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Analytic Expressions for Static Electric Fields in an Infinite Plane Condenser with One or Three Homogeneous Layers

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Abstract

Expressions for the electrostatic field of a point charge in an infinite plane condenser comprising one or three homogeneous isolating parallel dielectric layers are presented. These solutions are essential for detector physics simulations of Parallel Plate Chambers (PPCs) and Resistive Plate Chambers (RPCs). In addition, expressions for the weighting field of a strip electrode are presented which allow calculation of induced signals and crosstalk in these detectors. A detailed discussion of the derivation of these solutions can be found in [1].

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Static Electric Fields in an Infinite Plane Condenser with One or Three Homogeneous Layers

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1 Introduction

The electric field solutions of a point charge in an infinite plane condenser is necessary for detector simulations of various kinds of particle detectors. Fig.1 shows the two geometries described in this report. The point charge is at position x', y', z'. The signal induced on a strip electrode (Fig. 2) by the movement of a charge in the condenser can be calculated by a so called weighting field, i.e. the electric field in the condenser if the electrode is put to a potential V while all the other electrodes are grounded. These solutions are also given in this report.

The capacitor with one homogeneous layer resembles e.g. the geometry of a noble liquid calorimeter cell [2] or Parallel Plate Chamber [3] while the structure with three homogeneous layers resembles the geometry of a Resistive Plate Chamber [4].

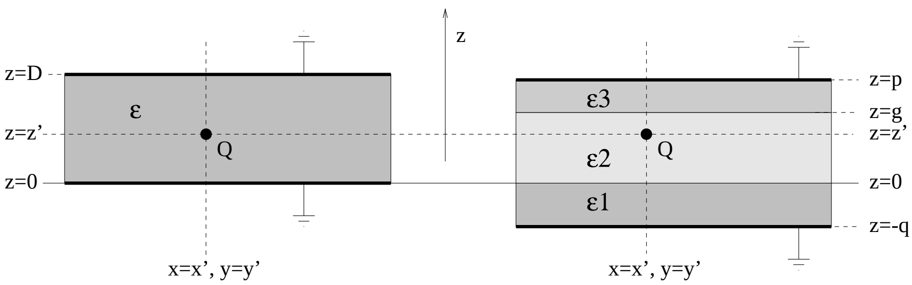

Figure 1: The two geometries discussed in this report. The point charge Q is at position x', y', z'.

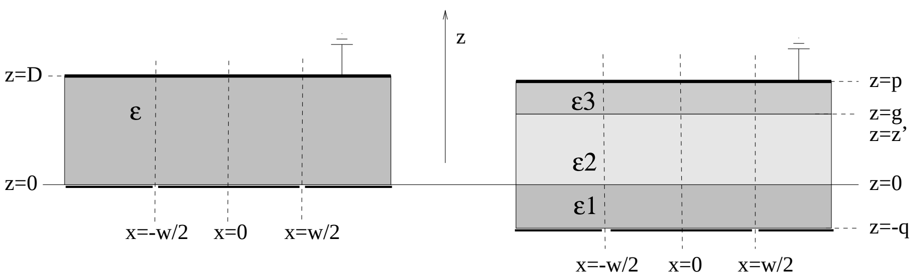

Figure 2: Readout strip geometries discussed in this report. The strips are infinitely long in y and the gap between the strips is assumed to be zero.

We will use cylindrical coordinates and write the distance between point charge (at $\vec{r}\,'$) and point of observation (at $\vec{r}$) as

$$\begin{aligned} R^2 = |\vec{r} - \vec{r}\,'|^2 &= (x-x')^2 + (y-y')^2 + (z-z')^2 = \\ &= \rho^2 - 2\rho\rho' \cos(\phi - \phi') + \rho'^2 + (z-z')^2 = \\ &= P^2 + (z-z')^2. \end{aligned} \quad (1)$$

2 Potential of a Point Charge in a Condenser Filled by a Homogeneous Dielectric

In this geometry it is no longer possible to represent the potential by a closed analytical expression. One must use superpositions of particular solutions of the homogeneous potential equation. These may be infinite series or integrals.

2.1 Infinite Series Representation

The Eigenfunction expansion of the potential [5] is given by

$$\Phi(\rho,\phi,z;\rho',\phi',z') \;=\; \frac{Q}{\epsilon\pi D}\sum_{n=1}^{\infty}\sin(k_n\,z)\;\sin(k_n\,z')\;K_0(k_n\,P) \qquad k_n=\frac{n\pi}{D} \tag{2}$$

where K_0 is the modified Bessel function. This representation does not exist on the line $P = 0$ passing through the source point since K_0 diverges. Convergence will be slow near to this line. For sufficiently large values of P, the convergence will be excellent due to the exponential decay of K_0.

2.2 Integral Representation

An integral representation is given by

$$\begin{aligned}\Phi(\rho,\phi,z;\rho',\phi',z') \;&=\; \frac{Q}{4\pi\epsilon}\Big[\frac{1}{\sqrt{P^2\;+\;(z-z')^2}}\;-\;\frac{1}{\sqrt{P^2\;+\;(z+z')^2}}\\ &\quad -\frac{1}{\sqrt{P^2\;+\;(2D-z-z')^2}}\;+\;\int_0^{\infty}d\kappa\;J_0(\kappa\,P)\;g(\kappa;z,z')\Big]\end{aligned} \tag{3}$$

with

$$g(\kappa,z,z')\;=\;\frac{e^{-\kappa(2D-z+z')}\;+\;e^{-\kappa(2D+z-z')}\;-\;e^{-\kappa(2D+z+z')}\;-\;e^{-\kappa(4D-z-z')}}{1-e^{-2\kappa D}}. \tag{4}$$

This modified integral representation works fine since g decreases rather fast with increasing κ, and this quite independent of the arguments P, z and z'.

The singularities arising from a possible coalescence of the point of observation with the source point or with the first image just outside each electrode are accounted for by the first three terms. The final, fast converging integral is obtained by subtracting the Sommerfeld integrals [6] corresponding to the first three terms from the common integral representation of the potential for the plane condensor.

2.3 The Weighting Field of a Strip Electrode

The current induced on an electrode i by a point charge q moving with velocity $\vec{v}$ may be calculated from Ramo's theorem [7] and may be expressed as [8]:

$$I_i\;=\;-\,q\,\vec{v}\cdot\frac{\vec{E}_i(\vec{r})}{V_i}\;=\;q\,\vec{v}\cdot\frac{1}{V_i}\,\vec{\nabla}\Phi_i(\vec{r}), \tag{5}$$

where V_i is the voltage applied to the electrode i generating the electric field $\vec{E}_i(\vec{r})$ in the absence of the charge q and having all the other electrodes grounded. $\vec{E}_i(\vec{r})$ is called the weighting field [9] and can be derived from a scalar potential $\Phi_i(\vec{r})$, which we shall call the weighting potential. We get for $0 < z < D$:

$$\begin{aligned}\Phi_1(x,z)\;&=\;\frac{V_1}{\pi}\Big[\quad\arctan\Big(\cot\big(\frac{z\pi}{2D}\big)\;\tanh\big(\pi\,\frac{x+w/2}{2D}\big)\Big)\\ &\quad -\;\arctan\Big(\cot\big(\frac{z\pi}{2D}\big)\;\tanh\big(\pi\,\frac{x-w/2}{2D}\big)\Big)\;\Big].\end{aligned} \tag{6}$$

The two components of the weighting field are derived from this as:

$$E_{1x} = V_1 \frac{1}{2D} \left[\frac{\sin\left(\frac{z\pi}{D}\right)}{\cosh\left(\pi \frac{x-w/2}{D}\right) - \cos\left(\frac{z\pi}{D}\right)} - \frac{\sin\left(\frac{z\pi}{D}\right)}{\cosh\left(\pi \frac{x+w/2}{D}\right) - \cos\left(\frac{z\pi}{D}\right)} \right]; \quad (7)$$

$$E_{1z} = -V_1 \frac{1}{2D} \left[\frac{\sinh\left(\pi \frac{x-w/2}{D}\right)}{\cosh\left(\pi \frac{x-w/2}{D}\right) - \cos\left(\frac{z\pi}{D}\right)} - \frac{\sinh\left(\pi \frac{x+w/2}{D}\right)}{\cosh\left(\pi \frac{x+w/2}{D}\right) - \cos\left(\frac{z\pi}{D}\right)} \right]. \quad (8)$$

3 Potential of a Point Charge for the Three-Layer Problem

In this section the plane condenser comprising three homogeneous isolating dielectric layers is treated. The configuration is shown in Fig.1. The electrodes are at $z = -q < 0$ and at $z = p > g > 0$. The gas gap corresponds to layer 2 ($0 \leq z \leq g$) with a dielectric constant ε_2, the two planes ($-q < z < 0$), ($g < z < p$) have a dielectric constant ε_1, ε_3 respectively. The ε_i's represent the full dielectric constants, i.e. they are ε_0 times the relative dielectric constant.

3.1 Series Representations

In principle also series representations may be derived for the potential in a plane condenser comprising several homogeneous layers. This has been done before [10][11]. However this requires to find a large number of roots of a transcendental equation and it must be ensured that all roots with small values are found! The resulting series converge slowly or not at all (those for the field components) and must be summed and differentiated numerically. So such an approach has unfavourable auspices.

3.2 Integral Representation

An integral representation with good convergence for the potential in layer 2 of a point charge in layer 2 is given by

$$\begin{aligned}\Phi(\rho,\phi,z;\rho',\phi',z') = \frac{Q}{4\pi\varepsilon_2}\Bigg[&\frac{1}{\sqrt{P^2+(z-z')^2}} - \frac{(\varepsilon_1-\varepsilon_2)}{(\varepsilon_1+\varepsilon_2)\sqrt{P^2+(z+z')^2}} \\ &- \frac{(\varepsilon_3-\varepsilon_2)}{(\varepsilon_2+\varepsilon_3)\sqrt{P^2+(2g-z-z')^2}} \\ &+ \frac{1}{(\varepsilon_1+\varepsilon_2)(\varepsilon_2+\varepsilon_3)} \int_0^\infty d\kappa\, J_0(\kappa P)\, \frac{R(\kappa,z,z')}{D(\kappa)} \Bigg], \; 0 \leq z \leq g\end{aligned} \quad (9)$$

where the denominator $D(\kappa)$ is given by

$$\begin{aligned} D(\kappa) = \; & (\varepsilon_1+\varepsilon_2)(\varepsilon_2+\varepsilon_3)\,(1-e^{-2\kappa\,(p+q)}) \\ & -(\varepsilon_1-\varepsilon_2)(\varepsilon_2+\varepsilon_3)(e^{-2\kappa\,p}-e^{-2\kappa\,q}) \\ & -(\varepsilon_1+\varepsilon_2)(\varepsilon_2-\varepsilon_3)(e^{-2\kappa\,(p-g)}-e^{-2\kappa\,(q+g)}) \\ & +(\varepsilon_1-\varepsilon_2)(\varepsilon_2-\varepsilon_3)(e^{-2\kappa g}-e^{-2\kappa\,(p+q-g)}) \end{aligned} \quad (10)$$

and the numerator is

$$
\begin{aligned}
R(\kappa; z, z') \ =& \\
& (\varepsilon_1+\varepsilon_2)^2(\varepsilon_2+\varepsilon_3)^2 \left[e^{\kappa(-2p-2q+z-z')} \ + \ e^{\kappa(-2p-2q-z+z')}\right] \\
& - (\varepsilon_1+\varepsilon_2)^2 \, (\varepsilon_2-\varepsilon_3)^2 \, e^{\kappa(-4g-2q+z+z')} \\
& - 4\varepsilon_1 \, \varepsilon_2(\varepsilon_2+\varepsilon_3)^2 \, e^{\kappa(-2q-z-z')} \ - \ (\varepsilon_1-\varepsilon_2)^2 \, (\varepsilon_2+\varepsilon_3)^2 \, e^{\kappa(-2p-z-z')} \\
& - \ \left({\varepsilon_1}^2-{\varepsilon_2}^2\right)(\varepsilon_2-\varepsilon_3)^2 \, e^{\kappa(-4g+z+z')} \\
& + \ \left({\varepsilon_1}^2-{\varepsilon_2}^2\right)(\varepsilon_2+\varepsilon_3)^2 \ \left[-e^{\kappa(-2p-2q-z-z')} + e^{\kappa(-2p+z-z')} + e^{\kappa(-2p-z+z')}\right] \\
& - 4 \ \left({\varepsilon_1}^2-{\varepsilon_2}^2\right) \ \varepsilon_2 \, \varepsilon_3 \, e^{\kappa(-2p-2q+z+z')} \ - 4\,(\varepsilon_1+\varepsilon_2)^2 \varepsilon_2 \, \varepsilon_3 \, e^{\kappa(-2p+z+z')} \\
& + (\varepsilon_1-\varepsilon_2)^2 \left({\varepsilon_2}^2-{\varepsilon_3}^2\right) \ e^{\kappa(-2g-z-z')} \ + \ 4\,\varepsilon_1 \, \varepsilon_2 \ \left({\varepsilon_2}^2-{\varepsilon_3}^2\right) \ e^{\kappa(2g-2p-2q-z-z')} \\
& + (\varepsilon_1+\varepsilon_2)^2 \left({\varepsilon_2}^2-{\varepsilon_3}^2\right) \ \left[-e^{\kappa(-2g-2q+z-z')} - e^{\kappa(-2g-2q-z+z')} + e^{\kappa(-2g-2p-2q+z+z')}\right] \\
& + \ \left({\varepsilon_1}^2-{\varepsilon_2}^2\right) \ \left({\varepsilon_2}^2-{\varepsilon_3}^2\right) \left[e^{\kappa(-2g-2q-z-z')} - e^{\kappa(-2g+z-z')} - e^{\kappa(-2g-z+z')} + e^{\kappa(-2g-2p+z+z')}\right].
\end{aligned}
\tag{11}
$$

3.3 Computation of the Electric Field Components

The expressions for the electric fields are found from the potentials by derivation. For the radial electric field component E_ρ the integrals contain $-\kappa J_1(\kappa\rho)$ in place of $J_0(\kappa\rho)$. Since this κ is multiplied by exponentials with negative exponents the convergence may be a bit slower than that of the integrals of the potential but will still be satisfactory. The very last remark applies also to the component E_z.

3.4 The Weighting Field in the Gas Gap

The weighting potential in the gas gap for a strip electrode is given by

$$\Phi_1(x,z) = \ V_1 \, \varepsilon_1 \, \frac{2}{\pi} \int_0^\infty \ d\kappa \ \cos(\kappa \, x) \ \sin\left(\kappa \, \frac{w}{2}\right) \frac{1}{\kappa} \, F_1(\kappa,z) \tag{12}$$

The two components of the weighting field are

$$
\begin{aligned}
E_x(x,z) \ &= \ -V_1 \, \varepsilon_1 \, \frac{2}{\pi} \int_0^\infty \ d\kappa \ \sin(\kappa \, x) \ \sin\left(\kappa \, \frac{w}{2}\right) \, F_1(\kappa,z) && (13) \\
E_z(x,z) \ &= \ \ V_1 \, \varepsilon_1 \, \frac{2}{\pi} \int_0^\infty \ d\kappa \ \cos(\kappa \, x) \ \sin\left(\kappa \, \frac{w}{2}\right) \, F_2(\kappa,z) && (14)
\end{aligned}
$$

with

$$
\begin{aligned}
& F_1(\kappa,z) = \\
& \tfrac{2}{D(\kappa)} \left[(\varepsilon_2+\varepsilon_3)\left(e^{-\kappa(q+z)} \ - \ e^{-\kappa(2p+q-z)}\right) \ + \ (\varepsilon_2-\varepsilon_3)\left(e^{-\kappa(2g+q-z)} \ - \ e^{-\kappa(2p+q-2g+z)}\right)\right]
\end{aligned}
\tag{15}
$$

and

$$
\begin{aligned}
& F_2(\kappa,z) = \\
& -\tfrac{2}{D(\kappa)}\left[(\varepsilon_2+\varepsilon_3) \ \left(e^{-\kappa(q+z)} \ + \ e^{-\kappa(2p+q-z)}\right) \ - (\varepsilon_2-\varepsilon_3)\left(e^{-\kappa(q+2g-z)} \ + \ e^{-\kappa(2p+q-2g+z)}\right)\right].
\end{aligned}
\tag{16}
$$

The above integrals must be evaluated numerically. If the strip is very wide, the field E_z in the center of the strip ($x = 0$) approaches

$$E_z = \frac{V_1 \varepsilon_1 \varepsilon_3}{\varepsilon_2 \varepsilon_3 \, q + \varepsilon_1 \varepsilon_2 \, p + (\varepsilon_1 \varepsilon_3 - \varepsilon_1 \varepsilon_2) \, g} \tag{17}$$

independent of z.

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