Approximate next-to-next-to-leading-order threshold resummation in heavy flavor decays

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We present an approximate next-to-next-to-leading-order evaluation of the QCD form factor resumming large logarithmic perturbative contributions in semi-inclusive heavy flavor decays.

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I. INTRODUCTION

In this paper we present an approximate next-to-next-to-leading order (NNLO) evaluation of the QCD form factor resumming threshold logarithmic contributions in semi-inclusive heavy flavor decays.

In semi-inclusive processes, final gluon radiation is strongly inhibited in the phase space regions where the observed final state obtains its maximum energy, therefore opening the way to soft and collinear singularities. The perturbative calculation of the differential cross section is plagued, in that limit, by large logarithms. In order to improve the reliability of the perturbative calculation, these large logarithms need to be resummed.

Let us consider the rate of semi-inclusive heavy flavor decays. The Mellin N-moments of the rate contain double logarithmic contributions and have an expansion of the form

$$\frac{1}{\Gamma_B}\Gamma_N(\alpha_S)\equiv\int_0^1 dx\, x^{N-1}\frac{1}{\Gamma_B}\frac{d\Gamma}{dx}(x;\alpha_S) \tag{1}$$

$$\begin{aligned}&=1+\sum_{n=1}^{\infty}\sum_{m=0}^{2n}G_{n\,m}\alpha_S^nL^m+\sum_{l=1}^{\infty}\alpha_S^lR_N^{(l)}\\&=1+G_{12}\alpha_SL^2+G_{11}\alpha_SL+G_{10}\alpha_S+G_{24}\alpha_S^2L^4\\&\quad+G_{23}\alpha_S^2L^3+G_{22}\alpha_S^2L^2+\cdots+\alpha_SR_N^{(1)}\\&\quad+\alpha_S^2R_N^{(2)}+\alpha_S^3R_N^{(3)}+\cdots,\end{aligned} \tag{2}$$

where

$$L\equiv\ln N. \tag{3}$$

$R_N(\alpha_S)$ is a remainder function, which does not contain large logarithms and has a perturbative expansion of the form

$$R_N(\alpha_S)\equiv\sum_{l=1}^{\infty}\alpha_S^lR_N^{(l)}. \tag{4}$$

The running coupling is evaluated at a general renormalization scale $\mu\neq Q$, where Q is the hard scale:

$$\alpha_S=\alpha_S(\mu^2). \tag{5}$$

The logarithmic terms have an exponential structure, so one can write [1,2]

$$\frac{1}{\Gamma_B}\Gamma_N(\alpha_S)=C(\alpha_S)f_N(\alpha_S) \tag{6}$$

where the form factor $f_N(\alpha_S)$ reads

$$\begin{aligned}f_N(\alpha_S)&=\exp\left[\sum_{n=1}^{\infty}\sum_{k=1}^{n+1}c_{n\,m}\alpha_S^nL^k\right]\\&=\exp[c_{12}\alpha_SL^2+c_{11}\alpha_SL+c_{23}\alpha_S^2L^3+c_{22}\alpha_S^2L^2\\&\quad+c_{2\,1}\alpha_S^2L+c_{34}\alpha_S^3L^4+\cdots].\end{aligned} \tag{7}$$

Note that the exponent contains only the first term α_SL^2 of the double-logarithmic series $(\alpha_SL^2)^n$. The advantage of the exponentiation is therefore that we can predict $\ln f_N$ reliably for $\alpha_SL\ll1$, that is, for a larger region than $\alpha_SL^2\ll1$, where the perturbative expansion of f_N holds. In fact, the other terms of $(\alpha_SL^2)^n$ come purely from the expansion of the exponential function, as can be seen in formula (2).

The prefactor in Eq. (6) is the coefficient function, having an expansion in powers of α_S:

$$C(\alpha_S)=1+\sum_{n=1}^{\infty}C_n\alpha_S^n=1+C_1\alpha_S+C_2\alpha_S^2+\cdots. \tag{8}$$

The double sum in the exponent is usually organized as a series of functions, which resum "strips" in the (n,k) plane:

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$$f_N(\alpha_S)=\exp\left[L\, g_1(\beta_0\alpha_S L)+\sum_{n=2}^{\infty}\alpha_S^{n-2}g_n(\beta_0\alpha_S L)\right]$$

$$=\exp[L\, g_1(\beta_0\alpha_S L)+g_2(\beta_0\alpha_S L)+\alpha_S g_3(\beta_0\alpha_S L)+\alpha_S^2 g_4(\beta_0\alpha_S L)+\cdots]. \qquad (9)$$

The functions $g_i(\lambda)$ have a power-series expansion:

$$g_i(\lambda)=\sum_{n=1}^{\infty}g_{in}\lambda^n \qquad (10)$$

where $\lambda=\beta_0\alpha_S L$. They are all homogeneous functions: $g_i(0)=0$. This property ensures the normalization of the form factor $f_{N=1}=1$. The resumming at LO, NLO and NNLO is referred, respectively, to series in $L(\alpha_S L)^n$, $(\alpha_S L)^n$ and $\alpha_S(\alpha_S L)^n$.

We have computed the function $g_3(\lambda)$, which is necessary for the NNLO resumming. Not all the quantities determining $g_3(\lambda)$ are known exactly. A_3 is only known numerically from a fit of the three-loop Altarelli-Parisi (AP) splitting function to the known moments. B_2 is unknown and we approximate it with the infrared-regular part of the two-loop AP splitting function. The coefficient functions and the remainder functions for radiative and semileptonic B decays have been computed to NLO in Ref. [3]. A complete NNLO computation of the distributions requires also the knowledge of the two-loop coefficient function, which at present is unknown for any distribution. After the resummation of the threshold logarithms in N-space, we have returned to the form factor in x-space, by inverting $f_N(\alpha_S)$ with analytic and numerical procedures.

II. QCD FORM FACTOR AT NNLO

Let us briefly describe the derivation of the NNLO form factor in N-space. By NNLO accuracy, we mean the resummation of all the infrared logarithms up to and including

$$\text{NNLO:}\quad \alpha_S^n L^{n-1}. \qquad (11)$$

The general expression of the form factor is

$$\log f_N(\alpha_S)=\int_0^1 dz\,\frac{z^{N-1}-1}{1-z}\left\{\int_{Q^2(1-z)^2}^{Q^2(1-z)}\frac{dk^2}{k^2}A[\alpha_S(k^2)]+D[\alpha_S(Q^2(1-z)^2)]+B[\alpha_S(Q^2(1-z))]\right\}. \qquad (12)$$

Let us recall one difference between annihilation processes, such as the Drell-Yan process, and other processes, such as the present one or deep inelastic scattering (DIS). In the annihilation processes, the initial partons reduce their momenta by irradiation, before actually annihilating; when $\tau=Q^2/s\to 1$, only the emission of soft partons is allowed. Likewise, in the other processes, such as, for instance, DIS, and in the limit $x=Q^2/2Q\cdot p\to 1$, only soft emission is allowed, before the scattering; however, after the scattering, the parton fragments with the only kinematical constraint of having a low virtuality and collinear emission is no longer forbidden.

The functions $A(\alpha_S)$, $D(\alpha_S)$ and $B(\alpha_S)$ have a perturbative expansion in powers of α_S:

$$A(\alpha_S)=\sum_{n=1}^{\infty}A_n\alpha_S^n=A_1\alpha_S+A_2\alpha_S^2+A_3\alpha_S^3+\cdots;$$

$$D(\alpha_S)=\sum_{n=1}^{\infty}D_n\alpha_S^n=D_1\alpha_S+D_2\alpha_S^2+\cdots; \qquad (13)$$

$$B(\alpha_S)=\sum_{n=1}^{\infty}B_n\alpha_S^n=B_1\alpha_S+B_2\alpha_S^2+\cdots.$$

Let us observe that, in general, the transverse momentum rule is a guess, as it has not been proven to such an accuracy. The universality of B_2, in general, is a debated problem. If we neglect the variation of the coupling with the scale (frozen coupling), we find logarithmic terms of the form

$$A_1\alpha_S L^2,\ \ A_2\alpha_S^2L^2,\ \ A_3\alpha_S^3L^2,\ \ldots$$

$$D_1\alpha_S L,\ \ D_2\alpha_S^2 L,\ \ldots \qquad (14)$$

$$B_1\alpha_S L,\ \ B_2\alpha_S^2 L,\ \ldots.$$

Then, to NNLO accuracy, one needs the first three terms in the expansion of $A(\alpha_S)$ and the first two terms of the functions $D(\alpha_S)$ and $B(\alpha_S)$.

The three-loop coupling, according to the definition given in the PDG [4], reads

$$\alpha_s(\mu^2)=\frac{1}{\beta_0\log\mu^2/\Lambda^2}-\frac{\beta_1}{\beta_0^3}\frac{\log\log\mu^2/\Lambda^2}{\log^2\mu^2/\Lambda^2}+\frac{\beta_1^2}{\beta_0^5}\frac{\log^2\log\mu^2/\Lambda^2-\log\log\mu^2/\Lambda^2-1}{\log^3\mu^2/\Lambda^2}+\frac{\beta_2}{\beta_0^4}\frac{1}{\log^3\mu^2/\Lambda^2}. \qquad (15)$$

The asymptotic expansion of the coupling is basically an expansion in inverse powers of $\log\mu^2/\Lambda^2$. The first three coefficients of the β function, defined as

$$\frac{d\alpha_s}{d\log\mu^2}=-\beta_0\alpha_S^2-\beta_1\alpha_S^3-\beta_2\alpha_S^4-\cdots, \qquad (16)$$

read

$$\beta_0=\frac{11C_A-2n_F}{12\pi}=\frac{33-2n_F}{12\pi}=0.87535-0.05305\,n_F,$$

$$\beta_1=\frac{17C_A^2-5C_An_F-3C_Fn_F}{24\pi^2}=\frac{153-19\,n_F}{24\pi^2}$$
$$=0.64592-0.08021\,n_F, \tag{17}$$

$$\beta_2=\frac{2857-5033/9\,n_F+325/27\,n_F^2}{128\pi^3}$$
$$=0.71986-0.140904\,n_F+0.003032\,n_F^2.$$

Let us note that β_0 and β_1 are renormalization-scheme independent, while β_2 is not and we have given its value in the modified minimal subtraction ($\overline{\rm MS}$) scheme [5].

Integrating the β-function Eq. (16) on both sides, one obtains

$$\alpha_S(k^2)=\alpha_S(Q^2)-\beta_0\alpha_S^2(Q^2)\log\frac{k^2}{Q^2}-\beta_1\alpha_S^3(Q^2)\log\frac{k^2}{Q^2}$$
$$-\beta_2\alpha_S^4(Q^2)\log\frac{k^2}{Q^2}+(\text{iterations}). \tag{18}$$

Substituting the above expression into Eq. (12), one sees that a β_0 insertion corresponds to the additional factor $\alpha_S L$, that of β_1 to $\alpha_S^2 L$ and that of β_2 to $\alpha_S^3 L$:

$$\beta_0:\alpha_S L,\quad \beta_1:\alpha_S^2L,\quad \beta_2:\alpha_S^3L. \tag{19}$$

Therefore, in the terms containing NNLO coefficients, the coupling can be replaced with the one-loop one and in the NLO terms the coupling can be replaced with the two-loop one, so that one has

$$A(\alpha_S)=A_1\alpha_{S,3L}+A_2\alpha_{S,2L}^2+A_3\alpha_{S,1L}^3;$$

$$D(\alpha_S)=D_1\alpha_{S,2L}+D_2\alpha_{S,1L}^2; \tag{20}$$

$$B(\alpha_S)=B_1\alpha_{S,2L}+B_2\alpha_{S,1L}^2.$$

Furthermore, the β_1^2 term in $A_2\alpha_{S,2L}^2$ can be neglected, as it is a N^3LO contribution. After replacing Eqs. (20) into Eq. (12), one performs a straightforward integration over k^2, the gluon transverse momentum. The integration over z, the longitudinal gluon momentum, is easily done using the approximation [1]

$$z^{N-1}-1\simeq-\theta\left(1-z-\frac{1}{n}\right). \tag{21}$$

This approximation misses the term proportional to $A_1\zeta_2$, where $\zeta_2=\pi^2/6$, which can be obtained in the following way. Using the large-N approximation derived in [6], one obtains, in the n variable,

$$\int_0^1 dz\,\frac{z^{N-1}-1}{1-z}\int_{Q^2(1-z)^2}^{Q^2(1-z)}\frac{dk^2}{k^2}A_1\alpha_S(k^2)$$
$$=\sum_{k=1}^{\infty}c_k\int_0^1 dz\,\frac{z^{N-1}-1}{1-z}\log^k(1-z)$$
$$\simeq\sum_{k=1}^{\infty}c_k\frac{(-1)^{k+1}}{k+1}\left[\log^{k+1}n+\frac{\zeta_2}{2}k(k+1)\log^{k-1}n\right]$$
$$=[\text{LO}]+[\text{NNLO}]. \tag{22}$$

Therefore,

$$[\text{NNLO}]=\frac{\zeta_2}{2}\frac{\partial^2}{\partial(\log n)^2}[\text{LO}]. \tag{23}$$

We have used the variable $n=N/N_0$ instead of N, where $N_0\equiv e^{-\gamma_E}=0.561459\ldots$. Within this alternative representation, the terms proportional to γ_E and to γ_E^2 disappear. This scheme is probably more accurate as Feynman diagram computation directly in N-space brings factors containing $\Gamma(1-\epsilon)$ with $D=4-2\epsilon$ the space-time dimension.

The advantage of the variable N is that the total rate is directly reproduced by setting $N=1$, while in the variable n it is given by $f_{n=1/N_0}$. These two variables differ by terms of higher order in γ_E.

The lowest-order term, computed within the approximation (21), reads

$$[\text{LO}]=-\frac{A_1}{2\beta_0}\left[\log\frac{s}{n^2}\log\log\frac{s}{n^2}+\log s\,\log\log s\right.$$
$$\left.-2\,\log\frac{s}{n}\log\log\frac{s}{n}\right]. \tag{24}$$

One then obtains

$$\log f_N(\alpha_S) = -\frac{A_1}{2\beta_0}[\log s/n^2 \ \log\log s/n^2 + \log s \ \log\log s - 2\ \log s/n \ \log\log s/n] + \frac{\beta_0 A_2 - \beta_1 A_1}{2\beta_0^3}[\log\log s - 2\ \log\log s/n$$
$$+\log\log s/n^2] - \frac{\beta_1 A_1}{4\beta_0^3}[\log^2 \ \log s/n^2 - 2\ \log^2 \ \log s/n + \log^2 \ \log s] + \frac{D_1}{2\beta_0}[\log\log s/n^2 - \log\log s]$$
$$+\frac{B_1}{\beta_0}[\log\log s/n - \log\log s] - \frac{A_3}{4\beta_0^3}\left[\frac{1}{\log s/n^2} - \frac{2}{\log s/n} + \frac{1}{\log s}\right] - \frac{A_1\zeta_2}{2\beta_0}\left[\frac{2}{\log s/n^2} - \frac{1}{\log s/n} - \frac{1}{\log s}\right]$$
$$-\frac{A_1\beta_2}{8\beta_0^4}\left[\frac{1}{\log s/n^2} - \frac{2}{\log s/n} + \frac{1}{\log s}\right] + \frac{A_2\beta_1}{4\beta_0^4}\left[2\frac{\log\log s/n^2}{\log s/n^2} - 4\frac{\log\log s/n}{\log s/n} + 2\frac{\log\log s}{\log s} + \frac{3}{\log s/n^2} - \frac{6}{\log s/n}\right.$$
$$\left.+\frac{3}{\log s}\right] - \frac{A_1\beta_1^2}{4\beta_0^5}\left[\frac{\log^2 \ \log s/n^2}{\log s/n^2} - 2\frac{\log^2 \ \log s/n}{\log s/n} + \frac{\log^2 \ \log s}{\log s} + 2\frac{\log\log s/n^2}{\log s/n^2} - 4\frac{\log\log s/n}{\log s/n} + 2\frac{\log\log s}{\log s}\right.$$
$$\left.+\frac{1}{\log s/n^2} - \frac{2}{\log s/n} + \frac{1}{\log s}\right] + \frac{D_1\beta_1}{2\beta_0^3}\left[\frac{\log\log s/n^2}{\log s/n^2} - \frac{\log\log s}{\log s} + \frac{1}{\log s/n^2} - \frac{1}{\log s}\right] - \frac{D_2}{2\beta_0^2}\left[\frac{1}{\log s/n^2} - \frac{1}{\log s}\right]$$
$$+\frac{B_1\beta_1}{\beta_0^3}\left[\frac{\log\log s/n}{\log s/n} - \frac{\log\log s}{\log s} + \frac{1}{\log s/n} - \frac{1}{\log s}\right] - \frac{B_2}{\beta_0^2}\left[\frac{1}{\log s/n} - \frac{1}{\log s}\right]. \tag{25}$$

The next step is to write the above result as a function of the three-loop coupling. The inverse of Eq. (15) reads

$$\log s = \frac{1}{\beta_0\alpha_S} + \frac{\beta_1}{\beta_0^2}\log(\beta_0\alpha_S) - \left(\frac{\beta_1^2}{\beta_0^3} - \frac{\beta_2}{\beta_0^2}\right)\alpha_S + O(\alpha_S^2), \tag{26}$$

where s is the square of the hard scale in units of the QCD scale,

$$s \equiv \frac{Q^2}{\Lambda^2}. \tag{27}$$

After the replacement (26), terms of higher order with respect to NNLO are generated, which must be discarded. One finally makes the expansion in γ_E up to second order to obtain Eq. (40).

The quantities A_1 and A_2 are known analytically [1,7]:

$$A_1 = \frac{C_F}{\pi} = 0.424413,$$
$$A_2 = \frac{C_F}{\pi^2}\left[C_A\left(\frac{67}{36} - \frac{\pi^2}{12}\right) - \frac{5}{9}n_F T_R\right] = 0.42095 - 0.03753\, n_F, \tag{28}$$

where $C_A = N_c = 3$, $T_R = 1/2$ and $n_F = 3$ is the number of active quark flavors in b decay. The value for A_2 is given in the $\overline{\text{MS}}$ scheme for the coupling constant. At present, only a numerical estimate of A_3 is available [6,8]:

$$A_3 = 0.59413 - 0.09272\, n_F - 0.00040\, n_F^2. \tag{29}$$

The soft quantities D_1 and D_2 [9] are known analytically:

$$D_1 = -\frac{C_F}{\pi} = -0.424413,$$
$$D_2 = -\frac{C_F}{\pi^2}\left[\left(\frac{37}{108} + \frac{7}{18}\pi^2 - \frac{9}{4}\zeta(3)\right)C_A + \left(\frac{1}{27} - \frac{1}{9}\pi^2\right)T_R\, n_F\right]$$
$$= -0.59826 - 0.07157\, n_F. \tag{30}$$

Numerically, $\zeta(3) \cong 1.20206$. The constant D_1 is renormalization-scheme independent, while D_2 is not and we have given its value in the $\overline{\text{MS}}$ scheme. The latter quantity turns out to be the most important one to determine the NNLO effects.

The collinear quantity B_1 is known analytically,

$$B_1 = -\frac{3}{4}\frac{C_F}{\pi} = -0.31831. \tag{31}$$

The two-loop quantity B_2 is unknown and we approximate it with the infrared-regular part in the two-loop Altarelli-Parisi splitting function $P_{qq}(z)$. In general, the latter is naturally decomposed in the soft limit as

$$P_{qq}(z) = \left[\frac{A(\alpha_S(Q^2(1-z)))}{1-z}\right]_+ - K(z;\alpha_S) + K(\alpha_S)\,\delta(1-z). \tag{32}$$

The functions $K(z;\alpha_S)$ and $K(\alpha_S)$ have an expansion in powers of α_S:

$$K(\alpha_S) = K_1\alpha_S + K_2\alpha_S^2 + \cdots,$$
$$K(z;\alpha_S) = K_1(z)\,\alpha_S + K_2(z)\,\alpha_S^2 + \cdots. \tag{33}$$

Since the splitting function has a vanishing first moment,

$$K(\alpha_S)=\int_0^1 K(z;\alpha_S)dz, \tag{34}$$

it holds that

$$K_1=\int_0^1 K_1(z)dz$$

and

$$K_2=\int_0^1 K_2(z)dz. \tag{35}$$

In leading order, it holds that $B_1=K_1$. Our approximation then is

$$B_2\approx K_2 \tag{36}$$

where, in the $\overline{\mathrm{MS}}$ scheme [10],

$$K_2=-\frac{C_F}{\pi^2}\left[C_F\left(\frac{3}{32}-\frac{\pi^2}{8}+\frac{3}{2}\zeta(3)\right)+C_A\left(\frac{17}{96}+\frac{11}{72}\pi^2-\frac{3}{4}\zeta(3)\right)-\left(\frac{1}{24}+\frac{\pi^2}{18}\right)T_R\,n_F\right]$$
$$=-0.43695+0.03985\,n_F. \tag{37}$$

A. Results

The functions g_1 and g_2 have the following expressions [11,12]:

$$g_1\left(\lambda;\frac{\mu^2}{Q^2}\right)=-\frac{A_1}{2\beta_0}\frac{1}{\lambda}[(1-2\lambda)\log(1-2\lambda)-2(1-\lambda)\log(1-\lambda)], \tag{38}$$

$$\begin{aligned}g_2\left(\lambda;\frac{\mu^2}{Q^2}\right)=&+\frac{A_2}{2\beta_0^2}[\log(1-2\lambda)-2\log(1-\lambda)]\\&+\frac{A_1\gamma_E}{\beta_0}[\log(1-2\lambda)-\log(1-\lambda)]\\&-\frac{\beta_1A_1}{4\beta_0^3}[\log^2(1-2\lambda)-2\log^2(1-\lambda)+2\log(1-2\lambda)-4\log(1-\lambda)]\\&+\frac{D_1}{2\beta_0}\log(1-2\lambda)+\frac{B_1}{\beta_0}\log(1-\lambda)\\&+\frac{A_1}{2\beta_0}[\log(1-2\lambda)-2\log(1-\lambda)]\log\frac{\mu^2}{Q^2}.\end{aligned} \tag{39}$$

Our result for the NNLO function g_3 reads

$$\begin{aligned}g_3\left(\lambda;\frac{\mu^2}{Q^2}\right)=&-\frac{A_3}{2\beta_0^2}\left[\frac{\lambda}{1-2\lambda}-\frac{\lambda}{1-\lambda}\right]-\frac{A_1\zeta_2}{2}\left[\frac{4\lambda}{1-2\lambda}-\frac{\lambda}{1-\lambda}\right]-\frac{A_1\beta_2}{4\beta_0^3}\left[\frac{2\lambda}{1-2\lambda}-\frac{2\lambda}{1-\lambda}+\log(1-2\lambda)-2\log(1-\lambda)\right]\\&+\frac{A_2\beta_1}{2\beta_0^3}\left[\frac{\log(1-2\lambda)}{1-2\lambda}-\frac{2\log(1-\lambda)}{1-\lambda}+\frac{3\lambda}{1-2\lambda}-\frac{3\lambda}{1-\lambda}\right]-\frac{A_1\beta_1^2}{2\beta_0^4}\left[\frac{1}{2}\frac{\log^2(1-2\lambda)}{1-2\lambda}-\frac{\log^2(1-\lambda)}{1-\lambda}\right.\\&\left.+\frac{\log(1-2\lambda)}{1-2\lambda}-\frac{2\log(1-\lambda)}{1-\lambda}+\frac{\lambda}{1-2\lambda}-\frac{\lambda}{1-\lambda}-\log(1-2\lambda)+2\log(1-\lambda)\right]+\frac{D_1\beta_1}{2\beta_0^2}\left[\frac{\log(1-2\lambda)}{1-2\lambda}\right.\\&\left.+\frac{2\lambda}{1-2\lambda}\right]+\frac{B_1\beta_1}{\beta_0^2}\left[\frac{\log(1-\lambda)}{1-\lambda}+\frac{\lambda}{1-\lambda}\right]-\frac{D_2}{\beta_0}\frac{\lambda}{1-2\lambda}-\frac{B_2}{\beta_0}\frac{\lambda}{1-\lambda}-\frac{A_1\gamma_E^2}{2}\left[\frac{4\lambda}{1-2\lambda}-\frac{\lambda}{1-\lambda}\right]\\&+\frac{A_1\beta_1\gamma_E}{\beta_0^2}\left[\frac{\log(1-2\lambda)}{1-2\lambda}-\frac{\log(1-\lambda)}{1-\lambda}+\frac{1}{1-2\lambda}-\frac{1}{1-\lambda}\right]-\frac{A_2\gamma_E}{\beta_0}\left[\frac{1}{1-2\lambda}-\frac{1}{1-\lambda}\right]-\frac{D_1\gamma_E2\lambda}{1-2\lambda}-\frac{B_1\gamma_E\lambda}{1-\lambda}\\&-\frac{A_1}{2\beta_0}\left[\frac{2\lambda^2}{1-2\lambda}-\frac{\lambda^2}{1-\lambda}\right]\log^2\frac{\mu^2}{Q^2}-\frac{A_2}{\beta_0^2}\left[\frac{\lambda}{1-2\lambda}-\frac{\lambda}{1-\lambda}\right]\log\frac{\mu^2}{Q^2}-\frac{A_1\gamma_E}{\beta_0}\left[\frac{2\lambda}{1-2\lambda}-\frac{\lambda}{1-\lambda}\right]\log\frac{\mu^2}{Q^2}\\&-\frac{D_1}{\beta_0}\frac{\lambda}{1-2\lambda}\log\frac{\mu^2}{Q^2}-\frac{B_1}{\beta_0}\frac{\lambda}{1-\lambda}\log\frac{\mu^2}{Q^2}+\frac{A_1\beta_1}{\beta_0^3}\left[\frac{\lambda\log(1-2\lambda)}{1-2\lambda}-\frac{\lambda\log(1-\lambda)}{1-\lambda}+\frac{\lambda}{1-2\lambda}-\frac{\lambda}{1-\lambda}\right.\\&\left.+\frac{1}{2}\log(1-2\lambda)-\log(1-\lambda)\right]\log\frac{\mu^2}{Q^2}.\end{aligned} \tag{40}$$

Arbitrary constants have been added to the function g_3 in order to make it homogenous. The quantity $\gamma_E = 0.577216\ldots$ is the Euler constant and $\zeta(n)$ is the Riemann zeta function,

$$\zeta(n) \equiv \sum_{k=1}^{\infty} \frac{1}{k^n}. \tag{41}$$

$\zeta(2) = \pi^2/6 = 1.64493$. The functions g_2 and g_3 depend on the renormalization scale μ, while g_1 does not.

The expansion up to order α_S^3 reads

$$\begin{aligned}\log f_N = & -\frac{1}{2}A_1\alpha_S L^2 - D_1\alpha_S L - B_1\alpha_S L - \frac{1}{2}A_1\beta_0\alpha_S^2 L^3 \\ & -\left(\frac{1}{2}A_2 + \beta_0 D_1 + \frac{1}{2}\beta_0 B_1\right)\alpha_S^2 L^2 \\ & -\left(D_2 + B_2 + \frac{3}{2}A_1\zeta_2\beta_0\right)\alpha_S^2 L - \frac{7}{12}A_1\beta_0^2\alpha_S^3 L^4 \\ & -\left(A_2\beta_0 + \frac{1}{2}A_1\beta_1 + \frac{4}{3}D_1\beta_0^2 + \frac{1}{3}B_1\beta_0^2\right)\alpha_S^3 L^3 \\ & -\left(\frac{1}{2}A_3 + \frac{7}{2}A_1\zeta_2\beta_0^2 + 2D_2\beta_0 + B_2\beta_0 + D_1\beta_1 \right. \\ & \left. + \frac{1}{2}B_1\beta_1\right)\alpha_S^3 L^2 + O(\alpha_S^4). \end{aligned} \tag{42}$$

Let us note that β_2 appears only at order $O(\alpha_S^4)$.

The functions g_i become singular when

$$\lambda \to \frac{1}{2}^{-}. \tag{43}$$

Since

$$\lambda = \beta_0\alpha_S(\mu^2)L, \tag{44}$$

this means that a singularity in N-space occurs when

$$N \to \exp\left[\frac{1}{2\beta_0\alpha_S(\mu^2)}\right] \approx \frac{\mu}{\Lambda}, \tag{45}$$

where Λ is the QCD scale. Let us observe that, in general, this singularity signals non-perturbative effects but its precise position is completely unphysical, as we can move it with a change of renormalization scale.

B. Renormalization-scale dependence

In this section we consider renormalization-scale dependence. In principle, such scale μ should not appear in the cross sections, as it does not correspond to any fundamental constant or kinematical scale in the problem. The completely resummed perturbative expansion of an observable[1] is indeed formally independent on μ. In practice, truncated perturbative expansions exhibit a residual scale dependence, because of neglected higher orders.

We start with the form factor as a function of $\alpha_S(Q^2)$ and we derive its expression as a function of $\alpha_S(\mu^2)$ and μ^2/Q^2.

Since

$$a(Q) = a(\mu) + c\, a^2(\mu) + c'\, a^3(\mu) + O(a^4), \tag{46}$$

where $a \equiv \beta_0\alpha_S$ and [from Eq. (18)]

$$c = \log\frac{\mu^2}{Q^2}, \qquad c' = \log^2\frac{\mu^2}{Q^2} + \frac{\beta_1}{\beta_0^2}\log\frac{\mu^2}{Q^2}, \tag{47}$$

one has, where now $\lambda = \lambda(\mu)$,

$$\begin{aligned} L\, g_1[\lambda + ac\lambda + a^2c'\lambda] = & L\, g_1[\lambda] + c\lambda^2 g_1'[\lambda] + ac'\lambda^2 g_1'[\lambda] \\ & + a\frac{1}{2}c^2\lambda^3 g_1''[\lambda] + \cdots \end{aligned} \tag{48}$$

$$g_2[\lambda + ac\lambda + a^2c'\lambda] = g_2[\lambda] + a\, c\lambda\, g_2'[\lambda] + \cdots.$$

The additional terms in the functions g_i, to (partially) compensate for the scale change $Q^2 \to \mu^2$, therefore read

$$\begin{aligned} \delta g_1\left[\lambda, \frac{\mu^2}{Q^2}\right] &= 0 \\ \delta g_2\left[\lambda, \frac{\mu^2}{Q^2}\right] &= \lambda^2 g_1'[\lambda]\log\frac{\mu^2}{Q^2} \\ \delta g_3\left[\lambda, \frac{\mu^2}{Q^2}\right] &= \frac{1}{2}\lambda^3 g_1''[\lambda]\log^2\frac{\mu^2}{Q^2} + \lambda^2 g_1'[\lambda] \\ & \quad \times\left(\log^2\frac{\mu^2}{Q^2} + \frac{\beta_1}{\beta_0^2}\log\frac{\mu^2}{Q^2}\right) + \lambda\, g_2'[\lambda]\log\frac{\mu^2}{Q^2} \\ &= \frac{1}{2}\lambda\frac{d}{d\lambda}(\lambda^2 g_1'[\lambda])\log^2\frac{\mu^2}{Q^2} \\ & \quad + \left(\frac{\beta_1}{\beta_0^2}\lambda^2 g_1'[\lambda] + \lambda\, g_2'[\lambda]\right)\log\frac{\mu^2}{Q^2}. \end{aligned} \tag{49}$$

Therefore, the leading function g_1 does not depend explicitly on the renormalization scale, analogously to the coefficient of the leading term in the expansion of inclusive observables (such as the α_S/π term in the total e^+e^- hadronic cross section). The higher order functions $g_{i>1}$ instead explicitly depend on μ. We want to show the effect on $\log[N]$ of the variations in Eq. (49). Therefore, in Fig. 1 and in Fig. 2, we show the dependence of $\log[N]$ on μ at NLO and at NNLO, keeping α_S fixed at 0.21; we notice a NNLO improvement at high values of N. At lower values of N ($N \leqslant 10$), there is an appreciable improvement only by going at much higher en-

[1]We do not mean here the resummation of towers of logarithmic contributions, but the resummation of the whole series in α_S.

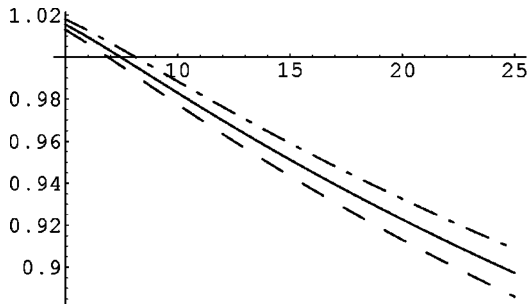

FIG. 1. We show the dependence of log[N] on the scale μ at NLO, with $\mu \simeq m_B$ (solid line), $\mu \simeq 2m_B$ (dashed line) and $\mu \simeq m_B/2$ (dot-dashed line), at fixed $\alpha_S(Q^2)=0.21$ and $5<N<25$.

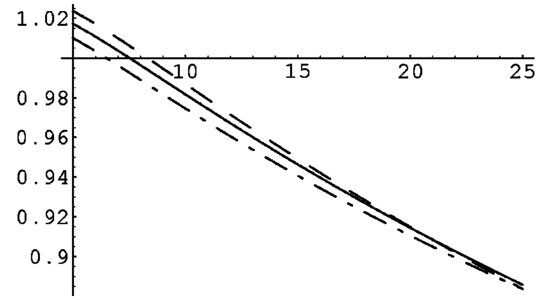

FIG. 2. We show the dependence of log[N] on the scale μ at NNLO, with $\mu \simeq m_B$ (solid line), $\mu \simeq 2m_B$ (dashed line) and $\mu \simeq m_B/2$ (dot-dashed line), at fixed $\alpha_S(Q^2)=0.21$ and $5<N<25$.

ergies (f.i. $\alpha_S=0.1$). In general, by varying also α_S with the scale μ around m_b, there is no significant improvement at NNLO.

III. ANALYTIC INVERSE MELLIN TRANSFORM

The N moments are physical quantities, but in practice a measure of the moments for large N is difficult. It is therefore convenient to perform the inverse transform back to momentum space.

Let us derive an analytical expression of the inverse Mellin transform at NNLO. We start by introducing the inverse Mellin transform of $f_N(\alpha_S)/N$:

$$G(\alpha_S;x)=\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}\frac{dN}{N}x^{-N}f_N(\alpha_S,L). \tag{50}$$

The inverse Mellin transform of $f_N(\alpha_S)$ is the logarithmic derivative of $G(\alpha_S,L;x)$:

$$f(x)=-x\frac{d}{dx}G(x). \tag{51}$$

After the substitutions $x\equiv e^{-s}$ and $u\equiv Ns$, we have

$$\begin{aligned}G(\alpha_S;x)&=\frac{1}{2\pi i}\int_{C'-i\infty}^{C'+i\infty}\frac{du}{u}e^u f_N(\alpha_S,L)\\&=\frac{1}{2\pi i}\int_{C'-i\infty}^{C'+i\infty}\frac{du}{u}e^u\exp\left[L\,g_1(\beta_0\alpha_S L)\right.\\&\quad\left.+\sum_{n=2}^{\infty}\alpha_S^{n-2}g_n(\beta_0\alpha_S L)\right].\end{aligned} \tag{52}$$

We Taylor expand the exponent with respect to L around $L=l=\ln 1/s$ [13]. In the x variable, $l\equiv-\ln(-\ln x)$. Note that $l\to-\ln(1-x)$ when $x\to1$.

We have, at NNLO,

$$G(\alpha_S;x)=\frac{1}{2\pi i}\int_{C'-i\infty}^{C'+i\infty}\frac{du}{u}\times e^{u+F_0(l)+F_1(l)\ln u+(1/2)F_2(l)\ln^2u+\cdots} \tag{53}$$

where, at scale Q^2 and at the same NNLO,

$$F_0(l)=l\,g_1(\beta_0\alpha_S l)+g_2(\beta_0\alpha_S l)+\alpha_S\,g_3(\beta_0\alpha_S l),$$

$$F_1(l)=g_1(\beta_0\alpha_S l)+\beta_0\alpha_S\,l\,g_1'(\beta_0\alpha_S l)+\beta_0\alpha_S\,g_2'(\beta_0\alpha_S l),$$

$$F_2(l)=2\beta_0\alpha_S\,g_1'(\beta_0\alpha_S l)+\beta_0^2\alpha_S^2\,l\,g_1''(\beta_0\alpha_S l).$$

By keeping in the exponent terms up to NLO, while expanding the NNLO terms, we obtain

$$G(\alpha_S;x)=\frac{e^{F_0(l)}}{2\pi i}\int_{C'-i\infty}^{C'+i\infty}du\,e^{u-[1-F_1^{NL}(l)]\ln u}\times\left[1+F_1^{N^2L}(l)\ln u+\frac{1}{2}F_2(l)\ln^2u+\cdots\right] \tag{54}$$

where

$$\begin{aligned}F_1(l)&\equiv F_1^{NL}(l)+F_1^{N^2L}(l)\\F_1^{NL}(l)&\equiv g_1(\beta_0\alpha_S l)+\beta_0\alpha_S\,l\,g_1'(\beta_0\alpha_S l)\\F_1^{N^2L}(l)&\equiv\beta_0\alpha_S g_2'(\beta_0\alpha_S l).\end{aligned} \tag{55}$$

By using the result

$$\frac{1}{2\pi i}\int_C du\,\log^k u\,e^{u-[1-F_1^{NL}(l)]\log u}=\frac{d^k}{dF_1^{NL\,k}}\frac{1}{\Gamma(1-F_1^{NL})}, \tag{56}$$

where Γ is the Euler Gamma function, we obtain, after integration,

$$
\begin{aligned}
G(\alpha_S;x) &= \frac{e^{F_0(l)}}{\Gamma(1-F_1^{NL})}\left[1+F_1^{N^2L}\psi(1-F_1^{NL})+\frac{1}{2}F_2(l)\big(\psi^2(1-F_1^{NL})-\psi'(1-F_1^{NL})\big)\right] \\
&= \frac{e^{l\,g_1(\beta_0\alpha_S l)+g_2(\beta_0\alpha_S l)+\alpha_S g_3(\beta_0\alpha_S l)}}{\Gamma(1-g_1(\beta_0\alpha_S l)-\beta_0\alpha_S l\,g_1'(\beta_0\alpha_S l))}\Big[1+\beta_0\alpha_S g_2'(\beta_0\alpha_S l)\,\psi(1-g_1(\beta_0\alpha_S l)-\beta_0\alpha_S l\,g_1'(\beta_0\alpha_S l)) \\
&\quad +\frac{1}{2}F_2(l)\big(\psi^2(1-g_1(\beta_0\alpha_S l)-\beta_0\alpha_S l\,g_1'(\beta_0\alpha_S l))-\psi'(1-g_1(\beta_0\alpha_S l)-\beta_0\alpha_S l\,g_1'(\beta_0\alpha_S l))\big)\Big]
\end{aligned}
\tag{57}
$$

with $\psi(x)=d\log\Gamma(x)/dx$, the digamma function.

At the end, we find the explicit analytic formula for the inverse Mellin transform at NNLO:

$$
\begin{aligned}
f(x) &= -x\frac{d}{dx}G(\alpha_S;x) \\
&= -x\frac{d}{dx}\left\{\frac{e^{l\,g_1(\beta_0\alpha_S l)+g_2(\beta_0\alpha_S l)+\alpha_S\,g_3(\beta_0\alpha_S l)}}{\Gamma(1-g_1(\beta_0\alpha_S l)-\beta_0\alpha_S l\,g_1'(\beta_0\alpha_S l))}\Big[1+\beta_0\alpha_S\,g_2'(\beta_0\alpha_S l)\,\psi(1-g_1(\beta_0\alpha_S l)-\beta_0\alpha_S l\,g_1'(\beta_0\alpha_S l))\right. \\
&\quad \left.+\frac{1}{2}F_2(l)\big(\psi^2(1-g_1(\beta_0\alpha_S l)-\beta_0\alpha_S l\,g_1'(\beta_0\alpha_S l))-\psi'(1-g_1(\beta_0\alpha_S l)-\beta_0\alpha_S l\,g_1'(\beta_0\alpha_S l))\big)\Big]\right\}.
\end{aligned}
\tag{58}
$$

Discussion and conclusions

There are two physical effects involved in the inverse Mellin transform: (1) *exact longitudinal momentum conservation*; the terms which enforce longitudinal momentum conservation are formally subleading in the infrared logarithm counting; (2) *infrared pole in the coupling*; the inverse transform involves an integration over all moments and arbitrarily large values of $|N|$ enter. This implies that a prescription for the infrared pole in the coupling has to be given. In general, the transformation mixes all the momentum scales in the problem.

One can study one problem at a time; for example, one can study the problem (1). only, by considering the frozen coupling approximation for the distributions in N-space. In one loop, this is equivalent to considering the QED case.

The frozen coupling approximation means neglecting the variation of α_S with the scale. Let us start from formula (12) at NNLO:

$$
\begin{aligned}
\log f_N(\alpha_S) &= \int_0^1 dz\frac{z^{N-1}-1}{1-z}\left\{\int_{Q^2(1-z)^2}^{Q^2(1-z)}\frac{dk^2}{k^2}\right. \\
&\quad \times[A_1\alpha_S+A_2\alpha_S^2+A_3\alpha_S^3+\cdots]+B_1\alpha_S+B_2\alpha_S^2 \\
&\quad \left.+\cdots+D_1\alpha_S+D_2\alpha_S^2+\cdots+\right\} \\
&\simeq \int_0^1 dz\frac{z^{N-1}-1}{1-z}\left\{(A_1\alpha_S+A_2\alpha_S^2+A_3\alpha_S^3)\ln\frac{1}{1-z}\right. \\
&\quad \left.+(B_1+D_1)\alpha_S+(B_2+D_2)\alpha_S^2\right\}.
\end{aligned}
\tag{59}
$$

After integration in z, at the lowest order in $\lambda=\beta_0\alpha_2\ln N$, we have

$$g_1=-\frac{A_1}{2\beta_0}\lambda \tag{60}$$

$$g_2=\left(-\frac{B_1}{\beta_0}-\frac{D_1}{\beta_0}-\frac{A_1\gamma_E}{\beta_0}\right)\lambda \tag{61}$$

$$g_3=\left(-\frac{B_2}{\beta_0}-\frac{D_2}{\beta_0}-\frac{A_2\gamma_E}{\beta_0}\right)\lambda. \tag{62}$$

We have two ways, analytical and numerical, to compute the inverse Mellin transform of $f_N(\alpha_S)$:

$$f(\alpha_S;x)=\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}dN\,x^{-N}f_N(\alpha_S,L). \tag{63}$$

In the case of frozen coupling, the g_i are linear in λ and therefore the numerical path does not include the Landau pole; the numerical integration is therefore exact. We have compared the inverse Mellin transform calculated numerically, directly from the definition (63), with the analytical result (58). We find a very good agreement, up to energies around the charm scale, provided that we go to NNLO. The exact longitudinal momentum conservation does not seem to spoil the reliability of the perturbative series.

In the real world, we cannot use linearized g_i, but we have to use the g_i computed in Sec. II; therefore, problems due to infrared poles come into play.

Let us first make an important observation: the degree of singularity of the functions g_i for $\lambda\to 1/2$, and therefore also of the form factor, increases with the order of the function, i.e. with i [14].

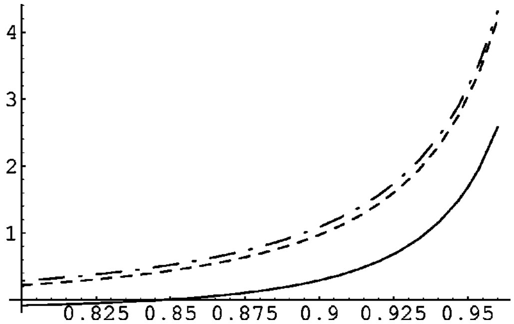

FIG. 3. mp numerical distribution (solid line), NLO (dashed line) and NNLO (dot-dashed line) analytic distributions of $f(x)$, at $\alpha_S = 0.1$.

At LO, g_1 has at most a logarithmic singularity of the form

$$g_1: \quad (1-2\lambda)\log(1-2\lambda); \tag{64}$$

at NLO, g_2 has at most a singularity of the form

$$g_2: \quad \log^2(1-2\lambda). \tag{65}$$

The prefactor $1-2\lambda$ in front of the logarithm, vanishing for $\lambda = 1/2$, is now absent, but the singularity is still logarithmic. At NNLO, g_3 has at most a singularity of the form

$$g_3: \quad \frac{\log^2(1-2\lambda)}{1-2\lambda}. \tag{66}$$

The latter is basically a pole singularity, no longer a logarithmic singularity, i.e. a much stronger singularity. Note that the above term is proportional to $A_1\beta_1^2$ and so it is scheme-independent.

Another observation concerns the size of the two-loop soft term. For three active flavors, the size of the two-loop correction with respect to the one-loop one is rather large,

$$\frac{D_2}{D_1}\alpha_S \approx 2\ \alpha_S \approx 40\%, \tag{67}$$

since $\alpha_S(m_B) \simeq 0.21$. The inclusion of D_2 is therefore important. The size of the two-loop term compared to the one-loop term is expected on general ground to be of order

$$\frac{D_2}{D_1} \approx \frac{C_A}{C_F} = 2.25, \tag{68}$$

since in two-loop order gluons start to be radiated by gluons instead of quarks, the former having a larger color charge C_A instead of C_F.

Let us now consider the inverse Mellin transform (63), releasing the frozen coupling approximation; in formula (12) the coupling runs over the whole integration range.

The form factor in N-space is computed in such a way that infrared-pole effects appear in a sharp way for $N > N_c$, where it acquires an unphysical (imaginary) part. In other terms, the numerical distribution is not real for any value of N because of the integration over the Landau pole. An exact numerical evaluation of the inverse transform then requires a prescription for the pole. An alternative strategy is to give a prescription for the infrared pole directly in N-space, in such a way that the form factors are well-defined for any N. In general, this results in a softening of the form factor for large N. It is then not necessary to give a prescription for the pole in the inverse transform.

We have used the minimal prescription (mp) [15], over two different paths (with the same results); the first path was made by two straight lines parallel to the negative real axes, closed by a half-circle centered around the origin and crossing the positive axes between the origin and the first Landau pole; the second path was composed by two lines almost vertical, meeting on the positive real axes between the origin and the first Landau pole. The precise crossing point is irrelevant, as long as it is before the Landau pole.

The inverse Mellin transform can also be derived analytically, as seen in Sec. III. We want to compare the analytic and numerical results, as we did before in the frozen coupling approximation. In this case, however, the Landau pole contribution plays an important role and we need to reach regions of higher energy ($\alpha_S = 0.1$), where the first Landau pole [see Eq. (43)] occurs at larger values of N ($N = e^{1/2\beta_0\alpha_S} \simeq 3000$–$4000$). In these regions, the perturbative resumming keeps under control the infrared divergences and cancels the oscillatory behavior. We compare the analytic (NLO and NNLO) and numerical (NNLO) plots in Fig. 3 (at $\alpha_S = 0.1$).

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