DESCRIPTION OF SOLAR ACOUSTIC WAVES WITH LOCALLY PLANE WAVES

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ABSTRACT

The concept of locally plane wave is used to describe as physically as possible the acoustic waves in the Sun. The conditions at the internal and external limits of the acoustic cavity are determined with rudimentary hypotheses. The eigenfrequencies so obtained are compared to numerical results in order to quantitatively characterize the performances and the limits of such an approach.

MOTIVATIONS

For linear oscillations with spherical symmetry and in the adiabatic approximation, there exists 4 differential equations which govern the fluid motion, see for example [CD98]. By solving numerically these 4 equations, one can obtain the eigenfrequencies of the Sun's cavity and the eigenfunctions describing the amplitude and phase of the standing wave.

Beside these numerical solutions, approximate but more physical models are also necessary to better identify the main physical phenomenon involved. Since many years, numerous such approaches exist: Cowling's approximation, Lamb's theory for atmosphere, functional analysis, JWKB analysis, Duvall's law, asymptotic theory following Gough, Tassoul, or Vorontsov , etc…[CD98, LTC94 and references therein]. These different theories are often valid only in a limited domain, and use different formalisms not totally coherent between them. One can note that the turning points are not always the same, and that the constant L related to the horizontal eigenvalue is either $l(l+1)$ or $(l+0.5)^2$.

The aim of this work is to provide an overall understanding of the properties of the acoustic modes by means of rudimentary concepts describing as closely as possible the wave physics.

WAVE STRUCTURE

Hypotheses. The following hypotheses are employed:

- Spherical symmetry for the equilibrium structure and the wave.
- Adiabatic approximation.
- Linear oscillation. The small and constant amplitude is neither damped nor amplified.
- The wave is purely acoustic.

Only the last hypothesis is not classic and needs additional comments. This approximation is even simpler than Cowling's. It means that the environment looks homogeneous to the wave. That may be partly justified by arguing that an oscillation at a small scale -acoustic wave- cannot be very perturbed by a larger scale phenomena -gravity wave-. The propagation condition can be expressed as:

$$\lambda = \frac{2\pi}{k} \leq aH_p \qquad (1)$$

where λ is the wavelength, k is the wavenumber , H_p is the pressure scale height and a is a constant to be determined. When the condition (1) is no longer valid, the propagation environment can no longer be considered as homogeneous and the acoustic wave is reflected. If the pressure is decreasing, a 'pressure vacuum' is encountered, and if it is increasing, a 'pressure wall' is encountered.

Present approach. One merely propagates the wave in known surroundings coming from a code calculating independently the structure. Due to its small amplitude, there is no feedback between the Sun structure and the wave perturbation. The purely acoustic assumption implies that during the propagation, the wave frequency ν stays constant while the wavenumber continuously follows the slight environment change, as given by the dispersion relation

$$k(r) = \frac{\omega}{c(r)} \qquad (2)$$

where $\omega = 2\pi\nu$ is the angular frequency, and c(r) is the radially dependent sound speed.

In the presence of spherical symmetry, one knows [CD98] that the vector $\vec{k}$ has a length given by (2) while its horizontal and radial components can be expressed following the degree l (number of nodal lines on a spherical surface) like

$$k_r = \sqrt{\frac{\omega^2}{c^2} - \frac{l(l+1)}{r^2}}$$
$$k_h = \sqrt{\frac{l(l+1)}{r^2}} \qquad (3)$$

This wave with a spherical wavesurface can be considered from the formal point of view as a locally plane wave because the radial projection of the eigenvector is simply

$$y = A\cos(\omega t - \int_{r_1}^{r_2} k_r dr + \phi) \qquad (4)$$

where $\int_{r_1}^{r_2} k_r dr$ is the phase advance between two points r_1, r_2, and A, ϕ arbitrary amplitude and phase.

Kinetic energy. The wavenumber plays a key role during the propagation. One can see a first example with the radial density of kinetic energy ε_c. As the wave amplitude is constant, the kinetic energy inside each wavelength is the same, and the one inside each length unity is proportional to the number of wavelengths by unit length, thus one has simply

$$\varepsilon_c \propto k_r. \qquad (5)$$

Indeed, k_r given by (3) and Fig. 1, is simply the envelope of the 'normalized eigenfunctions' given in the litterature, see for example [CD98].

Standing wave. When the wave is reflected at the external and internal limits of the Sun, there is formation of a standing wave, and in this cavity only determined frequencies are privileged, which are the eigenvalues of the differential equations. In the radial direction, the treatment formally looks very close to the one for plane standing wave. Nodes and antinodes are separated by π in phase, thus for the total propagation path, the phase advance is (see Fig. 2)

$$\int_{r_1}^{r_2} k_r dr = (n + \alpha)\pi \qquad (6)$$

where n is the number of radial nodes called the mode order, α is a number to adjust according to the phase conditions in r_1 and r_2. If for example there are two antinodes at the two ends, $\alpha = 0$, whereas if there are one node and one antinode, $\alpha = -0.5$. In our case, α will lay between –0.5 and 0. The relation (6) is a Duvall-like law, it will be used here to calculate the eigenfrequencies, but one needs before to determine r_1, r_2, α, i. e. the turning points and the related phases.

EXTERNAL TURNING POINT (r_2)

At the external limit of the Sun, the pressure is strongly decreasing, and one can consider that a 'pressure vacuum' is encountered when the condition (1) is no longer fulfilled. There, in the absence of any constraint, it is an antinode for the standing wave.

In fact, to be more precise, it is not H_p that must be used but its projection on the direction of propagation, that is on the vector $\vec{k}$. As a result, (1) becomes

$$\frac{1}{k_r}\frac{\omega^2}{c^2} \geq \frac{2\pi}{a_E H_p} \qquad (7)$$

which depends on the degree l. In the case of very high frequency, (7) turns back to (1) and is independent of l.

One can find out a_E empirically by using the fact that the highest frequency detected is ~ 5000 μHz. From Fig. 3 that presents the condition (1) with the [BTCZ99] solar model, when $a_E \sim 12.7$, the mode l=0, ν = 5000 μHz is the last one trapped in the Sun cavity.

The external conditions are thus totally determined. One can also note that for very low frequencies, the relation (7) could also be invalid in an internal point where the pressure is increasing. If it is the case, low l, low n modes could be not at their classic frequencies.

INTERNAL TURNING POINT (r_1)

Modes l ≠ 0. For the non radial modes, the internal turning point is given by the classical condition

$$k_r = 0 \iff \frac{c^2}{r_1^2} = \frac{\omega^2}{l(l+1)}. \qquad (8)$$

One can furthermore stress that this turning point is not due to the refraction of a plane wave which has an internal edge experiencing a higher speed than the external one, because the wave is not plane but spherical, so there is no external nor internal edge. The reason lays in fact in the spherical symmetry hypothesis. Indeed, in this case $\vec{k}$ is given by (3), and k_h increasing indefinitely to the center implies that k_r goes inevitably to zero, whether the sound speed increases or decreases towards the center. Physically speaking, as the surface pattern is independent of r, there will be a location before the center where the propagation consists only in describing the surface pattern and the wave no longer goes through.

What about the phase at the turning point? One can no longer invoke H_p because its projection on $\vec{k}$ is zero. In fact only the conditions at the two extremes in frequency are easy to determine. At the turning point,

$$k^2 = k_h^2 = \left(\frac{\omega}{c_{r1}}\right)^2 = \frac{l(l+1)}{r_1^2}. \qquad (9)$$

When the frequency goes to infinity, so does k_h^2, thus after (1) the wave is in the 'pressure vacuum' condition whatever the surroundings. So there is an antinode and as it is already the case at the external limit, α must go to zero. One can search for α a simple expression like

$$\alpha = \frac{b}{k_h^2} \qquad (10)$$

where b is a constant to be determined.

Now for the lowest frequency, that is for the fundamental mode where l = 1, n = 1, as the external limit is already an antinode, the internal one is necessarily a node to obtain a standing wave. Thus α must be –0.5. So we have every element to completely calculate this mode and then to determine b. But seeing the rudimentary aspect of expression (10), we can be satisfied to make a rough calculation. If one admits that the lowest frequency is slightly less than 300 μHz, then b is slightly less than $-8/R^2$, we can choose $b = -10/R^2$, where R is the solar radius. Thus

$$\alpha = -10\frac{(r_1/R)^2}{l(l+1)}. \qquad (11)$$

Modes l=0. For the radial modes, $k = k_r = \omega/c$. There is no turning point like above and one could think that the waves reach the center. Because of symmetry the center cannot move, there will be a node at that point, so $r_1 = 0$ and $\alpha = -0.5$ for all n. But there is a paradox with the present linear theory: the kinetic energy density ε_c is not zero at the center ! To overcome this paradox, one can invoke the internal pressure of the wave. Indeed, due to the spherical symmetry and the fact that the maximum amplitude is constant along the radius, the wave sees its own pressure increasing indefinitely to the center like $1/r^2$. It is equivalent to an internal pressure scale height

$$H_p = r/2 \tag{12}$$

The condition (1) can then be used to calculate the turning point corresponding now to a 'pressure wall', thus

$$\alpha = -0.5. \tag{13}$$

Let a_{l0} be the constant (internal turning point for l = 0) to be determined in (1). Knowing that the lowest frequency for l = 0 is slightly less than the one for l = 1, and that the two modes have α about –0.5, r_1 of the first mode is slightly less than for the second one. That leads to $a_{l0} \geq 8.89$ and let us choose $a_{l0} = 10$. So r_1 is given by

$$\frac{c^2}{r_1^2} = \frac{\omega^2}{1.6} \tag{14}$$

Remark: with the formalism using l+0.5 instead of l(l+1), one employs a coefficient 0.5 instead of 1.6.

RESULTS

Everything is now provided to calculate the eigenfrequencies and all helioseismic quantities. The calculation consists in solving the system of two equations (6) and (11) or (13) related to two unknowns ω and α, with the definitions of k_r, r_1, r_2 given by (3), (8) or (14), and (7). These equations compose the whole set of equations to be considered. One can point out the extreme simplicity of the equations used and note that the two structure functions involved are c(r) and $H_p(r)$.
Calculations have been performed for the modes l = 0 to 10, covering the frequencies 400 to 5000 μHz, using the [BTCZ99] model. The results are displayed in Fig. 4 to 12 with the following convention: crosses for l = 0, continuous line for l = 1, to dot line for l = 10.
The external turning points exhibit a remarkable behaviour: they depend mostly on the frequency and very little on l. That means that every global behaviour of the modes which follows this rule must be due to the external conditions. Using this property, the inversion of the surface conditions should be straightforward, even with low degree modes, this is what we will do hereafter.
The seismic parameters can be compared to the results of [LTC94]. For the large separation, except the absence of the bump at 2000 μHz, the discrepancies are globally limited to only about 10 μHz. For the small separation, except for the modes l = 0, 1, the remaining results are very close. For the second difference, there is not here exactly the same oscillation aspect, but the value range is correct.
Finally the eigenvalues are compared in Fig. 10 to the ones obtained by solving numerically the differential equations. There is no asymptotic behaviour like when this kind of approximation is used. The discrepancies reach +10, -55 μHz for l = 0, and +25, -40μHz for the other degrees. But one can note a global oscillation with ν for every l. As discussed above, that is necessarily due to surface conditions. The negative peak at 2500 μHz is related to r_2 around r/R = 0.999 and the positive peak at 4500 μHz around r/R = 0.9999. A rough manual inversion of H_p (one could do it indifferently on c) like in Fig. 11 leads easily to an important improvement for $\nu - \nu_{num}$ as shown in Fig. 12: the discrepancy is now only +10, -35 μHz for l = 0, and except for about the first ten of modes l = 0, 1, where it is +10, +25 μHz, for the remaining modes, it is better than ±10 μHz. The correction in fact has merely softened the transition Sun-Atmosphere at r/R=1. There remains a small global oscillation that could also be removed by the same method and that would further reduce the above discrepancies by a factor of two.
One can conclude that in spite of very simple concepts and expressions employed here, the agreement with numerical calculations is very good.

ACKNOWLEDGEMENTS

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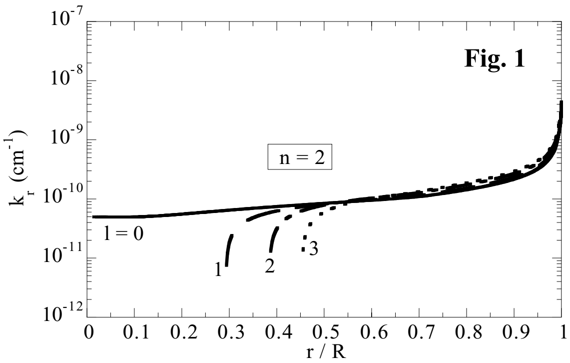

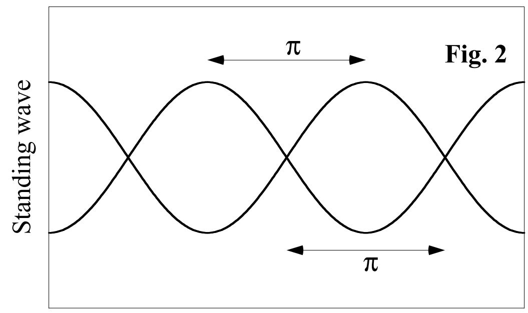

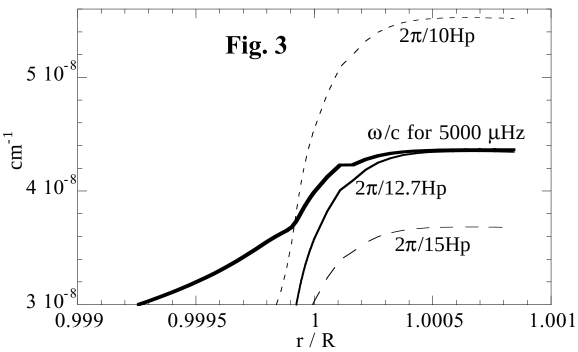
Fig. 3
$2\pi/10Hp$
ω/c for 5000 μHz
$2\pi/12.7Hp$
$2\pi/15Hp$
cm^{-1}
$5\ 10^{-8}$
$4\ 10^{-8}$
$3\ 10^{-8}$
0.999 0.9995 1 1.0005 1.001
r / R

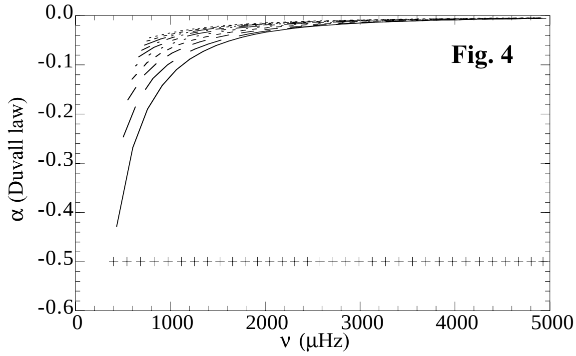
Fig. 4
α (Duvall law)
0.0 -0.1 -0.2 -0.3 -0.4 -0.5 -0.6
0 1000 2000 3000 4000 5000
ν (μHz)

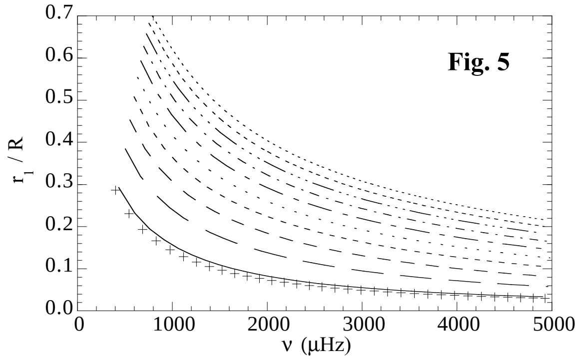
Fig. 5
r_1 / R
0.7 0.6 0.5 0.4 0.3 0.2 0.1 0.0
0 1000 2000 3000 4000 5000
ν (μHz)

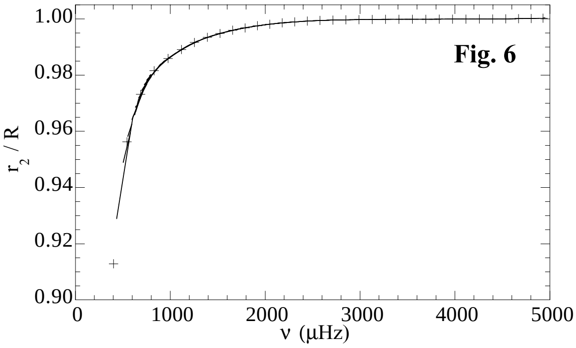
Fig. 6
r_2 / R
1.00 0.98 0.96 0.94 0.92 0.90
0 1000 2000 3000 4000 5000
ν (μHz)

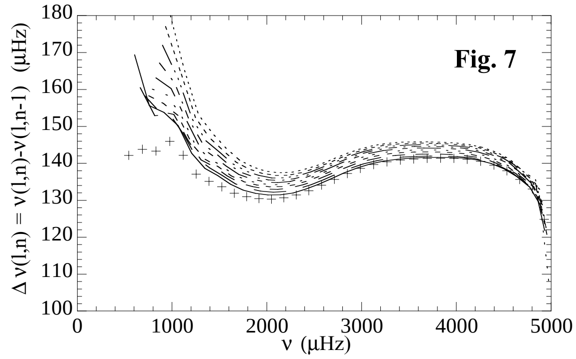
Fig. 7
$\Delta\nu(l,n) = \nu(l,n)-\nu(l,n-1)$ (μHz)
180 170 160 150 140 130 120 110 100
0 1000 2000 3000 4000 5000
ν (μHz)

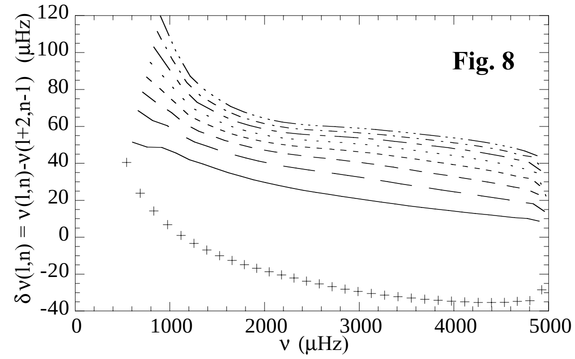
Fig. 8
$\delta\nu(l,n) = \nu(l,n)-\nu(l+2,n-1)$ (μHz)
120 100 80 60 40 20 0 -20 -40
0 1000 2000 3000 4000 5000
ν (μHz)

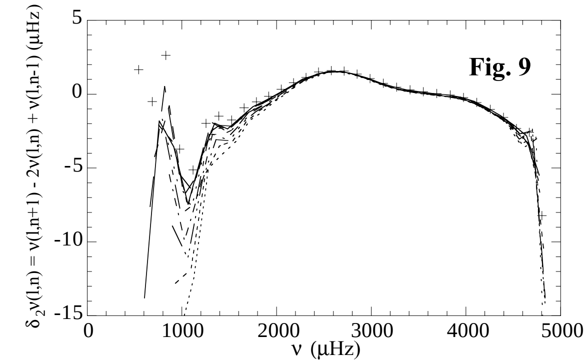
Fig. 9
$\delta_2\nu(l,n) = \nu(l,n+1) - 2\nu(l,n) + \nu(l,n-1)$ (μHz)
5 0 -5 -10 -15
0 1000 2000 3000 4000 5000
ν (μHz)

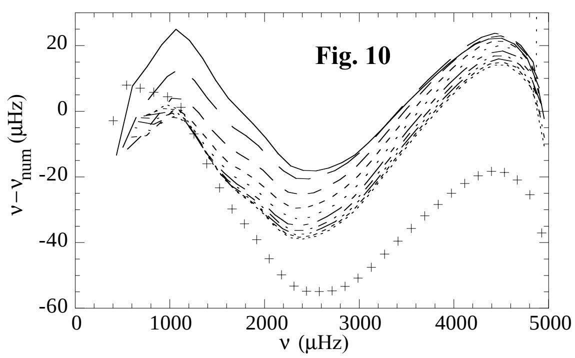
Fig. 10
$\nu-\nu_{num}$ (μHz)
20 0 -20 -40 -60
0 1000 2000 3000 4000 5000
ν (μHz)

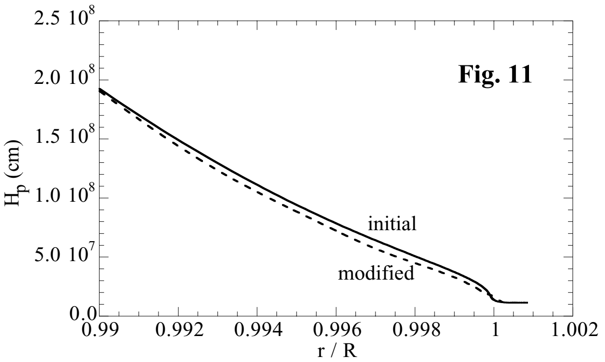
Fig. 11
H_p (cm)
$2.5\ 10^8$
$2.0\ 10^8$
$1.5\ 10^8$
$1.0\ 10^8$
$5.0\ 10^7$
0.0
initial
modified
0.99 0.992 0.994 0.996 0.998 1 1.002
r / R

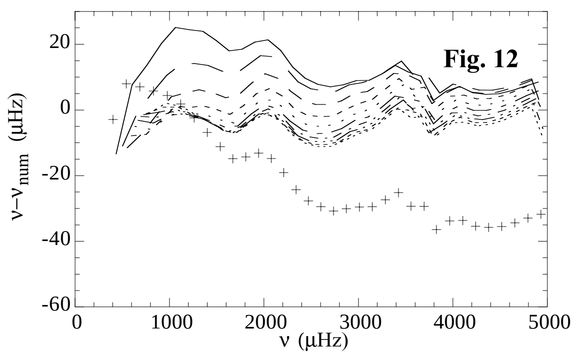
Fig. 12
$\nu-\nu_{num}$ (μHz)
20 0 -20 -40 -60
0 1000 2000 3000 4000 5000
ν (μHz)