time queueing models [3, 4, 5, 10]. The primary interest of analysing a queueing system is the congestion that may develop and the distribution of and the number of customers in the system at different time points are important measures. We consider here a single server queue where the customers are discouraged by the queue length and the arrivals are geared (or could be controlled) in accordance with the availability of service.

In this section, the time-dependent system size probabilities are obtained using the continued fraction methodology for this discouraged arrivals queueing system in discrete time. Here, when there are n customers in the system, the arrivals occur according to a Bernoulli process with probability $\frac{\lambda}{n+1}$ and service completions occur according to the geometric distribution with probability μ during any time slot. We assume that the probability of more than one arrival (departure) or that arrivals and departures occur simultaneously during a given slot is zero and that the events in different slots are mutually independent. Therefore, $\frac{\lambda}{n+1}+\mu<1$ for all $n \geq 1$. This is ensured if we assume that $\frac{\lambda}{2}+\mu<1$.

Let X_r be the random variable denoting the number of customers in the system at discrete time epoch r. Then $\{X_r, r=0,1,2,\dots\}$ is a discrete time Markov chain. Let

$$P_r(n)=P(X_r=n|X_0=0), \quad n=0,1,2,\dots.$$

The function $P_r(n)$ satisfies the following difference equations:

$$\begin{aligned} P_{r+1}(0) &= (1-\lambda)P_r(0)+\mu P_r(1) \\ P_{r+1}(n) &= \frac{\lambda}{n}P_r(n-1)+(1-\frac{\lambda}{n+1}-\mu)P_r(n)+\mu P_r(n+1), \quad n \geq 1. \end{aligned}$$

If $G_z(n)=\sum_{r=0}^{\infty} P_r(n)z^r$; $|z|<1$, then the above system can be written as

$$\begin{aligned} \left[\frac{1}{z}-(1-\lambda)\right]G_z(0)-\mu G_z(1) &= \frac{1}{z} \\ -\frac{\lambda}{n}G_z(n-1)+\left[\frac{1}{z}-(1-\frac{\lambda}{n+1}-\mu)\right]G_z(n)-\mu G_z(n+1) &= 0, \quad n \geq 1. \end{aligned}$$

For the sake of simplicity we make the transformation $s=\frac{1}{z}-1$ and denote $G_z(n)$ by $G_s(n)$. Then the above system reduces to

$$\begin{aligned} (s+\lambda)G_s(0)-\mu G_s(1) &= s+1, \\ -\tfrac{\lambda}{n}G_s(n-1)+(s+\tfrac{\lambda}{n+1}+\mu)G_s(n)-\mu G_s(n+1) &= 0, \qquad n \geq 1. \end{aligned} \tag{6.1}$$

From the first equation of (6.1), we obtain

$$G_s(0)=\frac{s+1}{s+\lambda-\mu\frac{G_s(1)}{G_s(0)}} \tag{6.2}$$

and the second equation of (6.1) can be written as, for $n \geq 1$,

$$\frac{G_s(n)}{G_s(n-1)} = \frac{\frac{\lambda}{n}}{s+\frac{\lambda}{n+1}+\mu-\mu\frac{G_s(n+1)}{G_s(n)}}. \tag{6.3}$$

Iterating this equation, we get

$$\frac{G_s(n)}{G_s(n-1)} = \frac{\frac{\lambda}{n}}{s+\frac{\lambda}{n+1}+\mu-}\;\frac{\frac{\lambda}{n+1}\mu}{s+\frac{\lambda}{n+2}+\mu-}\;\cdots. \tag{6.4}$$

By substituting (6.4) in (6.2) we get a continued fraction expression for $G_s(0)$:

$$G_s(0) = \frac{s+1}{s+\lambda-}\;\frac{\lambda\mu}{s+\frac{\lambda}{2}+\mu-}\;\frac{\frac{\lambda}{2}\mu}{s+\frac{\lambda}{3}+\mu-}\;\cdots. \tag{6.5}$$

By making use of (2.5) the above equation simplifies to

$$\begin{aligned} G_s(0) &= (s+1)\left\{ s+\lambda+(s+\mu)\left[\left(\frac{s\lambda}{(s+\mu)^2}+1\right)\frac{{}_1F_1\left(1;\frac{s\lambda}{(s+\mu)^2}+1;\frac{-\lambda\mu}{(s+\mu)^2}\right)}{{}_1F_1\left(2;\frac{s\lambda}{(s+\mu)^2}+2;\frac{-\lambda\mu}{(s+\mu)^2}\right)}\right.\right. \\ &\qquad \left.\left. -\left(\frac{\lambda}{s+\mu}+1\right)\right]\right\}^{-1} \\ &= (s+1)\left\{(s+\mu)\left(\frac{s\lambda}{(s+\mu)^2}+1\right)\frac{{}_1F_1\left(1;\frac{s\lambda}{(s+\mu)^2}+1;\frac{-\lambda\mu}{(s+\mu)^2}\right)}{{}_1F_1\left(2;\frac{s\lambda}{(s+\mu)^2}+2;\frac{-\lambda\mu}{(s+\mu)^2}\right)}-\mu\right\}^{-1}. \end{aligned}$$

Use of (2.6) yields

$$\begin{aligned} G_s(0) &= \frac{s+1}{s}\left\{\left(\frac{s\lambda}{(s+\mu)^2}+1\right)\frac{{}_1F_1\left(1;\frac{s\lambda}{(s+\mu)^2}+1;\frac{-\lambda\mu}{(s+\mu)^2}\right)}{{}_1F_1\left(2;\frac{s\lambda}{(s+\mu)^2}+2;\frac{-\lambda\mu}{(s+\mu)^2}\right)}\right. \\ &\qquad \left. +\frac{\lambda\mu}{(s+\mu)^2}\frac{{}_1F_1\left(1;\frac{s\lambda}{(s+\mu)^2}+2;\frac{-\lambda\mu}{(s+\mu)^2}\right)}{{}_1F_1\left(2;\frac{s\lambda}{(s+\mu)^2}+2;\frac{-\lambda\mu}{(s+\mu)^2}\right)}\right\}^{-1}. \end{aligned}$$

Again, making use of (2.7) and using the fact that ${}_1F_1(0;c;z)=1$, we obtain

$$G_s(0) = \frac{s+1}{s\left(\frac{s\lambda}{(s+\mu)^2}+1\right)}\,{}_1F_1\left(2;\frac{s\lambda}{(s+\mu)^2}+2;\frac{-\lambda\mu}{(s+\mu)^2}\right). \tag{6.6}$$

In a similar way, we will find $G_s(n)$. Making use of (2.6), (2.7) and (2.8), we can write (6.4) as, for $n\geq 1$,

$$\frac{G_s(n)}{G_s(n-1)} = \frac{1}{n}\,\frac{\frac{(n+1)\lambda}{s+\mu}}{\left(\frac{s\lambda}{(s+\mu)^2}+n+1\right)}\,\frac{{}_1F_1\left(n+2;\frac{s\lambda}{(s+\mu)^2}+n+2;\frac{-\lambda\mu}{(s+\mu)^2}\right)}{{}_1F_1\left(n+1;\frac{s\lambda}{(s+\mu)^2}+n+1;\frac{-\lambda\mu}{(s+\mu)^2}\right)}.$$

Iterating this equation and using (6.6), we get

$$\begin{aligned} G_s(n) &= G_s(0)\prod_{i=1}^{n}\frac{G_s(i)}{G_s(i-1)} \\ &= \frac{(s+1)\frac{n+1}{s}\left(\frac{\lambda}{s+\mu}\right)^n}{\left(\frac{s\lambda}{(s+\mu)^2}+1\right)\ldots\left(\frac{s\lambda}{(s+\mu)^2}+n+1\right)} \\ &\qquad \times\,{}_1F_1\left(n+2;\frac{s\lambda}{(s+\mu)^2}+n+2;\frac{-\lambda\mu}{(s+\mu)^2}\right). \end{aligned} \tag{6.7}$$

We can extract the coefficients of z from (6.7) by first expanding $G_s(n)$ and then returning to the variable z. We then get

$$\begin{aligned} G_z(n) &= \sum_{k=0}^{\infty}\frac{\lambda^{n+k+1}(-\mu)^k}{k!\,n!}\frac{z^{n+2k}}{\lambda}\sum_{m=0}^{\infty}z^m\sum_{j=0}^{m}\binom{n+2k-1+j}{j}(1-\mu)^j \\ &\quad +\sum_{k=0}^{\infty}\frac{\lambda^{n+k+1}(-\mu)^k}{k!\,n!}\sum_{i=1}^{n+k+1}\binom{n+k+1}{i}(-1)^i z^{n+2k} \\ &\qquad \times\sum_{m=1}^{\infty}z^m\sum_{j=0}^{m-1}\binom{n+2k-1+j}{j}\frac{(1-\mu)^j}{i(s_1-s_2)}\left[(s_1+1)^{m-j}-(s_2+1)^{m-j}\right] \end{aligned}$$

where s_1, s_2 are given as $-\left(\frac{\lambda}{2i}+\mu\right)+\sqrt{\frac{\lambda^2}{4i^2}+\frac{\lambda\mu}{i}}, -\left(\frac{\lambda}{2i}+\mu\right)-\sqrt{\frac{\lambda^2}{4i^2}+\frac{\lambda\mu}{i}}$ respectively. Note that $|s_j+1| \leq 1, j=1,2$ under the assumption $\frac{\lambda}{2}+\mu<1$. Further simplification of $G_z(n)$ yields

$$G_z(n) = z^n \sum_{m=0}^{\infty} \alpha(n,m) z^m + z^n \sum_{m=1}^{\infty} \beta(n,m) z^m \tag{6.8}$$

where

$$\alpha(n,m) = \sum_{k=0}^{\left[\frac{m}{2}\right]} \frac{\lambda^{n+k}(-\mu)^k}{k!\,n!} \sum_{j=0}^{m-2k} \binom{n+2k-1+j}{j} (1-\mu)^j \tag{6.9}$$

$$\begin{aligned}\beta(n,m) &= \sum_{k=0}^{\left[\frac{m-1}{2}\right]} \frac{\lambda^{n+k+1}(-\mu)^k}{k!\,n!} \sum_{i=1}^{n+k+1} \binom{n+k+1}{i} (-1)^i \sum_{j=0}^{m-2k-1} \binom{n+2k-1+j}{j} \\ &\quad \times \frac{(1-\mu)^j}{i(s_1-s_2)}\left[(s_1+1)^{m-2k-j}-(s_2+1)^{m-2k-j}\right]\end{aligned} \tag{6.10}$$

with $[x]$ denoting the integral part of x.

Thus, the system size probabilities are given by

$$P_r(n) = \begin{cases} 0, & r<n \\ \dfrac{\lambda^n}{n!}, & r=n \\ \alpha(n,r-n)+\beta(n,r-n), & r>n \end{cases} \tag{6.11}$$

where $\alpha(n,m)$ and $\beta(n,m)$ are as given above.

Since $s=\frac{1}{z}-1$, using the final value theorem of z-transforms, we can obtain the steady state solutions for $n=0,1,2,\ldots$ as

$$\begin{aligned}\lim_{r\to\infty} P_r(n) &= \lim_{s\to 0} s\, G_s(n) \\ &= exp\left\{\frac{-\lambda}{\mu}\right\} \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!}, \quad \text{using (6.7).}\end{aligned}$$

7 Busy Period in Discrete Time

Let an arrival to an empty system start a busy period and define T as the time until the server becomes idle for the first time. Thus T is what commonly referred to as the length or duration of a busy period.

Busy period of a single server queue is the first passage time to state zero. In view of this we modify the actual process by making state 0 the absorbing state, *i.e.*, $\lambda_0=0$. Then $\tilde{P}_r(0)$ gives the distribution function of the busy period and hence $\tilde{P}_r(0)=Pr(T\leq r)$. The function $\tilde{P}_r(n)$ satisfies the following difference equations:

$$\tilde{P}_{r+1}(0) = \tilde{P}_r(0)+\mu\tilde{P}_r(1) \tag{7.1}$$

$$\tilde{P}_{r+1}(1) = (1-\frac{\lambda}{2}-\mu)\tilde{P}_r(1)+\mu\tilde{P}_r(2) \tag{7.2}$$

$$\tilde{P}_{r+1}(n) = \frac{\lambda}{n}\tilde{P}_r(n-1)+(1-\frac{\lambda}{n+1}-\mu)\tilde{P}_r(n)+\mu\tilde{P}_r(n+1), \quad n\geq 2. \tag{7.3}$$

Considering (7.2) and (7.3) and by an analysis similar to the one in the previous section, the generating function of $\tilde{P}_r(1)$ for the queueing system under consideration is given by

$$G_z(1) = \frac{\frac{2}{1-(1-\mu)z}}{\left(\frac{z(1-z)\lambda}{[1-(1-\mu)z]^2}+2\right)} \frac{{}_1F_1\left(3; \frac{z(1-z)\lambda}{[1-(1-\mu)z]^2}+3; \frac{-\lambda\mu z^2}{[1-(1-\mu)z]^2}\right)}{{}_1F_1\left(2; \frac{z(1-z)\lambda}{[1-(1-\mu)z]^2}+2; \frac{-\lambda\mu z^2}{[1-(1-\mu)z]^2}\right)}. \tag{7.4}$$

Let $Q_r(0)$ be the probability mass function for the length of the busy period. Then $Q_{r+1}(0) = \tilde{P}_{r+1}(0) - \tilde{P}_r(0)$. Now, from (7.1), $Q_{r+1}(0)$ can be written as

$$Q_{r+1}(0) = \mu \tilde{P}_r(1), \qquad r = 0, 1, 2, \ldots$$

where $\tilde{P}_r(1)$ is the coefficient of z^r in (7.4). The mean busy period, say M, is given as

$$\begin{aligned} M &= 1 + \mu \left. \frac{dG_z(1)}{dz} \right|_{z=1} \\ &= 1 + \frac{\lambda + 2\mu}{2\mu^2} - 1 + \frac{\lambda^2}{6\mu^3} {}_1F_1\left(1; 4; \frac{\lambda}{\mu}\right) \\ &= \frac{1}{\mu} + \frac{\lambda}{2\mu^2} {}_1F_1\left(1; 3; \frac{\lambda}{\mu}\right), \text{ by using (2.7).} \end{aligned}$$

By making use of ${}_1F_1(1; 3; -z) = \frac{2}{z^2}(z + e^{-z} - 1)$, we obtain the mean busy period as

$$M = \frac{e^{\frac{\lambda}{\mu}} - 1}{\lambda}. \tag{7.5}$$

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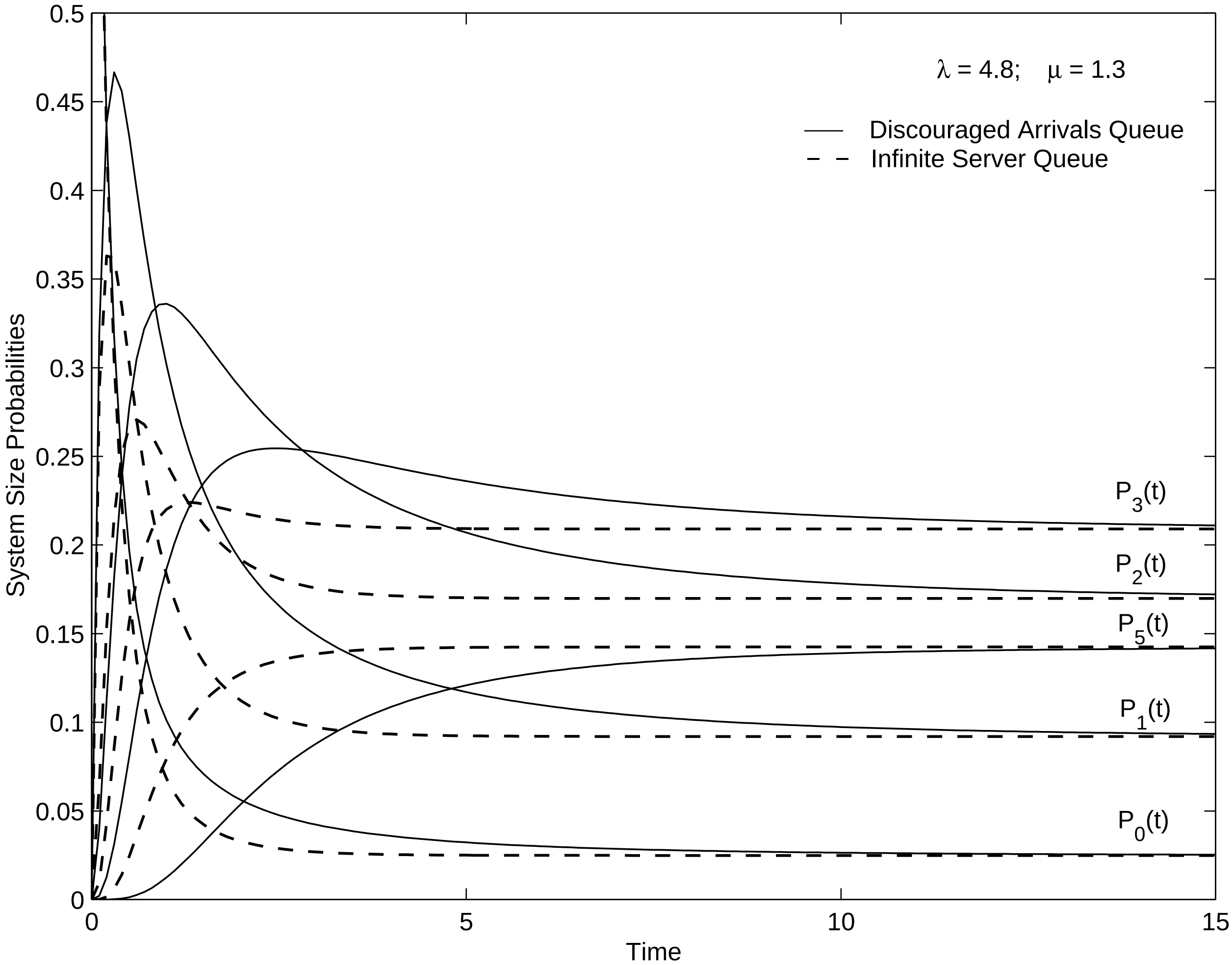

Figure 1: System size probabilities for the discouraged arrivals queue and the infinite server queue

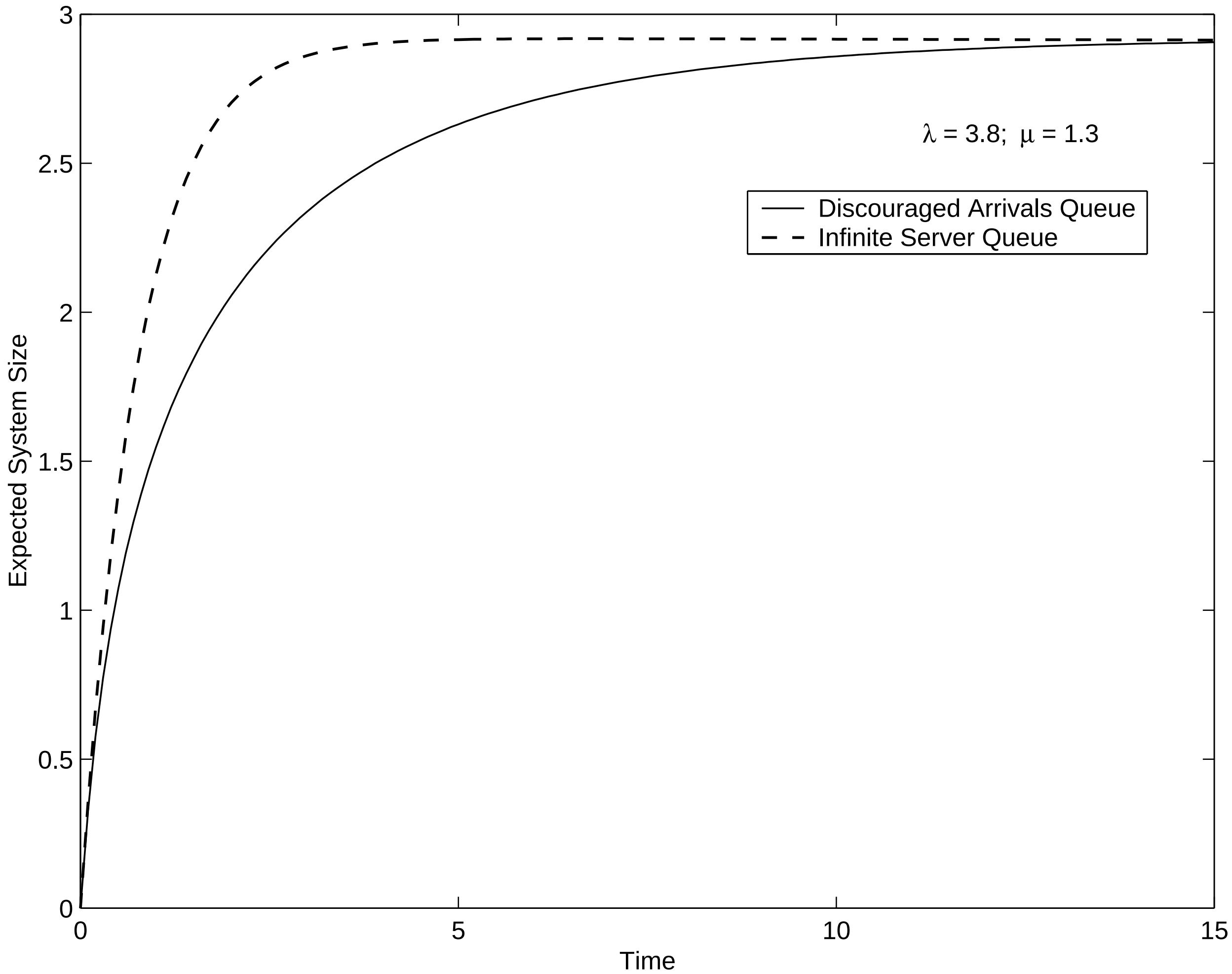

Figure 2: Expected system sizes for the discouraged arrivals queue and the infinite server queue