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Schwinger terms in gravitation in two dimensions as a consequence of the gravitational anomaly[1]

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ABSTRACT

We compute the Schwinger term in the gravitational constraints in two dimensions, starting from the path integral in Hamiltonian form and the Einstein anomaly.

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1 Introduction

A Yang–Mills theory with a non-abelian anomaly (gauge anomaly) leads to Schwinger terms (central charges) in the constraint algebra (the Gauss law operators) as well as in the algebra of currents (see e.g. [1],[2]). Theories of gravitation and matter that have a gravitational anomaly (Einstein or Lorentz) also lead to Schwinger terms in the constraints and currents (energy–momentum tensors). As we will see later, in two dimensions, the gravitational constraints reduce to the energy–momentum tensor and therefore the two Schwinger terms are equivalent. This is also the case that has been considered in the literature (see e.g. [3] – [6]).

In [7], Faddeev et al. found the following method to compute the Schwinger term in the algebra of the Gauss law operators in a Yang–Mills theory: starting with the path integral in Hamiltonian form, they make a gauge transformation and include the non-abelian anomaly. From the Ward identity in second order in the gauge parameter, one can then extract the Schwinger term by acting with a suitable operator. Our goal is to generalize this to gravitation.

2 Gravitation as a constrained Hamiltonian system

We start with Einstein's theory of gravitation and a massless chiral fermion, which are described by the action (we have eighter $P_+\psi = 0$ or $P_-\psi = 0$ with $P_\pm = \frac{1}{2}(1 \pm \gamma_5)$):

$$S = \int dx \, e \left[R + \frac{i}{2} e^{a\mu} \bar{\psi} \gamma_a \overleftrightarrow{\nabla}_\mu \psi \right]. \tag{1}$$

This action can also be written as a constrained Hamiltonian system (see e.g. [8] – [10]) (in d-dimensions i runs from 1 to $d-1$):

$$S = \int dx \left[\pi_a{}^i \dot{e}^a{}_i + i \, {}^{(3)}g^{1/2} \, \bar{\psi} \Gamma^\perp \dot{\psi} - N \tilde{\mathcal{H}}_\perp - N^i \tilde{\mathcal{H}}_i - \frac{1}{2} \omega_0{}^{ab} J_{ab} \right], \tag{2}$$

where ${}^{(3)}g^{ij}$ is the induced metric and $\pi_a{}^i$ is the canonical momentum to $e^a{}_i$ defined by $p^{ij} = \frac{1}{4}\left(\pi_a{}^i \, {}^{(3)}e^{aj} + \pi_a{}^j \, {}^{(3)}e^{ai} \right)$, where the canonical momentum to g_{ij} is expressed in terms of the extrinsic curvature K_{ij}

$$p^{ij} = {}^{(3)}g^{1/2} \, (K^{ij} - K \, {}^{(3)}g^{ij} \,). \tag{3}$$

The Lagrangian multipliers are the lapse N, the shift N_i, $N^i = {}^{(3)}g^{ij} \, N_j$, and the 0-component of the spin connection $\omega_0{}^{ab}$. The constraints are

$$\tilde{\mathcal{H}}_\perp = \mathcal{H}_\perp + \partial_k J^{\perp k} - i \, {}^{(3)}g^{1/2} \, \bar{\psi} \Gamma^i D_i \psi \tag{4}$$

$$\tilde{\mathcal{H}}_i = \mathcal{H}_i + \frac{1}{2} g_{ij} \partial_k J^{kj} + K_{ik} J^{\perp k} + i \Big[{}^{(3)}g^{1/2} \, \bar{\psi} \Gamma^\perp D_i \psi - \frac{1}{4} \partial_k \Big({}^{(3)}g^{1/2} \, \bar{\psi} [\Gamma_i, \Gamma^k] \Gamma^\perp \psi \Big) \Big] \tag{5}$$

$$J_{ab} = \pi_a{}^i e_{bi} - \pi_b{}^i e_{ai} - \frac{i}{4} \, {}^{(3)}g^{1/2} \, \bar{\psi} \Big(\Gamma^\perp \gamma_a \Gamma^i - \Gamma^i \gamma_a \Gamma^\perp \Big) e_{bi} + \frac{i}{4} \, {}^{(3)}g^{1/2} \, \bar{\psi} \Big(\Gamma^\perp \gamma_b \Gamma^i - \Gamma^i \gamma_b \Gamma^\perp \Big) e_{ai} \tag{6}$$

$$\Gamma^i = {}^{(3)}g^{ij} e^a{}_j \, \gamma_a = {}^{(3)}e^{\,ai} \gamma_a, \qquad \Gamma^\perp = -n^a \gamma_a. \tag{7}$$

Remember that the gravitational Hamiltonian is vanishing.

3 The gravitational path integral

The general coordinate transformations are the gauge transformations of gravitation. To define a well behaved path integral, we choose the gauge fixing $e^a{}_0 = \delta^a{}_0$ and introduce the coordinate ghosts using the Faddeev–Popov method. Under infinitesimal active coordinate transformations (Einstein transformations) the vielbein transforms as

$$\delta^c_\xi e^a{}_\mu(x) = e'^a_\mu(x) - e^a_\mu(x) = \xi^\nu \partial_\nu e^a{}_\mu(x) + e^a{}_\nu(x)\partial_\mu \xi^\nu. \tag{8}$$

This leads to the ghost action

$$\begin{aligned} S_{GH} &= \int dx\, dy\, e(x)\bar{v}^\nu(x) e_{a\nu}(x) \left.\frac{\delta^c_\xi(e^a{}_0 - \delta^a{}_0)}{\delta\xi_\mu(y)}\right|_{\xi=0} v_\mu(y) \\ &= \int dx\, e(x)\bar{v}_\nu(x)\left(e_a{}^\nu(x)\partial_\mu e^a{}_0(x) v^\mu(x) + \partial_0 v^\nu(x)\right). \end{aligned} \tag{9}$$

We find the path integral in Hamiltonian form for gravitation and a chiral fermion

$$\begin{aligned} Z &= \frac{1}{N}\int d\pi_a{}^i\, de^a{}_i\, de^a{}_0\, d\bar{\psi}\, d\psi\, d\bar{v}_\alpha dv^\alpha \delta(e^a{}_0 - \delta^a{}_0) \\ &\times \exp\Big\{ i\int dx \Big[\pi_a{}^i \dot{e}^a{}_i + i\, {}^{(3)}g^{1/2}\; \bar{\psi}\Gamma^\perp\dot{\psi} - N^\mu \tilde{\mathcal{H}}_\mu - \frac{1}{2}\omega_0{}^{ab} J_{ab} \\ &+ e\,\bar{v}_\nu\left(e_a{}^\nu \partial_\mu e^a{}_0 v^\mu + \partial_0 v^\nu\right)\Big]\Big\}. \end{aligned} \tag{10}$$

The inclusion of powers of e as weights in the fermionic measure, as it is explained in [11] and [12], would be no problem but we will not need it.

4 Schwinger terms in gravitation in two dimensions

In two dimensions the Einstein–Hilbert action is proportional to the Euler number of the two-dimensional manifold (see e.g. [13]). We choose a manifold where the Euler number vanishes. So in two dimensions there are no dynamical degrees of freedom for gravity in Einstein's theory. From (3) we see that the conjugate momenta $\pi_a{}^1$ identically vanish. Using $e^a{}_1 e^b{}_1[\gamma_a, \gamma_b] = 0$ and $\gamma_b\gamma_0\gamma_a - \gamma_a\gamma_0\gamma_b = 0$ the constraints reduce to

$$\tilde{\mathcal{H}}_\perp = -i\, {}^{(1)}g^{1/2}\; \bar{\psi}\Gamma^1\partial_1\psi \tag{11}$$

$$\tilde{\mathcal{H}}_1 = i\, {}^{(1)}g^{1/2}\; \bar{\psi}\Gamma^\perp\partial_1\psi \tag{12}$$

$$J_{ab} = 0. \tag{13}$$

We have

$$\Gamma^1 = {}^{(1)}e^{a1}\gamma_a = (g_{11})^{-1} e^a{}_1\gamma_a = e^{a1}\gamma_a - \frac{N^1}{N} n^a\gamma_a \tag{14}$$

$$\Gamma^\perp = N^\perp e^{a0}\gamma_a = -n^a\gamma_a \tag{15}$$

$$e = |g|^{1/2} = {}^{(1)}g^{1/2}\; N, \tag{16}$$

so that

$$N^{\mu}\mathcal{H}_{\mu} = -ie\,\bar{\psi}e^{a1}\gamma_a\partial_1\psi. \tag{17}$$

The path integral (10) reduces to

$$\begin{aligned} Z &= \frac{1}{N}\int d\pi_a{}^1 de^a{}_1\, de^a{}_0\, d\bar{\psi}\, d\psi\, d\bar{v}_\alpha dv^\alpha \delta(e^a{}_0 - \delta^a{}_0) \\ &\quad \times \exp\Big\{i\int dx\,\Big[\pi_a{}^1\dot{e}^a{}_1 + ie\,\bar{\psi}e^{a0}\gamma_a\partial_0\psi + ie\,\bar{\psi}e^{a1}\gamma_a\partial_1\psi \\ &\quad + e\,\bar{v}_\nu\,(e_a{}^\nu\partial_\mu e^a{}_0 v^\mu + \partial_0 v^\nu)\Big]\Big\}. \end{aligned} \tag{18}$$

We relabel all fields, $e^a{}_\mu(x) \to e'^a{}_\mu(x)$, as for the other fields. We interpret this as an active coordinate transformation and use the invariance of the classical action and the bosonic measure under coordinate transformations. The fermionic measure gives us the Einstein anomaly $G[\Lambda, \Gamma]$, and we are left with

$$\begin{aligned} Z &= \frac{1}{N}\int d\pi_a{}^1 de^a{}_1\, de'^a_0\, d\bar{\psi}\, d\psi\, d\bar{v}_\alpha dv^\alpha \delta(e'^a_0 - \delta^a{}_0) \\ &\quad \times \exp\Big\{i\int dx\,\Big[\pi_a{}^1\dot{e}^a{}_1 + ie\,\bar{\psi}e^{a0}\gamma_a\partial_0\psi + ie\,\bar{\psi}e^{a1}\gamma_a\partial_1\psi \\ &\quad + e\,\bar{v}_\nu\,(e_a{}^\nu\partial_\mu e^a{}_0 v^\mu + \partial_0 v^\nu)\Big]\Big\} G[\Lambda, \Gamma] \end{aligned} \tag{19}$$

with the explicit expression (see e.g. [14])

$$G[\Lambda, \Gamma] = \exp\Big\{\pm\frac{i}{96\pi}\Big[\frac{1}{3}\int_{\Gamma^+}\mathrm{tr}(d\Lambda\Lambda^{-1})^3 + \int_{M_2}\mathrm{tr}(d\Lambda\Lambda^{-1}\Gamma)\Big]\Big\}, \tag{20}$$

where $\Gamma^a{}_b$ is the Christoffel connexion 1-form, $\Lambda^a{}_b$ is the "gauge" element and $\partial\Gamma^+ = M_2$. For a general coordinate transformation $x' = x'(x)$ we define the "gauge" parameter ξ by

$$x'^\alpha = x^\alpha - \xi^\alpha(x) \tag{21}$$

$$(\Lambda^{-1})^\alpha{}_\beta = \frac{\partial x'^\alpha}{\partial x^\beta} = \delta^\alpha{}_\beta - \partial_\beta\xi^\alpha(x) \tag{22}$$

so that, in second order in ξ, we have

$$x^\alpha = x'^\alpha + \xi^\alpha(x') + \xi^\lambda\partial_\lambda\xi^\alpha(x') + O(\xi^3) \tag{23}$$

$$\Lambda^\alpha{}_\beta = \frac{\partial x^\alpha}{\partial x'^\beta} = \delta^\alpha{}_\beta + \partial_\beta\xi^\alpha(x) + \partial_\beta\xi^\lambda\partial_\lambda\xi^\alpha(x) + O(\xi^3). \tag{24}$$

The zweibein transforms under passive coordinate transformations as

$$e'^a_\mu(x') = \Lambda^\nu{}_\mu e^a{}_\nu(x) = e^a{}_\mu(x) + \partial_\mu\xi^\nu e^a{}_\nu(x) + \partial_\mu\xi^\lambda\partial_\lambda\xi^\nu e^a{}_\nu(x) + O(\xi^3) \tag{25}$$

and under active coordinate tranformations (Einstein transformations) as

$$\begin{aligned} e'^a_\mu(x) &= e^a{}_\mu(x) + \partial_\mu\xi^\nu e^a{}_\nu(x) + \xi^\alpha\partial_\alpha e^a{}_\mu(x) + \partial_\mu\xi^\lambda\partial_\lambda\xi^\nu e^a{}_\nu(x) \\ &\quad + \xi^\alpha\partial_\alpha\partial_\mu\xi^\nu e^a{}_\nu(x) + \xi^\alpha\partial_\mu\xi^\nu\partial_\alpha e^a{}_\nu(x) + \xi^\alpha\partial_\alpha\xi^\beta\partial_\beta e^a{}_\mu(x) \\ &\quad + \frac{1}{2}\xi^\alpha\xi^\beta\partial_\alpha\partial_\beta e^a{}_\mu(x) + O(\xi^3). \end{aligned} \tag{26}$$

The gauge fixing $e'^a_{\ 0}(x) = \delta^a_{\ 0}$ leads to a second order differential equation for $e^a_{\ 0}$. If we are in two dimensions and choose $\xi^1 = 0$, then we find

$$\delta^a_{\ 0} = \Big[1 + \partial_0\xi^0 + \xi^0\partial_0 + \partial_0\xi^0\partial_0\xi^0 + \xi^0\partial_0\partial_0\xi^0 + 2\xi^0\partial_0\xi^0\partial_0 + \frac{1}{2}\xi^0\xi^0\partial_0\partial_0\Big]e^a_{\ 0}. \tag{27}$$

Its solution is

$$e^a_{\ 0} = (1 - \partial_0\xi^0)\delta^a_{\ 0} + O(\xi^3). \tag{28}$$

Using (28) we express everything up to second order in ξ, in terms of $e^0_{\ 1}$ and $e^1_{\ 1}$. For the Einstein anomaly (20) we find

$$\begin{aligned} G[\xi,\Gamma] \ =\ & \exp\Big\{\pm\frac{i}{48\pi}\int dx\,\Big[\partial_0\partial_0\xi^0\partial_1\partial_0\xi^0(e^0_{\ 1})^2(e^1_{\ 1})^{-2} + \partial_1\partial_0\xi^0\partial_1\partial_1\xi^0(e^1_{\ 1})^{-2} \\ & -2\partial_1\partial_0\xi^0\partial_1\partial_0\xi^0 e^0_{\ 1}(e^1_{\ 1})^{-2} + \Big[\Big(\partial_0\partial_0\xi^0 + 2\partial_0\partial_0\xi^0\partial_0\xi^0\Big)(e^0_{\ 1})^2(e^1_{\ 1})^{-2} \\ & -2\Big(\partial_1\partial_0\xi^0 + \partial_1\partial_0\xi^0\partial_0\xi^0\Big)e^0_{\ 1}(e^1_{\ 1})^{-2} + \partial_1\partial_1\xi^0(e^1_{\ 1})^{-2}\Big]\partial_0 e^0_{\ 1} \\ & +\Big[-\Big(\partial_0\partial_0\xi^0 + 2\partial_0\partial_0\xi^0\partial_0\xi^0\Big)e^0_{\ 1}(e^1_{\ 1})^{-1} \\ & +\Big(\partial_1\partial_0\xi^0 + \partial_1\partial_0\xi^0\partial_0\xi^0\Big)(e^1_{\ 1})^{-1}\Big]\partial_0 e^1_{\ 1}\Big]\Big\} + O(\xi^3), \end{aligned} \tag{29}$$

where we use the convention $\varepsilon^{01} = 1$. As can be seen from (19), the momentum $\pi_a^{\ 1}$ is no longer conjugate to $e^a_{\ 1}$, since there are terms in (29) proportional to $\partial_0 e^0_{\ 1}$ and $\partial_0 e^1_{\ 1}$. Therefore we make a shift in $\pi_a^{\ 1}$ to absorb these terms. The functional determinant is simply 1 and its effect is to kill all terms proportional to $\partial_0 e^0_{\ 1}$ and $\partial_0 e^1_{\ 1}$ in (29). Using

$$\mathcal{L}_{GH} \ =\ e\,\bar v_\nu\,(e_a^{\ \nu}\partial_\mu e^a_{\ 0}v^\mu + \partial_0 v^\nu) = e\,\bar v_\nu\partial_0 v^\nu + e^1_{\ 1}\partial_\mu e^0_{\ 0}\bar v_0 v^\mu \tag{30}$$

$$e\,e^{a0}\gamma_a \ =\ -(e^1_{\ 1}\gamma_0 + e^0_{\ 1}\gamma_1), \qquad e\,e^{a1}\gamma_a = e\,(e^1_{\ 1})^{-1}\gamma_1 \tag{31}$$

we find the generating functional in second order in the gauge parameter

$$\begin{aligned} Z \ =\ & \frac{1}{N}\int d\pi_a^{\ 1}de^a_{\ 1}\,d\bar\psi\,d\psi\,d\bar v_\alpha dv^\alpha \exp\Big\{i\int dx\,\Big[\pi_a^{\ 1}\dot e^a_{\ 1} - i\bar\psi(e^1_{\ 1}\gamma_0 + e^0_{\ 1}\gamma_1)\partial_0\psi \\ & +\Big(1-\partial_0\xi^0\Big)i\bar\psi\gamma_1\partial_1\psi + \Big(1-\partial_0\xi^0\Big)e^1_{\ 1}\bar v_\nu\partial_0 v^\nu - \partial_\mu\partial_0\xi^0 e^1_{\ 1}\bar v_0 v^\mu \\ & \pm\frac{1}{48\pi}\Big[\partial_0\partial_0\xi^0\partial_1\partial_0\xi^0(e^0_{\ 1})^2(e^1_{\ 1})^{-2} - 2\partial_1\partial_0\xi^0\partial_1\partial_0\xi^0 e^0_{\ 1}(e^1_{\ 1})^{-2} \\ & +\partial_1\partial_0\xi^0\partial_1\partial_1\xi^0(e^1_{\ 1})^{-2}\Big]\Big]\Big\} + O(\xi^3). \end{aligned} \tag{32}$$

In $O((\xi^0)^2)$ we find the following Ward identity

$$\begin{aligned} 0 \ =\ & \Big\langle \pm\frac{i}{48\pi}\int dx\,\Big[\partial_0\partial_0\xi^0\partial_1\partial_0\xi^0(e^0_{\ 1})^2(e^1_{\ 1})^{-2} - 2\partial_1\partial_0\xi^0\partial_1\partial_0\xi^0 e^0_{\ 1}(e^1_{\ 1})^{-2} + \partial_1\partial_0\xi^0\partial_1\partial_1\xi^0(e^1_{\ 1})^{-2}\Big] \\ & +\frac{1}{2}\Big[\int dx\Big[\partial_0\xi^0\bar\psi\gamma_1\partial_1\psi - i\partial_0\xi^0 e^1_{\ 1}v_\nu\partial_0 v^\nu - i\partial_\mu\partial_0\xi^0 e^1_{\ 1}\bar v_0 v^\mu\Big]\Big]^2\Big\rangle. \end{aligned} \tag{33}$$

Acting with $\frac{\delta^2}{\delta\xi^0(x)\delta\xi^0(y)}$ on (33) we arrive at

$$0 \ =\ \Big\langle \pm\frac{i}{48\pi}\Big[\Big(2\partial_1^y\partial_0^y\partial_0^y\partial_0^y\delta(x-y) + 3\partial_1^y\partial_0^y\partial_0^y\delta(x-y)\partial_0^y + \partial_0^y\partial_0^y\partial_0^y\delta(x-y)\partial_1^y$$

$$+\partial_0^y\partial_0^y\delta(x-y)\partial_1^y\partial_0^y+\partial_1^y\partial_0^y\delta(x-y)\partial_0^y\partial_0^y\Big)\Big((e^0{}_1)^2(e^1{}_1)^{-2}\Big)$$
$$-4\Big(\partial_1^y\partial_1^y\partial_0^y\partial_0^y\delta(x-y)+\partial_1^y\partial_1^y\partial_0^y\delta(x-y)\partial_0^y+\partial_1^y\partial_0^y\partial_0^y\delta(x-y)\partial_1^y$$
$$+\partial_1^y\partial_0^y\delta(x-y)\partial_1^y\partial_0^y\Big)\Big(e^0{}_1(e^1{}_1)^{-2}\Big)+\Big(2\partial_1^y\partial_1^y\partial_1^y\partial_0^y\delta(x-y)$$
$$+\partial_1^y\partial_1^y\partial_1^y\delta(x-y)\partial_0^y+3\partial_1^y\partial_1^y\partial_0^y\delta(x-y)\partial_1^y+\partial_1^y\partial_1^y\delta(x-y)\partial_1^y\partial_0^y$$
$$+\partial_1^y\partial_0^y\delta(x-y)\partial_1^y\partial_1^y\Big)(e^1{}_1)^{-2}\Big]\Big\rangle$$
$$+\partial_0^x\partial_0^y\left\langle\Big(-\bar\psi\gamma_1\partial_1\psi(x)+ie^1{}_1\bar v_\nu\partial_0 v^\nu(x)+i\partial_\mu^x(e^1{}_1\bar v_0 v^\mu(x))\Big)\right.$$
$$\left.\times\Big(-\bar\psi\gamma_1\partial_1\psi(y)+ie^1{}_1\bar v_\nu\partial_0 v^\nu(y)+i\partial_\mu^y(e^1{}_1\bar v_0 v^\mu(y))\Big)\right\rangle. \tag{34}$$

The last two lines give the commutator times $\partial_0^y\delta(x^0-y^0)$:

$$\partial_0^y\delta(x-y)\Big[\bar\psi\gamma_1\partial_1\psi(x),\bar\psi\gamma_1\partial_1\psi(y)\Big], \tag{35}$$

plus terms that are proportional to $\delta(x^0-y^0)$ or regular as $y^0\to x^0$. Next we apply

$$\lim_{(p_0-q_0)\to\infty}\frac{p_0-q_0}{p_0q_0^3}\int dx^0\,dy^0\,e^{ip_0x^0+iq_0y^0} \tag{36}$$

on (34) to project onto terms proportional to $\partial_0^y\partial_0^y\partial_0^y\delta(x-y)$ and we find

$$0=\pm\frac{i}{48\pi}\langle\Big(2\partial_1^y\delta(x^1-y^1)+\delta(x^1-y^1)\partial_1^y\Big)\Big((e^0{}_1)^2(e^1{}_1)^{-2}\Big)\rangle. \tag{37}$$

Using this, we apply

$$\lim_{(p_0-q_0)\to\infty}\frac{p_0-q_0}{p_0q_0^2}\int dx^0\,dy^0\,e^{ip_0x^0+iq_0y^0} \tag{38}$$

on (34) to project onto terms proportional to $\partial_0^y\partial_0^y\delta(x-y)$. We obtain

$$\begin{aligned}0\;=\;&\Big\langle\pm\frac{i}{48\pi}\Big[\partial_1^y\delta(x^1-y^1)\partial_0^y\Big((e^0{}_1)^2(e^1{}_1)^{-2}\Big)-4\Big(\partial_1^y\partial_1^y\delta(x^1-y^1)\\&+\partial_1^y\delta(x^1-y^1)\partial_1^y\Big)\Big(e^0{}_1(e^1{}_1)^{-2}\Big)\Big]\Big\rangle.\end{aligned} \tag{39}$$

Using this and finally applying

$$\lim_{(p_0-q_0)\to\infty}\frac{p_0-q_0}{p_0q_0}\int dx^0\,dy^0\,e^{ip_0x^0+iq_0y^0} \tag{40}$$

on (34) to project onto terms proportional to $\partial_0^y\delta(x-y)$, we arrive at

$$\begin{aligned}\Big\langle\Big[-i\bar\psi\gamma_1\partial_1\psi(x),-i\bar\psi\gamma_1\partial_1\psi(y)\Big]\Big\rangle&=\pm\frac{i}{48\pi}\Big\langle\Big[2\partial_1^y\partial_1^y\partial_1^y\delta(x^1-y^1)\\&+3\partial_1^y\partial_1^y\delta(x^1-y^1)\partial_1^y+\partial_1^y\delta(x^1-y^1)\partial_1^y\partial_1^y\Big](e^1{}_1)^{-2}\Big\rangle.\end{aligned} \tag{41}$$

From the fermionic part of the action (1) we find the energy–momentum tensor

$$T_{\mu\nu}=-\frac{i}{4}\Big[\bar\psi\gamma_\mu\overleftrightarrow{\nabla}_\nu\psi+\bar\psi\gamma_\nu\overleftrightarrow{\nabla}_\mu\psi\Big]+g_{\mu\nu}\frac{i}{2}\bar\psi\gamma^\lambda\overleftrightarrow{\nabla}_\lambda\psi. \tag{42}$$

For flat space, we have $e^0{}_1=0, e^1{}_1=1$, and we find the Schwinger term in the energy–momentum tensor

$$\langle[T_{00}(x),T_{00}(y)]_{ET}\rangle=\pm\frac{i}{24\pi}\partial_1^y\partial_1^y\partial_1^y\delta(x^1-y^1). \tag{43}$$

This indeed agrees with [3] – [6].

5 Conclusion

From (17) we see that we found an elegant way to compute the Schwinger term in the gravitational constraints in two dimensions, which emphasizes its relation to the gravitational anomaly. The gravitational anomaly contributes in $4k + 2 = 2, 6, 10, \ldots$ dimensions and we expect that this method could be generalized to these higher dimensions. However, the calculation will become more complicated there.

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