Sergiu Vacaru and Panayiotis Stavrinos

SPINORS
and
SPACE–TIME ANISOTROPY

University of Athens

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ABOUT THE BOOK

This is the first monograph on the geometry of anisotropic spinor spaces and its applications in modern physics. The main subjects are the theory of gravity and matter fields in spaces provided with off–diagonal metrics and associated anholonomic frames and nonlinear connection structures, the algebra and geometry of distinguished anisotropic Clifford and spinor spaces, their extension to spaces of higher order anisotropy and the geometry of gravity and gauge theories with anisotropic spinor variables. The book summarizes the authors' results and can be also considered as a pedagogical survey on the mentioned subjects.

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ABOUT THE AUTHORS

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Contents

0.1 Preface vii
0.1.1 Historical remarks on spinor theory vii
0.1.2 Metric Spaces depending on Spinor Variables and Gauge Field Theories ix
0.1.3 Nonlinear connection geometry and physics x
0.1.4 Anholonomic frames and nonlinear connections in Einstein gravity xiv
0.1.5 The layout of the book xvi
0.1.6 Acknowledgments xvii
0.2 Notation xix

I Space–Time Anisotropy 1

1 Vector Bundles and N–Connections 3
1.1 Vector and Covector Bundles 4
1.1.1 Vector and tangent bundles 4
1.1.2 Covector and cotangent bundles 5
1.1.3 Higher order vector/covector bundles 6
1.2 Nonlinear Connections 10
1.2.1 N–connections in vector bundles 10
1.2.2 N–connections in covector bundles: 11
1.2.3 N–connections in higher order bundles 12
1.2.4 Anholonomic frames and N–connections 13
1.3 Distinguished connections and metrics 19
1.3.1 D–connections 19
1.3.2 Metric structure 22
1.3.3 Some remarkable d–connections 25
1.3.4 Amost Hermitian anisotropic spaces 27
1.4 Torsions and Curvatures 29
1.4.1 N–connection curvature 29
1.4.2 d–Torsions in v- and cv–bundles 30

1.4.3 d–Curvatures in v- and cv–bundles 31
1.5 Generalizations of Finsler Spaces 32
1.5.1 Finsler Spaces 32
1.5.2 Lagrange and Generalized Lagrange Spaces 34
1.5.3 Cartan Spaces 35
1.5.4 Generalized Hamilton and Hamilton Spaces 37
1.6 Gravity on Vector Bundles 38

2 Anholonomic Einstein and Gauge Gravity 41
2.1 Introduction 41
2.2 Anholonomic Frames 42
2.3 Higher Order Anisotropic Structures 48
2.3.1 Ha–frames and corresponding N–connections 48
2.3.2 Distinguished linear connections 52
2.3.3 Ha–torsions and ha–curvatures 54
2.3.4 Einstein equations with respect to ha–frames 55
2.4 Gauge Fields on Ha–Spaces 56
2.4.1 Bundles on ha–spaces 57
2.4.2 Yang-Mills equations on ha-spaces 60
2.5 Gauge Ha-gravity 63
2.5.1 Bundles of linear ha–frames 64
2.5.2 Bundles of affine ha–frames and Einstein equations . . 65
2.6 Nonlinear De Sitter Gauge Ha–Gravity 66
2.6.1 Nonlinear gauge theories of de Sitter group 67
2.6.2 Dynamics of the nonlinear de Sitter ha–gravity 69
2.7 An Ansatz for 4D d–Metrics 72
2.7.1 The h–equations 74
2.7.2 The v–equations 75
2.7.3 H–v equations 76
2.8 Anisotropic Cosmological Solutions 77
2.8.1 Rotation ellipsoid FRW universes 77
2.8.2 Toroidal FRW universes 79
2.9 Concluding Remarks 80

3 Anisotropic Taub NUT – Dirac Spaces 85
3.1 N–connections in General Relativity 85
3.1.1 Anholonomic Einstein–Dirac Equations 88
3.1.2 Anisotropic Taub NUT – Dirac Spinor Solutions 94
3.2 Anisotropic Taub NUT Solutions 96
3.2.1 A conformal transform of the Taub NUT metric 97

3.2.2 Anisotropic Taub NUT solutions with magnetic polarization . . . 99
3.3 Anisotropic Taub NUT–Dirac Fields . . . 101
3.3.1 Dirac fields and angular polarizations . . . 101
3.3.2 Dirac fields and extra dimension polarizations . . . 103
3.4 Anholonomic Dirac–Taub NUT Solitons . . . 104
3.4.1 Kadomtsev–Petviashvili type solitons . . . 105
3.4.2 (2+1) sine–Gordon type solitons . . . 106

II Anisotropic Spinors 109

4 Anisotropic Clifford Structures 113
4.1 Distinguished Clifford Algebras . . . 113
4.2 Anisotropic Clifford Bundles . . . 118
4.2.1 Clifford d-module structure . . . 118
4.2.2 Anisotropic Clifford fibration . . . 120
4.3 Almost Complex Spinors . . . 121

5 Spinors and Anisotropic Spaces 127
5.1 Anisotropic Spinors and Twistors . . . 128
5.2 Mutual Transforms of Tensors and Spinors . . . 133
5.2.1 Transformation of d-tensors into d-spinors . . . 133
5.2.2 Fundamental d–spinors . . . 134
5.3 Anisotropic Spinor Differential Geometry . . . 135
5.4 D-covariant derivation . . . 136
5.5 Infeld - van der Waerden coefficients . . . 138
5.6 D-spinors of Anisotropic Curvature and Torsion . . . 140

6 Anisotropic Spinors and Field Equations 143
6.1 Anisotropic Scalar Field Interactions . . . 143
6.2 Anisotropic Proca equations . . . 145
6.3 Anisotropic Gravitons and Backgrounds . . . 146
6.4 Anisotropic Dirac Equations . . . 146
6.5 Yang-Mills Equations in Anisotropic Spinor Form . . . 147

III Higher Order Anisotropic Spinors 149

7 Clifford Ha–Structures 153
7.1 Distinguished Clifford Algebras . . . 153
7.2 Clifford Ha–Bundles . . . 158

7.2.1 Clifford d–module structure in dv–bundles 158
7.2.2 Clifford fibration 160
7.3 Almost Complex Spinor Structures 161

8 Spinors and Ha–Spaces 165
8.1 D–Spinor Techniques 165
8.1.1 Clifford d–algebra, d–spinors and d–twistors 166
8.1.2 Mutual transforms of d-tensors and d-spinors 169
8.1.3 Transformation of d-tensors into d-spinors 169
8.1.4 Fundamental d–spinors 170
8.2 Differential Geometry of Ha–Spinors 171
8.2.1 D-covariant derivation on ha–spaces 172
8.2.2 Infeld–van der Waerden coefficients 174
8.2.3 D–spinors of ha–space curvature and torsion 176

9 Ha-Spinors and Field Interactions 179
9.1 Scalar field ha–interactions 179
9.2 Proca equations on ha–spaces 181
9.3 Higher order anisotropic Dirac equations 182
9.4 D–spinor Yang–Mills fields 183
9.5 D–spinor Einstein–Cartan Theory 184
9.5.1 Einstein ha–equations 184
9.5.2 Einstein–Cartan d–equations 185
9.5.3 Higher order anisotropic gravitons 185

IV Finsler Geometry and Spinor Variables 187

10 Metrics Depending on Spinor Variables 189
10.1 Lorentz Transformation 189
10.2 Curvature 193

11 Field Equations in Spinor Variables 199
11.1 Introduction 199
11.2 Derivation of the field equations 201
11.3 Generalized Conformally Flat Spaces 206
11.4 Geodesics and geodesic deviation 211
11.5 Conclusions 213

12 Gauge Gravity Over Sinor Bundles **215**
12.1 Introduction 215
12.2 Connections 217
12.2.1 Nonlinear connections 218
12.2.2 Lorentz transformation 221
12.3 Curvatures and torsions 222
12.4 Field equations 223
12.5 Bianchi identities 225
12.6 Yang-Mills fields 227
12.7 Yang-Mills-Higgs field 228

13 Spinors on Internal Deformed Systems **231**
13.1 Introduction 231
13.2 Connections 232
13.3 Curvatures and Torsions 235
13.4 Field Equations 237

14 Bianchi Identities and Deformed Bundles **241**
14.1 Introduction 241
14.2 Bianchi Identities 242
14.3 Yang-Mills-Higgs equations. 245
14.4 Field Equations of an Internal Deformed System 247

15 Tensor and Spinor Equivalence **251**
15.1 Introduction 251
15.2 Generalization Spinor–Tensor Equivalents 254
15.3 Adapted Frames and Linear Connections 256
15.4 Torsions and Curvatures 258

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0.1 Preface

0.1.1 Historical remarks on spinor theory

Spinors and Clifford algebras play a major role in the contemporary physics and mathematics. In their mathematical form spinors had been discovered by Èlie Cartan in 1913 in his researches on the representation group theory [43] who showed that spinors furnish a linear representation of the groups of rotations of a space of arbitrary dimensions. In 1927 the physicists Pauli [126] and Dirac [54] (respectively, for the three–dimensional and four–dimensional space–time) introduced spinors to represent wave functions.

The spinors studied by mathematicians and physicists are connected with the general theory of Clifford spaces introduced in 1876 [46].

In general relativity theory spinors and the Dirac equations on (pseudo) Riemannian spaces, were defined in 1929 by H. Weyl [206], V. Fock [60] and E. Schrödinger [138]. The book [127], by R. Penrose, and volumes 1 and 2 of the R. Penrose and W. Rindler monograph [128, 129] summarize the spinor and twistor methods in space–time geometry (see additiona references [65, 33, 119, 91, 154, 42] on Clifford structures and spinor theory).

Spinor variables were introduced in Finsler geometries by Y. Takano in 1983 [152] who considered anisotropic dependencies not only on vectors from the tangent bundle but on some spinor variables in a spinor bundle on a space–time manifold. That work was inspired from H. Yukawa's quantum theory of non–local fields in 1950 [211]; it was suggested that non–localization may be in Finsler like manner but on spinor variables. There was also a similarity with supersymmetric models (see, for instance, references from [204, 205]), which also used spinor variables. The Y. Takano's approach followed standard Finsler ideas and was not concerned with topics relating supersymmetries of interactions.

This direction of generalized Finsler geometry, with spinor variables, was developed by T. Ono and Y. Takano in a series of works during 1990–1993 [121, 122, 123, 124]. The next steps were investigations of anisotropic and deformed geometries with mixtures of spinor and vector variables and applications in gauge and gravity theories elaborated by P. Stavrinos and his students S. Koutroubis and P. Manouselis as well as with Professor V. Balan beginning 1994 [145, 147, 148, 142, 143]. In those works the authors assumed that some spinor variables may be introduced in a Finsler like manner, they do not related the Finlser metric to a Clifford structure and restricted the spinor–gauge Finsler constructions only for antisymmetric spinor metrics on two–spinor fibers with generalizations four dimensional Dirac spinors.

Isotopic spinors, related with $SU(2)$ internal structural groups, were considered in generalized Finsler gravity and gauge theories also by G. Asanov and S. Ponomarenko [19], in 1988. But in that book, as well in the another mentioned papers on Finsler geometry with spinor variables the authors had not investigated the problem if a rigorous mathematical definition of spinors is possible on spaces with generic local anisotropy.

An alternative approach to spinor differential geometry and generalized Finsler spaces was elaborated, beginning 1994, in a series of papers and communications by S. Vacaru with participation of S. Ostaf [189, 192, 190, 161]. This direction originates from Clifford algebras and Clifford bundles [83, 154] and Penrose's spinor and twistor space–time geometry [127, 128, 129] which were re–considered for the case of nearly autoparallel maps (generalized conformal transforms) in Refs. [156, 157, 158]. In the works [162, 163, 166], a rigorous definition of spinors for Finsler spaces, and their generalizations, was given. It was proven that a Finsler, or Lagrange, metric (in a tangent, or, more generally, in a vector bundle) induces naturally a distinguished Clifford (spinor) structure which is locally adapted to the nonlinear connection structure. Such spinor spaces could be defined for arbitrary dimensions of base and fiber subspaces, their spinor metrics are symmetric, antisymmetric or nonsymmetric (depending on corresponding base and fiber dimensions). In result it was formulated the spinor differential geometry of generalized Finsler spaces and developed a number of geometric applications the theory of gravitational and matter filed interactions with generic local anisotropy.

Further, the geometry of anisotropic spinors and of distinguished by nonlinear connections Clifford structures was elaborated for higher order anisotropic spaces spaces [165, 173, 172] and, recently, to Hamilton and Lagrange spaces [198].

Here it would be necessary to emphasize that the theory of anisotropic spinors may be related not only with generalized Finsler, Lagrange, Cartan and Hamilton spaces or their higher order generalizations. Anholonomic frames with associated nonlinear connections appear naturally even in (pseudo) Riemannian geometry if off–diagonal metrics are considered [176, 177, 179, 182, 183]. In order to construct exact solutions of Einstein equations in general relativity and extra dimension gravity (for lower dimensions see [175, 196, 197]), it is more convenient to diagonalize space–time metrics by using some anholonomic transforms. In result one induces locally anisotropic structures on space–time which are related to anholonomic (anisotropic) spinor structures.

The main purpose of this book is to present an exhaustive summary and new results on spinor differential geometry for generalized Finsler spaces and (pseudo) Riemannian space–times provided with anholonomic frame and associated nonlinear connection structure, to discuss and compare the existing approaches and to consider applications in modern gravity and gauge theories.

0.1.2 Metric Spaces depending on Spinor Variables and Gauge Field Theories

An interesting study of differential geometry of spaces whose metric tensor $g_{\mu\nu}$ depends on spinor variables ξ and $\overline{\xi}$ (its adjoint) as well as coordinates x^i, has been proposed by Y. Takano [152]. Then Y. Takano and T. Ono [121, 122, 123] had studied the above–mentioned spaces and they gave a generalization of these spaces in the case of the metric tensor depending on spinor variables ξ and $\overline{\xi}$ and vector variables y^i as well as coordinates x^i. Such spaces are considered as a generalization of Finsler spaces.

Latter P. Stavrinos and S. Koutroubis studied the Lorents transformations and the curvature of generalized spaces with metric tensor $g_{\mu\nu}(x, y, \xi, \overline{\xi})$ [145].

The gravitational field equations are derived in the framework of these spaces whose metric tensor depends also on spinor variables ξ and $\overline{\xi}$. The attempt is to describe gravity by a tetrad field and the Lorentz connection coefficients in a more generalized framework than that was developed by P. Ramond (cf. eg. [134]). An interesting case with generalized conformally flat spaces with metric $g_{\mu\nu}(x, \xi, \overline{\xi}) = \exp[2\sigma(x, \xi, \overline{\xi})]\eta_{\mu\nu}$ was studied and the deviation of geodesic equation in this space was derived.

In Chapter 12 we study the differential structure of a spinor bundle in spaces with metric tensor $g_{\mu\nu}(x, \xi, \bar{\xi})$ of the base manifold. Notions such as: gauge covariant derivatives of tensors, connections, curvatures, torsions and Bianchi identitities are presented in the context of a gauge approach due to the introduction of a Poincaré group and the use of d-connections [109, 116] in the spinor bundle $S^{(2)}M$. The introduction of basic 1-form fields ρ_μ and spinors ζ_α, $\bar{\zeta}^\alpha$ with values in the Lie algebra of the Poincaré group is also essential in our study. The gauge field equations are derived. Also we give the Yang-Mills and the Yang-Mills-Higgs equations in a form sufficiently generalized for our approach.

Using the Hilbert–Palatini technique for a Utiyama–type Lagrangian density in the deformed spinor bundle $S^{(2)}M \times R$, there are determined the explicit expressions of the field equations, generalizing previous results; also, the equivalence principle is shown to represent an extension for the corresponding one from $S^{(2)}M$.

In this chapter we studied the spinor bundle of order two $\tilde{S}^{(2)}(M)$, which is a foliation of the structure of the spinor bundle presented in [140, 148].

In the present approach the generalized tetrads and the spin-tetrads define, by means of the relations (13.8), a generalized principle of equivalence in the spinor bundle $\tilde{S}^{(2)}(M)$. Also, employing the Miron - type connections, we cover all the possibilities for the S - bundle connections, which represent the gauge potential in physical interpretation. These have, in the framework of our considerations, the remarkable property of isotopic spin conservation. The introduction of the internal deformed system (as a fibre) in $\tilde{S}^{(2)}M)$, is expected to produce as a natural consequence , for a definite value κ, ϕ^{α} (where κ is a constant and ϕ^{α} a scalar field), the Higgs field. This will be derived within the developed theory, in a forecoming paper.

In chapter 14 the Bianchi equations are determined for a deformed spinor bundle $\tilde{S}^{(2)}M = S^{(2)}M \times R$. Also the Yang-Mills-Higgs equations are derived, and a geometrical interpretation of the Higgs field is given [141].

1. We study the Bianchi identities choosing a Lagrangian density that contains the component φ of a g-valued spinor gauge field of mass $m \in R$. Also we derived the Yang-Mills-Higgs equations on $\tilde{S}^{(2)}M \times R$. When $m_0 \in R$ the gauge symmetry is spontaneous broken which is connected with Higgs field.

2. The introduction of d-connections in the internal (spinor) structures on $\tilde{S}^{(2)}M$-bundle provides the presentation of parallelism of the spin components constraints which satisfy by the field strengths.

3. In the metric G (relation (14.1)) of the bundle $\tilde{S}^{(2)}M$, the term $g^{\alpha\beta}D\xi^{\alpha}D\xi^{*}_{\beta}$ has a physical meaning since it expresses the measure of the number of particles to same point of the space.

4. The above mentioned approach can be combined with the phase transformations of the fibre $U(1)$ on a bundle $S^{(2)}M \times U(1)$ in the Higgs mechanism. This will be the subject of our future study.

In the last part of our monograph we establish the relation between spinor of $SL(2, C$ group and tensors in the framework of Lagrange spaces is studied. A geometrical extension to generalized metric tangent bundles is developed by means of spinor. Also, the spinorial equation of causality for the unique solution of the null-cone in the Finsler or Lagrange space is given explicitly [149].

0.1.3 Nonlinear connection geometry and physics

It was namely Èlie Cartan, in the 30th years of the previous century, who additionally to the mentioned first monograph on spinors wrote some fundamental books on the geometry of Riemannian, fibred and Finsler spaces by developing the moving frame method and the formalism of Pfaff forms for systems of first order partial differential equations [42, 41, 44]. The first examples of Finsler metrics and original definitions were given by B. Riemann [135] in 1854 and in Paul Finsler's thesis [59] written under the direction of Caratheodory in 1938. In those works one could found the origins of notions of locally trivial fiber bundle (which naturally generalize that of the manifold, the theory of these bundles was developed, by 20 years later, especially by Gh. Ehresmann) and of nonlinear connection (appearing as a set of coefficients in the book [41] and in a more explicit form in some papers by A. Kawaguchi [84]).

The global formulation of nonlinear connection is due to W. Barthel [25]; detailed investigations of nonlinear connection geometry in vector bundles and higher order tangent bundles, with applications to physics and mechanics, are contained in the monographs and works [108, 109, 106, 107, 110, 113] summarizing the investigations of Radu Miron school on Finsler and Lagrange geometry and generalizations. The geometry of nonlinear connections was developed in S. Vacaru's works and monograph for vector and higher order [169, 172] superbundles and anisotropic Clifford/spinor fibrations [189, 162, 163, 165, 166], with generalizations and applications in (super) gravity [184, 177, 179, 185, 185, 186, 194, 195] and string theories [170, 171] and noncommutative gravity [180]). There are a number of results on nonlinear connections and Finsler geometry, see for instance [136, 24, 96], with generalizations and applications in mechanics, physics and biology which can be found in references [5, 7, 8, 9, 12, 14, 16, 19, 27, 29, 37].

Finsler spaces and their generalizations have been also developed with the aim to propose applications in classical and higher order mechanics, optics, generalized Kaluza–Klein theories and gauge theories. But for a long period of time the Finsler geometry was considered as to be very sophisticate and less compatible with the standard paradigm of modern physics. The first objection was that on spaces with local anisotropy there are not even local groups of authomorphisms which made impossible to define local conservation laws, develop a theory of anisotropic random and kinetic processes and introduce spinor fields. The second objection was based on a confusion stating that in Finsler like gravity theories the local Lorentz symmetry is broken which is not compatible with the modern paradigms of particle physics and gravity [208]. Nevertheless, it was proven that there are not more conceptual problems with definition of local conservation laws than in the usual theory of gravity on pseudo–Riemannian spaces if Finsler like theories are formulated with respect to local frames adapted to the nonlinear connection structure: a variant of definition of conservation laws for locally anisotropic gravitational and matter field interactions being proposed by using chains of nearly autoparallel maps generalizing conformal transforms [164, 191, 193]. As to violations of the local Lorentz symmetries, one should be mentioned that really there were investigated some classes of such Finsler like metrics with the aim to revise the special and general theories of relativity (see, for instance, Refs. [18, 13, 37, 70]), but it is also possible to define Finsler like, and another type anisotropic, structures, even in the framework of general relativity theory. Such structures are described by some exact solutions of the Einstein equations if off–diagonal frames and anholonomic frames are introduced into consideration [176, 177, 179, 182, 183, 185]. We conclude that there are different classes of generalized Finsler like metrics: some of them posses broken Lorentz symmetries another ones do not have such properties and are compatible with the general relativity canons. Here should be emphasized that the violation of Lorentz geometry is not already a prohibited subject in modern physics, for instance, the effects induced by Lorentz violations are analyzed in brane physics [52] and non–commutative field theories [118, 40].

The third objection was induced by "absence" of a mathematical theory of stochastic processes and diffusion on spaces with generic local anisotropy. But this problem was also solved in a series of papers: The first results on diffusion processes on Finsler manifolds were announced in 1992 by P. Antonelly and T. Zastavniak [10, 11]; their formalism was not yet adapted to the nonlinear connection structure. In a communication at the Iaşi Academic Days (1994, Romania) [159] S. Vacaru suggested to develop the theory of stochastic differential equations as in the Riemannian spaces but on vector bundles provided with nonlinear connection structures. In result the theory of anisotropic processes was in parallel developed on vector bundles by S. Vacaru [159, 160, 167] (see Chapter 5 in [172] for supersymmetric anisotropic stochastic processes) and P. Antonelli, T. Zastavniak and D. Hrimiuc [10, 11, 68, 69, 6] (by the last three authors with a number of applications in biology and biophysics) following a theory of stochastic differential equations formulated on bundles provided with anholnomic frames and nonlinear connections. It was also possible to formulate a theory of anisotropic kinetic processes and thermodynamics [175, 178, 179] which applications in modern cosmology and astrophysics. So, the third difficulty for anisotropic physics, connected with the definition of random and kinetic models on spaces with generic local anisotropy was got over.

As a forth objection on acceptance by "physical community" of Finsler spaces was the arguments like "it is not clear how to supersymmetrize such theories and how to embed them in a modern string theory because at low energies from string theories one follows only (pseudo) Riemannian geometries and their supersymmetric generalizations". The question on definition of nonlinear connections in superbundles was solved in a series of preprints in 1996 [169] with the results included in the paper [171] and monograph [172]. It was formulated a new Finsler supergeometry with generalizations and applications in (super) gravity and string theories [184]. The works [170, 171] contained explicit proofs that we can embed in (super) string theories Finsler like geometris if we are dealing with anholonomomic (super) frame structures, at low energies we obtain anholonomic frames on (pseudo) Riemannian space–times or, alternatively, different type of Finsler like geometries.

The monograph [172] summarized the basic results on anisotropic (in general, supersymmetric) field interactions, stochastic processes and strings. It was the first book where the basic directions in modern physics were reconsidered on (super) spaces provided with nonlinear connection structure. It was proven that following the E. Cartan geometrical ideas and methods to vector bundles, spinors, moving frames, nonlinear connections, Finsler and (pseudo) Riemannian spaces the modern phyisical theories can be formulated in a unified manner both on spaces with generic local anisotropy and on locally isotropic spaces if local frames adapted to nonlinear connection structures are introduced into consideration.

This book covers a more restricted area, comparing with the monograph [172], connected in the bulk with the spinor geometry and physic, and is intended to provide the reader with a thorough background for the theory of anisotropic spinors in generalized Finsler spaces and for the theory of anholonomic spinor structures in (pseudo) Riemannian spaces. The required core of knowledge is that the reader is familiar to basic concepts from the theory of bundle spaces, spinor geometry, classical field theory and general relativity at a standard level for graduate students from mathematics and theoretical physics. The primary purpose of this book is to introduce the new geometrical ideas in the language of standard fiber bundle geometry and establish a working familiarity with the modern applications of spinor geometry, anholonomic frame method and nonlinear connections formalism in physics. These techniques are subsequently generalized and applied to gravity and gauge theories. The secondary purpose is to consider and compare different approaches which deal with spinors in Finsler like geometries.

0.1.4 Anholonomic frames and nonlinear connections in Einstein gravity

Let us consider a $(n+m)$–dimensional (pseudo) Riemannian spacetime $V^{(n+m)}$, being a paracompact and connected Hausdorff C^{∞}–manifold, enabled with a nonsigular metric

$$ds^2 = g_{\alpha\beta}\ du^{\alpha} \otimes du^{\beta}$$

with the coefficients

$$g_{\alpha\beta} = \left[\begin{array}{cc} g_{ij} + N_i^a N_j^b h_{ab} & N_j^e h_{ae} \\ N_i^e h_{be} & h_{ab} \end{array} \right]$$

parametrized with respect to a local coordinate basis $du^{\alpha} = (dx^i, dy^a)$, having its dual $\partial/u^{\alpha} = (\partial/x^i, \partial/y^a)$, where the indices of geometrical objects and local coordinate $u^{\alpha} = \left(x^k, y^a\right)$ run correspondingly the values: (for Greek indices)$\alpha, \beta, \ldots = n+m$; for (Latin indices) $i, j, k, ... = 1, 2, ..., n$ and $a, b, c, ... = 1, 2, ..., m$. Such off–diagonal ansatz for metric were considered, for instance, in Salam–Strathdee–Percacci–Randjbar-Daemi works on Kaluza–Klein theory [137, 130, 125] as well in four and five dimensional gravity [176, 177, 179, 194, 182, 183, 187, 188, 195, 181].

The metric ansatz can be rewritten equivalently in a block $(n \times n) + (m \times m)$ form

$$g_{\alpha\beta} = \left(\begin{array}{cc} g_{ij}(x^k, y^a) & 0 \\ 0 & h_{ab}(x^k, y^a) \end{array} \right)$$

with respect to a subclass of $n+m$ anholonomic frame basis (for four dimensions one used terms tetrads, or vierbiends) defined

$$\delta_{\alpha} = (\delta_i, \partial_a) = \frac{\delta}{\partial u^{\alpha}} = \left(\delta_i = \frac{\delta}{\partial x^i} = \frac{\partial}{\partial x^i} - N_i^b \left(x^j, y^c\right) \frac{\partial}{\partial y^b}, \partial_a = \frac{\partial}{\partial y^a} \right)$$

and

$$\delta^{\beta} = \left(d^i, \delta^a\right) = \delta u^{\beta} = \left(d^i = dx^i, \delta^a = \delta y^a = dy^a + N_k^a \left(x^j, y^b\right) dx^k \right),$$

called locally anisotropic bases (in brief, anisotropic bases) adapted to the coefficients N_j^a. The $n \times n$ matrix g_{ij} defines the so–called horizontal metric (in brief, h–metric) and the $m \times m$ matrix h_{ab} defines the vertical (v–metric) with respect to the associated nonlinear connection (N–connection) structure given by its coefficients $N_j^a(u^{\alpha})$, see for instance [109] where the geometry of N–connections is studied in detail for generalized Finsler and Lagrange spaces (the y–coordinates parametrizing fibers in a bundle).

Here we emphasize that a matter of principle we can consider that our ansatz and N–elongated bases are defined on a (pseudo) Riemannian manifold, and not on a bundle space. In this case we can treat that the x–coordinates are holonomic ones given with respect to a sub–basis not subjected to any constraints, but the y–coordinates are those defined with respcect to an anholonomic (constrained) sub–basis.

An anholonomic frame structure δ_α on $V^{(n+m)}$ is characterized by its anholonomy relations

$$\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha = w^{\gamma}{}_{\alpha\beta}\delta_\gamma.$$

with anholonomy coefficients $w^{\alpha}{}_{\beta\gamma}$. The elongation of partial derivatives (by N–coefficients) in the locally adapted partial derivatives reflects the fact that on the (pseudo) Riemannian space–time $V^{(n+m)}$ it is modeled a generic local anisotropy characterized by some anholonomy relations when the anholonomy coefficients are computed as follows

$$\begin{aligned} w^{k}{}_{ij} &= 0, w^{k}{}_{aj} = 0, w^{k}{}_{ia} = 0, w^{k}{}_{ab} = 0, w^{c}{}_{ab} = 0, \\ w^{a}{}_{ij} &= -\Omega^a_{ij}, w^{b}{}_{aj} = -\partial_a N^b_i, w^{b}{}_{ia} = \partial_a N^b_i, \end{aligned}$$

where

$$\Omega^a_{ij} = \partial_i N^a_j - \partial_j N^a_i + N^b_i \partial_b N^a_j - N^b_j \partial_b N^a_i$$

defines the coefficients of the N–connection curvature, in brief, N–curvature. On (pseudo) Riemannian space–times this is a characteristic of a chosen anholonomic system of reference.

For generic off–diagonal metrics we have two alternatives: The first one is to try to compute the connection coefficients and components of the Einstein tensor directly with respect to a usual coordinate basis. This is connected to a cumbersome tensor calculus and off–diagonal systems of partial differential equations which makes almost impossible to find exact solutions of Einstein equations. But we may try do diagonalize the metric by some anholonomic transforms to a suitable N–elongated anholonomic basis. Even this modifies the low of partial derivation (like in all tetradic theories of gravity) the procedure of computing the non–trivial components of the Ricci and Einstein tensor simplifies substantially, and for a very large class of former off–diagonal ansatz of metric, anholonomically diagonalized, the Einstein equations can be integrated in general form [176, 177, 179, 194, 182, 183].

So, we conclude that when generic off–diagonal metrics and anholonomic frames are introduces into consideration on (pseudo) Riemannian spaces the space–time geometry may be equivalently modeled as the geometry of moving anholonomic frames with associated nonlinear connection structure. In this case the problem of definition of anholonomic (anisotropic) spinor structures arises even in general relativity theory which points to the fact that the topic of anisotropic spinor differential geometry is not an exotic subject from Finsler differential geometry but a physical important problem which must be solved in order to give a spinor interpretation of space–times provided with off–diagonal metrics and anholonomic gravitational and matter field interactions.

0.1.5 The layout of the book

This book is organized in four Parts: the first three Parts each consisting of three or Chapters, the forth Part consisting from six Chapters.

The Part I has is a geometric introduction into the geometry of anisotropic spaces as well it outlines original results on the geometry of anholonomic frames with associated nonlinear connections structures in (pseudo) Riemannian spaces. In the Capter 1 we give the basic definitions from the theory of generalized Finsler, Lagrange, Cartan and Hamilton spaces on vector and co–vector (tangent and co–tangent spaces) and their generalizations for higher order vector–covector bundles following the monographs [109, 113, 172]. The next two Chapters are devoted to a discussion and explicit examples when anisotropic (Finsler like and more general ones) structures can be modeled on pseudo–Riemannian spacetimes and in gravitational theories. They are based on results of works ellaborated by S. Vacaru and co–authors [176, 177, 179, 182, 185, 194, 195, 199]

The Part II covers the algebra (Chapter 4) and geometry (Chapter 5) of Clifford and spinor structures in vector bundles provided with nonlinear connection structure. A spinor formulation of generalized Finsler gravity and anisotropic matter field interactions is given in Chapter 6. This Part originates from S. Vacaru and co–authors works [189, 190, 161, 162, 163, 165].

The Part III is a generalization of results on Clifford and spinor structures for higher order vector bundles (the Chapters 7–9 extend respectively the results of Chapters 4–6), which are based on S. Vacaru's papers [166, 173].

The Part IV (consisting from Chapters 10–15) summarizes the basic results on various extensions of Finsler like geometries by considering spinor variables. In the main, this Part originates from Y. Takano and T. Ono papers [152, 121, 122, 123, 124] and reflects the most important contributions by P. Stavrinos and co–authors [140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150].

Summing up, in this monograph we investigate anholonomic (anisotropic) spinor structures in space–times with generic local anisotropy (i. e. in generalized Finsler spaces) and in (pseudo) Riemannian spaces provided with off–diagonal metrics and anholonomic frame bases. It is addressed primarily to researches and other readers in theoretical and mathematical physics and differential geometry, both at the graduate student and more advances physicist and mathematical levels.

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0.2 Notation

The reader is advised to refer as and when necessary to the list below where there are set out the conventions that will be followed in this book with regard to the presentation of the various physical and mathematical expressions.

(1) *Equations.* For instance, equation (3.16) is the 16th equation in Chapter 3.

(2) *Indices.* It is impossible to satisfy everybody in matter of choice of labels of geometrical objects and coordinates. In general, we shall use Greek superscripts for labels on both vector bundles and superbundles. The reader will have to consult the first sections in every Chapter in order to understand the meaning of various types of boldface and/or underlined Greek or Latin letters for operators, distinguished spinors and tensors.

(3) *Differentiation.* Ordinary partial differentiation with respect to a coordinate x^i will either be denoted by the operator ∂_i or by subscript i following a comma, for instance, $\frac{\partial A^i}{\partial x^j} \equiv \partial_j A^i \equiv A^i{}_{,j}$. We shall use the denotation $\frac{\delta A^i}{\delta x^j} \equiv \delta_j A^i$ for partial derivations locally adapted to a nonlinear connection structure.

(4) *Summation convention.* We shall follow the Einstein summation rule for spinor and tensor indices.

(5) *References.* In the bibliography we cite the scientific journals in a generally accepted abbreviated form, give the volume, the year and the first page of the authors' articles; the monographs and collections of works are cited completely. For the author's works and communications, a part of them been published in not enough accessible issues, or being under consideration, the extended form (with the titles of articles and communications) is presented. We emphasize that the references are intended to give a sense of the book's scopes. We ask kindly the readers they do not feel offended by any omissions.

(6) *Introductions and Conclusions.* If it is considered necessary a Chapter starts with an introduction into the subject and ends with concluding remarks.

Part I

Space–Time Anisotropy

Chapter 1

Vector/Covector Bundles and Nonlinear Connections

In this Chapter the space–time geometry is modeled not only on a (pseudo) Riemannian manifold $V^{[n+m]}$ of dimension $n+m$ but it is considered on a vector bundle (or its dual, covector bundle) being, for simplicity, locally trivial with a base space M of dimension n and a typical fiber F (cofiber F^*) of dimension m, or as a higher order extended vector/covector bundle (we follow the geometric constructions and definitions of monographs [109, 108, 113, 106, 107] which were generalized for vector superbundles in Refs. [171, 172]). Such fibered space–times (in general, with extra dimensions and duality relations) are supposed to be provided with compatible structures of nonlinear and linear connections and (pseudo) Riemannian metric. For the particular cases when: a) the total space of the vector bundle is substituted by a pseudo–Riemannian manifold of necessary signature we can model the usual pseudo–Riemannian space–time from the Einstein gravity theory with field equations and geometric objects defined with respect to some classes of moving anholonomic frames with associated nonlinear connection structure; b) if the dimensions of the base and fiber spaces are identical, $n=m$, for the first order anisotropy, we obtain the tangent bundle TM.

Such both (pseudo) Riemanian spaces and vector/covector (in particular cases, tangent/cotangent) bundles of metric signature (-,+,...,+) enabled with compatible fibered and/or anholonomic structures, the metric in the total space being a solution of the Einstein equations, will be called **anisotropic space–times**. If the anholonomic structure with associated nonlinear connection is modeled on higher order vector/covector bundles we shall use the term of **higher order anisotropic space–time.**

The geometric constructions are outlined as to present the main concepts and formulas in a unique way for both type of vector and covector structures.

In this part of the book we usually shall omit proofs which can be found in the mentioned monographs [108, 109, 106, 107, 113, 172].

1.1 Vector and Covector Bundles

In this Section we introduce the basic definitions and denotations for vector and tangent (and theirs dual spaces) bundles and higher order vector/covector bundle geometry.

1.1.1 Vector and tangent bundles

A locally trivial **vector bundle**, in brief, **v–bundle**, $\mathcal{E}=(E,\pi,M,Gr,F)$ is introduced as a set of spaces and surjective map with the properties that a real vector space $F=\mathcal{R}^m$ of dimension m ($\dim F=m$, $\mathcal{R}$ denotes the real number field) defines the typical fibre, the structural group is chosen to be the group of automorphisms of $\mathcal{R}^m$, i. e. $Gr=GL\left(m,\mathcal{R}\right)$, and $\pi:E\rightarrow M$ is a differentiable surjection of a differentiable manifold E (total space, $\dim E=n+m$) to a differentiable manifold M (base space, $\dim M=n$). Local coordinates on $\mathcal{E}$ are denoted $u^\alpha=\left(x^i,y^a\right)$, or in brief $u=(x,y)$ (the Latin indices $i,j,k,...=1,2,...,n$ define coordinates of geometrical objects with respect to a local frame on base space M; the Latin indices $a,b,c,...$ =1, 2, ..., m define fibre coordinates of geometrical objects and the Greek indices $\alpha,\beta,\gamma,...$ are considered as cumulative ones for coordinates of objects defined on the total space of a v-bundle).

Coordinate transforms $u^{\alpha'}=u^{\alpha'}\left(u^{\alpha}\right)$ on a v–bundle $\mathcal{E}$ are defined as

$$\left(x^{i},y^{a}\right)\rightarrow\left(x^{i'},y^{a'}\right),$$

where

$$x^{i'}=x^{i'}\left(x^{i}\right),\qquad y^{a'}=K_{a}^{a'}(x^{i})y^{a} \tag{1.1}$$

and matrix $K_{a}^{a'}(x^{i})\in GL\left(m,\mathcal{R}\right)$ are functions of necessary smoothness class.

A local coordinate parametrization of v–bundle $\mathcal{E}$ naturally defines a coordinate basis

$$\partial_\alpha=\frac{\partial}{\partial u^\alpha}=\left(\partial_i=\frac{\partial}{\partial x^i},\ \partial_a=\frac{\partial}{\partial y^a}\right), \tag{1.2}$$

and the reciprocal to (1.2) coordinate basis

$$d^\alpha=du^\alpha=(d^i=dx^i,\ d^a=dy^a) \tag{1.3}$$

which is uniquely defined from the equations

$$d^{\alpha} \circ \partial_{\beta} = \delta^{\alpha}_{\beta},$$

where δ^{α}_{β} is the Kronecher symbol and by "$\circ$" we denote the inner (scalar) product in the tangent bundle $\mathcal{TE}$.

A **tangent bundle** (in brief, **t–bundle**) (TM, π, M) to a manifold M can be defined as a particular case of a v–bundle when the dimension of the base and fiber spaces (the last one considered as the tangent subspace) are identic, $n = m$. In this case both type of indices $i, k, ...$ and $a, b, ...$ take the same values $1, 2, ...n$. For t–bundles the matrices of fiber coordinates transforms from (1.1) can be written $K_{\ i}^{i'} = \partial x^{i'}/\partial x^{i}$.

We shall distinguish the base and fiber indices and values which is necessary for our further geometric and physical applications.

1.1.2 Covector and cotangent bundles

We shall also use the concept of **covector bundle**, (in brief, **cv–bundles)** $\breve{\mathcal{E}} = \left(\breve{E}, \pi^{*}, M, Gr, F^{*}\right)$, which is introduced as a dual vector bundle for which the typical fiber F^{*} (cofiber) is considered to be the dual vector space (covector space) to the vector space F. The fiber coordinates p_a of $\breve{E}$ are dual to y^a in E. The local coordinates on total space $\breve{E}$ are denoted $\breve{u} = (x, p) = (x^i, p_a)$. The coordinate transform on $\breve{E}$,

$$\breve{u} = (x^i, p_a) \rightarrow \breve{u}' = (x^{i'}, p_{a'}),$$

are written

$$x^{i'} = x^{\ i'} (x^{\ i}), \qquad p_{a'} = K_{\ a'}^{a} (x^{i}\ ) p_a. \qquad (1.4)$$

The coordinate bases on E^{*} are denoted

$$\breve{\partial}_{\alpha} = \frac{\breve{\partial}}{\partial u^{\alpha}} = \left(\partial_i = \frac{\partial}{\partial x^i}, \breve{\partial}^a = \frac{\breve{\partial}}{\partial p_a}\right) \qquad (1.5)$$

and

$$\breve{d}^{\alpha} = \breve{d}u^{\alpha} = \left(d^i = dx^i, \breve{d}_a = dp_a\right). \qquad (1.6)$$

We shall use "breve" symbols in order to distinguish the geometrical objects on a cv–bundle $\mathcal{E}^{*}$ from those on a v–bundle $\mathcal{E}$.

As a particular case with the same dimension of base space and cofiber one obtains the **cotangent bundle** $(T^{*}M, \pi^{*}, M)$, in brief, **ct–bundle,** being dual to TM. The fibre coordinates p_i of T^*M are dual to y^i in TM. The coordinate transforms (1.4) on T^*M are stated by some matrices $K^k_{\ k'}(x^i) = \partial x^k/\partial x^{k'}$.

In our further considerations we shall distinguish the base and cofiber indices.

1.1.3 Higher order vector/covector bundles

The geometry of higher order tangent and cotangent bundles provided with nonlinear connection structure was elaborated in Refs. [106, 107, 110, 113] following the aim of geometrization of higher order Lagrange and Hamilton mechanics. In this case we have base spaces and fibers of the same dimension. In order to develop the approach to modern high energy physics (in superstring and Kaluza–Klein theories) one had to introduce (in Refs [165, 173, 172, 171]) the concept of higher order vector bundle with the fibers consisting from finite 'shells" of vector, or covector, spaces of different dimensions not obligatory coinciding with the base space dimension.

Definition 1.1. *A distinguished vector/covector space, in brief dvc–space, of type*

$$\tilde{F} = F[v(1), v(2), cv(3), ..., cv(z-1), v(z)] \tag{1.7}$$

is a vector space decomposed into an invariant oriented direct summ

$$\tilde{F} = F_{(1)} \oplus F_{(2)} \oplus F^*_{(3)} \oplus ... \oplus F^*_{(z-1)} \oplus F_{(z)}$$

of vector spaces $F_{(1)}, F_{(2)}, ..., F_{(z)}$ *of respective dimensions*

$$dimF_{(1)} = m_1, dimF_{(2)} = m_2, ..., dimF_{(z)} = m_z$$

and of covector spaces $F^*_{(3)}, ..., F^*_{(z-1)}$ *of respective dimensions*

$$dimF^*_{(3)} = m^*_3, ..., dimF^*_{(z-1)} = m^*_{(z-1)}.$$

As a particular case we obtain a distinguished vector space, in brief dv–space (a distinguished covector space, in brief dcv–space), if all components of the sum are vector (covector) spaces. We note that we have fixed for simplicity an orientation of vector/covector subspaces like in (1.7); in general there are possible various type of orientations, number of subspaces and dimensions of subspaces.

Coordinates on $\tilde{F}$ are denoted

$$\tilde{y} = (y_{(1)}, y_{(2)}, p_{(3)}, ..., p_{(z-1)}, y_{(z)}) = \{y^{<\alpha_z>}\} = (y^{a_1}, y^{a_2}, p_{a_3}, ..., p_{a_{z-1}}, y^{a_z}),$$

where indices run corresponding values:

$$a_1 = 1, 2, ..., m_1; \ a_2 = 1, 2, ..., m_2, \ ..., a_z = 1, 2, ..., m_z.$$

Definition 1.2. *A higher order vector/covector bundle (in brief, hvc--bundle) of type* $\tilde{\mathcal{E}} = \tilde{\mathcal{E}}[v(1), v(2), cv(3), ..., cv(z-1), v(z)]$ *is a vector bundle* $\tilde{\mathcal{E}} = (\tilde{E}, p^{<d>}, \tilde{F}, M)$ *with corresponding total,* $\tilde{E}$*, and base,* M*, spaces, surjective projection* $p^{<d>}: \ \tilde{E} \to M$ *and typical fibre* $\tilde{F}$.

We define higher order vector (covector) bundles, in brief, hv–bundles (in brief, hcv–bundles), if the typical fibre is a dv–space (dcv–space) as particular cases of hvc–bundles.

A hvc–bundle is constructed as an oriented set of enveloping 'shell by shell' v–bundles and/or cv–bundles,

$$p^{<s>}: \ \tilde{E}^{<s>} \to \tilde{E}^{<s-1>},$$

where we use the index $<s>= 0, 1, 2, ..., z$ in order to enumerate the shells, when $\tilde{E}^{<0>} = M$. Local coordinates on $\tilde{E}^{<s>}$ are denoted

$$\begin{aligned} \tilde{u}_{(s)} &= (x, \tilde{y}_{<s>}) = (x, y_{(1)}, y_{(2)}, p_{(3)}, ..., y_{(s)}) \\ &= (x^i, y^{a_1}, y^{a_2}, p_{a_3}, ..., y^{a_s}). \end{aligned}$$

If $<s>=<z>$ we obtain a complete coordinate system on $\tilde{\mathcal{E}}$ denoted in brief

$$\tilde{u} = (x, \tilde{y}) = \tilde{u}^\alpha = (x^i = y^{a_0}, y^{a_1}, y^{a_2}, p_{a_3}, ..., p_{a_{z-1}}, y^{a_z}).$$

We shall use the general commutative indices $\alpha, \beta, ...$ for objects on hvc—bundles which are marked by tilde, like $\tilde{u}, \tilde{u}^\alpha, ..., \tilde{E}^{<s>}, ...$

The coordinate transforms for a hvc–bundle $\tilde{\mathcal{E}}$,

$$\tilde{u} = (x, \tilde{y}) \to \tilde{u}' = (x', \tilde{y}')$$

are given by recurrent formulas

$$\begin{aligned} x^{i'} &= x^{i'}\left(x^i\right), \ rank\left(\frac{\partial x^{i'}}{\partial x^i}\right) = n; \\ y^{a'_1} &= K^{a'_1}_{a_1}(x) y^{a_1}, K^{a'_1}_{a_1} \in GL(m_1, \mathcal{R}); \\ y^{a'_2} &= K^{a'_2}_{a_2}(x, y_{(1)}) y^{a_2}, K^{a'_2}_{a_2} \in GL(m_2, \mathcal{R}); \\ p_{a'_3} &= K^{a_3}_{a'_3}(x, y_{(1)}, y_{(2)}) p_{a_3}, K^{a_3}_{a'_3} \in GL(m_3, \mathcal{R}); \\ y^{a'_4} &= K^{a'_4}_{a_4}(x, y_{(1)}, y_{(2)}, p_{(3)}) y^{a_4}, K^{a'_4}_{a_4} \in GL(m_4, \mathcal{R}); \\ & \qquad .... \\ p_{a'_{z-1}} &= K^{a_{z-1}}_{a'_{z-1}}(x, y_{(1)}, y_{(2)}, p_{(3)}, ..., y_{(z-2)}) p_{a_{z-1}}, K^{a_{z-1}}_{a'_{z-1}} \in GL(m_{z-1}, \mathcal{R}); \\ y^{a'_z} &= K^{a'_z}_{a_z}(x, y_{(1)}, y_{(2)}, p_{(3)}, ..., y_{(z-2)}, p_{a_{z-1}}) y^{a_z}, K^{a'_z}_{a_z} \in GL(m_z, \mathcal{R}), \end{aligned}$$

where, for instance. by $GL(m_2, \mathcal{R})$ we denoted the group of linear transforms of a real vector space of dimension m_2.

The coordinate bases on $\tilde{\mathcal{E}}$ are denoted

$$\begin{aligned} \tilde{\partial}_\alpha &= \frac{\tilde{\partial}}{\partial u^\alpha} \\ &= \left(\partial_i = \frac{\partial}{\partial x^i}, \partial_{a_1} = \frac{\partial}{\partial y^{a_1}}, \partial_{a_2} = \frac{\partial}{\partial y^{a_2}}, \breve{\partial}^{a_3} = \frac{\breve{\partial}}{\partial p_{a_3}}, ..., \partial_{a_z} = \frac{\partial}{\partial y^{a_z}} \right) \end{aligned} \tag{1.8}$$

and

$$\begin{aligned} \tilde{d}^\alpha &= \tilde{d}u^\alpha \\ &= \left(d^i = dx^i, d^{a_1} = dy^{a_1}, d^{a_2} = dy^{a_2}, \breve{d}_{a_3} = dp_{a_3}, ..., d^{a_z} = dy^{a_z} \right). \end{aligned} \tag{1.9}$$

We end this subsection with two examples of higher order tangent / co-tangent bundles (when the dimensions of fibers/cofibers coincide with the dimension of bundle space, see Refs. [106, 107, 110, 113]).

Osculator bundle

The k–osculator bundle is identified with the k–tangent bundle $\left(T^k M, p^{(k)}, M\right)$ of a n–dimensional manifold M. We denote the local coordinates

$$\tilde{u}^\alpha = \left(x^i, y^i_{(1)}, ..., y^i_{(k)} \right),$$

where we have identified $y^i_{(1)} \simeq y^{a_1}, ..., y^i_{(k)} \simeq y^{a_k}, k = z$, in order to to have similarity with denotations from [113]. The coordinate transforms

$$\tilde{u}^{\alpha'} \to \tilde{u}^{\alpha'}\left(\tilde{u}^\alpha\right)$$

preserving the structure of such higher order vector bundles are parametrized

$$\begin{aligned} x^{i'} &= x^{i'}\left(x^i\right), \det\left(\frac{\partial x^{i'}}{\partial x^i}\right) \neq 0, \\ y^{i'}_{(1)} &= \frac{\partial x^{i'}}{\partial x^i} y^i_{(1)}, \\ 2y^{i'}_{(2)} &= \frac{\partial y^{i'}_{(1)}}{\partial x^i} y^i_{(1)} + 2\frac{\partial y^{i'}_{(1)}}{\partial y^i} y^i_{(2)}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ ky^{i'}_{(k)} &= \frac{\partial y^{i'}_{(1)}}{\partial x^i} y^i_{(1)} + ... + k\frac{\partial y^{i'}_{(k-1)}}{\partial y^i_{(k-1)}} y^i_{(k)}, \end{aligned}$$

where the equalities

$$\frac{\partial y^{i'}_{(s)}}{\partial x^i} = \frac{\partial y^{i'}_{(s+1)}}{\partial y^i_{(1)}} = ... = \frac{\partial y^{i'}_{(k)}}{\partial y^i_{(k-s)}}$$

hold for $s = 0, ..., k-1$ and $y^i_{(0)} = x^i$.

The natural coordinate frame on $\left(T^k M, p^{(k)}, M\right)$ is defined

$$\tilde{\partial}_\alpha = \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i_{(1)}}, ..., \frac{\partial}{\partial y^i_{(k)}}\right)$$

and the coframe is

$$\tilde{d}_\alpha = \left(dx^i, dy^i_{(1)}, ..., dy^i_{(k)}\right).$$

These formulas are respectively some particular cases of (1.8) and (1.9).

The dual bundle of k–osculator bundle

This higher order vector/covector bundle, denoted as $\left(T^{*k}M, p^{*k}, M\right)$, is defined as the dual bundle to the k–tangent bundle $\left(T^k M, p^k, M\right)$. The local coordinates (parametrized as in the previous paragraph) are

$$\tilde{u} = \left(x, y_{(1)}, ..., y_{(k-1)}, p\right) = \left(x^i, y^i_{(1)}, ..., y^i_{(k-1)}, p_i\right) \in T^{*k}M.$$

The coordinate transforms on $\left(T^{*k}M, p^{*k}, M\right)$ are

$$\begin{aligned}
x^{i'} &= x^{i'}\left(x^i\right), \det\left(\frac{\partial x^{i'}}{\partial x^i}\right) \neq 0,\\
y^{i'}_{(1)} &= \frac{\partial x^{i'}}{\partial x^i} y^i_{(1)},\\
2y^{i'}_{(2)} &= \frac{\partial y^{i'}_{(1)}}{\partial x^i} y^i_{(1)} + 2\frac{\partial y^{i'}_{(1)}}{\partial y^i} y^i_{(2)},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
(k-1)y^{i'}_{(k-1)} &= \frac{\partial y^{i'}_{(k-2)}}{\partial x^i} y^i_{(1)} + ... + k\frac{\partial y^{i'}_{(k-1)}}{\partial y^i_{(k-2)}} y^i_{(k-1)},\\
p_{i'} &= \frac{\partial x^i}{\partial x^{i'}} p_i,
\end{aligned}$$

where the equalities

$$\frac{\partial y^{i'}_{(s)}}{\partial x^i} = \frac{\partial y^{i'}_{(s+1)}}{\partial y^i_{(1)}} = ... = \frac{\partial y^{i'}_{(k-1)}}{\partial y^i_{(k-1-s)}}$$

hold for $s = 0, ..., k-2$ and $y^i_{(0)} = x^i$.

The natural coordinate frame on $\left(T^{*k}M, p^{*(k)}, M\right)$ is defined

$$\tilde{\partial}_\alpha = \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i_{(1)}}, ..., \frac{\partial}{\partial y^i_{(k-1)}}, \frac{\partial}{\partial p_i}\right)$$

and the coframe is

$$\tilde{d}_\alpha = \left(dx^i, dy^i_{(1)}, ..., dy^i_{(k-1)}, dp_i\right).$$

These formulas are respectively another particular cases of (1.8) and (1.9).

1.2 Nonlinear Connections

The concept of **nonlinear connection,** in brief, N-connection, is fundamental in the geometry of vector bundles and anisotropic spaces (see a detailed study and basic references in [108, 109]). A rigorous mathematical definition is possible by using the formalism of exact sequences of vector bundles.

1.2.1 N–connections in vector bundles

Let $\mathcal{E} = = (E, p, M)$ be a v–bundle with typical fibre $\mathcal{R}^m$ and $\pi^T : TE \to TM$ being the differential of the map P which is a fibre–preserving morphism of the tangent bundle $TE, \tau_E, E) \to E$ and of tangent bundle $(TM, \tau, M) \to M$. The kernel of the vector bundle morphism, denoted as (VE, τ_V, E), is called the **vertical subbundle** over E, which is a vector subbundle of the vector bundle (TE, τ_E, E).

A vector X_u tangent to a point $u \in E$ is locally written as

$$(x, y, X, Y) = (x^i, y^a, X^i, Y^a),$$

where the coordinates (X^i, Y^a) are defined by the equality

$$X_u = X^i \partial_i + Y^a \partial_a.$$

We have $\pi^T(x, y, X, Y) = (x, X)$. Thus the submanifold VE contains the elements which are locally represented as $(x, y, 0, Y)$.

Definition 1.3. *A nonlinear connection* $\mathbf{N}$ *in a vector bundle* $\mathcal{E} = (E, \pi, M)$ *is the splitting on the left of the exact sequence*

$$0 \mapsto VE \mapsto TE \mapsto TE/VE \mapsto 0$$

where TE/VE *is the factor bundle.*

By definition (1.3) it is defined a morphism of vector bundles $C: TE \rightarrow VE$ such the superposition of maps $C \circ i$ is the identity on VE, where $i: VE \mapsto VE$. The kernel of the morphism C is a vector subbundle of (TE, τ_E, E) which is the horizontal subbundle, denoted by (HE, τ_H, E). Consequently, we can prove that in a v-bundle $\mathcal{E}$ a N–connection can be introduced as a distribution

$$\{N: E_u \rightarrow H_u E, T_u E = H_u E \oplus V_u E\}$$

for every point $u \in E$ defining a global decomposition, as a Whitney sum, into horizontal,$H\mathcal{E}$, and vertical, $V\mathcal{E}$, subbundles of the tangent bundle $T\mathcal{E}$

$$T\mathcal{E} = H\mathcal{E} \oplus V\mathcal{E}. \tag{1.10}$$

Locally a N-connection in a v–bundle $\mathcal{E}$ is given by its coefficients $N_i^a(\ u) = N_i^a(x, y)$ with respect to bases (1.2) and (1.3)

$$\mathbf{N} = N_i^{\ a}(u) d^i \otimes \partial_a.$$

We note that a linear connection in a v–bundle $\mathcal{E}$ can be considered as a particular case of a N–connection when $N_i^{\ a}(x, y) = K_{bi}^a\,(x)\,y^b$, where functions $K_{ai}^b\,(x)$ on the base M are called the Christoffel coefficients.

1.2.2 N–connections in covector bundles:

A nonlinear connection in a cv–bundle $\breve{\mathcal{E}}$ (in brief a Ň–connection) can be introduces in a similar fashion as for v–bundles by reconsidering the corresponding definitions for cv–bundles. For instance, it is stated by a Whitney sum, into horizontal,$H\breve{\mathcal{E}}$, and vertical, $V\breve{\mathcal{E}}$, subbundles of the tangent bundle $T\breve{\mathcal{E}}$:

$$T\breve{\mathcal{E}} = H\breve{\mathcal{E}} \oplus V\breve{\mathcal{E}}. \tag{1.11}$$

Hereafter, for the sake of brevity we shall omit details on definition of geometrical objects on cv–bundles if they are very similar to those for v–bundles: we shall present only the basic formulas by emphasizing the most important particularities and differences.

Definition 1.4. *A $\breve{N}$–connection on $\breve{\mathcal{E}}$ is a differentiable distribution*

$$\breve{N}: \ \breve{\mathcal{E}} \rightarrow \breve{N}_u \in T_u^* \breve{\mathcal{E}}$$

which is suplimentary to the vertical distribution V, i. e.

$$T_u \breve{\mathcal{E}} = \breve{N}_u \oplus \breve{V}_u, \forall \breve{\mathcal{E}}.$$

The same definition is true for Ň–connections in ct–bundles, we have to change in the definition (1.4) the symbol $\breve{\mathcal{E}}$ into T^*M.

A Ň–connection in a cv–bundle $\breve{\mathcal{E}}$ is given locally by its coefficients $\breve{N}_{\ ia}(\ u) = \breve{N}_{ia}(x,p)$ with respect to bases (1.2) and (1.3)

$$\breve{\mathbf{N}} = \breve{N}_{ia}(u) d^i \otimes \breve{\partial}^a.$$

We emphasize that if a N–connection is introduced in a v–bundle (cv–bundle) we have to adapt the geometric constructions to the N–connection structure.

1.2.3 N–connections in higher order bundles

The concept of N–connection can be defined for higher order vector / covector bundle in a standard manner like in the usual vector bundles:

Definition 1.5. *A nonlinear connection* $\tilde{\mathbf{N}}$ *in hvc–bundle*

$$\tilde{\mathcal{E}} = \tilde{\mathcal{E}}[v(1), v(2), cv(3), ..., cv(z-1), v(z)]$$

is a splitting of the left of the exact sequence

$$0 \to V\tilde{\mathcal{E}} \to T\tilde{\mathcal{E}} \to T\tilde{\mathcal{E}}/V\tilde{\mathcal{E}} \to 0 \qquad (1.12)$$

We can associate sequences of type (1.12) to every mappings of intermediary subbundles. For simplicity, we present here the Whitney decomposition

$$T\tilde{\mathcal{E}} = H\tilde{\mathcal{E}} \oplus V_{v(1)}\tilde{\mathcal{E}} \oplus V_{v(2)}\tilde{\mathcal{E}} \oplus V^*_{cv(3)}\tilde{\mathcal{E}} \oplus .... \oplus V^*_{cv(z-1)}\tilde{\mathcal{E}} \oplus V_{v(z)}\tilde{\mathcal{E}}.$$

Locally a N–connection $\tilde{\mathbf{N}}$ in $\tilde{\mathcal{E}}$ is given by its coefficients

$$\begin{array}{llllll}
N_i^{\ a_1}, & N_i^{\ a_2}, & N_{ia_3}, & ..., & N_{ia_{z-1}}, & N_i^{\ a_z}, \\
0, & N_{a_1}^{\ a_2}, & N_{a_1 a_3}, & ..., & N_{a_1 a_{z-1}}, & N_{a_1}^{\ a_z}, \\
0, & 0, & N_{a_2 a_3}, & ..., & N_{a_2 a_{z-1}}, & N_{a_2}^{\ a_z}, \\
..., & ..., & ..., & ..., & ..., & ..., \\
0, & 0, & 0, & ..., & N_{a_{z-2}\ a_{z-1}}, & N_{a_{z-2}}^{\ a_z}, \\
0, & 0, & 0, & ..., & 0, & N^{a_{z-1} a_z},
\end{array} \qquad (1.13)$$

which are given with respect to the components of bases (1.8) and (1.9).

We end this subsection with two exemples of N–connections in higher order vector/covector bundles:

N–connection in osculator bundle

Let us consider the second order of osculator bundle (see subsection (1.1.3)) $T^2M = Osc^2M$. A N–connection $\tilde{\mathbf{N}}$ in Osc^2M is associated to a Whitney summ

$$TT^2M = NT^2M \oplus VT^2M$$

which defines in every point $\tilde{u} \in T^2M$ a distribution

$$T_uT^2M = N_0(\tilde{u}) \oplus N_1(\tilde{u}) \oplus VT^2M.$$

We can parametrize $\tilde{\mathbf{N}}$ with respect to natural coordinate bases as

$$\begin{array}{ll} N_i^{a_1}, & N_i^{a_2}, \\ 0, & N_{a_1}^{a_2}. \end{array} \tag{1.14}$$

As a particular case we can consider $N_{a_1}^{a_2} = 0$.

N–connection in dual osculator bundle

In a similar fashion we can take the bundle $(T^{*2}M, p^{*2}, M)$ being dual bundle to the Osc^2M (see subsection (1.1.3)). We have

$$T^{*2}M = TM \otimes T^*M.$$

The local coefficients of a N–connection in $(T^{*2}M, p^{*2}, M)$ are parametrizied

$$\begin{array}{ll} N_i{}^{a_1}, & N_{ia_2}, \\ 0, & N_{a_1a_2}. \end{array} \tag{1.15}$$

We can choose a particular case when $N_{a_1a_2} = 0$.

1.2.4 Anholonomic frames and N–connections

Having defined a N–connection structure in a (vector, covector, or higher order vector / covenctor) bundle we can adapt to this structure, (by 'N–elongation', the operators of partial derivatives and differentials and to consider decompositions of geometrical objects with respect to adapted bases and cobases.

Anholonomic frames in v–bundles

In a v–bunde $\mathcal{E}$ provided with a N-connection we can adapt to this structure the geometric constructions by introducing locally adapted basis (N–frame, or N–basis):

$$\delta_\alpha = \frac{\delta}{\delta u^\alpha} = \left(\delta_i = \frac{\delta}{\delta x^i} = \partial_i - N_i^{\ a}(u)\,\partial_a, \partial_a = \frac{\partial}{\partial y^a}\right), \tag{1.16}$$

and its dual N–basis, (N–coframe, or N–cobasis),

$$\delta^{\ \alpha} = \delta u^\alpha = \left(d^i = \delta x^i = dx^i, \delta^a = \delta y^a + N_i^{\ a}(u)\,dx^i\right). \tag{1.17}$$

The **anholonomic coefficients,** $\mathbf{w} = \{w^\alpha_{\beta\gamma}(u)\}$, of N–frames are defined to satisfy the relations

$$[\delta_\alpha, \delta_\beta] = \delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha = w^\alpha_{\beta\gamma}(u)\,\delta_\alpha. \tag{1.18}$$

A frame bases is holonomic is all anholonomy coefficients vanish (like for usual coordinate bases (1.3)), or anholonomic if there are nonzero values of $w^\alpha_{\beta\gamma}$.

So, we conclude that a N–connection structure splitting conventionally a v–bundle $\mathcal{E}$ into some horizontal $H\mathcal{E}$ and vertical $V\mathcal{E}$ subbundles can be modelled by an anholonomic frame structure with mixed holonomic $\{x^i\}$ and anholonomic $\{y^a\}$ variables. This case differs from usual, for instance, tetradic approach in general relativity when tetradic (frame) fields are stated to have only for holonomic or only for anholonomic variables. By using the N–connection formalism we can investigate geometrical and physical systems when some degees of freedoms (variables) are subjected to anholonomic constraints, the rest of variables being holonomic.

The operators (1.16) and (1.17) on a v–bundle $\mathcal{E}$ enabled with a N–connection can be considered as respective equivalents of the operators of partial derivations and differentials: the existence of a N–connection structure results in 'elongation' of partial derivations on x–variables and in 'elongation' of differentials on y–variables.

The **algebra of tensorial distinguished fields** $DT(\mathcal{E})$ (d–fields, d–tensors, d–objects) on $\mathcal{E}$ is introduced as the tensor algebra $\mathcal{T} = \{\mathcal{T}^{pr}_{qs}\}$ of the v–bundle

$$\mathcal{E}_{(d)} = \left(H\mathcal{E} \oplus V\mathcal{E}, p_d, \mathcal{E}\right),$$

where $p_d: \ H\mathcal{E} \oplus V\mathcal{E} \to \mathcal{E}.$

An element $\mathbf{t} \in \mathcal{T}^{pr}_{qs}$, d–tensor field of type $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$, can be written in local form as

$$\begin{aligned} \mathbf{t} &= t^{i_1 \ldots i_p a_1 \ldots a_r}_{j_1 \ldots j_q b_1 \ldots b_r} (u)\, \delta_{i_1} \otimes \ldots \otimes \delta_{i_p} \otimes \partial_{a_1} \otimes \ldots \otimes \partial_{a_r} \\ & \otimes d^{j_1} \otimes \ldots \otimes d^{j_q} \otimes \delta^{b_1} \ldots \otimes \delta^{b_r}. \end{aligned}$$

We shall respectively use the denotations $\mathcal{X}(\mathcal{E})$ (or $\mathcal{X}\ (M)$), $\Lambda^p(\mathcal{E})$ or $(\Lambda^p(M))$ and $\mathcal{F}(\mathcal{E})$ (or $\mathcal{F}\ (M)$) for the module of d–vector fields on $\mathcal{E}$ (or M), the exterior algebra of p–forms on $\mathcal{E}$ (or M) and the set of real functions on $\mathcal{E}$ (or M).

Anholonomic frames in cv–bundles

The anholnomic frames adapted to the Ň–connection structure are introduced similarly to (1.16) and (1.17):

the locally adapted basis (Ň–basis, or Ň–frame):

$$\breve{\delta}_\alpha = \frac{\breve{\delta}}{\delta u^\alpha} = \left(\delta_i = \frac{\delta}{\delta x^i} = \partial_i + \breve{N}_{ia}(\breve{u})\, \breve{\partial}^a, \breve{\partial}^a = \frac{\partial}{\partial p_a} \right), \tag{1.19}$$

and its dual (Ň–cobasis, or Ň–coframe) :

$$\breve{\delta}^\alpha = \breve{\delta} u^\alpha = \left(d^i = \delta x^i = dx^i,\ \breve{\delta}_a = \breve{\delta} p_a = dp_a - \breve{N}_{ia}(\breve{u})\, dx^i \right). \tag{1.20}$$

We note that for the signes of Ň–elongations are inverse to those for N–elongations.

The **anholonomic coefficients,** $\breve{\mathbf{w}} = \{\breve{w}^{\alpha}_{\beta\gamma}(\breve{u})\}$, of Ň–frames are defined by the relations

$$\left[\breve{\delta}_\alpha, \breve{\delta}_\beta\right] = \breve{\delta}_\alpha \breve{\delta}_\beta - \breve{\delta}_\beta \breve{\delta}_\alpha = \breve{w}^{\alpha}_{\beta\gamma}(\breve{u})\, \breve{\delta}_\alpha. \tag{1.21}$$

The **algebra of tensorial distinguished fields** $DT\left(\breve{\mathcal{E}}\right)$ (d–fields, d–tensors, d–objects) on $\breve{\mathcal{E}}$ is introduced as the tensor algebra $\breve{\mathcal{T}} = \{\breve{\mathcal{T}}^{pr}_{qs}\}$ of the cv–bundle

$$\breve{\mathcal{E}}_{(d)} = \left(H\breve{\mathcal{E}} \oplus V\breve{\mathcal{E}}, \breve{p}_d, \breve{\mathcal{E}} \right),$$

where $\breve{p}_d :\ H\breve{\mathcal{E}} \oplus V\breve{\mathcal{E}} \rightarrow \breve{\mathcal{E}}.$

An element $\breve{\mathbf{t}} \in \breve{\mathcal{T}}_{qs}^{pr}$, d–tensor field of type $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$, can be written in local form as

$$\breve{\mathbf{t}} = \breve{t}^{i_1 \ldots i_p a_1 \ldots a_r}_{j_1 \ldots j_q b_1 \ldots b_r} (\breve{u}) \breve{\delta}_{i_1} \otimes ... \otimes \breve{\delta}_{i_p} \otimes \breve{\partial}_{a_1} \otimes ... \otimes \breve{\partial}_{a_r} \otimes \breve{d}^{j_1} \otimes ... \otimes \breve{d}^{j_q} \otimes \breve{\delta}^{b_1} ... \otimes \breve{\delta}^{b_r}.$$

We shall respectively use the denotations $\mathcal{X}\left(\breve{E}\right)$ (or $\mathcal{X}(M)$), $\Lambda^p\left(\breve{\mathcal{E}}\right)$ or $(\Lambda^p(M))$ and $\mathcal{F}\left(\breve{E}\right)$ (or $\mathcal{F}(M)$) for the module of d–vector fields on $\breve{\mathcal{E}}$ (or M), the exterior algebra of p–forms on $\breve{\mathcal{E}}$ (or M) and the set of real functions on $\breve{\mathcal{E}}$ (or M).

Anholonomic frames in hvc–bundles

The anholnomic frames adapted to a N–connection in hvc–bundle $\tilde{\mathcal{E}}$ are defined by the set of coefficients (1.13); having restricted the constructions to a vector (covector) shell we obtain some generalizations of the formulas for corresponding N(or Ň)–connection elongations of partial derivatives defined by (1.16) (or (1.19)) and (1.17) (or (1.20)).

We introduce the adapted partial derivatives (anholonomic N–frames, or N–bases) in $\tilde{\mathcal{E}}$ by applying the coefficients (1.13)

$$\tilde{\delta}_\alpha = \frac{\tilde{\delta}}{\delta \tilde{u}^\alpha} = \left(\delta_i, \delta_{a_1}, \delta_{a_2}, \breve{\delta}^{a_3}, ..., \breve{\delta}^{a_{z-1}}, \partial_{a_z}\right),$$

where

$$\begin{aligned}
\delta_i &= \partial_i - N_i^{\ a_1}\partial_{a_1} - N_i^{\ a_2}\partial_{a_2} + N_{ia_3}\breve{\partial}^{a_3} - ... + N_{ia_{z-1}}\breve{\partial}^{a_{z-1}} - N_i^{\ a_z}\partial_{a_z}, \\
\delta_{a_1} &= \partial_{a_1} - N_{a_1}^{a_2}\partial_{a_2} + N_{a_1a_3}\breve{\partial}^{a_3} - ... + N_{a_1a_{z-1}}\breve{\partial}^{a_{z-1}} - N_{a_1}^{a_z}\partial_{a_z}, \\
\delta_{a_2} &= \partial_{a_2} + N_{a_2a_3}\breve{\partial}^{a_3} - ... + N_{a_2a_{z-1}}\breve{\partial}^{a_{z-1}} - N_{a_2}^{a_z}\partial_{a_z}, \\
\breve{\delta}^{a_3} &= \tilde{\partial}^{a_3} - N^{a_3a_4}\partial_{a_4} - ... + N^{a_3}_{\ a_{z-1}}\breve{\partial}^{a_{z-1}} - N^{\ a_3a_z}\partial_{a_z}, \\
&\ldots\ldots\ldots\ldots\ldots \\
\breve{\delta}^{a_{z-1}} &= \tilde{\partial}^{a_{z-1}} - N^{\ a_{z-1}a_z}\partial_{a_z}, \\
\partial_{a_z} &= \partial/\partial y^{a_z}.
\end{aligned}$$

These formulas can be written in the matrix form:

$$\tilde{\delta}_\bullet = \widehat{\mathbf{N}}(u) \times \tilde{\partial}_\bullet \qquad (1.22)$$

where

$$\tilde{\delta}_{\bullet} = \left(\begin{array}{l} \delta_i \\ \delta_{a_1} \\ \delta_{a_2} \\ \breve{\delta}^{a_3} \\ ... \\ \breve{\delta}^{a_{z-1}} \\ \partial_{a_z} \end{array} \right), \quad \tilde{\partial}_{\bullet} = \left(\begin{array}{l} \partial_i \\ \partial_{a_1} \\ \partial_{a_2} \\ \tilde{\partial}^{a_3} \\ ... \\ \tilde{\partial}^{a_{z-1}} \\ \partial_{a_z} \end{array} \right), \quad (1.23)$$

and

$$\widehat{\mathbf{N}} = \left(\begin{array}{llllllll} 1 & -N_i^{\ a_1} & -N_i^{\ a_2} & N_{ia_3} & -N_i^{\ a_4} & ... & N_{ia_{z-1}} & -N_i^{\ a_z} \\ 0 & 1 & -N_{a_1}^{\ a_2} & N_{a_1a_3} & -N_{a_1}^{\ a_4} & ... & N_{a_1a_{z-1}} & -N_{a_1}^{\ a_z} \\ 0 & 0 & 1 & N_{a_2a_3} & -N_{a_2}^{\ a_4} & ... & N_{a_2a_{z-1}} & -N_{a_2}^{\ a_z} \\ 0 & 0 & 0 & 1 & -N^{a_3a_4} & ... & N^{a_3}_{\ \ a_{z-1}} & -N^{\ a_3a_z} \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & 0 & ... & 1 & -N^{\ a_{z-1}a_z} \\ 0 & 0 & 0 & 0 & 0 & ... & 0 & 1 \end{array} \right).$$

The adapted differentials (anholonomic N–coframes, or N–cobases) in $\tilde{\mathcal{E}}$ are introduced in the symplest form by using matrix formalism: The respective dual matrices to (1.23)

$$\begin{aligned} \tilde{\delta}^{\bullet} &= \{\tilde{\delta}^{\alpha}\} = \left(\begin{array}{lllllll} d^i & \delta^{a_1} & \delta^{a_2} & \breve{\delta}_{a_3} & ... & \breve{\delta}_{a_{z-1}} & \delta^{a_z} \end{array} \right), \\ \tilde{d}^{\bullet} &= \{\tilde{\partial}^{\alpha}\} = \left(\begin{array}{lllllll} d^i & d^{a_1} & d^{a_2} & d_{a_3} & ... & d_{a_{z-1}} & d^{a_z} \end{array} \right) \end{aligned}$$

are related via a matrix relation

$$\tilde{\delta}^{\bullet} = \tilde{d}^{\bullet}\widehat{\mathbf{M}} \quad (1.24)$$

which defines the formulas for anholonomic N–coframes. The matrix $\widehat{\mathbf{M}}$ from (1.24) is the inverse to $\widehat{\mathbf{N}}$, i. e. satisfies the condition

$$\widehat{\mathbf{M}} \times \widehat{\mathbf{N}} = I. \quad (1.25)$$

The **anholonomic coefficients,** $\widetilde{\mathbf{w}} = \{\widetilde{w}^{\alpha}_{\beta\gamma}(\widetilde{u})\}$, on hcv–bundle $\tilde{\mathcal{E}}$ are expressed via coefficients of the matrix $\widehat{\mathbf{N}}$ and their partial derivatives following the relations

$$\left[\widetilde{\delta}_{\alpha}, \widetilde{\delta}_{\beta}\right] = \widetilde{\delta}_{\alpha}\widetilde{\delta}_{\beta} - \widetilde{\delta}_{\beta}\widetilde{\delta}_{\alpha} = \widetilde{w}^{\alpha}_{\beta\gamma}(\widetilde{u})\,\widetilde{\delta}_{\alpha}. \quad (1.26)$$

We omit the explicit formulas on shells.

A d–tensor formalism can be also developed on the space $\tilde{\mathcal{E}}$. In this case the indices have to be stipulated for every shell separately, like for v–bunles or cv–bundles.

Let us consider some examples for particular cases of hcv–bundles:

Anholonomic frames in osculator bundle

For the osculator bundle $T^2M = Osc^2M$ from subsection (1.2.3) the formulas (1.22) and (1.24) are written respectively in the form

$$\tilde{\delta}_\alpha = \left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^i_{(1)}}, \frac{\partial}{\partial y^i_{(2)}} \right),$$

where

$$\begin{aligned} \frac{\delta}{\delta x^i} &= \frac{\partial}{\partial x^i} - N_{(1)i}{}^{j} \frac{\partial}{\partial y^i_{(1)}} - N_{(2)i}{}^{j} \frac{\partial}{\partial y^i_{(2)}}, \\ \frac{\delta}{\delta y^i_{(1)}} &= \frac{\partial}{\partial y^i_{(1)}} - N_{(2)i}{}^{j} \frac{\partial}{\partial y^j_{(2)}}, \end{aligned}$$

and

$$\tilde{\delta}^\alpha = \left(dx^i, \delta y^i_{(1)}, \delta y^i_{(2)} \right), \tag{1.27}$$

where

$$\begin{aligned} \delta y^i_{(1)} &= dy^i_{(1)} + M^i_{(1)j} dx^j, \\ \delta y^i_{(2)} &= dy^i_{(2)} + M^i_{(1)j} dy^j_{(1)} + M^i_{(2)j} dx^j, \end{aligned}$$

with the dual coefficients $M^i_{(1)j}$ and $M^i_{(2)j}$ (see (1.25)) expressed via primary coefficients $N^i_{(1)j}$ and $N^i_{(2)j}$ as

$$M^i_{(1)j} = N^i_{(1)j}, M^i_{(2)j} = N_{(2)j}{}^{i} + N_{(1)m}{}^{i} N_{(1)j}{}^{m}.$$

Anholonomic frames in dual osculator bundle

Following the definitions for dual osculator bundle $(T^{*2}M, p^{*2}, M)$ in subsection (1.2.3) the formulas (1.22) and (1.24) are written respectively in the form

$$\tilde{\delta}_\alpha = \left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^i_{(1)}}, \frac{\partial}{\partial p_{(2)} i} \right),$$

where

$$\begin{aligned} \frac{\delta}{\delta x^i} &= \frac{\partial}{\partial x^i} - N_{(1)i}{}^{j} \frac{\partial}{\partial y^i_{(1)}} + N_{(2)ij} \frac{\partial}{\partial p_{(2)} j}, \\ \frac{\delta}{\delta y^i_{(1)}} &= \frac{\partial}{\partial y^i_{(1)}} + N_{(2)ij} \frac{\partial}{\partial p_{(2)j}}, \end{aligned}$$

and

$$\tilde{\delta}^{\alpha} = \left(dx^i, \delta y^i_{(1)}, \delta p_{(2)i}\right), \tag{1.28}$$

where

$$\begin{aligned} \delta y^i_{(1)} &= dy^i_{(1)} + N^i_{(1)j} dx^j, \\ \delta p_{(2)i} &= dp_{(2)i} - N_{(2)ij} dx^j, \end{aligned}$$

with the dual coefficients $M^i_{(1)j}$ and $M^i_{(2)j}$ (see (1.25)) were expressed via $N^i_{(1)j}$ and $N^i_{(2)j}$ like in Ref. [113].

1.3 Distinguished connections and metrics

In general, distinguished objects (d–objects) on a v–bundle $\mathcal{E}$ (or cv–bundle $\breve{\mathcal{E}}$) are introduced as geometric objects with various group and coordinate transforms coordinated with the N–connection structure on $\mathcal{E}$ (or $\breve{\mathcal{E}}$). For example, a distinguished connection (in brief, **d–connection)** D on $\mathcal{E}$ (or $\breve{\mathcal{E}}$) is defined as a linear connection D on E (or $\breve{E}$) conserving under a parallelism the global decomposition (1.10) (or (1.11)) into horizontal and vertical subbundles of $T\mathcal{E}$ (or $T\breve{\mathcal{E}}$). A covariant derivation associated to a d–connection becomes d–covariant. We shall give necessary formulas for cv–bundles in round backets.

1.3.1 D–connections

D–connections in v–bundles (cv–bundles)

A N–connection in a v–bundle $\mathcal{E}$ (cv–bundle $\breve{\mathcal{E}}$) induces a corresponding decomposition of d–tensors into sums of horizontal and vertical parts, for example, for every d–vector $X \in \mathcal{X}(\mathcal{E})$ ($\breve{X} \in \mathcal{X}\left(\breve{\mathcal{E}}\right)$) and 1–form $A \in \Lambda^1(\mathcal{E})$ ($\breve{A} \in \Lambda^1\left(\breve{\mathcal{E}}\right)$) we have respectively

$$\begin{aligned} X &= hX + vX \quad \text{and} \quad A = hA + vA, \\ (\breve{X} &= h\breve{X} + vX \quad \text{and} \quad \breve{A} = h\breve{A} + v\breve{A}) \end{aligned} \tag{1.29}$$

where

$$hX = X^i \delta_i, vX = X^a \partial_a \ (h\breve{X} = \breve{X}^i \tilde{\delta}_i, v\breve{X} = \breve{X}_a \breve{\partial}^a)$$

and

$$hA = A_i \delta^i, vA = A_a d^a \ (h\breve{A} = \breve{A}_i \breve{\delta}^i, v\breve{A} = \breve{A}^a \breve{d}_a).$$

In consequence, we can associate to every d–covariant derivation along the d–vector (1.29), $D_X = X \circ D$ $(D_{\breve{X}} = \breve{X} \circ D)$ two new operators of h- and v–covariant derivations

$$
\begin{array}{rcl}
D_X^{(h)} Y & = & D_{hX} Y \quad \text{and} \quad D_X^{(v)} Y = D_{vX} Y, \quad \forall Y \in \mathcal{X}(\mathcal{E}) \\
(D_{\breve{X}}^{(h)} \breve{Y} & = & D_{h\breve{X}} \breve{Y} \quad \text{and} \quad D_{\breve{X}}^{(v)} \breve{Y} = D_{v\breve{X}} \breve{Y}, \quad \forall \breve{Y} \in \mathcal{X}\left(\breve{\mathcal{E}}\right))
\end{array}
$$

for which the following conditions hold:

$$
\begin{array}{rcl}
D_X Y & = & D_X^{(h)} Y + D_X^{(v)} Y \\
(D_{\breve{X}} \breve{Y} & = & D_{\breve{X}}^{(h)} \breve{Y} + D_{\breve{X}}^{(v)} \breve{Y}),
\end{array}
\tag{1.30}
$$

where

$$
\begin{array}{rcl}
D_X^{(h)} f & = & (hX) f \quad \text{and} \quad D_X^{(v)} f = (vX) f, \quad X, Y \in \mathcal{X}(\mathcal{E}), f \in \mathcal{F}(M) \\
(\breve{D}_{\breve{X}}^{(h)} f & = & (h\breve{X}) f \quad \text{and} \quad \breve{D}_{\breve{X}}^{(v)} f = (v\breve{X}) f, \quad \breve{X}, \breve{Y} \in \mathcal{X}\left(\breve{\mathcal{E}}\right), f \in \mathcal{F}(M)).
\end{array}
$$

The components $\Gamma^{\alpha}_{\beta\gamma}$ ($\breve{\Gamma}^{\alpha}_{\beta\gamma}$)of a d–connection $\breve{D}_{\alpha} = (\breve{\delta}_{\alpha} \circ D)$, locally adapted to the N—connection structure with respect to the frames (1.16) and (1.17) ((1.19) and (1.20)), are defined by the equations

$$
D_{\alpha} \delta_{\beta} = \Gamma^{\gamma}_{\alpha\beta} \delta_{\gamma} \ (\breve{D}_{\alpha} \breve{\delta}_{\beta} = \breve{\Gamma}^{\gamma}_{\alpha\beta} \breve{\delta}_{\gamma} \ ),
$$

from which one immediately follows

$$
\Gamma^{\gamma}_{\alpha\beta}(u) = (D_{\alpha} \delta_{\beta}) \circ \delta^{\gamma} \quad (\breve{\Gamma}^{\gamma}_{\alpha\beta}(\breve{u}) = \left(\breve{D}_{\alpha} \breve{\delta}_{\beta}\right) \circ \breve{\delta}^{\gamma}). \tag{1.31}
$$

The coefficients of operators of h- and v–covariant derivations,

$$
\begin{array}{rcl}
D_k^{(h)} & = & \{L^i_{jk}, L^a_{bk}\} \text{ and } D_c^{(v)} = \{C^i_{jk}, C^a_{bc}\} \\
(\breve{D}_k^{(h)} & = & \{\breve{L}^i_{jk}, \breve{L}_{ak}^{\ \ b}\} \text{ and } \breve{D}^{(v)c} = \{\breve{C}^{i\ c}_{\ j}, \breve{C}_a^{\ bc}\})
\end{array}
$$

(see (1.30)), are introduced as corresponding h- and v–parametrizations of (1.31)

$$
\begin{array}{rcl}
L^i_{jk} & = & (D_k \delta_j) \circ d^i, \quad L^a_{bk} = (D_k \partial_b) \circ \delta^a \\
(\breve{L}^i_{jk} & = & \left(\breve{D}_k \breve{\delta}_j\right) \circ d^i, \quad \breve{L}_{ak}^{\ \ b} = \left(\breve{D}_k \breve{\partial}^b\right) \circ \breve{\delta}_a)
\end{array}
\tag{1.32}
$$

and

$$
\begin{array}{rcl}
C^i_{jc} & = & (D_c \delta_j) \circ d^i, \quad C^a_{bc} = (D_c \partial_b) \circ \delta^a \\
(\breve{C}^{i\ c}_{\ j} & = & \left(\breve{D}^c \breve{\delta}_j\right) \circ d^i, \quad \breve{C}_a^{\ bc} = \left(\breve{D}^c \breve{\partial}^b\right) \circ \breve{\delta}_a).
\end{array}
\tag{1.33}
$$

A set of components (1.32) and (1.33)

$$\Gamma^{\gamma}_{\alpha\beta} = [L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc}] \ \left(\breve{\Gamma}^{\gamma}_{\alpha\beta} = [\breve{L}^i_{jk}, \breve{L}_{ak}^{\ b}, \breve{C}^{i\ c}_{\ j}, \breve{C}_a^{\ bc}]\right)$$

completely defines the local action of a d—connection D in $\mathcal{E}$ ($\breve{D}$ in $\breve{\mathcal{E}}$).

For instance, having taken on $\mathcal{E}$ ($\breve{\mathcal{E}}$) a d—tensor field of type $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$,

$$\begin{aligned} \mathbf{t} &= t^{ia}_{jb}\delta_i \otimes \partial_a \otimes d^j \otimes \delta^b, \\ \tilde{\mathbf{t}} &= \breve{t}^{ib}_{ja}\breve{\delta}_i \otimes \breve{\partial}^a \otimes d^j \otimes \breve{\delta}_b, \end{aligned}$$

and a d–vector $\mathbf{X}$ ($\breve{\mathbf{X}}$) we obtain

$$\begin{aligned} D_X\mathbf{t} &= D^{(h)}_X\mathbf{t} + D^{(v)}_X\mathbf{t} = \left(X^k\breve{t}^{ia}_{jb|k} + X^c t^{ia}_{jb\perp c}\right)\delta_i \otimes \partial_a \otimes d^j \otimes \delta^b, \\ (\breve{D}_{\breve{X}}\tilde{\mathbf{t}} &= \breve{D}^{(h)}_{\breve{X}}\tilde{\mathbf{t}} + \breve{D}^{(v)}_{\breve{X}}\tilde{\mathbf{t}} = \left(\breve{X}^k\breve{t}^{ib}_{ja|k} + \breve{X}_c\breve{t}^{ib\perp c}_{ja}\right)\breve{\delta}_i \otimes \breve{\partial}^a \otimes d^j \otimes \breve{\delta}_b) \end{aligned}$$

where the h–covariant derivative is written

$$\begin{aligned} t^{ia}_{jb|k} &= \delta_k t^{ia}_{jb} + L^i_{hk}t^{ha}_{jb} + L^a_{ck}t^{ic}_{jb} - L^h_{jk}t^{ia}_{hb} - L^c_{bk}t^{ia}_{jc} \\ (\breve{t}^{ib}_{ja|k} &= \breve{\delta}_k\breve{t}^{ib}_{ja} + \breve{L}^i_{hk}\breve{t}^{hb}_{ja} + \breve{L}_{ck}^{\ b}\breve{t}^{ic}_{ja} - \breve{L}^h_{jk}\breve{t}^{ib}_{ha} - \breve{L}_{ck}^{\ b}\breve{t}^{ic}_{ja}) \end{aligned}$$

and the v-covariant derivative is written

$$\begin{aligned} t^{ia}_{jb\perp c} &= \partial_c t^{ia}_{jb} + C^i_{hc}t^{ha}_{jb} + C^a_{dc}t^{id}_{jb} - C^h_{jc}t^{ia}_{hb} - C^d_{bc}t^{ia}_{jd} && (1.34) \\ (\breve{t}^{ib\perp c}_{ja} &= \breve{\partial}^c\breve{t}^{ib}_{ja} + \breve{C}^{i\ c}_{\ j}\breve{t}^{hb}_{ja} + \breve{C}_a^{\ dc}\breve{t}^{ib}_{jd} - \breve{C}^{i\ c}_{\ j}\breve{t}^{ib}_{ha} - \breve{C}_d^{\ bc}\breve{t}^{id}_{ja}). && (1.35) \end{aligned}$$

For a scalar function $f \in \mathcal{F}(\mathcal{E})$ ($f \in \mathcal{F}\left(\breve{\mathcal{E}}\right)$) we have

$$\begin{aligned} D^{(h)}_k &= \frac{\delta f}{\delta x^k} = \frac{\partial f}{\partial x^k} - N^a_k\frac{\partial f}{\partial y^a} \text{ and } D^{(v)}_c f = \frac{\partial f}{\partial y^c} \\ (\breve{D}^{(h)}_k &= \frac{\breve{\delta} f}{\delta x^k} = \frac{\partial f}{\partial x^k} + N_{ka}\frac{\partial f}{\partial p_a} \text{ and } \breve{D}^{(v)c} f = \frac{\partial f}{\partial p_c}). \end{aligned}$$

D–connections in hvc–bundles

The theory of connections in higher order anisotropic vector superbundles and vector bundles was elaborated in Refs. [171, 173, 172]. Here we re–formulate that formalism for the case when some shells of higher order anisotropy could be covector spaces by stating the general rules of covariant derivation compatible with the N–connection structure in hvc–bundle $\tilde{\mathcal{E}}$ and omit details and combersome formulas.

For a hvc–bundle of type $\tilde{\mathcal{E}} = \tilde{\mathcal{E}}[v(1), v(2), cv(3), ..., cv(z-1), v(z)]$ a d–connection $\tilde{\Gamma}^{\gamma}_{\alpha\beta}$ has the next shell decomposition of components (on induction being on the p-th shell, considered as the base space, which in this case a hvc–bundle, we introduce in a usual manner, like a vector or covector fibre, the $(p+1)$-th shell)

$$\begin{aligned}\tilde{\Gamma}^{\gamma}_{\alpha\beta} = & \{\Gamma^{\gamma_1}_{\alpha_1\beta_1} = [L^{i_1}_{j_1k_1}, L^{a_1}_{b_1k_1}, C^{i_1}_{j_1c_1}, C^{a_1}_{b_1c_1}], \\ & \Gamma^{\gamma_2}_{\alpha_2\beta_2} = [L^{i_2}_{j_2k_2}, L^{a_2}_{b_2k_2}, C^{i_2}_{j_2c_2}, C^{a_2}_{b_2c_2}], \\ & \breve{\Gamma}^{\gamma_3}_{\alpha_3\beta_3} = [\breve{L}^{i_3}_{j_3k_3}, \breve{L}^{\ \ b_3}_{a_3k_3}, \breve{C}^{i_3\ c_3}_{\ j_3}, \breve{C}^{\ \ b_3c_3}_{a_3}], \\ & ...., \\ & \breve{\Gamma}^{\gamma_{z-1}}_{\alpha_{z-1}\beta_{z-1}} = [\breve{L}^{i_{z-1}}_{j_{z-1}k_{z-1}}, \breve{L}^{\ \ b_{z-1}}_{a_{z-1}k_{z-1}}, \breve{C}^{i_{z-1}\ c_{z-1}}_{\ j_{z-1}}, \breve{C}^{\ \ b_{z-1}c_{z-1}}_{a_{z-1}}], \\ & \Gamma^{\gamma_z}_{\alpha_z\beta_z} = [L^{i_z}_{j_zk_z}, L^{a_z}_{b_zk_z}, C^{i_z}_{j_zc_z}, C^{a_z}_{b_zc_z}]\}.\end{aligned}$$

These coefficients determine the rules of a covariant derivation $\tilde{D}$ on $\tilde{\mathcal{E}}$.

For example, let us consider a d–tensor $\tilde{\mathbf{t}}$ of type

$$\begin{pmatrix} 1 & 1_1 & 1_2 & \breve{1}_3 & ... & 1_z \\ 1 & 1_1 & 1_2 & \breve{1}_3 & ... & 1_z \end{pmatrix}$$

with corresponding tensor product of components of anholonomic N–frames (1.22) and (1.24)

$$\begin{aligned}\tilde{\mathbf{t}} = & \tilde{t}^{ia_1a_2\breve{b}_3...\breve{b}_{z-1}a_z}_{jb_1b_2\breve{a}_3...\breve{a}_{z-1}b_z}\delta_i \otimes \partial_{a_1} \otimes d^j \otimes \delta^{b_1} \otimes \partial_{a_2} \otimes \delta^{b_2} \otimes \breve{\partial}^{a_3} \otimes \breve{\delta}_{b_3}, \\ & ... \otimes \breve{\partial}^{a_{z-1}} \otimes \breve{\delta}_{bz-1} \otimes \partial_{a_z} \otimes \delta^{b_z}.\end{aligned}$$

The d–covariant derivation $\tilde{D}$ of $\tilde{\mathbf{t}}$ is to be performed separately for every shall according the rule (1.34) if a shell is defined by a vector subspace, or according the rule (1.35) if the shell is defined by a covector subspace.

1.3.2 Metric structure

D–metrics in v–bundles

We define a **metric structure G** in the total space E of a v–bundle $\mathcal{E} = (E, p, M)$ over a connected and paracompact base M as a symmetric covariant tensor field of type $(0, 2)$,

$$\mathbf{G} = G_{\alpha\beta}du^{\alpha} \otimes du^{\beta}$$

being non degenerate and of constant signature on E.

Nonlinear connection $\mathbf{N}$ and metric $\mathbf{G}$ structures on $\mathcal{E}$ are mutually compatible it there are satisfied the conditions:

$$\mathbf{G}\left(\delta_i,\partial_a\right)=0,\text{or equivalently, } G_{ia}\left(u\right)-N_i^b\left(u\right)h_{ab}\left(u\right)=0, \tag{1.36}$$

where $h_{ab}=\mathbf{G}\left(\partial_a,\partial_b\right)$ and $G_{ia}=\mathbf{G}\left(\partial_i,\partial_a\right)$, which gives

$$N_i^b\left(u\right)=h^{ab}\left(u\right)G_{ia}\left(u\right) \tag{1.37}$$

(the matrix h^{ab} is inverse to h_{ab}). In consequence one obtains the following decomposition of metric:

$$\mathbf{G}(X,Y)=\mathbf{hG}(X,Y)+\mathbf{vG}(X,Y), \tag{1.38}$$

where the d–tensor $\mathbf{hG}(X,Y)=\mathbf{G}(hX,hY)$ is of type $\begin{pmatrix}0&0\\2&0\end{pmatrix}$ and the d–tensor $\mathbf{vG}(X,Y)=\mathbf{G}(vX,vY)$ is of type $\begin{pmatrix}0&0\\0&2\end{pmatrix}$. With respect to anholonomic basis (1.16) the d–metric (1.38) is written

$$\mathbf{G}=g_{\alpha\beta}\left(u\right)\delta^\alpha\otimes\delta^\beta=g_{ij}\left(u\right)d^i\otimes d^j+h_{ab}\left(u\right)\delta^a\otimes\delta^b, \tag{1.39}$$

where $g_{ij}=\mathbf{G}\left(\delta_i,\delta_j\right)$.

A metric structure of type (1.38) (equivalently, of type (1.39)) or a metric on E with components satisfying constraints (1.36), (equivalently (1.37)) defines an adapted to the given N–connection inner (d–scalar) product on the tangent bundle $\mathcal{TE}$.

We shall say that a d–connection $\widehat{D}_X$ is compatible with the d-scalar product on $\mathcal{TE}$ (i. e. it is a standard d–connection) if

$$\widehat{D}_X\left(\mathbf{X}\cdot\mathbf{Y}\right)=\left(\widehat{D}_X\mathbf{Y}\right)\cdot\mathbf{Z}+\mathbf{Y}\cdot\left(\widehat{D}_X\mathbf{Z}\right),\forall\mathbf{X},\mathbf{Y},\mathbf{Z}\in\mathcal{X}\left(\mathcal{E}\right).$$

An arbitrary d–connection D_X differs from the standard one $\widehat{D}_X$ by an operator $\widehat{P}_X\left(u\right)=\{X^\alpha\widehat{P}_{\alpha\beta}^\gamma\left(u\right)\}$, called the deformation d-tensor with respect to $\widehat{D}_X$, which is just a d-linear transform of $\mathcal{E}_u$, $\forall\, u\in\mathcal{E}$. The explicit form of $\widehat{P}_X$ can be found by using the corresponding axiom defining linear connections [91]

$$\left(D_X-\widehat{D}_X\right)fZ=f\left(D_X-\widehat{D}_X\right)Z,$$

written with respect to N–elongated bases (1.16) and (1.17). From the last expression we obtain

$$\widehat{P}_X\left(u\right)=\left[(D_X-\widehat{D}_X)\delta_\alpha\left(u\right)\right]\delta^\alpha\left(u\right),$$

therefore

$$D_X Z = \widehat{D}_X Z + \widehat{P}_X Z. \qquad (1.40)$$

A d–connection D_X is **metric** (or **compatible** with metric $\mathbf{G}$) on $\mathcal{E}$ if

$$D_X \mathbf{G} = 0, \forall X \in \mathcal{X}(\mathcal{E}).$$

With respect to anholonomic frames these conditions are written

$$D_\alpha g_{\beta\gamma} = 0, \qquad (1.41)$$

where by $g_{\beta\gamma}$ we denote the coefficients in the block form (1.39).

D–metrics in cv– and hvc–bundles

The presented considerations on self–consisten definition of N–connection, d–connection and metric structures in v–bundles can reformulated in a similar fashion for another types of anisotropic space–times, on cv–bundles and on shells of hvc–bundles. For symplicity, we give here only the anagolous formulas for the metric d–tensor (1.39):

- On cv–bundle $\breve{\mathcal{E}}$ we write

$$\breve{\mathbf{G}} = \breve{g}_{\alpha\beta}(\breve{u})\, \breve{\delta}^\alpha \otimes \breve{\delta}^\beta = \breve{g}_{ij}(\breve{u})\, d^i \otimes d^j + \breve{h}^{ab}(\breve{u})\, \breve{\delta}_a \otimes \breve{\delta}_b, \qquad (1.42)$$

  where $\breve{g}_{ij} = \breve{\mathbf{G}}\left(\breve{\delta}_i, \breve{\delta}_j\right)$ and $\breve{h}^{ab} = \breve{\mathbf{G}}\left(\breve{\partial}^a, \breve{\partial}^b\right)$ and the N–coframes are given by formulas (1.20).

  For simplicity, we shall consider that the metricity conditions are satisfied, $\breve{D}_\gamma \breve{g}_{\alpha\beta} = 0$.

- On hvc–bundle $\tilde{\mathcal{E}}$ we write

$$\begin{aligned} \tilde{\mathbf{G}} &= \tilde{g}_{\alpha\beta}(\tilde{u})\, \tilde{\delta}^\alpha \otimes \tilde{\delta}^\beta = \tilde{g}_{ij}(\tilde{u})\, d^i \otimes d^j + \tilde{h}_{a_1 b_1}(\tilde{u})\, \delta^{a_1} \otimes \delta^{b_1} \qquad (1.43) \\ & + \tilde{h}_{a_2 b_2}(\tilde{u})\, \delta^{a_2} \otimes \delta^{b_2} + \tilde{h}^{a_3 b_3}(\tilde{u})\, \breve{\delta}_{a_3} \otimes \breve{\delta}_{b_3} + ... \\ & + \tilde{h}^{a_{z-1} b_{z-1}}(\tilde{u})\, \breve{\delta}_{a_{z-1}} \otimes \breve{\delta}_{b_{z-1}} + \tilde{h}_{a_z b_z}(\tilde{u})\, \delta^{a_z} \otimes \delta^{b_z}, \end{aligned}$$

  where $\tilde{g}_{ij} = \tilde{\mathbf{G}}\left(\tilde{\delta}_i, \tilde{\delta}_j\right)$ and $\tilde{h}_{a_1 b_1} = \tilde{\mathbf{G}}(\partial_{a_1}, \partial_{b_1})$, $\tilde{h}_{a_2 b_2} = \tilde{\mathbf{G}}(\partial_{a_2}, \partial_{b_2})$, $\tilde{h}^{a_3 b_3} = \tilde{\mathbf{G}}\left(\breve{\partial}^{a_3}, \breve{\partial}^{b_3}\right), ...$ and the N–coframes are given by formulas (1.24).

  The metricity conditions are $\tilde{D}_\gamma \tilde{g}_{\alpha\beta} = 0$.

- On osculator bundle $T^2M = Osc^2M$ we have a particular case of (1.43) when

$$\begin{aligned}\tilde{\mathbf{G}} &= \tilde{g}_{\alpha\beta}(\tilde{u})\,\tilde{\delta}^{\alpha}\otimes\tilde{\delta}^{\beta} \\ &= \tilde{g}_{ij}(\tilde{u})\,d^{i}\otimes d^{j}+\tilde{h}_{ij}(\tilde{u})\,\delta y^{i}_{(1)}\otimes\delta y^{i}_{(1)}+\tilde{h}_{ij}(\tilde{u})\,\delta y^{i}_{(2)}\otimes\delta y^{i}_{(2)}\end{aligned} \quad (1.44)$$

  where the N–coframes are given by (1.27).

- On dual osculator bundle $(T^{*2}M, p^{*2}, M)$ we have another particular case of (1.43) when

$$\begin{aligned}\tilde{\mathbf{G}} &= \tilde{g}_{\alpha\beta}(\tilde{u})\,\tilde{\delta}^{\alpha}\otimes\tilde{\delta}^{\beta} \\ &= \tilde{g}_{ij}(\tilde{u})\,d^{i}\otimes d^{j}+\tilde{h}_{ij}(\tilde{u})\,\delta y^{i}_{(1)}\otimes\delta y^{i}_{(1)}+\tilde{h}^{ij}(\tilde{u})\,\delta p^{(2)}_{i}\otimes\delta p^{(2)}_{i}\end{aligned} \quad (1.45)$$

  where the N–coframes are given by (1.28).

1.3.3 Some remarkable d–connections

We emphasize that the geometry of connections in a v–bundle $\mathcal{E}$ is very reach. If a triple of fundamental geometric objects $\left(N_i^a(u), \Gamma^{\alpha}_{\beta\gamma}(u), g_{\alpha\beta}(u)\right)$ is fixed on $\mathcal{E}$, a multi–connection structure (with corresponding different rules of covariant derivation, which are, or not, mutually compatible and with the same, or not, induced d–scalar products in $\mathcal{TE}$) is defined on this v–bundle. We can give a priority to a connection structure following some physical arguments, like the reduction to the Christoffel symbols in the holonomic case, mutual compatibility between metric and N–connection and d–connection structures and so on.

In this subsection we enumerate some of the connections and covariant derivations in v–bundle $\mathcal{E}$, cv–bundle $\breve{\mathcal{E}}$ and in some hvc–bundles which can present interest in investigation of locally anisotropic gravitational and matter field interactions :

1. Every N–connection in $\mathcal{E}$ with coefficients $N_i^a(x,y)$ being differentiable on y–variables, induces a structure of linear connection $N^{\alpha}_{\beta\gamma}$, where

$$N^a_{bi} = \frac{\partial N^a_i}{\partial y^b} \text{ and } N^a_{bc}(x,y) = 0. \quad (1.46)$$

   For some $Y(u) = Y^i(u)\,\partial_i + Y^a(u)\,\partial_a$ and $B(u) = B^a(u)\,\partial_a$ one introduces a covariant derivation as

$$D_Y^{(\tilde{N})}B = \left[Y^i\left(\frac{\partial B^a}{\partial x^i} + N^a_{bi}B^b\right) + Y^b\frac{\partial B^a}{\partial y^b}\right]\frac{\partial}{\partial y^a}.$$

2. The d–connection of Berwald type [32] on v–bundle $\mathcal{E}$ (cv–bundle $\breve{\mathcal{E}}$)

$$\Gamma^{(B)\alpha}_{\beta\gamma} = \left(L^i_{jk}, \frac{\partial N^a_k}{\partial y^b}, 0, C^a_{bc} \right), \tag{1.47}$$
$$(\breve{\Gamma}^{(B)\alpha}_{\beta\gamma} = \left(\breve{L}^i_{jk}, -\frac{\partial \breve{N}_{ka}}{\partial p_b}, 0, \breve{C}_a^{\ bc} \right))$$

where

$$L^i_{.jk}(x,y) = \frac{1}{2} g^{ir} \left(\frac{\delta g_{jk}}{\delta x^k} + \frac{\delta g_{kr}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^r} \right), \tag{1.48}$$
$$C^a_{.bc}(x,y) = \frac{1}{2} h^{ad} \left(\frac{\partial h_{bd}}{\partial y^c} + \frac{\partial h_{cd}}{\partial y^b} - \frac{\partial h_{bc}}{\partial y^d} \right)$$
$$(\breve{L}^i_{.jk}(x,p) = \frac{1}{2} \breve{g}^{ir} \left(\frac{\breve{\delta} \breve{g}_{jk}}{\delta x^k} + \frac{\breve{\delta} \breve{g}_{kr}}{\delta x^j} - \frac{\breve{\delta} \breve{g}_{jk}}{\delta x^r} \right),$$
$$\breve{C}_a^{\ bc}(x,p) = \frac{1}{2} \breve{h}_{ad} \left(\frac{\partial \breve{h}^{bd}}{\partial p_c} + \frac{\partial \breve{h}^{cd}}{\partial p_b} - \frac{\partial \breve{h}^{bc}}{\partial p_d} \right)),$$

which is hv—metric, i.e. there are satisfied the conditions $D^{(B)}_k g_{ij} = 0$ and $D^{(B)}_c h_{ab} = 0$ ($\breve{D}^{(B)}_k \breve{g}_{ij} = 0$ and $\breve{D}^{(B)c} \breve{h}^{ab} = 0$).

3. The canonical d–connection $\Gamma^{(\mathbf{c})}$ (or $\breve{\Gamma}^{(\mathbf{c})}$) on a v–bundle (or cv–bundle) is associated to a metric $\mathbf{G}$ (or $\breve{\mathbf{G}}$) of type (1.39) (or (1.42)),

$$\Gamma^{(c)\alpha}_{\beta\gamma} = [L^{(c)i}_{jk}, L^{(c)a}_{bk}, C^{(c)i}_{jc}, C^{(c)a}_{bc}] \ (\breve{\Gamma}^{(c)\alpha}_{\beta\gamma} = [\breve{L}^{(c)i}_{jk}, \breve{L}^{(c).b}_{\ a\ .k}, \breve{C}^{(c)i\ c}_{\ j}, \breve{C}^{(c)\ bc}_{a}])$$

with coefficients

$$L^{(c)i}_{jk} = L^i_{.jk}, C^{(c)a}_{bc} = C^a_{.bc} \ (\breve{L}^{(c)i}_{jk} = \breve{L}^i_{.jk}, \breve{C}^{(c)\ bc}_{a} = \breve{C}_a^{\ bc}), \text{ (see (1.48))}$$
$$L^{(c)a}_{bi} = \frac{\partial N^a_i}{\partial y^b} + \frac{1}{2} h^{ac} \left(\frac{\delta h_{bc}}{\delta x^i} - \frac{\partial N^d_i}{\partial y^b} h_{dc} - \frac{\partial N^d_i}{\partial y^c} h_{db} \right)$$
$$(\breve{L}^{(c).b}_{\ a\ .i} = -\frac{\partial \breve{N}^a_i}{\partial p_b} + \frac{1}{2} \breve{h}_{ac} \left(\frac{\breve{\delta} \breve{h}^{bc}}{\delta x^i} + \frac{\partial \breve{N}_{id}}{\partial p_b} \breve{h}^{dc} + \frac{\partial \breve{N}_{id}}{\partial p_c} \breve{h}^{db} \right)),$$
$$C^{(c)i}_{jc} = \frac{1}{2} g^{ik} \frac{\partial g_{jk}}{\partial y^c} \ (\breve{C}^{(c)i\ c}_{\ j} = \frac{1}{2} \breve{g}^{ik} \frac{\partial \breve{g}_{jk}}{\partial p_c}). \tag{1.49}$$

This is a metric d–connection which satisfies conditions

$$D^{(c)}_k g_{ij} = 0, D^{(c)}_c g_{ij} = 0, D^{(c)}_k h_{ab} = 0, D^{(c)}_c h_{ab} = 0$$
$$(\breve{D}^{(c)}_k \breve{g}_{jk} = 0, \breve{D}^{(c)c} \breve{g}_{jk} = 0, \breve{D}^{(c)}_k \breve{h}^{bc} = 0, \breve{D}^{(c)c} \breve{h}^{ab} = 0).$$

In physical applications we shall use the canonical connection and for symplicity we chall omit the index (c). The coefficients (1.49) are to be extended to higher order if we are dealing with derivations of geometrical objects with "shell" indices. In this case the fiber indices are to be stipulated for every type of shell into consideration.

4. We can consider the N–adapted Christoffel d–symbols

$$\widetilde{\Gamma}^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\tau} \left(\delta_{\gamma} g_{\tau\beta} + \delta_{\beta} g_{\tau\gamma} - \delta g_{\beta\gamma} \right), \qquad (1.50)$$

which have the components of d–connection $\widetilde{\Gamma}^{\alpha}_{\beta\gamma} = \left(L^{i}_{jk}, 0, 0, C^{a}_{bc} \right)$, with L^{i}_{jk} and C^{a}_{bc} as in (1.48) if $g_{\alpha\beta}$ is taken in the form (1.39).

Arbitrary linear connections on a v-bundle $\mathcal{E}$ can be also characterized by theirs deformation tensors (see (1.40)) with respect, for instance, to the d–connection (1.50):

$$\Gamma^{(B)\alpha}_{\beta\gamma} = \widetilde{\Gamma}^{\alpha}_{\beta\gamma} + P^{(B)\alpha}_{\beta\gamma}, \Gamma^{(c)\alpha}_{\beta\gamma} = \widetilde{\Gamma}^{\alpha}_{\beta\gamma} + P^{(c)\alpha}_{\beta\gamma}$$

or, in general,

$$\Gamma^{\alpha}_{\beta\gamma} = \widetilde{\Gamma}^{\alpha}_{\beta\gamma} + P^{\alpha}_{\beta\gamma},$$

where $P^{(B)\alpha}_{\beta\gamma}, P^{(c)\alpha}_{\beta\gamma}$ and $P^{\alpha}_{\beta\gamma}$ are respectively the deformation d-tensors of d–connections (1.47), (1.49) or of a general one. Similar deformation d–tensors can be introduced for d–connections on cv–bundles and hvc–bundles. We omit explicit formulas.

1.3.4 Amost Hermitian anisotropic spaces

The are possible very interesting particular constructions [108, 109, 113] on t–bundle TM provided with N–connection which defines a N–adapted frame structure $\delta_{\alpha} = (\delta_i, \dot{\partial}_i)$ (for the same formulas (1.16) and (1.17) but with identified fiber and base indices). We are using the 'dot' symbol in order to distinguish the horizontal and vertical operators because on t–bundles the indices could take the same values both for the base and fiber objects. This allow us to define an almost complex structure $\mathbf{J} = \{J_{\alpha}{}^{\beta}\}$ on TM as follows

$$\mathbf{J}(\delta_i) = -\dot{\partial}_i, \ \mathbf{J}(\dot{\partial}_i) = \delta_i. \qquad (1.51)$$

It is obvious that $\mathbf{J}$ is well–defined and $\mathbf{J}^2 = -I$.

For d–metrics of type (1.39), on TM, we can consider the case when $g_{ij}(x,y) = h_{ab}(x,y)$, i. e.

$$\mathbf{G}_{(t)} = g_{ij}(x,y)dx^i \otimes dx^j + g_{ij}(x,y)\delta y^i \otimes \delta y^j, \tag{1.52}$$

where the index (t) denotes that we have geometrical object defined on tangent space.

An almost complex structure $J_\alpha^{\ \beta}$ is compatible with a d–metric of type (1.52) and a d–connection D on tangent bundle TM if the conditions

$$J_\alpha^{\ \beta} J_\gamma^{\ \delta} g_{\beta\delta} = g_{\alpha\gamma} \text{ and } D_\alpha J_{\ \beta}^{\gamma} = 0$$

are satisfied.

The pair $(\mathbf{G}_{(t)}, \mathbf{J})$ is an almost Hermitian structure on TM.

One can introduce an almost sympletic 2–form associated to the almost Hermitian structure $(\mathbf{G}_{(t)}, \mathbf{J})$,

$$\theta = g_{ij}(x,y)\delta y^i \wedge dx^j. \tag{1.53}$$

If the 2–form (1.53), defined by the coefficients g_{ij}, is closed, we obtain an almost Kählerian structure in TM.

Definition 1.6. *An almost Kähler metric connection is a linear connection $D^{(H)}$ on $\tilde{TM} = TM \setminus \{0\}$ with the properties:*

1. *$D^{(H)}$ preserve by parallelism the vertical distribution defined by the N–connection structure;*

2. *$D^{(H)}$ is compatible with the almost Kähler structure $(\mathbf{G}_{(t)}, \mathbf{J})$, i. e.*

$$D_X^{(H)} g = 0,\ D_X^{(H)} J = 0,\ \forall X \in \mathcal{X}\left(\tilde{TM}\right).$$

By straightforward calculation we can prove that a d–connection $D\Gamma = \left(L^i_{jk}, L^i_{jk}, C^i_{jc}, C^i_{jc}\right)$ with the coefficients defined by

$$\begin{aligned} D^{(H)}_{\delta_i}\delta_j &= L^i_{jk}\delta_i,\ D^{(H)}_{\delta_i}\dot{\partial}_j = L^i_{jk}\dot{\partial}_i, \\ D^{(H)}_{\delta_i}\delta_j &= C^i_{jk}\delta_i,\ D^{(H)}_{\delta_i}\dot{\partial}_j = C^i_{jk}\dot{\partial}_i, \end{aligned} \tag{1.54}$$

where L^i_{jk} and $C^e_{ab} \to C^i_{jk}$, on TM are defined by the formulas (1.48), define a torsionless (see the next section on torsion structures) metric d–connection which satisfy the compatibility conditions (1.41).

Almost complex structures and almost Kähler models of Finsler, Lagrange, Hamilton and Cartan geometries (of first an higher orders) are investigated in details in Refs. [106, 107, 113, 172].

1.4 Torsions and Curvatures

In this section we outline the basic definitions and formulas for the torsion and curvature structures in v–bundles and cv–bundles provided with N–connection structure.

1.4.1 N–connection curvature

1. The curvature Ω of a nonlinear connection $\mathbf{N}$ in a v–bundle $\mathcal{E}$ can be defined in local form as [108, 109]:

$$\Omega = \frac{1}{2}\Omega^a_{ij} d^i \bigwedge d^j \otimes \partial_a,$$

where

$$\begin{aligned} \Omega^a_{ij} &= \delta_j N^a_i - \delta_i N^a_j \\ &= \partial_j N^a_i - \partial_i N^a_j + N^b_i N^a_{bj} - N^b_j N^a_{bi}, \end{aligned} \tag{1.55}$$

N^a_{bi} being that from (1.46).

2. For the curvature $\breve{\Omega}$, of a nonlinear connection $\breve{\mathbf{N}}$ in a cv–bundle $\breve{\mathcal{E}}$ we introduce

$$\breve{\Omega} = \frac{1}{2}\breve{\Omega}_{ija} d^i \bigwedge d^j \otimes \breve{\partial}^a,$$

where

$$\begin{aligned} \breve{\Omega}_{ija} &= -\breve{\delta}_j \breve{N}_{ia} + \breve{\delta}_i \breve{N}_{ja} \\ &= -\partial_j \breve{N}_{ia} + \partial_i \breve{N}_{ja} + \breve{N}_{ib} \breve{N}_{ja}^{\ \ b} - \breve{N}_{jb} \breve{N}_{ja}^{\ \ b}, \\ \breve{N}_{ja}^{\ \ b} &= \breve{\partial}^b \breve{N}_{ja} = \partial \breve{N}_{ja} / \partial p_b. \end{aligned} \tag{1.56}$$

3. Curvatures $\tilde{\Omega}$ of different type of nonlinear connections $\tilde{\mathbf{N}}$ in higher order anisotropic bundles were analyzed for different type of higher order tangent/dual tangent bundles and higher order prolongations of generalized Finsler, Lagrange and Hamiloton spaces in Refs. [106, 107, 113] and for higher order anisotropic superspaces and spinor bundles in Refs. [172, 165, 173, 171]: For every higher order anisotropy shell we shall define the coefficients (1.55) or (1.56) in dependence of the fact with type of subfiber we are considering (a vector or covector fiber).

1.4.2 d–Torsions in v- and cv–bundles

The torsion $\mathbf{T}$ of a d–connection $\mathbf{D}$ in v–bundle $\mathcal{E}$ (cv–bundle $\check{\mathcal{E}}$) is defined by the equation

$$\mathbf{T}(\mathbf{X},\mathbf{Y}) = \mathbf{X}\mathbf{Y}^{\circ}_{\circ}\mathbf{T} \doteq D_X\mathbf{Y} - D_Y\mathbf{X} - [\mathbf{X},\mathbf{Y}]. \quad (1.57)$$

One holds the following h- and v–decompositions

$$\mathbf{T}(\mathbf{X},\mathbf{Y}) = \mathbf{T}(\mathbf{hX},\mathbf{hY}) + \mathbf{T}(\mathbf{hX},\mathbf{vY}) + \mathbf{T}(\mathbf{vX},\mathbf{hY}) + \mathbf{T}(\mathbf{vX},\mathbf{vY}).$$

We consider the projections:

$$\mathbf{hT}(\mathbf{X},\mathbf{Y}), \mathbf{vT}(\mathbf{hX},\mathbf{hY}), \mathbf{hT}(\mathbf{hX},\mathbf{hY}), ...$$

and say that, for instance, $\mathbf{hT}(\mathbf{hX},\mathbf{hY})$ is the h(hh)–torsion of $\mathbf{D}$, $\mathbf{vT}(\mathbf{hX},\mathbf{hY})$ is the v(hh)–torsion of $\mathbf{D}$ and so on.

The torsion (1.57) in v-bundle is locally determined by five d–tensor fields, torsions, defined as

$$\begin{aligned} T^i_{jk} &= \mathbf{hT}(\delta_k,\delta_j)\cdot d^i, \quad T^a_{jk} = \mathbf{vT}(\delta_k,\delta_j)\cdot\delta^a, \\ P^i_{jb} &= \mathbf{hT}(\partial_b,\delta_j)\cdot d^i, \quad P^a_{jb} = \mathbf{vT}(\partial_b,\delta_j)\cdot\delta^a, \\ S^a_{bc} &= \mathbf{vT}(\partial_c,\partial_b)\cdot\delta^a. \end{aligned} \quad (1.58)$$

Using formulas (1.16), (1.17), (1.55) and (1.57) we can computer [108, 109] in explicit form the components of torsions (1.58) for a d–connection of type (1.32) and (1.33):

$$\begin{aligned} T^i_{.jk} &= T^i_{jk} = L^i_{jk} - L^i_{kj}, \quad T^i_{ja} = C^i_{.ja}, T^i_{aj} = -C^i_{ja}, \\ T^i_{.ja} &= 0, \qquad T^a_{.bc} = S^a_{.bc} = C^a_{bc} - C^a_{cb}, \\ T^a_{.ij} &= \delta_j N^a_i - \delta_j N^a_j, \quad T^a_{.bi} = P^a_{.bi} = \partial_b N^a_i - L^a_{.bj}, \quad T^a_{.ib} = -P^a_{.bi}. \end{aligned} \quad (1.59)$$

Formulas similar to (1.58) and (1.59) hold for cv–bundles:

$$\begin{aligned} \check{T}^i_{jk} &= \mathbf{hT}(\delta_k,\delta_j)\cdot d^i, \quad \check{T}_{jka} = \mathbf{vT}(\delta_k,\delta_j)\cdot\check{\delta}_a, \\ \check{P}^{i\quad b}_{j} &= \mathbf{hT}\left(\check{\partial}^b,\delta_j\right)\cdot d^i, \quad \check{P}_{aj}^{\quad b} = \mathbf{vT}\left(\check{\partial}^b,\delta_j\right)\cdot\check{\delta}_a, \\ \check{S}_a^{\ \ bc} &= \mathbf{vT}\left(\check{\partial}^c,\check{\partial}^b\right)\cdot\check{\delta}_a. \end{aligned} \quad (1.60)$$

and

$$\begin{aligned} \check{T}^i_{.jk} &= \check{T}^i_{jk} = L^i_{jk} - L^i_{kj}, \quad \check{T}^{ia}_{j} = \check{C}^{i\ \ a}_{.j}, \check{T}^{ia}_{\ \ j} = -\check{C}^{i\ \ a}_{j}, \\ \check{T}^{i\ \ a}_{.j} &= 0, \qquad \check{T}_a^{\ \ bc} = \check{S}_a^{\ \ bc} = \check{C}_a^{\ \ bc} - \check{C}_a^{\ \ cb}, \\ \check{T}_{.ija} &= -\delta_j\check{N}_{ia} + \delta_j\check{N}_{ja}, \quad \check{T}_a^{\ \ bi} = \check{P}_a^{\ \ bi} = -\check{\partial}^b\check{N}_{ia} - \check{L}_a^{\ \ bi}, \quad \check{T}_{a\ \ b}^{\ \ j} = -\check{P}_{a\ \ b}^{\ \ j}. \end{aligned} \quad (1.61)$$

The formulas for torsion can be generalized for hvc–bundles (on every shell we must write (1.59) or (1.61) in dependence of the type of shell, vector or co-vector one, we are dealing).

1.4.3 d–Curvatures in v- and cv–bundles

The curvature $\mathbf{R}$ of a d–connection in v–bundle $\mathcal{E}$ is defined by the equation

$$\mathbf{R}(X,Y)Z = XY\!\!\overset{\bullet}{\bullet} R \bullet Z = D_X D_Y\ Z - D_Y D_X Z - D_{[X,Y]}Z.$$

One holds the next properties for the h- and v–decompositions of curvature:

$$\begin{aligned} \mathbf{vR}(X,Y)hZ &= 0,\ \mathbf{hR}(X,Y)vZ = 0, \\ \mathbf{R}(X,Y)Z &= \mathbf{hR}(X,Y)hZ + \mathbf{vR}(X,Y)vZ. \end{aligned} \tag{1.62}$$

From (1.62) and the equation $\mathbf{R}(\mathbf{X},\mathbf{Y}) = -\mathbf{R}(\mathbf{Y},\mathbf{X})$ we get that the curvature of a d–connection $\mathbf{D}$ in $\mathcal{E}$ is completely determined by the following six d–tensor fields:

$$\begin{aligned} R_{h.jk}^{.i} &= d^i \cdot \mathbf{R}(\delta_k,\delta_j)\delta_h,\ R_{b.jk}^{.a} = \delta^a \cdot \mathbf{R}(\delta_k,\delta_j)\partial_b, \\ P_{j.kc}^{.i} &= d^i \cdot \mathbf{R}(\partial_c,\partial_k)\delta_j,\ P_{b.kc}^{.a} = \delta^a \cdot \mathbf{R}(\partial_c,\partial_k)\partial_b, \\ S_{j.bc}^{.i} &= d^i \cdot \mathbf{R}(\partial_c,\partial_b)\delta_j,\ S_{b.cd}^{.a} = \delta^a \cdot \mathbf{R}(\partial_d,\partial_c)\partial_b. \end{aligned} \tag{1.63}$$

By a direct computation, using (1.16),(1.17),(1.32),(1.33) and (1.63) we get:

$$\begin{aligned} R_{h.jk}^{.i} &= \delta_h L_{.hj}^i - \delta_j L_{.hk}^i + L_{.hj}^m L_{mk}^i - L_{.hk}^m L_{mj}^i + C_{.ha}^i R_{.jk}^a, \\ R_{b.jk}^{.a} &= \delta_k L_{.bj}^a - \delta_j L_{.bk}^a + L_{.bj}^c L_{.ck}^a - L_{.bk}^c L_{.cj}^a + C_{.bc}^a R_{.jk}^c, \\ P_{j.ka}^{.i} &= \partial_a L_{.jk}^i - \left(\delta_k C_{.ja}^i + L_{.lk}^i C_{.ja}^l - L_{.jk}^l C_{.la}^i - L_{.ak}^c C_{.jc}^i\right) + C_{.jb}^i P_{.ka}^b, \\ P_{b.ka}^{.c} &= \partial_a L_{.bk}^c - \left(\delta_k C_{.ba}^c + L_{.dk}^c C_{.ba}^d - L_{.bk}^d C_{.da}^c - L_{.ak}^d C_{.bd}^c\right) + C_{.bd}^c P_{.ka}^d, \\ S_{j.bc}^{.i} &= \partial_c C_{.jb}^i - \partial_b C_{.jc}^i + C_{.jb}^h C_{.hc}^i - C_{.jc}^h C_{hb}^i, \\ S_{b.cd}^{.a} &= \partial_d C_{.bc}^a - \partial_c C_{.bd}^a + C_{.bc}^e C_{.ed}^a - C_{.bd}^e C_{.ec}^a. \end{aligned} \tag{1.64}$$

We note that d–torsions (1.59) and d–curvatures (1.64) are computed in explicit form by particular cases of d–connections (1.47), (1.49) and (1.50).

For cv–bundles we have

$$\begin{aligned} \check{R}_{h.jk}^{.i} &= d^i \cdot \mathbf{R}(\delta_k,\delta_j)\delta_h,\ \check{R}^b{}_{a.jk} = \check{\delta}_a \cdot \mathbf{R}(\delta_k,\delta_j)\check{\partial}^b, \\ \check{P}_{j.k}^{.i\ \ c} &= d^i \cdot \mathbf{R}\left(\check{\partial}^c,\partial_k\right)\delta_j,\ \check{P}^b{}_{a.k}^{\ \ c} = \check{\delta}_a \cdot \mathbf{R}\left(\check{\partial}^c,\partial_k\right)\check{\partial}^b, \\ \check{S}_{j.}^{.ibc} &= d^i \cdot \mathbf{R}\left(\check{\partial}^c,\check{\partial}^b\right)\delta_j,\ \check{S}_{.a}^{b.cd} = \check{\delta}_a \cdot \mathbf{R}\left(\check{\partial}^d,\check{\partial}^c\right)\check{\partial}^b. \end{aligned} \tag{1.65}$$

and

$$\begin{aligned}
\check{R}^{.i}_{h.jk} &= \check{\delta}_h L^i_{.hj} - \check{\delta}_j L^i_{.hk} + L^m_{.hj} L^i_{mk} - L^m_{.hk} L^i_{mj} + C^{i\ a}_{.h} \check{R}_{.ajk}, \qquad (1.66)\\
\check{R}^{b.}_{.ajk} &= \check{\delta}_k \check{L}_{a.j}^{\ \ b} - \check{\delta}_j \check{L}^{a}_{\ b\ k} + \check{L}_{cj}^{\ b} \check{L}_{.ak}^{\ \ c} - \check{L}^b_{ck} \check{L}_{a.j}^{\ \ c} + \check{C}_a^{\ bc} \check{R}_{c.jk},\\
\check{P}^{.i\ a}_{j.k} &= \check{\partial}^a L^i_{.jk} - \left(\check{\delta}_k \check{C}^{i\ a}_{.j} + L^i_{.lk} \check{C}^{l\ a}_{.j} - L^l_{.jk} \check{C}^{i\ a}_{.l} - \check{L}^{\ a}_{ck} \check{C}^{i\ c}_{.j}\right) + \check{C}^{i\ b}_{.j} \check{P}_{bk}^{\quad a},\\
\check{P}^{b\ a}_{ck} &= \check{\partial}^a \check{L}_{c.k}^{\ \ b} - (\check{\delta}_k \check{C}_{c.}^{\ ba} + \check{L}^{bd}_{c.k} \check{C}_d^{\ ba} - \check{L}_{d.k}^{\ \ b} \check{C}_{c.}^{\ ad}),\\
&\quad - \check{L}_{dk}^{\ \ a} \check{C}_{c.}^{\ bd}) + \check{C}_{c.}^{\ bd} \check{P}_{d.k}^{\ \ a},\\
\check{S}^{.ibc}_{j.} &= \check{\partial}^c \check{C}^{i\ b}_{.j} - \check{\partial}^b \check{C}^{i\ c}_{.j} + \check{C}^{h\ b}_{.j} \check{C}^{i\ c}_{.h} - \check{C}^{h\ c}_{.j} \check{C}^{i\ b}_{h},\\
\check{S}^{b\ cd}_{\ a.} &= \check{\partial}^d \check{C}_{a.}^{\ bc} - \check{\partial}^c \check{C}_{a.}^{\ bd} + \check{C}_{e.}^{\ bc} \check{C}_{a.}^{\ ed} - \check{C}_{e.}^{\ bd} \check{C}_{.a}^{\ ec}.
\end{aligned}$$

The formulas for curvature can be also generalized for hvc–bundles (on every shell we must write (1.59) or (1.60) in dependence of the type of shell, vector or co-vector one, we are dealing).

1.5 Generalizations of Finsler Spaces

We outline the basic definitions and formulas for Finsler, Lagrange and generalized Lagrange spaces (constructed on tangent bundle) and for Cartan, Hamilton and generalized Hamilton spaces (constructed on cotangent bundle). The original results are given in details in monographs [108, 109, 113]

1.5.1 Finsler Spaces

The Finsler geometry is modeled on tangent bundle TM.

Definition 1.7. *A Finsler space (manifold) is a pair* $F^n = (M, F(x,y))$ *where* M *is a real* n*–dimensional differentiable manifold and* $F : TM \to \mathcal{R}$ *is a scalar function which satisfy the following conditions:*

1. F *is a differentiable function on the manifold* $\widetilde{TM} = TM \backslash \{0\}$ *and* F *is continous on the null section of the projection* $\pi : TM \to M$*;*

2. F *is a positive function, homogeneous on the fibers of the* TM*, i. e.* $F(x, \lambda y) = \lambda F(x,y), \lambda \in \mathcal{R}$*;*

3. *The Hessian of* F^2 *with elements*

$$g^{(F)}_{ij}(x,y) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j} \qquad (1.67)$$

is positively defined on $\widetilde{TM}$*.*

The function $F(x,y)$ and $g_{ij}(x,y)$ are called respectively the fundamental function and the fundamental (or metric) tensor of the Finsler space F.

One considers "anisotropic" (depending on directions y^i) Christoffel symbols, for simplicity we write $g_{ij}^{(F)} = g_{ij}$,

$$\gamma^i{}_{jk}(x,y) = \frac{1}{2} g^{ir} \left(\frac{\partial g_{rk}}{\partial x^j} + \frac{\partial g_{jr}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^r} \right),$$

which are used for definition of the Cartan N–connection,

$$N^i_{(c)\ j} = \frac{1}{2} \frac{\partial}{\partial y^j} \left[\gamma^i{}_{nk}(x,y) y^n y^k \right]. \tag{1.68}$$

This N–connection can be used for definition of an almost complex structure like in (1.51) and to define on TM a d–metric

$$\mathbf{G}_{(F)} = g_{ij}(x,y) dx^i \otimes dx^j + g_{ij}(x,y) \delta y^i \otimes \delta y^j, \tag{1.69}$$

with $g_{ij}(x,y)$ taken as (1.67).

Using the Cartan N–connection (1.68) and Finsler metric tensor (1.67) (or, equivalently, the d–metric (1.69)) we can introduce the canonical d–connection

$$D\Gamma\left(N_{(c)}\right) = \Gamma^{\alpha}_{(c)\beta\gamma} = \left(L^i_{(c)\ jk}, C^i_{(c)\ jk}\right)$$

with the coefficients computed like in (1.54) and (1.48) with $h_{ab} \to g_{ij}$. The d–connection $D\Gamma\left(N_{(c)}\right)$ has the unique property that it is torsionless and satisfies the metricity conditions both for the horizontal and vertical components, i. e. $D_\alpha g_{\beta\gamma} = 0$.

The d–curvatures

$$\check{R}^{.i}_{h.jk} = \{\check{R}^{.i}_{h.jk}, \check{P}^{.i}_{j.k}\ \ {}^{l}, S^{.i}_{(c)j.kl}\}$$

on a Finsler space provided with Cartan N–connection and Finsler metric structures are computed following the formulas (1.64) when the $a,b,c...$ indices are identified with $i,j,k,...$ indices. It should be emphasized that in this case all values $g_{ij,}\Gamma^{\alpha}_{(c)\beta\gamma}$ and $R^{.\alpha}_{(c)\beta.\gamma\delta}$ are defined by a fundamental function $F\left(x,y\right).$

In general, we can consider that a Finsler space is provided with a metric $g_{ij} = \partial^2 F^2 / 2\partial y^i \partial y^j$, but the N–connection and d–connection are be defined in a different manner, even not be determined by F.

1.5.2 Lagrange and Generalized Lagrange Spaces

The notion of Finsler spaces was extended by J. Kern [86] and R. Miron [99]. It is widely developed in monographs [108, 109] and exteded to superspaces by S. Vacaru [169, 171, 172].

The idea of extension was to consider instead of the homogeneous fundamental function $F(x,y)$ in a Finsler space a more general one, a Lagrangian $L\,(x,y)$, defined as a differentiable mapping $L:(x,y)\in TM\to L(x,y)\in\mathcal{R}$, of class C^{∞} on manifold $\widetilde{TM}$ and continous on the null section $0:M\to TM$ of the projection $\pi:TM\to M$. A Lagrangian is regular if it is differentiable and the Hessian

$$g_{ij}^{(L)}(x,y)=\frac{1}{2}\frac{\partial^2 L^2}{\partial y^i\partial y^j} \tag{1.70}$$

is of rank n on M.

Definition 1.8. *A Lagrange space is a pair $L^n=(M,L(x,y))$ where M is a smooth real n–dimensional manifold provided with regular Lagrangian $L(x,y)$ structure $L:TM\to\mathcal{R}$ for which $g_{ij}(x,y)$ from (1.70) has a constant signature over the manifold $\widetilde{TM}$.*

The fundamental Lagrange function $L(x,y)$ defines a canonical N–connection

$$N_{(cL)\ j}^{i}=\frac{1}{2}\frac{\partial}{\partial y^j}\left[g^{ik}\left(\frac{\partial^2 L^2}{\partial y^k\partial y^h}y^h-\frac{\partial L}{\partial x^k}\right)\right]$$

as well a d-metric

$$\mathbf{G}_{(L)}=g_{ij}(x,y)dx^i\otimes dx^j+g_{ij}(x,y)\delta y^i\otimes\delta y^j, \tag{1.71}$$

with $g_{ij}(x,y)$ taken as (1.70). As well we can introduce an almost Kählerian structure and an almost Hermitian model of L^n, denoted as H^{2n} as in the case of Finsler spaces but with a proper fundamental Lagange function and metric tensor g_{ij}. The canonical metric d–connection $D\Gamma\left(N_{(cL)}\right)=\Gamma^{\alpha}_{(cL)\beta\gamma}=\left(L^i_{(cL)\ jk},C^i_{(cL)\ jk}\right)$ is to computed by the same formulas (1.54) and (1.48) with $h_{ab}\to g_{ij}^{(L)}$, for $N^i_{(cL)\ j}$. The d–torsions (1.59) and d–curvatures (1.64) are defined, in this case, by $L^i_{(cL)\ jk}$ and $C^i_{(cL)\ jk}$. We also note that instead of $N^i_{(cL)\ j}$ and $\Gamma^{\alpha}_{(cL)\beta\gamma}$ one can consider on a L^n–space arbitrary N–connections $N^i{}_j$, d–connections $\Gamma^{\alpha}_{\beta\gamma}$ which are not defined only by $L(x,y)$ and $g_{ij}^{(L)}$ but can be metric, or non–metric with respect to the Lagrange metric.

The next step of generalization is to consider an arbitrary metric $g_{ij}\,(x,y)$ on TM instead of (1.70) which is the second derivative of "anisotropic" coordinates y^i of a Lagrangian [99, 100].

Definition 1.9. *A generalized Lagrange space is a pair* $GL^n = (M, g_{ij}(x,y))$ *where* $g_{ij}(x,y)$ *is a covariant, symmetric d–tensor field, of rank* n *and of constant signature on* $\widetilde{TM}$.

One can consider different classes of N– and d–connections on TM, which are compatible (metric) or non compatible with (1.71) for arbitrary $g_{ij}(x,y)$. We can apply all formulas for d–connections, N-curvatures, d-torsions and d-curvatures as in a v–bundle $\mathcal{E}$, but reconsidering them on TM, by changing $h_{ab} \to g_{ij}(x,y)$ and $N_i^a \to N^k{}_i$.

1.5.3 Cartan Spaces

The theory of Cartan spaces (see, for instance, [136, 85]) was formulated in a new fashion in R. Miron's works [101, 102] by considering them as duals to the Finsler spaces (see details and references in [113]). Roughly, a Cartan space is constructed on a cotangent bundle T^*M like a Finsler space on the corresponding tangent bundle TM.

Consider a real smooth manifold M, the cotangent bundle (T^*M, π^*, M) and the manifold $\widetilde{T^*M} = T^*M \backslash \{0\}$.

Definition 1.10. *A Cartan space is a pair* $C^n = (M, K(x,p))$ *such that* $K : T^*M \to \mathcal{R}$ *is a scalar function which satisfy the following conditions:*

1. *K is a differentiable function on the manifold $\widetilde{T^*M} = T^*M\backslash\{0\}$ and continous on the null section of the projection $\pi^* : T^*M \to M$;*
2. *K is a positive function, homogeneous on the fibers of the T^*M, i. e. $K(x, \lambda p) = \lambda F(x,p), \lambda \in \mathcal{R}$;*
3. *The Hessian of K^2 with elements*

$$\check{g}^{ij}_{(K)}(x,p) = \frac{1}{2}\frac{\partial^2 K^2}{\partial p_i \partial p_j} \tag{1.72}$$

is positively defined on $\widetilde{T^*M}$.

The function $K(x,y)$ and $\check{g}^{ij}(x,p)$ are called respectively the fundamental function and the fundamental (or metric) tensor of the Cartan space C^n. We use symbols like "$\check{g}$" as to emphasize that the geometrical objects are defined on a dual space.

One considers "anisotropic" (depending on directions, momenta, p_i) Christoffel symbols, for symplicty, we write the inverse to (1.72) as $g_{ij}^{(K)} = \check{g}_{ij}$,

$$\check{\gamma}^i{}_{jk}(x,p) = \frac{1}{2}\check{g}^{ir}\left(\frac{\partial \check{g}_{rk}}{\partial x^j} + \frac{\partial \check{g}_{jr}}{\partial x^k} - \frac{\partial \check{g}_{jk}}{\partial x^r}\right),$$

which are used for definition of the canonical N–connection,

$$\check{N}_{ij} = \check{\gamma}^k{}_{ij}p_k - \frac{1}{2}\gamma^k{}_{nl}p_k p^l \check{\partial}^n \check{g}_{ij},\ \check{\partial}^n = \frac{\partial}{\partial p_n}. \quad (1.73)$$

This N–connection can be used for definition of an almost complex structure like in (1.51) and to define on T^*M a d–metric

$$\check{\mathbf{G}}_{(k)} = \check{g}_{ij}(x,p)dx^i \otimes dx^j + \check{g}^{ij}(x,p)\delta p_i \otimes \delta p_j, \quad (1.74)$$

with $\check{g}^{ij}(x,p)$ taken as (1.72).

Using the canonical N–connection (1.73) and Finsler metric tensor (1.72) (or, equivalently, the d–metric (1.74) we can introduce the canonical d–connection

$$D\check{\Gamma}\left(\check{N}_{(k)}\right) = \check{\Gamma}^{\alpha}_{(k)\beta\gamma} = \left(\check{H}^i_{(k)\ jk}, \check{C}_{(k)\ i}^{\ \ jk}\right)$$

with the coefficients are computed

$$\begin{aligned} \check{H}^i_{(k)\ jk} &= \frac{1}{2}\check{g}^{ir}\left(\check{\delta}_j \check{g}_{rk} + \check{\delta}_k \check{g}_{jr} - \check{\delta}_r \check{g}_{jk}\right), \\ \check{C}_{(k)\ i}^{\ \ jk} &= \check{g}_{is}\check{\partial}^s \check{g}^{jk}, \end{aligned}$$

The d–connection $D\check{\Gamma}\left(\check{N}_{(k)}\right)$ has the unique property that it is torsionless and satisfies the metricity conditions both for the horizontal and vertical components, i. e. $\check{D}_\alpha \check{g}_{\beta\gamma} = 0.$

The d–curvatures

$$\check{R}^{.\alpha}_{(k)\beta.\gamma\delta} = \{R^{.i}_{(k)h.jk}, P^{.i}_{(k)j.km}, \check{S}^{.ikl}_{j.}\}$$

on a Finsler space provided with Cartan N–connection and Finsler metric structures are computed following the formulas (1.66) when the $a, b, c...$ indices are identified with $i, j, k, ...$ indices. It should be emphasized that in this case all values $\check{g}_{ij}, \check{\Gamma}^{\alpha}_{(k)\beta\gamma}$ and $\check{R}^{.\alpha}_{(k)\beta.\gamma\delta}$ are defined by a fundamental function $K(x,p)$.

In general, we can consider that a Cartan space is provided with a metric $\check{g}^{ij} = \partial^2 K^2/2\partial p_i \partial p_j$, but the N–connection and d–connection could be defined in a different manner, even not be determined by K.

1.5.4 Generalized Hamilton and Hamilton Spaces

The geometry of Hamilton spaces was defined and investigated by R. Miron in the papers [105, 104, 103] (see details and references in [113]). It was developed on the cotangent bundel as a dual geometry to the geometry of Lagrange spaces. Here we start with the definition of generalized Hamilton spaces and then consider the particular case.

Definition 1.11. *A generalized Hamilton space is a pair* $GH^n = (M, \breve{g}^{ij}(x,p))$ *where* M *is a real* n*–dimensional manifold and* $\breve{g}^{ij}(x,p)$ *is a contravariant, symmetric, nondegenerate of rank* n *and of constant signature on* $\widetilde{T^*M}$*.*

The value $\breve{g}^{ij}(x,p)$ is called the fundamental (or metric) tensor of the space GH^n. One can define such values for every paracompact manifold M. In general, a N–connection on GH^n is not determined by $\breve{g}^{ij}$. Therefore we can consider arbitrary coefficients $\check{N}_{ij}(x,p)$ and define on T^*M a d–metric like (1.42)

$$\breve{\mathbf{G}} = \breve{g}_{\alpha\beta}(\breve{u})\, \breve{\delta}^{\alpha} \otimes \breve{\delta}^{\beta} = \breve{g}_{ij}(\breve{u})\, d^i \otimes d^j + \breve{g}^{ij}(\breve{u})\, \breve{\delta}_i \otimes \breve{\delta}_j, \tag{1.75}$$

This N–coefficients $\check{N}_{ij}(x,p)$ and d–metric structure (1.75) allow to define an almost Kähler model of generalized Hamilton spaces and to define canonical d–connections, d–torsions and d-curvatures (see respectively the formulas (1.48), (1.49), (1.61) and (1.64) with the fiber coefficients redefined for the cotangent bundle T^*M).

A generalized Hamilton space $GH^n = (M, \breve{g}^{ij}(x,p))$ is called reducible to a Hamilton one if there exists a Hamilton function $H(x,p)$ on T^*M such that

$$\breve{g}^{ij}(x,p) = \frac{1}{2}\frac{\partial^2 H}{\partial p_i \partial p_j}. \tag{1.76}$$

Definition 1.12. *A Hamilton space is a pair* $H^n = (M, H(x,p))$ *such that* $H : T^*M \to \mathcal{R}$ *is a scalar function which satisfy the following conditions:*

1. *H is a differentiable function on the manifold $\widetilde{T^*M} = T^*M \backslash \{0\}$ and continous on the null section of the projection $\pi^* : T^*M \to M$;*

2. *The Hessian of H with elements (1.76) is positively defined on $\widetilde{T^*M}$ and $\breve{g}^{ij}(x,p)$ is nondegenerate matrix of rank n and of constant signature.*

For Hamilton spaces the canonical N–connection (defined by H and its Hessian) exists,

$$\check{N}_{ij} = \frac{1}{4}\{\check{g}_{ij}, H\} - \frac{1}{2}\left(\check{g}_{ik}\frac{\partial^2 H}{\partial p_k \partial x^j} + \check{g}_{jk}\frac{\partial^2 H}{\partial p_k \partial x^i}\right),$$

where the Poisson brackets, for arbitrary functions f and g on T^*M, act as

$$\{f, g\} = \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x^i} - \frac{\partial g}{\partial p_i}\frac{\partial p}{\partial x^i}.$$

The canonical d–connection $D\check{\Gamma}\left(\check{N}_{(c)}\right) = \check{\Gamma}^{\alpha}_{(c)\beta\gamma} = \left(\check{H}^i_{(c)\ jk}, \check{C}_{(c)\ i}^{\ jk}\right)$is defined by the coefficients

$$\begin{aligned}
\check{H}^i_{(c)\ jk} &= \frac{1}{2}\check{g}^{is}\left(\check{\delta}_j \check{g}_{sk} + \check{\delta}_k \check{g}_{js} - \check{\delta}_s \check{g}_{jk}\right), \\
\check{C}_{(c)\ i}^{\ jk} &= -\frac{1}{2}\check{g}_{is}\check{\partial}^j \check{g}^{sk}.
\end{aligned}$$

In result we can compute the d–torsions and d–curvatures like on cv–bundle or on Cartan spaces. On Hamilton spaces all such objects are defined by the Hamilton function $H(x, p)$ and indeces have to be reconsidered for co–fibers of the co-tangent bundle.

1.6 Gravity on Vector Bundles

The components of the Ricci d–tensor

$$R_{\alpha\beta} = R^{\cdot\tau}_{\alpha.\beta\tau}$$

with respect to a locally adapted frame (1.17) are as follows:

$$\begin{aligned}
R_{ij} &= R^{\cdot k}_{i.jk}, \quad R_{ia} = -{}^2P_{ia} = -P^{\cdot k}_{i.ka}, \\
R_{ai} &= {}^1P_{ai} = P^{\cdot b}_{a.ib}, \quad R_{ab} = S^{\cdot c}_{a.bc}.
\end{aligned} \tag{1.77}$$

We point out that because, in general, ${}^1P_{ai} \neq {}^2P_{ia}$ the Ricci d-tensor is non symmetric.

Having defined a d-metric of type in $\mathcal{E}$ we can introduce the scalar curvature of d–connection $\mathbf{D}$:

$$\overleftarrow{R} = G^{\alpha\beta}R_{\alpha\beta} = R + S, \tag{1.78}$$

where $R = g^{ij}R_{ij}$ and $S = h^{ab}S_{ab}$.

For our further considerations it will be also useful to use an alternative way of definition torsion (1.57) and curvature (1.62) by using the commutator

$$\Delta_{\alpha\beta} \doteq \bigtriangledown_\alpha \bigtriangledown_\beta - \bigtriangledown_\beta \bigtriangledown_\alpha = 2 \bigtriangledown_{[\alpha} \bigtriangledown_{\beta]}.$$

For components of d–torsion we have

$$\Delta_{\alpha\beta} f = T^{\gamma}_{.\alpha\beta} \bigtriangledown_\gamma f$$

for every scalar function f on $\mathcal{E}$. Curvature can be introduced as an operator acting on arbitrary d-vector V^δ :

$$(\Delta_{\alpha\beta} - T^{\gamma}_{.\alpha\beta} \bigtriangledown_\gamma) V^\delta = R^{\cdot\delta}_{\gamma.\alpha\beta} V^\gamma \qquad (1.79)$$

(we note that in this section we shall follow conventions of Miron and Anastasiei [108, 109] on d-tensors; we can obtain corresponding Penrose and Rindler abstract index formulas [128, 129] just for a trivial N-connection structure and by changing denotations for components of torsion and curvature in this manner: $T^{\gamma}_{.\alpha\beta} \to T_{\alpha\beta}{}^{\gamma}$ and $R^{\cdot\delta}_{\gamma.\alpha\beta} \to R_{\alpha\beta\gamma}{}^{\delta}$).

Here we also note that torsion and curvature of a d-connection on $\mathcal{E}$ satisfy generalized for locally anisotropic spaces Ricci and Bianchi identities [108, 109] which in terms of components (1.79) are written respectively as

$$R^{\cdot\delta}_{[\gamma.\alpha\beta]} + \bigtriangledown_{[\alpha} T^{\delta}_{.\beta\gamma]} + T^{\nu}_{.[\alpha\beta} T^{\delta}_{.\gamma]\nu} = 0 \qquad (1.80)$$

and

$$\bigtriangledown_{[\alpha} R^{\cdot\sigma}_{|\nu|\beta\gamma]} + T^{\delta}_{\cdot[\alpha\beta} R^{\cdot\sigma}_{|\nu|.\gamma]\delta} = 0. \qquad (1.81)$$

Identities (1.80) and (1.81) can be proved similarly as in [128] by taking into account that indices play a distinguished character.

We can also consider a la-generalization of the so-called conformal Weyl tensor (see, for instance, [128]) which can be written as a d-tensor in this form:

$$C^{\gamma\delta}{}_{\alpha\beta} = R^{\gamma\delta}{}_{\alpha\beta} - \frac{4}{n+m-2} R^{[\gamma}{}_{[\alpha}\ \delta^{\delta]}{}_{\beta]} + \frac{2}{(n+m-1)(n+m-2)} \overleftarrow{R}\ \delta^{[\gamma}{}_{[\alpha}\ \delta^{\delta]}{}_{\beta]}. \qquad (1.82)$$

This object is conformally invariant on locally anisotropic spaces provided with d-connection generated by d-metric structures.

The Einstein equations and conservation laws on v-bundles provided with N-connection structures are studied in detail in [108, 109, 2, 3]. In Ref. [186]

we proved that the locally anisotropic gravity can be formulated in a gauge like manner and analyzed the conditions when the Einstein locally anisotropic gravitational field equations are equivalent to a corresponding form of Yang-Mills equations. In this subsection we write the locally anisotropic gravitational field equations in a form more convenient for theirs equivalent reformulation in locally anisotropic spinor variables.

We define d-tensor $\Phi_{\alpha\beta}$ as to satisfy conditions

$$-2\Phi_{\alpha\beta} \doteq R_{\alpha\beta} - \frac{1}{n+m} \overleftarrow{R} g_{\alpha\beta} \tag{1.83}$$

which is the torsionless part of the Ricci tensor for locally isotropic spaces [128, 129], i.e. $\Phi_{\alpha}{}^{\alpha} \doteq 0$. The Einstein equations on locally anisotropic spaces

$$\overleftarrow{G}_{\alpha\beta} + \lambda g_{\alpha\beta} = \kappa E_{\alpha\beta}, \tag{1.84}$$

where

$$\overleftarrow{G}_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} \overleftarrow{R} g_{\alpha\beta} \tag{1.85}$$

is the Einstein d-tensor, λ and κ are correspondingly the cosmological and gravitational constants and by $E_{\alpha\beta}$ is denoted the locally anisotropic energy-momentum d-tensor [108, 109], can be rewritten in equivalent form:

$$\Phi_{\alpha\beta} = -\frac{\kappa}{2}(E_{\alpha\beta} - \frac{1}{n+m} E_{\tau}{}^{\tau}\ g_{\alpha\beta}). \tag{1.86}$$

Because the locally anisotropic spaces generally have nonzero torsions we shall add to (1.86) (equivalently to (1.84)) a system of algebraic d-field equations with the source $S^{\alpha}{}_{\beta\gamma}$ being the locally anisotropic spin density of matter (if we consider a variant of locally anisotropic Einstein-Cartan theory):

$$T^{\gamma}{}_{\alpha\beta} + 2\delta^{\gamma}{}_{[\alpha} T^{\delta}{}_{\beta]\delta} = \kappa S^{\gamma}{}_{\alpha\beta.} \tag{1.87}$$

From (1.80) and (1.87) one follows the conservation law of locally anisotropic spin matter:

$$\bigtriangledown_{\gamma} S^{\gamma}{}_{\alpha\beta} - T^{\delta}{}_{\delta\gamma} S^{\gamma}{}_{\alpha\beta} = E_{\beta\alpha} - E_{\alpha\beta}.$$

Finally, in this section, we remark that all presented geometric constructions contain those elaborated for generalized Lagrange spaces [108, 109] (for which a tangent bundle TM is considered instead of a v-bundle $\mathcal{E}$). We also note that the Lagrange (Finsler) geometry is characterized by a metric with components parametized as $g_{ij} = \frac{1}{2}\frac{\partial^2 \mathcal{L}}{\partial y^i \partial y^j}$ $\left(g_{ij} = \frac{1}{2}\frac{\partial^2 \Lambda^2}{\partial y^i \partial y^j}\right)$ and $h_{ij} = g_{ij}$, where $\mathcal{L} = \mathcal{L}\ (x,y)$ $(\Lambda = \Lambda\,(x,y))$ is a Lagrangian (Finsler metric) on TM (see details in [108, 109, 96, 27]).

Chapter 2

Anholonomic Einstein and Gauge Gravity

We analyze local anisotropies induced by anholonomic frames and associated nonlinear connections in general relativity and extensions to affine–Poincaré and de Sitter gauge gravity and different types of Kaluza–Klein theories. We construct some new classes of cosmological solutions of gravitational field equations describing Friedmann–Robertson–Walker like universes with rotation (ellongated and flattened) ellipsoidal or torus symmetry [185].

2.1 Introduction

The search for exact solutions with generic local anisotropy in general relativity, gauge gravity and non–Riemannian extensions has its motivation from low energy limits in modern string and Kaluza–Klein theories. Such classes of solutions constructed by using moving anholonomic frame fields (tetrads, or vierbeins; we shall use the term frames for higher dimensions) reflect a new type of constrained dynamics and locally anisotropic interactions of gravitational and matter fields [177].

What are the requirements of such constructions and their physical treatment? We believe that such solutions should have the properties: (i) they satisfy the Einstein equations in general relativity and are locally anisotropic generalizations of some known solutions in isotropic limits with a well posed Cauchy problem; (ii) the corresponding geometrical and physical values are defined, as a rule, with respect to an anholonomic system of reference which reflects the imposed constraints and supposed symmetry of locally anisotropic interactions; the reformulation of results for a coordinate frame is also possible; (iii) by applying the method of moving frames of reference, we can generalize the solutions to some analogous in metric–affine and/or gauge gravity, in higher dimension and string theories.

Comparing with the previos results [163, 170, 173, 172, 186] on definition of self–consistent field theories incorporating various possible anisotropic, inhomogeneous and stochastic manifestations of classical and quantum interactions on locally anisotropic and higher order anisotropic spaces, we emphasize that, in this Chapter, we shall be interested not in some extensions of the well known gravity theories with locally isotropic space–times ((pseudo) Riemannian or Riemanian–Cartan–Weyl ones, in brief, RCW space–times) to Finsler geometry and its generalizations. We shall present a proof that locally anisotropic structures (Finsler, Lagrange and higher order developments [59, 41, 136, 96, 14, 109, 106, 27, 70]) could be induced by anholonomic frames on locally isotropic spaces, even in general relativity and its metric–affine and gauge like modifications [63, 153, 132, 133, 98, 53, 186, 131, 202].

To evolve some new (frame anholonomy) features of locally isotropic gravity theories we shall apply the methods of the geometry of anholonomic frames and associated nonlinear connection (in brief, N–connection) structures elaborated in details for bundle spaces and generalized Finsler spaces in monographs [109, 106, 27] with further developments for spinor differential geometry, superspaces and stochastic calculus in [163, 171, 173, 172]. The first rigorous global definition of N–connections is due to W. Barthel [25] but the idea and some rough constructions could be found in the E. Cartan's works [41]. We note that the point of this paper is to emphasize the generic locally anisotropic geometry and physics and apply the N–connection method for ´non–Finslerian´(pseudo) Riemannian and RCW spacetimes. Here, it should be mentioned that anholonomic frames are considered in detail, for instance, in monographs [56, 117, 128, 129] and with respect to geometrization of gauge theories in [98, 131] but not concerning the topic on associated N–connection structures which grounds our geometric approach to anisotropies in physical theories and developing of a new method of integrating gravitational field equations.

2.2 Anholonomic Frames on (Pseudo) Riemannian Spaces

For definiteness, we consider a $(n+m)$–dimensional (pseudo) Riemannian spacetime $V^{(n+m)}$, being a paracompact and connected Hausdorff C^∞–manifold, enabled with a nonsigular metric

$$ds^2 = \widetilde{g}_{\alpha\beta}\ du^\alpha \otimes du^\beta$$

with the coefficients

$$\widetilde{g}_{\alpha\beta} = \left[\begin{array}{cc} g_{ij} + N_i^a N_j^b h_{ab} & N_j^e h_{ae} \\ N_i^e h_{be} & h_{ab} \end{array} \right] \qquad (2.1)$$

parametrized with respect to a local coordinate basis $du^\alpha = (dx^i, dy^a)$, having its dual $\partial/u^\alpha = (\partial/x^i, \partial/y^a)$, where the indices of geometrical objects and local coordinate $u^\alpha = \left(x^k, y^a\right)$ run correspondingly the values: (for Greek indices)$\alpha, \beta, \ldots = n + m$; for (Latin indices) $i, j, k, ... = 1, 2, ..., n$ and $a, b, c, ... = 1, 2, ..., m$. We shall use 'tilds' if would be necessary to emphasize that a value is defined with respect to a coordinate basis.

The metric (2.1) can be rewritten in a block $(n \times n) + (m \times m)$ form

$$g_{\alpha\beta} = \left(\begin{array}{cc} g_{ij}(x^k, y^a) & 0 \\ 0 & h_{ab}(x^k, y^a) \end{array} \right) \qquad (2.2)$$

with respect to a subclass of $n+m$ anholonomic frame basis (for four dimensions one used terms tetrads, or vierbiends) defined

$$\delta_\alpha = (\delta_i, \partial_a) = \frac{\delta}{\partial u^\alpha} = \left(\delta_i = \frac{\delta}{\partial x^i} = \frac{\partial}{\partial x^i} - N_i^b \left(x^j, y^c\right) \frac{\partial}{\partial y^b}, \partial_a = \frac{\partial}{\partial y^a} \right) \qquad (2.3)$$

and

$$\delta^\beta = \left(d^i, \delta^a\right) = \delta u^\beta = \left(d^i = dx^i, \delta^a = \delta y^a = dy^a + N_k^a \left(x^j, y^b\right) dx^k\right), \qquad (2.4)$$

called the locally anisotropic bases (in brief, la–bases) adapted to the coefficients N_j^a. The $n \times n$ matrix g_{ij} defines the so–called horizontal metric (in brief, h–metric) and the $m \times m$ matrix h_{ab} defines the vertical (v–metric) with respect to the associated nonlinear connection (N–connection) structure given by its coefficients $N_j^a(u^\alpha)$ from (2.3) and (2.4). The geometry of N–connections is studied in detail in [25, 109]; here we shall consider its applications with respect to anholonomic frames in general relativity and its locally isotropic generalizations.

A frame structure δ_α (2.3) on $V^{(n+m)}$ is characterized by its anholonomy relations

$$\delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha = w^\gamma{}_{\alpha\beta} \delta_\gamma. \qquad (2.5)$$

with anholonomy coefficients $w^\alpha{}_{\beta\gamma}$.The elongation of partial derivatives (by N–coefficients) in the locally adapted partial derivatives (2.3) reflects the fact that on the (pseudo) Riemannian space–time $V^{(n+m)}$ it is modeled a generic local anisotropy characterized by the anholonomy relations (2.5) when the anholonomy coefficients are computed as follows

$$
\begin{aligned}
w^{k}{}_{ij} &= 0, w^{k}{}_{aj} = 0, w^{k}{}_{ia} = 0, w^{k}{}_{ab} = 0, w^{c}{}_{ab} = 0, \\
w^{a}{}_{ij} &= -\Omega^{a}_{ij}, w^{b}{}_{aj} = -\partial_a N^b_i, w^{b}{}_{ia} = \partial_a N^b_i,
\end{aligned}
$$

where

$$
\Omega^a_{ij} = \partial_i N^a_j - \partial_j N^a_i + N^b_i \partial_b N^a_j - N^b_j \partial_b N^a_i
$$

defines the coefficients of the N–connection curvature, in brief, N–curvature. On (pseudo) Riemannian space–times this is a characteristic of a chosen anholonomic system of reference.

A N–connection N defines a global decomposition,

$$
N : V^{(n+m)} = H^{(n)} \oplus V^{(m)},
$$

of spacetime $V^{(n+m)}$ into a n–dimensional horizontal subspace $H^{(n)}$ (with holonomic x–coordinates) and into a m–dimensional vertical subspace $V^{(m)}$ (with anisotropic, anholonomic, y–coordinates). This form of parametrizations of sets of mixt holonomic–anholonomic frames is very useful for investigation, for instance, of kinetic and thermodynamic systems in general relativity, spinor and gauge field interactions in curved space–times and for definition of non–trivial reductions from higher dimension to lower dimension ones in Kaluza–Klein theories. In the last case the N–connection could be treated as a 'splitting' field into base's and extra dimensions with the anholonomic (equivalently, anisotropic) structure defined from some prescribed types of symmetries and constraints (imposed on a physical system) or, for a different class of theories, with some dynamical field equations following in the low energy limit of string theories [170, 171] or from Einstein equations on a higher dimension space.

The locally anisotropic spacetimes, anisoropic spacetimes, to be investigated in this section are considered to be some (pseudo) Riemannian manifolds $V^{(n+m)}$ enabled with a frame, in general, anholonomic structures of basis vector fields, $\delta^\alpha = (\delta^i, \delta^a)$ and theirs duals $\delta_\alpha = (\delta_i, \delta_a)$ (equivalently to an associated N–connection structure), adapted to a symmetric metric field $g_{\alpha\beta}$ (2.2) of necessary signature and to a linear, in general nonsymmetric, connection $\Gamma^{\alpha}{}_{\beta\gamma}$ defining the covariant derivation D_α satisfying the metricity conditions $D_\alpha g_{\beta\gamma} = 0$. The term anisotropic points to a prescribed type of anholonomy structure. As a matter of principle, on a (pseudo) Riemannian space–time, we can always, at least locally, remove our considerations with respect to a coordinate basis. In this case the geometric anisotopy is modelled by metrics of type (2.1). Such ansatz for metrics are largely applied in modern Kaluza–Klein theory [125] where the N–conection structures have been not pointed out because in the simplest approximation on topological compactification of extra dimensions the N–connection geometry is trivial. A rigorous analysis of systems with mixed holonomic–anholonomic variables was not yet provided for general relativity, extra dimension and gauge like gravity theories..

A $n+m$ anholonomic structure distinguishes (d) the geometrical objects into h– and v–components. Such objects are briefly called d–tensors, d–metrics and/or d–connections. Their components are defined with respect to a locally anisotropic basis of type (2.3), its dual (2.4), or their tensor products (d–linear or d–affine transforms of such frames could also be considered). For instance, a covariant and contravariant d–tensor Z, is expressed

$$Z = Z^{\alpha}_{\ \beta}\delta_{\alpha} \otimes \delta^{\beta} = Z^{i}_{\ j}\delta_i \otimes d^j + Z^{i}_{\ a}\delta_i \otimes \delta^a + Z^{b}_{\ j}\partial_b \otimes d^j + Z^{b}_{\ a}\partial_b \otimes \delta^a.$$

A linear d–connection D on locally anisotropic space–time $V^{(n+m)}$,

$$D_{\delta_{\gamma}}\delta_{\beta} = \Gamma^{\alpha}_{\ \beta\gamma}(x,y)\,\delta_{\alpha},$$

is parametrized by non–trivial h–v–components,

$$\Gamma^{\alpha}_{\ \beta\gamma} = \left(L^{i}_{\ jk}, L^{a}_{\ bk}, C^{i}_{\ jc}, C^{a}_{\ bc}\right). \tag{2.6}$$

A metric on $V^{(n+m)}$ with $(m\times m)+(n\times n)$ block coefficients (2.2) is written in distinguished form, as a metric d–tensor (in brief, d–metric), with respect to a locally anisotropic base (2.4)

$$\delta s^2 = g_{\alpha\beta}(u)\,\delta^{\alpha}\otimes\delta^{\beta} = g_{ij}(x,y)dx^i dx^j + h_{ab}(x,y)\delta y^a \delta y^b. \tag{2.7}$$

Some d–connection and d–metric structures are compatible if there are satisfied the conditions

$$D_{\alpha}g_{\beta\gamma} = 0.$$

For instance, a canonical compatible d–connection

$$^{c}\Gamma^{\alpha}_{\ \beta\gamma} = \left(^{c}L^{i}_{\ jk}, ^{c}L^{a}_{\ bk}, ^{c}C^{i}_{\ jc}, ^{c}C^{a}_{\ bc}\right)$$

is defined by the coefficients of d–metric (2.7), $g_{ij}(x,y)$ and $h_{ab}(x,y)$, and by the N–coefficients,

$$\begin{aligned}
^{c}L^{i}_{\ jk} &= \frac{1}{2}g^{in}\left(\delta_k g_{nj} + \delta_j g_{nk} - \delta_n g_{jk}\right), \qquad (2.8)\\
^{c}L^{a}_{\ bk} &= \partial_b N^a_k + \frac{1}{2}h^{ac}\left(\delta_k h_{bc} - h_{dc}\partial_b N^d_i - h_{db}\partial_c N^d_i\right),\\
^{c}C^{i}_{\ jc} &= \frac{1}{2}g^{ik}\partial_c g_{jk},\\
^{c}C^{a}_{\ bc} &= \frac{1}{2}h^{ad}\left(\partial_c h_{db} + \partial_b h_{dc} - \partial_d h_{bc}\right)
\end{aligned}$$

The coefficients of the canonical d–connection generalize for locally anisotropic space–times the well known Christoffel symbols; on a (pseudo) Riemannian spacetime with a fixed anholonomic frame the d–connection coefficients transform exactly into the metric connection coefficients.

For a d–connection (2.6) the components of torsion,

$$\begin{aligned} T\left(\delta_\gamma,\delta_\beta\right) &= T^{\alpha}_{\ \beta\gamma}\delta_\alpha, \\ T^{\alpha}_{\ \beta\gamma} &= \Gamma^{\alpha}_{\ \beta\gamma}-\Gamma^{\alpha}_{\ \gamma\beta}+w^{\alpha}_{\ \beta\gamma} \end{aligned}$$

are expressed via d–torsions

$$\begin{aligned} T^{i}_{.jk} &= -T^{i}_{.kj}=L^{i}_{jk}-L^{i}_{kj}, \quad T^{i}_{ja}=C^{i}_{.ja}, T^{i}_{aj}=-C^{i}_{ja}, \\ T^{i}_{.ab} &= 0, \quad T^{a}_{.bc}=S^{a}_{.bc}=C^{a}_{bc}-C^{a}_{cb}, \\ T^{a}_{.ij} &= -\Omega^{a}_{ij}, \quad T^{a}_{.bi}=\partial_b N^{a}_{i}-L^{a}_{.bj}, \quad T^{a}_{.ib}=-T^{a}_{.bi}. \end{aligned} \tag{2.9}$$

We note that for symmetric linear connections the d–torsions are induced as a pure anholonomic effect. They vanish with respect to a coordinate frame of reference.

In a similar manner, putting non–vanishing coefficients (2.6) into the formula for curvature,

$$\begin{aligned} & R\left(\delta_\tau,\delta_\gamma\right)\delta_\beta = R_{\beta\ \gamma\tau}^{\ \alpha}\delta_\alpha, \\ & R_{\beta\ \gamma\tau}^{\ \alpha} = \delta_\tau\Gamma^{\alpha}_{\ \beta\gamma}-\delta_\gamma\Gamma^{\alpha}_{\ \beta\delta}+\Gamma^{\varphi}_{\ \beta\gamma}\Gamma^{\alpha}_{\ \varphi\tau}-\Gamma^{\varphi}_{\ \beta\tau}\Gamma^{\alpha}_{\ \varphi\gamma}+\Gamma^{\alpha}_{\ \beta\varphi}w^{\varphi}_{\ \gamma\tau}, \end{aligned}$$

we can compute the components of d–curvatures

$$\begin{aligned} R^{.i}_{h.jk} &= \delta_k L^{i}_{.hj}-\delta_j L^{i}_{.hk}+L^{m}_{.hj}L^{i}_{mk}-L^{m}_{.hk}L^{i}_{mj}-C^{i}_{.ha}\Omega^{a}_{.jk}, \\ R^{.a}_{b.jk} &= \delta_k L^{a}_{.bj}-\delta_j L^{a}_{.bk}+L^{c}_{.bj}L^{a}_{.ck}-L^{c}_{.bk}L^{a}_{.cj}-C^{a}_{.bc}\Omega^{c}_{.jk}, \\ P^{.i}_{j.ka} &= \partial_k L^{i}_{.jk}+C^{i}_{.jb}T^{b}_{.ka}-(\delta_k C^{i}_{.ja}+L^{i}_{.lk}C^{l}_{.ja}-L^{l}_{.jk}C^{i}_{.la}-L^{c}_{.ak}C^{i}_{.jc}), \\ P^{.c}_{b.ka} &= \partial_a L^{c}_{.bk}+C^{c}_{.bd}T^{d}_{.ka}-(\delta_k C^{c}_{.ba}+L^{c}_{.dk}C^{d}_{.ba}-L^{d}_{.bk}C^{c}_{.da}-L^{d}_{.ak}C^{c}_{.bd}), \\ S^{.i}_{j.bc} &= \partial_c C^{i}_{.jb}-\partial_b C^{i}_{.jc}+C^{h}_{.jb}C^{i}_{.hc}-C^{h}_{.jc}C^{i}_{hb}, \\ S^{.a}_{b.cd} &= \partial_d C^{a}_{.bc}-\partial_c C^{a}_{.bd}+C^{e}_{.bc}C^{a}_{.ed}-C^{e}_{.bd}C^{a}_{.ec}. \end{aligned}$$

The Ricci tensor

$$R_{\beta\gamma}=R_{\beta\ \gamma\alpha}^{\ \alpha}$$

has the d–components

$$\begin{aligned} R_{ij} &= R^{.k}_{i.jk}, \quad R_{ia}=-{}^{2}P_{ia}=-P^{.k}_{i.ka}, \\ R_{ai} &= {}^{1}P_{ai}=P^{.b}_{a.ib}, \quad R_{ab}=S^{.c}_{a.bc}. \end{aligned} \tag{2.10}$$

We point out that because, in general, ${}^1P_{ai} \neq {}^2P_{ia}$, the Ricci d–tensor is non symmetric.

Having defined a d-metric of type (2.7) in $V^{(n+m)}$ we can compute the scalar curvature

$$\overleftarrow{R} = g^{\beta\gamma} R_{\beta\gamma}$$

of a d-connection D,

$$\overleftarrow{R} = \widehat{R} + S, \qquad (2.11)$$

where $\widehat{R} = g^{ij} R_{ij}$ and $S = h^{ab} S_{ab}$.

Now, by introducing the values (2.10) and (2.11) into the Einstein's equations

$$R_{\beta\gamma} - \frac{1}{2} g_{\beta\gamma} \overleftarrow{R} = k \Upsilon_{\beta\gamma},$$

we can write down the system of field equations for locally anisotropic gravity with anholonomic (N–connection) structure:

$$\begin{aligned} R_{ij} - \frac{1}{2}\left(\widehat{R} + S\right) g_{ij} &= k\Upsilon_{ij}, \qquad (2.12)\\ S_{ab} - \frac{1}{2}\left(\widehat{R} + S\right) h_{ab} &= k\Upsilon_{ab},\\ {}^1P_{ai} &= k\Upsilon_{ai},\\ {}^2P_{ia} &= -k\Upsilon_{ia}, \end{aligned}$$

where $\Upsilon_{ij}, \Upsilon_{ab}, \Upsilon_{ai}$ and Υ_{ia} are the components of the energy–momentum d–tensor field $\Upsilon_{\beta\gamma}$ (which includes possible cosmological constants, contributions of anholonomy d–torsions (2.9) and matter) and k is the coupling constant.

The h- v- decomposition of gravitational field equations (2.12) was introduced by Miron and Anastasiei [109] in their N–connection approach to generalized Finsler and Lagrange spaces. It holds true as well on (pseudo) Riemannian spaces, in general gravity; in this case we obtain the usual form of Einstein equations if we transfer considerations with respect to coordinate frames. If the N–coefficients are prescribed by fixing the anholonomic frame of reference, different classes of solutions are to be constructed by finding the h– and v–components, g_{ij} and h_{ab}, of metric (2.1), or its equivalent (2.2). A more general approach is to consider the N–connection as 'free' but subjected to the condition that its coefficients along with the d–metric components are chosen to solve the Einsten equations in the form (2.12) for some suggested symmetries, configurations of horizons and type of singularities and well defined Cauchy problem. This way one can construct new classes of metrics with generic local anisotropy (see [177]).

2.3 Higher Order Anisotropic Structures

Miron and Atanasiu [110, 106, 107] developed the higher order Lagrange and Finsler geometry with applications in mechanics in order to geometrize the concepts of classical mechanics on higher order tangent bundles. The work [171] was a proof that higher order anisotropies (in brief, one writes abbreviations like ha–, ha–superspace, ha–spacetime, ha–geometry and so on) can be induced alternatively in low energy limits of (super) string theories and a higher order superbundle N–connection formalism was proposed. There were developed the theory of spinors [173], proposed models of ha–(super)gravity and matter interactions on ha–spaces and defined the supersymmetric stochastic calculus in ha–superspaces which were summarized in the monograph [172] containing a local (super) geometric approach to so called ha–superstring and generalized Finsler–Kaluza–Klein (super) gravities.

The aim of this section is to proof that higher order anisotropic (ha–structures) are induced by respective anholonomic frames in higher dimension Einstein gravity, to present the basic geometric background for a such moving frame formalism and associated N–connections and to deduce the system of gravitational field equations with respect to ha–frames.

2.3.1 Ha–frames and corresponding N–connections

Let us consider a (pseudo) Riemannian spacetime $V^{(\overline{n})} = V^{(n+\overline{m})}$ where the anisotropic dimension $\overline{m}$ is split into z sub–dimensions m_p, $(p = 1, 2, ..., z)$, i. e. $\overline{m} = m_1 + m_2 + ... + m_z$. The local coordinates on a such higher dimension curved space–time will be denoted as to take into account the m–decomposition,

$$\begin{aligned} u &= \{u^{\overline{\alpha}} \equiv u^{\alpha_z} = (x^i, y^{a_1}, y^{a_2}, \ldots, y^{a_p}, \ldots y^{a_z})\}, \\ u^{\alpha_p} &= (x^i, y^{a_1}, y^{a_2}, \ldots, y^{a_p}) = (u^{\alpha_{p-1}}, y^{a_p}) \,. \end{aligned}$$

The la–constructions from the previous Section are considered to describe anholonomic structures of first order; for $z = 1$ we put $u^{\alpha_1} = (x^i, y^{a_1}) = u^{\alpha} = (x^i, y^{a_1})$. The higher order anisotropies are defined inductively, 'shell by shell', starting from the first order to the higher order, z–anisotropy. In order to distinguish the components of geometrical objects with respect to a p–shell we provide both Greek and Latin indices with a corresponding subindex like $\alpha_p = (\alpha_{p-1}, a_p)$, and $a_p = (1, 2, ..., m_p)$, i. e. one holds a shell parametrization for coordinates,

$$y^{a_p} = (y^1_{(p)} = y^1, y^2_{(p)} = y^2, ..., y^{m_p}_{(p)} = y^{m_p}).$$

We shall overline some indices, for instance, $\overline{\alpha}$ and $\overline{a}$, if would be necessary to point that it could be split into shell components and omit the p–shell mark (p) if this does not lead to misunderstanding. Such decompositions of indices and geometrical and physical values are introduced with the aim for a further modelling of (in general, dynamical) spllittings of higher dimension spacetimes, step by step, with 'interior' subspaces being of different dimension, to lower dimensions, with nontrival topology and anholonomic (anisotropy) structures in generalized Kaluza–Klein theories.

The coordinate frames are denoted

$$\partial_{\overline{\alpha}} = \partial/u^{\overline{\alpha}} = \left(\partial/x^i, \partial/y^{a_1}, ..., \partial/y^{a_z}\right)$$

with the dual ones

$$d^{\overline{\alpha}} = du^{\overline{\alpha}} = \left(dx^i, dy^{a_1}, ..., dy^{a_z}\right),$$

when

$$\partial_{\alpha_p} = \partial/u^{\alpha_p} = \left(\partial/x^i, \partial/y^{a_1}, ..., \partial/y^{a_p}\right)$$

and

$$d^{\alpha_p} = du^{\alpha_p} = \left(dx^i, dy^{a_1}, ..., dy^{a_p}\right)$$

if considerations are limited to the p-th shell.

With respect to a coordinate frame a nonsigular metric

$$ds^2 = \widetilde{g}_{\overline{\alpha}\overline{\beta}}\, du^{\overline{\alpha}} \otimes du^{\overline{\beta}}$$

with coefficients $\widetilde{g}_{\overline{\alpha}\overline{\beta}}$ defined on induction,

$$\begin{aligned}
\widetilde{g}_{\alpha_1\beta_1} &= \left[\begin{array}{cc} g_{ij} + M_i^{a_1} M_j^{b_1} h_{a_1 b_1} & M_j^{e_1} h_{a_1 e_1} \\ M_i^{e_1} h_{b_1 e_1} & h_{a_1 b_1} \end{array}\right], \qquad (2.13)\\
&\vdots \\
\widetilde{g}_{\alpha_p\beta_p} &= \left[\begin{array}{cc} g_{\alpha_{p-1}\beta_{p-1}} + M_{\alpha_{p-1}}^{a_p} M_{\beta_{p-1}}^{b_p} h_{a_p b_p} & M_{\beta_{p-1}}^{e_p} h_{a_p e_p} \\ M_{\alpha_{p-1}}^{e_p} h_{b_p e_p} & h_{a_p b_p} \end{array}\right], \\
&\vdots \\
\widetilde{g}_{\overline{\alpha}\overline{\beta}} = \widetilde{g}_{\alpha_z\beta_z} &= \left[\begin{array}{cc} g_{\alpha_{z-1}\beta_{z-1}} + M_{\alpha_{z-1}}^{a_z} M_{\beta_{z-1}}^{b_z} h_{a_z b_z} & M_{\beta_{z-1}}^{e_z} h_{a_z e_z} \\ M_{\alpha_{z-1}}^{e_z} h_{b_z e_z} & h_{a_z b_z} \end{array}\right],
\end{aligned}$$

where indices are split as $\alpha_1 = (i_1, a_1)$, $\alpha_2 = (\alpha_1, a_2)$, $\alpha_p = (\alpha_{p-1}, a_p)$; $p = 1, 2, ...z$.

The metric (2.13) on $V^{(\overline{n})}$ splits into symmetric blocks of matrices of dimensions

$$(n \times n) \oplus (m_1 \times m_1) \oplus ... \oplus (m_z \times m_z),$$

$n+m$ form

$$g_{\alpha\beta} = \begin{pmatrix} g_{ij}(u) & 0 & \dots & 0 \\ 0 & h_{a_1 b_1} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & h_{a_z b_z} \end{pmatrix} \quad (2.14)$$

with respect to an anholonomic frame basis defined on induction

$$\begin{aligned} \delta_{\alpha_p} &= (\delta_{\alpha_{p-1}}, \partial_{a_p}) = \left(\delta_i, \delta_{a_1}, ..., \delta_{a_{p-1}}, \partial_{a_p}\right) \quad (2.15) \\ &= \frac{\delta}{\partial u^{\alpha_p}} = \left(\frac{\delta}{\partial u^{\alpha_{p-1}}} = \frac{\partial}{\partial u^{\alpha_{p-1}}} - N^{b_p}_{\alpha_{p-1}}(u) \frac{\partial}{\partial y^{b_p}}, \frac{\partial}{\partial y^{a_p}}\right), \end{aligned}$$

and

$$\begin{aligned} \delta^{\beta_p} &= \left(d^i, \delta^{\overline{a}_p}\right) = \left(d^i, \delta^{a_1}, ..., \delta^{a_{p-1}}, \delta^{a_p}\right) \quad (2.16) \\ &= \delta u^{\beta_p} = \left(d^i = dx^i, \delta^{\overline{a}_p} = \delta y^{\overline{a}_p} = dy^{\overline{a}_p} + M^{\overline{a}_p}_{\alpha_{p-1}}(u)\, du^{\alpha_{p-1}}\right), \end{aligned}$$

where $\overline{a}_p = (a_1, a_2, ..., a_p)$, are called the locally anisotropic bases (in brief la–bases) adapted respectively to the N–coefficients

$$N^{a_p}_{\alpha_{p-1}} = \left\{N^{a_p}_i, N^{a_p}_{a_1}, ..., N^{a_p}_{a_{p-2}}, N^{a_p}_{a_{p-1}}\right\}$$

and M–coefficients

$$M^{a_p}_{\alpha_{p-1}} = \left\{M^{a_p}_i, M^{a_p}_{a_1}, ..., M^{a_p}_{a_{p-2}}, M^{a_p}_{a_{p-1}}\right\};$$

the coefficients $M^{a_p}_{\alpha_{p-1}}$ are related via some algebraic relations with $N^{a_p}_{\alpha_{p-1}}$ in order to be satisfied the locally anisotropic basis duality conditions

$$\delta_{\alpha_p} \otimes \delta^{\beta_p} = \delta^{\beta_p}_{\alpha_p},$$

where $\delta^{\beta_p}_{\alpha_p}$ is the Kronecker symbol, for every shell.

The geometric structure of N– and M–coefficients of a higher order nonlinear connection becomes more explicit if we write the relations (2.15) and (2.16) in matrix form, respectively,

$$\delta_\bullet = \widehat{N}(u) \times \partial_\bullet$$

and

$$\delta^{\bullet} = d^{\bullet} \times M(u),$$

where

$$\delta_{\bullet} = \delta_{\overline{\alpha}} = \begin{pmatrix} \delta_i \\ \delta_{a_1} \\ \delta_{a_2} \\ \dots \\ \delta_{a_z} \end{pmatrix} = \begin{pmatrix} \delta/\partial x^i \\ \delta/\partial y^{a_1} \\ \delta/\partial y^{a_2} \\ \dots \\ \delta/\partial y^{a_z} \end{pmatrix}, \partial_{\bullet} = \partial_{\overline{\alpha}} = \begin{pmatrix} \partial_i \\ \partial_{a_1} \\ \partial_{a_2} \\ \dots \\ \partial_{a_z} \end{pmatrix} = \begin{pmatrix} \partial/\partial x^i \\ \partial/\partial y^{a_1} \\ \partial/\partial y^{a_2} \\ \dots \\ \partial/\partial y^{a_z} \end{pmatrix},$$

$$\delta^{\bullet} = \begin{pmatrix} dx^i & \delta y^{a_1} & \delta y^{a_2} & \dots & \delta y^{a_z} \end{pmatrix}, \; d^{\bullet} = \begin{pmatrix} dx^i & dy^{a_1} & dy^{a_2} & \dots & dy^{a_z} \end{pmatrix},$$

and

$$\widehat{N} = \begin{pmatrix} 1 & -N_i^{a_1} & -N_i^{a_2} & \dots & -N_i^{a_z} \\ 0 & 1 & -N_{a_1}^{a_2} & \dots & -N_{a_1}^{a_z} \\ 0 & 0 & 1 & \dots & -N_{a_2}^{a_z} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

$$M = \begin{pmatrix} 1 & M_i^{a_1} & M_i^{a_2} & \dots & M_i^{a_z} \\ 0 & 1 & M_{a_1}^{a_2} & \dots & M_{a_1}^{a_z} \\ 0 & 0 & 1 & \dots & M_{a_2}^{a_z} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

The $n \times n$ matrix g_{ij} defines the horizontal metric (in brief, h–metric) and the $m_p \times m_p$ matrices $h_{a_p b_p}$ defines the vertical, v_p–metrics with respect to the associated nonlinear connection (N–connection) structure given by its coefficients $N_{\alpha_{p-1}}^{a_p}$ from (2.15). The geometry of N–connections on higher order tangent bundles is studied in detail in [110, 106, 107], for vector (super)bundles there it was proposed the approach from [171, 172]; the approach and denotations elaborated in this work is adapted to further applications in higher dimension Einstein gravity and its non–Riemannian locally anisotropic extensions.

A ha–basis $\delta_{\overline{\alpha}}$ (2.4) on $V^{(\overline{n})}$ is characterized by its anholonomy relations

$$\delta_{\overline{\alpha}}\delta_{\overline{\beta}} - \delta_{\overline{\beta}}\delta_{\overline{\alpha}} = w^{\overline{\gamma}}{}_{\overline{\alpha}\overline{\beta}}\delta_{\overline{\gamma}}. \tag{2.17}$$

with anholonomy coefficients $w^{\overline{\gamma}}_{\ \overline{\alpha}\overline{\beta}}$. The anholonomy coefficients are computed

$$\begin{aligned}
w^{k}_{\ ij} &= 0; w^{k}_{\ a_p j} = 0; w^{k}_{\ i a_p} = 0; w^{k}_{\ a_p b_p} = 0; w^{c_p}_{\ a_p b_p} = 0; \\
w^{a_p}_{\ ij} &= -\Omega^{a_p}_{ij}; w^{b_p}_{\ a_p j} = -\delta_{a_p} N^{b_p}_i; w^{b_p}_{\ i a_p} = \delta_{a_p} N^{b_p}_i; \\
w^{k_p}_{\ a_p b_p} &= 0; w^{c_f}_{\ a_p b_f} = 0, f < p; w^{c_f}_{\ b_f a_p} = 0, f < p; w^{c_f}_{\ a_p b_p} = 0, f < p; \\
w^{a_p}_{\ c_f d_s} &= -\Omega^{a_p}_{c_f d_s}, (f, s < p); \\
w^{b_p}_{\ a_p c_f} &= -\delta_{a_p} N^{b_p}_{c_f}, f < p; w^{b_p}_{\ c_f a_p} = \delta_{a_p} N^{b_p}_{c_f}, f < p;
\end{aligned}$$

where

$$\begin{aligned}
\Omega^{a_p}_{ij} &= \partial_i N^{a_p}_j - \partial_j N^{a_p}_i + N^{b_p}_i \delta_{b_p} N^{a_p}_j - N^{b_p}_j \delta_{b_p} N^{a_p}_i, \\
\Omega^{a_p}_{\alpha_f \beta_s} &= \partial_{\alpha_f} N^{a_p}_{\beta_s} - \partial_{\beta_s} N^{a_p}_{\alpha_f} + N^{b_p}_{\alpha_f} \delta_{b_p} N^{a_p}_{\beta_s} - N^{b_p}_{\beta_s} \delta_{b_p} N^{a_p}_{\alpha_f},
\end{aligned} \tag{2.18}$$

for $1 \leq s, f < p$, are the coefficients of higher order N–connection curvature (N–curvature).

A higher order N–connection N defines a global decomposition

$$N: \ V^{(\overline{n})} = H^{(n)} \oplus V^{(m_1)} \oplus V^{(m_2)} \oplus ... \oplus V^{(m_z)},$$

of space–time $V^{(\overline{n})}$ into a n–dimensional horizontal subspace $H^{(n)}$ (with holonomic x–components) and into m_p–dimensional vertical subspaces $V^{(m_p)}$ (with anisotropic, anholonomic, $y_{(p)}$–components).

2.3.2 Distinguished linear connections

In this section we consider fibered (pseudo) Riemannian manifolds $V^{(\overline{n})}$ enabled with anholonomic frame structures of basis vector fields, $\delta^{\overline{\alpha}} = (\delta^i, \delta^{\overline{a}})$ and theirs duals $\delta_{\overline{\alpha}} = (\delta_i, \delta_{\overline{a}})$ with associated N–connection structure, adapted to a symmetric metric field $g_{\overline{\alpha}\overline{\beta}}$ (2.14) and to a linear, in general nonsymmetric, connection $\Gamma^{\overline{\alpha}}_{\ \overline{\beta}\overline{\gamma}}$ defining the covariant derivation $D_{\overline{\alpha}}$ satisfying the metricity conditions $D_{\overline{\alpha}} g_{\overline{\beta}\overline{\gamma}} = 0$. Such space–times are provided with anholonomic higher order anisotropic structures and, in brief, are called ha–spacetimes.

A higher order N–connection distinguishes (d) the geometrical objects into h– and v_p–components (d–tensors, d–metrics and/or d–connections). For instance, a d-tensor field of type $\begin{pmatrix} p & r_1 & ... & r_p & ... & r_z \\ q & s_1 & ... & s_p & ... & s_z \end{pmatrix}$ is written in local form as

$$\begin{aligned}\mathbf{t} &= t^{i_1...i_p a_1^{(1)}...a_{r_1}^{(1)}...a_1^{(p)}...a_{r_p}^{(p)}...a_1^{(z)}...a_{r_z}^{(z)}}_{j_1...j_q b_1^{(1)}...b_{r_1}^{(1)}...b_1^{(p)}...b_{r_p}^{(p)}...b_1^{(z)}...b_{r_z}^{(z)}}(u)\,\delta_{i_1}\otimes...\otimes\delta_{i_p}\otimes d^{j_1}\otimes...\otimes d^{j_q}\otimes \\ & \delta_{a_1^{(1)}}\otimes...\otimes\delta_{a_{r_1}^{(1)}}\otimes\delta^{b_1^{(1)}}...\otimes\delta^{b_{s_1}^{(1)}}\otimes...\otimes\delta_{a_1^{(p)}}\otimes...\otimes\delta_{a_{r_p}^{(p)}}\otimes...\otimes \\ & \delta^{b_1^{(p)}}...\otimes\delta^{b_{s_p}^{(p)}}\otimes\delta_{a_1^{(z)}}\otimes...\otimes\delta_{a_{r_z}^{(z)}}\otimes\delta^{b_1^{(z)}}...\otimes\delta^{b_{s_z}^{(z)}}.\end{aligned}$$

A linear d–connection D on ha–spacetime $V^{(\overline{n})}$,

$$D_{\delta_{\overline{\gamma}}}\delta_{\overline{\beta}} = \Gamma^{\overline{\alpha}}_{\ \overline{\beta}\overline{\gamma}}(u)\,\delta_{\overline{\alpha}},$$

is defined by its non–trivial h–v–components,

$$\Gamma^{\overline{\alpha}}_{\ \overline{\beta}\overline{\gamma}} = \left(L^i_{\ jk}, L^{\overline{a}}_{\ \overline{b}k}, C^i_{\ j\overline{c}}, C^{\overline{a}}_{\ \overline{b}\overline{c}}, K^{a_p}_{\ \ b_p c_p}, K^{a_p}_{\ \ b_s c_f}, Q^{a_f}_{\ \ b_f c_p}\right), \tag{2.19}$$

for $f < p, s$.

A metric with block coefficients (2.14) is written as a d–metric, with respect to a la–base (2.16)

$$\delta s^2 = g_{\overline{\alpha}\overline{\beta}}(u)\,\delta^{\overline{\alpha}}\otimes\delta^{\overline{\beta}} = g_{ij}(u)dx^i dx^j + h_{a_p b_p}(u)\delta y^{a_p}\delta y^{b_p}, \tag{2.20}$$

where $p = 1, 2, ..., z$.

A d–connection and a d–metric structure are compatible if there are satisfied the conditions

$$D_{\overline{\alpha}} g_{\overline{\beta}\overline{\gamma}} = 0.$$

The canonical d–connection ${}^c\Gamma^{\overline{\alpha}}_{\ \overline{\beta}\overline{\gamma}}$ is defined by the coefficients of d–metric (2.20), and by the higher order N–coefficients,

$$\begin{aligned}{}^cL^i_{\ jk} &= \frac{1}{2}g^{in}\left(\delta_k g_{nj} + \delta_j g_{nk} - \delta_n g_{jk}\right), \qquad (2.21)\\ {}^cL^{\overline{a}}_{\ \overline{b}k} &= \delta_{\overline{b}}N^{\overline{a}}_k + \frac{1}{2}h^{\overline{a}\overline{c}}\left(\partial_k h_{\overline{b}\overline{c}} - h_{\overline{d}\overline{c}}\delta_{\overline{b}}N^{\overline{d}}_k - h_{\overline{d}\overline{b}}\delta_{\overline{c}}N^{\overline{d}}_k\right), \\ {}^cC^i_{\ j\overline{c}} &= \frac{1}{2}g^{ik}\delta_{\overline{c}}g_{jk}, \\ {}^cC^{\overline{a}}_{\ \overline{b}\overline{c}} &= \frac{1}{2}h^{\overline{a}\overline{d}}\left(\delta_{\overline{c}}h_{\overline{d}\overline{b}} + \delta_{\overline{b}}h_{\overline{d}\overline{c}} - \delta_{\overline{d}}h_{\overline{b}\overline{c}}\right), \\ {}^cK^{a_p}_{\ \ b_p c_p} &= \frac{1}{2}g^{a_p e_p}\left(\delta_{c_p}g_{e_p b_p} + \delta_{b_p}g_{e_p c_p} - \delta_{e_p}g_{b_p c_p}\right), \\ {}^cK^{a_p}_{\ \ b_s e_f} &= \delta_{b_s}N^{a_p}_{e_f} + \frac{1}{2}h^{a_p c_p}\left(\partial_{e_f}h_{b_s c_p} - h_{d_p c_p}\delta_{b_s}N^{d_p}_{e_f} - h_{d_s b_s}\delta_{c_p}N^{d_s}_{e_f}\right), \\ {}^cQ^{a_f}_{\ \ b_f c_p} &= \frac{1}{2}h^{a_f e_f}\delta_{c_p}h_{b_f e_f},\end{aligned}$$

where $f < p, s$. They transform into usual Christoffel symbols with respect to a coordinate base.

2.3.3 Ha–torsions and ha–curvatures

For a higher order anisotropic d–connection (2.19) the components of torsion,

$$\begin{aligned} T\left(\delta_{\overline{\gamma}}, \delta_{\overline{\beta}}\right) &= T^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}} \delta_{\overline{\alpha}}, \\ T^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}} &= \Gamma^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}} - \Gamma^{\overline{\alpha}}_{\overline{\gamma}\overline{\beta}} + w^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}} \end{aligned}$$

are expressed via d–torsions

$$\begin{aligned} T^{i}_{.jk} &= -T^{i}_{.kj} = L^{i}_{jk} - L^{i}_{kj}, \quad T^{i}_{j\overline{a}} = -T^{i}_{\overline{a}j} = C^{i}_{.j\overline{a}}, \\ T^{i}_{.\overline{a}\overline{b}} &= 0, \quad T^{\overline{a}}_{.\overline{b}\overline{c}} = S^{\overline{a}}_{.\overline{b}\overline{c}} = C^{\overline{a}}_{\overline{b}\overline{c}} - C^{\overline{a}}_{\overline{c}\overline{b}}, \\ T^{\overline{a}}_{.ij} &= -\Omega^{\overline{a}}_{ij}, \quad T^{\overline{a}}_{.\overline{b}i} = -T^{\overline{a}}_{.\overline{b}i} = \delta_{\overline{b}} N^{\overline{a}}_{i} - L^{\overline{a}}_{.\overline{b}j}, \\ T^{a_f}_{.b_f c_f} &= -T^{a_f}_{.c_f b_f} = K^{a_f}_{.b_f c_f} - K^{a_f}_{.c_f b_f}, \\ T^{a_f}_{.a_p b_s} &= 0, \quad T^{a_f}_{.b_f a_p} = -T^{a_f}_{.a_p b_f} = Q^{a_f}_{.b_f a_p}, \\ T^{a_p}_{.a_f b_f} &= -\Omega^{a_p}_{.a_f b_f}, \quad T^{a_p}_{.b_s a_f} = -T^{a_p}_{.a_f b_s} = \delta_{b_s} N^{a_p}_{a_f} - K^{a_p}_{.b_s a_f}. \end{aligned} \tag{2.22}$$

We note that for symmetric linear connections the d–torsion is induced as a pure anholonomic effect.

In a similar manner, putting non–vanishing coefficients (2.6) into the formula for curvature,

$$\begin{aligned} & R\left(\delta_{\overline{\tau}}, \delta_{\overline{\gamma}}\right) \delta_{\overline{\beta}} = R_{\overline{\beta}\ \overline{\gamma}\overline{\tau}}^{\ \overline{\alpha}} \delta_{\overline{\alpha}}, \\ & R_{\overline{\beta}\ \overline{\gamma}\overline{\tau}}^{\ \overline{\alpha}} = \delta_{\overline{\tau}} \Gamma^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}} - \delta_{\overline{\gamma}} \Gamma^{\overline{\alpha}}_{\overline{\beta}\overline{\delta}} + \Gamma^{\overline{\varphi}}_{\overline{\beta}\overline{\gamma}} \Gamma^{\overline{\alpha}}_{\overline{\varphi}\overline{\tau}} - \Gamma^{\overline{\varphi}}_{\overline{\beta}\overline{\tau}} \Gamma^{\overline{\alpha}}_{\overline{\varphi}\overline{\gamma}} + \Gamma^{\overline{\alpha}}_{\overline{\beta}\overline{\varphi}} w^{\overline{\varphi}}_{\overline{\gamma}\overline{\tau}}, \end{aligned}$$

we can compute the components of d–curvatures

$$\begin{aligned} R^{.i}_{h.jk} &= \delta_k L^{i}_{.hj} - \delta_j L^{i}_{.hk} + L^{m}_{.hj} L^{i}_{mk} - L^{m}_{.hk} L^{i}_{mj} - C^{i}_{.h\overline{a}} \Omega^{\overline{a}}_{.jk}, \\ R^{.\overline{a}}_{\overline{b}.jk} &= \delta_k L^{\overline{a}}_{.\overline{b}j} - \delta_j L^{\overline{a}}_{.\overline{b}k} + L^{\overline{c}}_{.\overline{b}j} L^{\overline{a}}_{.\overline{c}k} - L^{\overline{c}}_{.\overline{b}k} L^{\overline{a}}_{.\overline{c}j} - C^{\overline{a}}_{.\overline{b}\overline{c}} \Omega^{\overline{c}}_{.jk}, \\ P^{.i}_{j.k\overline{a}} &= \partial_k L^{i}_{.jk} + C^{i}_{.j\overline{b}} T^{\overline{b}}_{.k\overline{a}} - (\delta_k C^{i}_{.j\overline{a}} + L^{i}_{.lk} C^{l}_{.j\overline{a}} - L^{l}_{.jk} C^{i}_{.l\overline{a}} - L^{\overline{c}}_{.\overline{a}k} C^{i}_{.j\overline{c}}), \\ P^{.\overline{c}}_{\overline{b}.k\overline{a}} &= \delta_{\overline{a}} L^{\overline{c}}_{.\overline{b}k} + C^{\overline{c}}_{.\overline{b}\overline{d}} T^{\overline{d}}_{.k\overline{a}} - (\delta_k C^{\overline{c}}_{.\overline{b}\overline{a}} + L^{\overline{c}}_{.\overline{d}k} C^{\overline{d}}_{.\overline{b}\overline{a}} - L^{\overline{d}}_{.\overline{b}k} C^{\overline{c}}_{.\overline{d}\overline{a}} - L^{\overline{d}}_{.\overline{a}k} C^{\overline{c}}_{.\overline{b}\overline{d}}), \\ S^{.i}_{j.\overline{b}\overline{c}} &= \delta_{\overline{c}} C^{i}_{.j\overline{b}} - \delta_{\overline{b}} C^{i}_{.j\overline{c}} + C^{h}_{.j\overline{b}} C^{i}_{.h\overline{c}} - C^{h}_{.j\overline{c}} C^{i}_{h\overline{b}}, \\ S^{.\overline{a}}_{\overline{b}.\overline{c}\overline{d}} &= \delta_{\overline{d}} C^{\overline{a}}_{.\overline{b}\overline{c}} - \delta_{\overline{c}} C^{\overline{a}}_{.\overline{b}\overline{d}} + C^{\overline{e}}_{.\overline{b}\overline{c}} C^{\overline{a}}_{.\overline{e}\overline{d}} - C^{\overline{e}}_{.\overline{b}\overline{d}} C^{\overline{a}}_{.\overline{e}\overline{c}}, \end{aligned} \tag{2.23}$$

$$
\begin{aligned}
W^{.a_f}_{b_f.c_fe_f} &= \delta_{e_f} K^{a_f}_{.b_fc_f} - \delta_{c_f} K^{a_f}_{.b_fe_f} + K^{h_f}_{.b_fc_f} K^{a_f}_{h_fe_f} \\
& -K^{h_f}_{.b_fe_f} K^{a_f}_{h_fc_f} - Q^{a_f}_{.b_fa_p} \Omega^{a_p}_{.c_fe_f}, \\
W^{.a_p}_{b_s.c_fe_f} &= \delta_{e_f} K^{a_p}_{.b_sc_f} - \delta_{c_f} K^{a_p}_{.b_se_f} + K^{c_p}_{.b_sc_f} K^{a_p}_{.c_pe_f} \\
& -K^{c_p}_{.b_se_f} L^{a_p}_{.c_pc_f} - K^{a_p}_{.b_sc_p} \Omega^{c_p}_{.c_fe_f}, \\
Z^{.a_f}_{b_f.c_fe_f} &= \partial_{e_p} K^{a_f}_{.b_fc_f} + Q^{a_f}_{.b_fb_p} T^{b_p}_{.c_fe_p} \\
& -(\delta_{c_f} Q^{a_f}_{.b_fe_p} + K^{a_f}_{.h_fc_f} Q^{h_f}_{.b_fc_p} - K^{h_f}_{.b_fc_f} Q^{a_f}_{.h_fe_p} - K^{c_p}_{.e_pc_f} C^{a_f}_{.b_fc_p}), \\
Z^{.c_s}_{b_r.c_fe_p} &= \delta_{e_p} K^{c_s}_{.b_rc_f} + K^{c_s}_{.b_rd_f} T^{d_f}_{.c_fe_p} \\
& -(\delta_{c_f} C^{c_s}_{.b_re_p} + K^{c_s}_{.d_fc_f} C^{d_f}_{.b_re_p} - K^{d_t}_{.b_rc_f} C^{c_s}_{.d_te_p} - K^{d_t}_{.e_pc_f} C^{c_s}_{.b_rd_t}), \\
Y^{.a_f}_{b_f.c_pe_p} &= \delta_{e_p} Q^{a_f}_{.b_fc_p} - \delta_{c_p} Q^{a_f}_{.b_fe_p} + Q^{d_f}_{.b_fc_p} Q^{a_f}_{.d_fe_p} - Q^{d_f}_{.b_fe_p} Q^{a_f}_{d_fc_p}.
\end{aligned}
$$

where $f < p, s, r, t$.

2.3.4 Einstein equations with respect to ha–frames

The Ricci tensor

$$R_{\overline{\beta}\overline{\gamma}} = R_{\overline{\beta}\ \overline{\gamma}\overline{\alpha}}^{\ \overline{\alpha}}$$

has the d–components

$$
\begin{aligned}
R_{ij} &= R^{.k}_{i.jk}, \quad R_{i\overline{a}} = -{}^2P_{i\overline{a}} = -P^{.k}_{i.k\overline{a}}, \\
R_{\overline{a}i} &= {}^1P_{\overline{a}i} = P^{.\overline{b}}_{\overline{a}.i\overline{b}}, \quad R_{\overline{a}\overline{b}} = S^{.\overline{c}}_{\overline{a}.\overline{b}\overline{c}} \\
R_{b_fc_f} &= W^{.a_f}_{b_f.c_fa_f}, \quad R_{e_pb_f} = -{}^2P_{b_fe_p} = -Z^{.a_f}_{b_f.a_fe_p}, \\
R_{b_rc_f} &= {}^1P_{b_rc_f} = Z^{.e_s}_{b_r.c_fe_s}.
\end{aligned}
\tag{2.24}
$$

The Ricci d-tensor is non symmetric.

If a higher order d-metric of type (2.20) is defined in $V^{(\overline{n})}$, we can compute the scalar curvature

$$\overline{R} = g^{\overline{\beta}\overline{\gamma}} R_{\overline{\beta}\overline{\gamma}}.$$

of a d-connection D,

$$\overline{R} = \widehat{R} + \overline{S}, \tag{2.25}$$

where $\widehat{R} = g^{ij} R_{ij}$ and $\overline{S} = h^{\overline{a}\overline{b}} S_{\overline{a}\overline{b}}$.

The h-v parametrization of the gravitational field equations in ha–spacetimes is obtained by introducing the values (2.24) and (2.25) into the Einstein's equations

$$R_{\overline{\beta}\overline{\gamma}} - \frac{1}{2} g_{\overline{\beta}\overline{\gamma}} \overline{R} = k \Upsilon_{\overline{\beta}\overline{\gamma}},$$

and written

$$\begin{aligned} R_{ij} - \frac{1}{2}\left(\widehat{R} + \overline{S}\right) g_{ij} &= k\Upsilon_{ij}, \\ S_{\overline{a}\overline{b}} - \frac{1}{2}\left(\widehat{R} + \overline{S}\right) h_{\overline{a}\overline{b}} &= k\Upsilon_{\overline{a}\overline{b}}, \\ {}^1P_{\overline{a}i} = k\Upsilon_{\overline{a}i}, \quad {}^1P_{a_p b_f} &= k\Upsilon_{a_p b_f} \\ {}^2P_{i\overline{a}} = -k\Upsilon_{i\overline{a}}, \quad {}^2P_{a_s b_f} &= -k\Upsilon_{a_f b_p}, \end{aligned} \tag{2.26}$$

where $\Upsilon_{ij}, \Upsilon_{\overline{a}\overline{b}}, \Upsilon_{\overline{a}i}, \Upsilon_{i\overline{a}}, \Upsilon_{a_p b_f}, \Upsilon_{a_f b_p}$ are the h-v–components of the energy–momentum d–tensor field $\Upsilon_{\overline{\beta}\overline{\gamma}}$ (which includes possible cosmological constants, contributions of anholonomy d–torsions (2.22) and matter) and k is the coupling constant.

We note that, in general, the ha–torsions are not vanishing. Nevetheless, for a (pseudo)–Riemannian spacetime with induced anholonomic anisotropies it is not necessary to consider an additional to (2.26) system of equations for torsion becouse in this case the torsion structure is an anholonomic effect wich becames trivial with respect to holonomic frames of reference.

If a ha–spacetime structure is associated to a generic nonzero torsion, we could consider additionally, for instance, as in [186], a system of algebraic d–field equations with a source $S^{\overline{\alpha}}_{\ \overline{\beta}\overline{\gamma}}$ for a locally anisotropic spin density of matter (if we construct a variant of higher order anisotropic Einstein–Cartan theory):

$$T^{\overline{\gamma}}_{\ \overline{\alpha}\overline{\beta}} + 2\delta^{\overline{\gamma}}_{\ [\overline{\alpha}} T^{\overline{\delta}}_{\ \overline{\beta}]\overline{\delta}} = \kappa S^{\overline{\gamma}}_{\ \overline{\alpha}\overline{\beta}.}$$

In a more general case we have to introduce some new constraints and/or dynamical equations for torsions and nonlinear connections which are induced from (super) string theory and/ or higher order anisotropic supergravity [170, 171]. Two variants of gauge dynamical field equations with both frame like and torsion variables will be considered in the Section 5 and 6 of this paper.

2.4 Gauge Fields on Ha–Spaces

This section is devoted to gauge field theories on spacetimes provided with higher order anisotropic anholonomic frame structures.

2.4.1 Bundles on ha–spaces

Let us consider a principal bundle $\left(\mathcal{P},\pi,Gr,V^{(\overline{n})}\right)$ over a ha–spacetime $V^{(\overline{n})}$ ($\mathcal{P}$ and $V^{(\overline{n})}$ are called respectively the base and total spaces) with the structural group Gr and surjective map $\pi:\mathcal{P}\rightarrow V^{(\overline{n})}$ (on geometry of bundle spaces see, for instance, [35, 109, 132]). At every point $u=(x,y_{(1)},...,y_{(z)})\in V^{(\overline{n})}$ there is a vicinity $\mathcal{U}\subset V^{(\overline{n})},u\in\mathcal{U}$, with trivializing $\mathcal{P}$ diffeomorphisms f and φ :

$$\begin{aligned} f_{\mathcal{U}}:\ \pi^{-1}(\mathcal{U}) &\rightarrow \mathcal{U}\times Gr, \qquad f(p)=(\pi(p),\varphi(p)), \\ \varphi_{\mathcal{U}}:\ \pi^{-1}(\mathcal{U}) &\rightarrow Gr, \varphi(pq)=\varphi(p)\,q \end{aligned}$$

for every group element $q\in Gr$ and point $p\in\mathcal{P}$. We remark that in the general case for two open regions

$$\mathcal{U},\mathcal{V}\subset V^{(\overline{n})},\mathcal{U}\cap\mathcal{V}\neq\emptyset, f_{\mathcal{U}|_p}\neq f_{\mathcal{V}|_p},\ \text{even } p\in\mathcal{U}\cap\mathcal{V}.$$

Transition functions $g_{\mathcal{U}\mathcal{V}}$ are defined

$$g_{\mathcal{U}\mathcal{V}}:\mathcal{U}\cap\mathcal{V}\rightarrow Gr, g_{\mathcal{U}\mathcal{V}}(u)=\varphi_{\mathcal{U}}(p)\left(\varphi_{\mathcal{V}}(p)^{-1}\right),\pi(p)=u.$$

Hereafter we shall omit, for simplicity, the specification of trivializing regions of maps and denote, for example, $f\equiv f_{\mathcal{U}},\varphi\equiv\varphi_{\mathcal{U}}$, $s\equiv s_{\mathcal{U}}$, if this will not give rise to ambiguities.

Let θ be the canonical left invariant 1-form on Gr with values in algebra Lie $\mathcal{G}$ of group Gr uniquely defined from the relation $\theta(q)=q$, for every $q\in\mathcal{G}$, and consider a 1-form ω on $\mathcal{U}\subset V^{(\overline{n})}$ with values in $\mathcal{G}$. Using θ and ω, we can locally define the connection form Θ in $\mathcal{P}$ as a 1-form:

$$\Theta=\varphi^{*}\theta+Ad\ \varphi^{-1}\left(\pi^{*}\omega\right) \tag{2.27}$$

where $\varphi^{*}\theta$ and $\pi^{*}\omega$ are, respectively, 1–forms induced on $\pi^{-1}(\mathcal{U})$ and $\mathcal{P}$ by maps φ and π and $\omega=s^{*}\Theta$. The adjoint action on a form λ with values in $\mathcal{G}$ is defined as

$$\left(Ad\ \varphi^{-1}\lambda\right)_p=\left(Ad\ \varphi^{-1}(p)\right)\lambda_p$$

where λ_p is the value of form λ at point $p\in\mathcal{P}$.

Introducing a basis $\{\Delta_{\widehat{a}}\}$ in $\mathcal{G}$ (index $\widehat{a}$ enumerates the generators making up this basis), we write the 1-form ω on $V^{(\overline{n})}$ as

$$\omega=\Delta_{\widehat{a}}\omega^{\widehat{a}}(u),\ \omega^{\widehat{a}}(u)=\omega_{\overline{\mu}}^{\widehat{a}}(u)\,\delta u^{\overline{\mu}} \tag{2.28}$$

where $\delta u^{\overline{\mu}} = (dx^i, \delta y^{\overline{a}})$ and the Einstein summation rule on indices $\widehat{a}$ and $\overline{\mu}$ is used. Functions $\omega_{\overline{\mu}}^{\widehat{a}}(u)$ from (2.28) are called the components of Yang-Mills fields on ha-spacetime $V^{(\overline{n})}$. Gauge transforms of ω can be interpreted as transition relations for $\omega_{\mathcal{U}}$ and $\omega_{\mathcal{V}}$, when $u \in \mathcal{U} \cap \mathcal{V}$,

$$\left(\omega_{\mathcal{U}}\right)_u = \left(g_{\mathcal{U}\mathcal{V}}^{*}\theta\right)_u + Ad\ g_{\mathcal{U}\mathcal{V}}\left(u\right)^{-1}\left(\omega_{\mathcal{V}}\right)_u . \tag{2.29}$$

To relate $\omega_{\overline{\mu}}^{\widehat{a}}$ with a covariant derivation we shall consider a vector bundle Ξ associated to $\mathcal{P}$. Let $\rho : Gr \to GL(\mathcal{R}^s)$ and $\rho' : \mathcal{G} \to End(E^s)$ be, respectively, linear representations of group Gr and Lie algebra $\mathcal{G}$ (where $\mathcal{R}$ is the real number field). Map ρ defines a left action on Gr and associated vector bundle $\Xi = P \times \mathcal{R}^s/Gr,\ \pi_E : E \to V^{(\overline{n})}$. Introducing the standard basis $\xi_{\underline{i}} = \{\xi_{\underline{1}}, \xi_{\underline{2}}, ..., \xi_{\underline{s}}\}$ in $\mathcal{R}^s$, we can define the right action on $P\times \mathcal{R}^s, ((p,\xi)\,q = (pq, \rho(q^{-1})\,\xi)\,, q \in Gr)$, the map induced from $\mathcal{P}$

$$p : \mathcal{R}^s \to \pi_E^{-1}(u)\,, \quad (p(\xi) = (p\xi)\,Gr, \xi \in \mathcal{R}^s, \pi(p) = u)$$

and a basis of local sections $e_{\underline{i}} : U \to \pi_E^{-1}(U)\,,\ e_{\underline{i}}(u) = s(u)\,\xi_{\underline{i}}$. Every section $\varsigma : V^{(\overline{n})} \to \Xi$ can be written locally as $\varsigma = \varsigma^i e_i, \varsigma^i \in C^{\infty}(\mathcal{U})$. To every vector field X on $V^{(\overline{n})}$ and Yang-Mills field $\omega^{\widehat{a}}$ on $\mathcal{P}$ we associate operators of covariant derivations:

$$\bigtriangledown_X \zeta = e_{\underline{i}}\left[X\zeta^{\underline{i}} + B\left(X\right)_{\underline{j}}^{\underline{i}}\zeta^{\underline{j}}\right],\ B\left(X\right) = \left(\rho' X\right)_{\widehat{a}}\omega^{\widehat{a}}\left(X\right). \tag{2.30}$$

The transform (2.29) and operators (2.30) are inter–related by these transition transforms for values $e_{\underline{i}}, \zeta^{\underline{i}}$, and $B_{\overline{\mu}}$:

$$\begin{aligned} e_{\underline{i}}^{\mathcal{V}}(u) &= [\rho g_{\mathcal{U}\mathcal{V}}(u)]_{\underline{i}}^{\underline{j}}\, e_{\underline{i}}^{\mathcal{U}},\ \zeta_{\mathcal{U}}^{\underline{i}}(u) = [\rho g_{\mathcal{U}\mathcal{V}}(u)]_{\underline{i}}^{\underline{j}}\, \zeta_{\mathcal{V}}^{\underline{i}}, \\ B_{\overline{\mu}}^{\mathcal{V}}(u) &= [\rho g_{\mathcal{U}\mathcal{V}}(u)]^{-1}\,\delta_{\overline{\mu}}\,[\rho g_{\mathcal{U}\mathcal{V}}(u)] + [\rho g_{\mathcal{U}\mathcal{V}}(u)]^{-1}\, B_{\overline{\mu}}^{\mathcal{U}}(u)\,[\rho g_{\mathcal{U}\mathcal{V}}(u)]\,, \end{aligned} \tag{2.31}$$

where $B_{\overline{\mu}}^{\mathcal{U}}(u) = B^{\overline{\mu}}(\delta/du^{\overline{\mu}})(u)$.

Using (2.31), we can verify that the operator $\bigtriangledown_X^{\mathcal{U}}$, acting on sections of $\pi_\Xi : \Xi \to V^{(\overline{n})}$ according to definition (2.30), satisfies the properties

$$\begin{aligned} \bigtriangledown_{f_1 X + f_2 Y}^{\mathcal{U}} &= f_1 \bigtriangledown_X^{\mathcal{U}} + f_2 \bigtriangledown_X^{\mathcal{U}},\ \ \bigtriangledown_X^{\mathcal{U}}(f\zeta) = f \bigtriangledown_X^{\mathcal{U}}\zeta + (Xf)\,\zeta, \\ \bigtriangledown_X^{\mathcal{U}}\zeta &= \bigtriangledown_X^{\mathcal{V}}\zeta, \quad u \in \mathcal{U} \cap \mathcal{V}, f_1, f_2 \in C^{\infty}(\mathcal{U}). \end{aligned}$$

So, we can conclude that the Yang–Mills connection in the vector bundle $\pi_\Xi : \Xi \to V^{(\overline{n})}$ is not a general one, but is induced from the principal bundle $\pi : \mathcal{P} \to V^{(\overline{n})}$ with structural group Gr.

The curvature $\mathcal{K}$ of connection Θ from (2.27) is defined as

$$\mathcal{K} = D\Theta,\ D = \widehat{H} \circ d \tag{2.32}$$

where d is the operator of exterior derivation acting on $\mathcal{G}$-valued forms as

$$d\left(\Delta_{\widehat{a}} \otimes \chi^{\widehat{a}}\right) = \Delta_{\widehat{a}} \otimes d\chi^{\widehat{a}}$$

and $\widehat{H}$ is the horizontal projecting operator acting, for example, on the 1-form λ as $\left(\widehat{H}\lambda\right)_P (X_p) = \lambda_p (H_p X_p)$, where H_p projects on the horizontal subspace

$$\mathcal{H}_p \in P_p \left[X_p \in \mathcal{H}_p \text{ is equivalent to } \Theta_p (X_p) = 0\right].$$

We can express (2.32) locally as

$$\mathcal{K} = Ad\ \varphi_{\mathcal{U}}^{-1} (\pi^* \mathcal{K}_{\mathcal{U}}) \tag{2.33}$$

where

$$\mathcal{K}_{\mathcal{U}} = d\omega_{\mathcal{U}} + \frac{1}{2}\left[\omega_{\mathcal{U}}, \omega_{\mathcal{U}}\right]. \tag{2.34}$$

The exterior product of $\mathcal{G}$-valued form (2.34) is defined as

$$\left[\Delta_{\widehat{a}} \otimes \lambda^{\widehat{a}}, \Delta_{\widehat{b}} \otimes \xi^{\widehat{b}}\right] = \left[\Delta_{\widehat{a}}, \Delta_{\widehat{b}}\right] \otimes \lambda^{\widehat{a}} \bigwedge \xi^{\widehat{b}},$$

where the anti–symmetric tensorial product is denoted $\lambda^{\widehat{a}} \bigwedge \xi^{\widehat{b}} = \lambda^{\widehat{a}}\xi^{\widehat{b}} - \xi^{\widehat{b}}\lambda^{\widehat{a}}$.

Introducing structural coefficients $f_{\widehat{b}\widehat{c}}^{\quad \widehat{a}}$ of $\mathcal{G}$ satisfying

$$\left[\Delta_{\widehat{b}}, \Delta_{\widehat{c}}\right] = f_{\widehat{b}\widehat{c}}^{\quad \widehat{a}} \Delta_{\widehat{a}}$$

we can rewrite (2.34) in a form more convenient for local considerations:

$$\mathcal{K}_{\mathcal{U}} = \Delta_{\widehat{a}} \otimes \mathcal{K}_{\overline{\mu}\overline{\nu}}^{\widehat{a}} \delta u^{\overline{\mu}} \bigwedge \delta u^{\overline{\nu}} \tag{2.35}$$

where

$$\mathcal{K}_{\overline{\mu}\overline{\nu}}^{\widehat{a}} = \frac{\delta \omega_{\overline{\nu}}^{\widehat{a}}}{\partial u^{\overline{\mu}}} - \frac{\delta \omega_{\overline{\mu}}^{\widehat{a}}}{\partial u^{\overline{\nu}}} + \frac{1}{2} f_{\widehat{b}\widehat{c}}^{\quad \widehat{a}} \left(\omega_{\overline{\mu}}^{\widehat{b}} \omega_{\overline{\nu}}^{\widehat{c}} - \omega_{\overline{\nu}}^{\widehat{b}} \omega_{\overline{\mu}}^{\widehat{c}}\right).$$

This subsection ends by considering the problem of reduction of the local anisotropic gauge symmetries and gauge fields to isotropic ones. For local trivial considerations we can consider that with respect to holonomic frames the higher order anisotropic Yang-Mills fields reduce to usual ones on (pseudo) Riemannian spaces.

2.4.2 Yang-Mills equations on ha-spaces

Interior gauge symmetries are associated to semisimple structural groups. On the principal bundle $\left(\mathcal{P},\pi,Gr,V^{(\overline{n})}\right)$ with nondegenerate Killing form for semisimple group Gr we can define the generalized bundle metric

$$h_p\left(X_p,Y_p\right)=G_{\pi(p)}\left(d\pi_P X_P, d\pi_P Y_P\right)+K\left(\Theta_P\left(X_P\right),\Theta_P\left(X_P\right)\right), \qquad (2.36)$$

where $d\pi_P$ is the differential of map $\pi:\mathcal{P}\to V^{(\overline{n})}$, $G_{\pi(p)}$ is locally generated as the ha-metric (2.20), and K is the Killing form on $\mathcal{G}$:

$$K\left(\Delta_{\widehat{a}},\Delta_{\widehat{b}}\right)=f_{\widehat{b}\widehat{d}}^{\ \ \widehat{c}} f_{\widehat{a}\widehat{c}}^{\ \ \widehat{d}}=K_{\widehat{a}\widehat{b}}.$$

Using the metric $g_{\overline{\alpha}\overline{\beta}}$ on $V^{(\overline{n})}$ (respectively, $h_P\left(X_P,Y_P\right)$ on $\mathcal{P}$) we can introduce operators $*_G$ and $\widehat{\delta}_G$ acting in the space of forms on $V^{(\overline{n})}$ ($*_H$ and $\widehat{\delta}_H$ acting on forms on $\mathcal{P}$)). Let $e_{\overline{\mu}}$ be an orthonormalized frame on $\mathcal{U}\subset V^{(\overline{n})}$, locally adapted to the N–connection structure, i. .e. being related via some local distinguisherd linear transforms to a ha–frame (2.15) and $e^{\overline{\mu}}$ be the adjoint coframe. Locally

$$G=\sum_{\overline{\mu}}\eta\left(\overline{\mu}\right)e^{\overline{\mu}}\otimes e^{\overline{\mu}},$$

where $\eta_{\overline{\mu}\overline{\mu}}=\eta\left(\overline{\mu}\right)=\pm 1$, $\overline{\mu}=1,2,...,\overline{n}$, and the Hodge operator $*_G$ can be defined as $*_G:\Lambda'\left(V^{(\overline{n})}\right)\to\Lambda^{\overline{n}}\left(V^{(\overline{n})}\right)$, or, in explicit form, as

$$*_G\left(e^{\overline{\mu}_1}\bigwedge ...\bigwedge e^{\overline{\mu}_r}\right)=\eta\left(\overline{\nu}_1\right)...\eta\left(\overline{\nu}_{\overline{n}-r}\right)\times \qquad (2.37)$$

$$sign\begin{pmatrix} 1 & 2 & \dots & r & r+1 & \dots & \overline{n} \\ \overline{\mu}_1 & \overline{\mu}_2 & \dots & \overline{\mu}_r & \overline{\nu}_1 & \dots & \overline{\nu}_{\overline{n}-r}\end{pmatrix} e^{\overline{\nu}_1}\bigwedge ...\bigwedge e^{\overline{\nu}_{\overline{n}-r}}.$$

Next, we define the operator

$$*_G^{-1}=\eta\left(1\right)...\eta\left(\overline{n}\right)\left(-1\right)^{r(\overline{n}-r)}*_G$$

and introduce the scalar product on forms $\beta_1,\beta_2,...\subset\Lambda^r\left(V^{(\overline{n})}\right)$ with compact carrier:

$$\left(\beta_1,\beta_2\right)=\eta\left(1\right)...\eta\left(n_E\right)\int\beta_1\bigwedge *_G\beta_2.$$

The operator $\widehat{\delta}_G$ is defined as the adjoint to d associated to the scalar product for forms, specified for r-forms as

$$\widehat{\delta}_G=\left(-1\right)^r *_G^{-1}\circ d\circ *_G. \qquad (2.38)$$

We remark that operators $*_H$ and δ_H acting in the total space of $\mathcal{P}$ can be defined similarly to (2.37) and (2.38), but by using metric (2.36). Both these operators also act in the space of $\mathcal{G}$-valued forms:

$$* \left(\Delta_{\widehat{a}} \otimes \varphi^{\widehat{a}}\right) = \Delta_{\widehat{a}} \otimes (*\varphi^{\widehat{a}}),$$

$$\widehat{\delta} \left(\Delta_{\widehat{a}} \otimes \varphi^{\widehat{a}}\right) = \Delta_{\widehat{a}} \otimes (\widehat{\delta}\varphi^{\widehat{a}}).$$

The form λ on $\mathcal{P}$ with values in $\mathcal{G}$ is called horizontal if $\widehat{H}\lambda = \lambda$ and equivariant if $R^* (q) \lambda = Ad\ q^{-1}\varphi,\ \forall g \in Gr, R(q)$ being the right shift on $\mathcal{P}$. We can verify that equivariant and horizontal forms also satisfy the conditions

$$\lambda = Ad\ \varphi_{\mathcal{U}}^{-1} (\pi^* \lambda), \qquad \lambda_{\mathcal{U}} = S_{\mathcal{U}}^* \lambda,$$

$$(\lambda_{\mathcal{V}})_{\mathcal{U}} = Ad\ \ (g_{\mathcal{UV}} (u))^{-1} (\lambda_{\mathcal{U}})_u .$$

Now, we can define the field equations for curvature (2.33) and connection (2.27):

$$\Delta \mathcal{K} = 0, \tag{2.39}$$

$$\bigtriangledown \mathcal{K} = 0, \tag{2.40}$$

where $\Delta = \widehat{H} \circ \widehat{\delta}_H$. Equations (2.39) are similar to the well-known Maxwell equations and for non-Abelian gauge fields are called Yang-Mills equations. The structural equations (2.40) are called the Bianchi identities.

The field equations (2.39) do not have a physical meaning because they are written in the total space of the bundle Ξ and not on the base anisotropic spacetime $V^{(\overline{n})}$. But this difficulty may be obviated by projecting the mentioned equations on the base. The 1-form $\Delta\mathcal{K}$ is horizontal by definition and its equivariance follows from the right invariance of metric (2.36). So, there is a unique form $(\Delta\mathcal{K})_{\mathcal{U}}$ satisfying

$$\Delta\mathcal{K} = Ad\ \varphi_{\mathcal{U}}^{-1}\pi^*(\Delta\mathcal{K})_{\mathcal{U}}.$$

The projection of (2.39) on the base can be written as $(\Delta\mathcal{K})_{\mathcal{U}} = 0$. To calculate $(\Delta\mathcal{K})_{\mathcal{U}}$, we use the equality [35, 133]

$$d \left(Ad\ \varphi_{\mathcal{U}}^{-1}\lambda\right) = Ad\ \ \varphi_{\mathcal{U}}^{-1}\ d\lambda - \left[\varphi_{\mathcal{U}}^*\theta, Ad\ \varphi_{\mathcal{U}}^{-1}\lambda\right]$$

where λ is a form on $\mathcal{P}$ with values in $\mathcal{G}$. For r-forms we have

$$\widehat{\delta} \left(Ad\ \varphi_{\mathcal{U}}^{-1}\lambda\right) = Ad\ \varphi_{\mathcal{U}}^{-1}\widehat{\delta}\lambda - (-1)^r *_H \{\left[\varphi_{\mathcal{U}}^*\theta, *_H Ad\ \varphi_{\mathcal{U}}^{-1}\lambda\right]$$

and, as a consequence,

$$\widehat{\delta}\mathcal{K} = Ad\ \varphi_{\mathcal{U}}^{-1}\{\widehat{\delta}_H\pi^*\mathcal{K}_{\mathcal{U}} + *_H^{-1}[\pi^*\omega_{\mathcal{U}}, *_H\pi^*\mathcal{K}_{\mathcal{U}}]\} - *_H^{-1}\left[\Theta, Ad\ \varphi_{\mathcal{U}}^{-1} *_H (\pi^*\mathcal{K})\right]. \tag{2.41}$$

By using straightforward calculations in the adapted dual basis on $\pi^{-1}(\mathcal{U})$ we can verify the equalities

$$\left[\Theta, Ad\ \varphi_{\mathcal{U}}^{-1}\ *_H(\pi^*\mathcal{K}_{\mathcal{U}})\right] = 0, \widehat{H}\delta_H(\pi^*\mathcal{K}_{\mathcal{U}}) = \pi^*\left(\widehat{\delta}_G\mathcal{K}\right), \tag{2.42}$$

$$*_H^{-1}\left[\pi^*\omega_{\mathcal{U}}, *_H(\pi^*\mathcal{K}_{\mathcal{U}})\right] = \pi^*\{*_G^{-1}\left[\omega_{\mathcal{U}}, *_G\mathcal{K}_{\mathcal{U}}\right]\}.$$

From (2.41) and (2.42) one follows that

$$(\Delta\mathcal{K})_{\mathcal{U}} = \widehat{\delta}_G\mathcal{K}_{\mathcal{U}} + *_G^{-1}\left[\omega_{\mathcal{U}}, *_G\mathcal{K}_{\mathcal{U}}\right]. \tag{2.43}$$

Taking into account (2.43) and (2.38), we prove that projection on the base of equations (2.39) and (2.40) can be expressed respectively as

$$*_G^{-1}\circ d\circ *_G\mathcal{K}_{\mathcal{U}} + *_G^{-1}\left[\omega_{\mathcal{U}}, *_G\mathcal{K}_{\mathcal{U}}\right] = 0. \tag{2.44}$$

$$d\mathcal{K}_{\mathcal{U}} + [\omega_{\mathcal{U}}, \mathcal{K}_{\mathcal{U}}] = 0.$$

Equations (2.44) (see (2.43)) are gauge–invariant because

$$(\Delta\mathcal{K})_{\mathcal{U}} = Ad\ g_{\mathcal{U}\mathcal{V}}^{-1}(\Delta\mathcal{K})_{\mathcal{V}}.$$

By using formulas (2.35)-(2.38) we can rewrite (2.44) in coordinate form

$$D_{\overline{\nu}}\left(G^{\overline{\nu}\overline{\lambda}}\mathcal{K}^{\widehat{a}}{}_{\overline{\lambda}\overline{\mu}}\right) + f_{\widehat{b}\widehat{c}}{}^{\widehat{a}}g^{\overline{\nu}\overline{\lambda}}\omega_{\overline{\lambda}}^{\widehat{b}}\mathcal{K}^{\widehat{c}}{}_{\overline{\nu}\overline{\mu}} = 0, \tag{2.45}$$

where $D_{\overline{\nu}}$ is a compatible with metric covariant derivation on ha-spacetime (2.45).

We point out that for our bundles with semisimple structural groups the Yang-Mills equations (2.39) (and, as a consequence, their horizontal projections (2.44), or (2.45)) can be obtained by variation of the action

$$I = \int \mathcal{K}^{\widehat{a}}{}_{\overline{\mu}\overline{\nu}}\mathcal{K}^{\widehat{b}}{}_{\overline{\alpha}\overline{\beta}}G^{\overline{\mu}\overline{\alpha}}g^{\overline{\nu}\overline{\beta}}K_{\widehat{a}\widehat{b}}\left|g_{\overline{\alpha}\overline{\beta}}\right|^{1/2}dx^1...dx^n\delta y^1_{(1)}...\delta y^{m_1}_{(1)}...\delta y^1_{(z)}...\delta y^{m_z}_{(z)}. \tag{2.46}$$

Equations for extremals of (2.46) have the form

$$K_{\widehat{r}\widehat{b}} g^{\overline{\lambda}\overline{\alpha}} g^{\overline{\kappa}\overline{\beta}} D_{\overline{\alpha}} \mathcal{K}^{\widehat{b}}{}_{\overline{\lambda}\overline{\beta}} - K_{\widehat{a}\widehat{b}} g^{\overline{\kappa}\overline{\alpha}} g^{\overline{\nu}\overline{\beta}} f_{\widehat{r}\widehat{l}}{}^{\widehat{a}} \omega^{\widehat{l}}_{\overline{\nu}} \mathcal{K}^{\widehat{b}}{}_{\overline{\alpha}\overline{\beta}} = 0,$$

which are equivalent to "pure" geometric equations (2.45) (or (2.44)) due to nondegeneration of the Killing form $K_{\widehat{r}\widehat{b}}$ for semisimple groups.

To take into account gauge interactions with matter fields (sections of vector bundle Ξ on $V^{(\overline{n})}$) we have to introduce a source 1–form $\mathcal{J}$ in equations (2.39) and to write them

$$\Delta\mathcal{K} = \mathcal{J} \tag{2.47}$$

Explicit constructions of $\mathcal{J}$ require concrete definitions of the bundle Ξ; for example, for spinor fields an invariant formulation of the Dirac equations on ha–spaces is necessary. We omit spinor considerations in this paper (see [163, 173]).

2.5 Gauge Ha-gravity

A considerable body of work on the formulation of gauge gravitational models on isotropic spaces is based on application of nonsemisimple groups, for example, of Poincare and affine groups, as structural gauge groups (see critical analysis and original results in [53, 186, 98, 63, 202, 153, 131]). The main impediment to developing such models is caused by the degeneration of Killing forms for nonsemisimple groups, which make it impossible to construct consistent variational gauge field theories (functional (2.46) and extremal equations are degenerate in these cases). There are at least two possibilities to get around the mentioned difficulty. The first is to realize a minimal extension of the nonsemisimple group to a semisimple one, similar to the extension of the Poincare group to the de Sitter group considered in [132, 133, 153]. The second possibility is to introduce into consideration the bundle of adapted affine frames on locally anisotropic space $V^{(\overline{n})}$, to use an auxiliary nondegenerate bilinear form $a_{\widehat{a}\widehat{b}}$ instead of the degenerate Killing form $K_{\widehat{a}\widehat{b}}$ and to consider a "pure" geometric method, illustrated in the previous section, of definition of gauge field equations. Projecting on the base $V^{(\overline{n})}$, we shall obtain gauge gravitational field equations on a ha–space having a form similar to Yang-Mills equations.

The goal of this section is to prove that a specific parametrization of components of the Cartan connection in the bundle of adapted affine frames on $V^{(\overline{n})}$ establishes an equivalence between Yang-Mills equations (2.47) and Einstein equations (2.26) on ha–spaces.

2.5.1 Bundles of linear ha–frames

Let $(X_{\overline{\alpha}})_u = (X_i, X_{\overline{a}})_u = (X_i, X_{a_1}, ..., X_{a_z})_u$ be a frame locally adapted to the N–connection structure at a point $u \in V^{(\overline{n})}$. We consider a local right distinguished action of matrices

$$A_{\overline{\alpha}'}{}^{\overline{\alpha}} = \begin{pmatrix} A_{i'}{}^{i} & 0 & ... & 0 \\ 0 & B_{a_1'}{}^{a_1} & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & B_{a_z'}{}^{a_z} \end{pmatrix} \subset GL_{\overline{n}} =$$

$$GL(n, \mathcal{R}) \oplus GL(m_1, \mathcal{R}) \oplus ... \oplus GL(m_z, \mathcal{R}).$$

Nondegenerate matrices $A_{i'}{}^{i}$ and $B_{j'}{}^{j}$, respectively, transform linearly $X_{i|u}$ into $X_{i'|u} = A_{i'}{}^{i} X_{i|u}$ and $X_{a_p'|u}$ into $X_{a_p'|u} = B_{a_p'}{}^{a_p} X_{a_p|u}$, where $X_{\overline{\alpha}'|u} = A_{\overline{\alpha}'}{}^{\overline{\alpha}} X_{\overline{\alpha}}$ is also an adapted frame at the same point $u \in V^{(\overline{n})}$. We denote by $La\left(V^{(\overline{n})}\right)$ the set of all adapted frames $X_{\overline{\alpha}}$ at all points of $V^{(\overline{n})}$ and consider the surjective map π from $La\left(V^{(\overline{n})}\right)$ to $V^{(\overline{n})}$ transforming every adapted frame $X_{\overline{\alpha}|u}$ and point u into the point u. Every $X_{\overline{\alpha}'|u}$ has a unique representation as $X_{\overline{\alpha}'} = A_{\overline{\alpha}'}{}^{\overline{\alpha}} X_{\overline{\alpha}}^{(0)}$, where $X_{\overline{\alpha}}^{(0)}$ is a fixed distinguished basis in tangent space $T\left(V^{(\overline{n})}\right)$. It is obvious that $\pi^{-1}(\mathcal{U}), \mathcal{U} \subset V^{(\overline{n})}$, is bijective to $\mathcal{U} \times GL_{\overline{n}}(\mathcal{R})$. We can transform $La\left(V^{(\overline{n})}\right)$ in a differentiable manifold taking $\left(u^{\overline{\beta}}, A_{\overline{\alpha}'}{}^{\overline{\alpha}}\right)$ as a local coordinate system on $\pi^{-1}(\mathcal{U})$. Now, it is easy to verify that

$$\mathcal{L}a(V^{(\overline{n})}) = (La(V^{(\overline{n})}), V^{(\overline{n})}, GL_{\overline{n}}(\mathcal{R}))$$

is a principal bundle. We call $\mathcal{L}a(V^{(\overline{n})})$ the bundle of linear adapted frames on $V^{(\overline{n})}$.

The next step is to identify the components of, for simplicity, compatible d-connection $\Gamma^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}}$ on $V^{(\overline{n})}$, with the connection in $\mathcal{L}a(V^{(\overline{n})})$

$$\Theta^{\widehat{a}}_{\mathcal{U}} = \omega^{\widehat{a}} = \{\omega^{\widehat{\alpha}\widehat{\beta}}{}_{\overline{\lambda}} \doteq \Gamma^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}}\}. \quad (2.48)$$

Introducing (2.48) in (2.43), we calculate the local 1-form

$$\left(\Delta\mathcal{R}^{(\Gamma)}\right)_{\mathcal{U}} = \Delta_{\widehat{\alpha}\widehat{\alpha}_1} \otimes (g^{\overline{\nu}\overline{\lambda}} D_{\overline{\lambda}} \mathcal{R}^{\widehat{\alpha}\widehat{\gamma}}{}_{\overline{\nu}\overline{\mu}} + f^{\widehat{\alpha}\widehat{\gamma}}{}_{\widehat{\beta}\widehat{\delta}\widehat{\gamma}\widehat{\varepsilon}} g^{\overline{\nu}\overline{\lambda}} \omega^{\widehat{\beta}\widehat{\delta}}{}_{\lambda} \mathcal{R}^{\widehat{\gamma}\widehat{\varepsilon}}{}_{\overline{\nu}\overline{\mu}}) \delta u^{\overline{\mu}}, \quad (2.49)$$

where

$$\Delta_{\widehat{\alpha}\widehat{\beta}} = \begin{pmatrix} \Delta_{\widehat{i}\widehat{j}} & 0 & ... & 0 \\ 0 & \Delta_{\widehat{a}_1\widehat{b}_1} & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \Delta_{\widehat{a}_z\widehat{b}_z} \end{pmatrix}$$

is the standard distinguished basis in the Lie algebra of matrices $\mathcal{G}l_{\overline{n}}(\mathcal{R})$ with $(\Delta_{\widehat{i}\widehat{k}})_{jl} = \delta_{ij}\delta_{kl}$ and $\left(\Delta_{\widehat{a}_p\widehat{c}_p}\right)_{b_p d_p} = \delta_{a_p b_p}\delta_{c_p d_p}$ defining the standard bases in $\mathcal{G}l\left(\mathcal{R}^{\overline{n}}\right)$. We have denoted the curvature of connection (2.48), considered in (2.49), as

$$\mathcal{R}_{\mathcal{U}}^{(\Gamma)} = \Delta_{\widehat{\alpha}\widehat{\alpha}_1} \otimes \mathcal{R}^{\widehat{\alpha}\widehat{\alpha}_1}{}_{\overline{\nu}\overline{\mu}} X^{\overline{\nu}} \bigwedge X^{\overline{\mu}},$$

where $\mathcal{R}^{\widehat{\alpha}\widehat{\alpha}_1}{}_{\overline{\nu}\overline{\mu}} = R_{\overline{\alpha}_1}{}^{\overline{\alpha}}{}_{\overline{\nu}\overline{\mu}}$ (see curvatures (2.23)).

2.5.2 Bundles of affine ha–frames and Einstein equations

Besides the bundles $\mathcal{L}a\left(V^{(\overline{n})}\right)$ on ha-spacetime $V^{(\overline{n})}$, there is another bundle, the bundle of adapted affine frames with structural group $Af_{n_E}(\mathcal{R}) = GL_{n_E}\left(V^{(\overline{n})}\right) \otimes \mathcal{R}^{\overline{n}}$, which can be naturally related to the gravity models on (pseudo) Riemannian spaces. Because as a linear space the Lie Algebra $af_{\overline{n}}(\mathcal{R})$ is a direct sum of $\mathcal{G}l_{\overline{n}}(\mathcal{R})$ and $\mathcal{R}^{\overline{n}}$, we can write forms on $\mathcal{A}a\left(V^{(\overline{n})}\right)$ as $\Phi = (\Phi_1, \Phi_2)$, where Φ_1 is the $\mathcal{G}l_{\overline{n}}(\mathcal{R})$ component and Φ_2 is the $\mathcal{R}^{\overline{n}}$ component of the form Φ. The connection (2.48), Θ in $\mathcal{L}a\left(V^{(\overline{n})}\right)$, induces the Cartan connection $\overline{\Theta}$ in $\mathcal{A}a\left(V^{(\overline{n})}\right)$; see the isotropic case in [132, 133, 35]. There is only one connection on $\mathcal{A}a\left(V^{(\overline{n})}\right)$ represented as $i^*\overline{\Theta} = (\Theta, \chi)$, where χ is the shifting form and $i : \mathcal{A}a \rightarrow \mathcal{L}a$ is the trivial reduction of bundles. If $s_{\mathcal{U}}^{(a)}$ is a local adapted frame in $\mathcal{L}a\left(V^{(\overline{n})}\right)$, then $\overline{s}_{\mathcal{U}}^{(0)} = i \circ s_{\mathcal{U}}$ is a local section in $\mathcal{A}a\left(V^{(\overline{n})}\right)$ and

$$\left(\overline{\Theta}_{\mathcal{U}}\right) = s_{\mathcal{U}}\Theta = (\Theta_{\mathcal{U}}, \chi_{\mathcal{U}}), \tag{2.50}$$

where $\chi = e_{\widehat{\alpha}} \otimes \chi^{\widehat{\alpha}}{}_{\overline{\mu}} X^{\overline{\mu}}$, $g_{\overline{\alpha}\overline{\beta}} = \chi^{\widehat{\alpha}}{}_{\overline{\alpha}} \chi^{\widehat{\beta}}{}_{\overline{\beta}} \eta_{\widehat{\alpha}\widehat{\beta}}$ ($\eta_{\widehat{\alpha}\widehat{\beta}}$ is diagonal with $\eta_{\widehat{\alpha}\widehat{\alpha}} = \pm 1$) is a frame decomposition of metric (2.20) on $V^{(\overline{n})}$, $e_{\widehat{\alpha}}$ is the standard distinguished basis on $\mathcal{R}^{\overline{n}}$, and the projection of torsion , $T_{\mathcal{U}}$, on the base $V^{(\overline{n})}$ is defined as

$$T_{\mathcal{U}} = d\chi_{\mathcal{U}} + \Omega_{\mathcal{U}} \bigwedge \chi_{\mathcal{U}} + \chi_{\mathcal{U}} \bigwedge \Omega_{\mathcal{U}} = e_{\widehat{\alpha}} \otimes \sum_{\overline{\mu}\overline{\nu}} T^{\widehat{\alpha}}{}_{\overline{\mu}\overline{\nu}} X^{\overline{\mu}} \bigwedge X^{\overline{\nu}}. \tag{2.51}$$

For a fixed locally adapted basis on $\mathcal{U} \subset V^{(\overline{n})}$ we can identify components $T^{\widehat{\alpha}}{}_{\overline{\mu}\overline{\nu}}$ of torsion (2.51) with components of torsion (2.22) on $V^{(\overline{n})}$, i.e. $T^{\widehat{\alpha}}{}_{\overline{\mu}\overline{\nu}} = T^{\overline{\alpha}}{}_{\overline{\mu}\overline{\nu}}$. By straightforward calculation we obtain

$$(\Delta\overline{\mathcal{R}})_{\mathcal{U}} = [(\Delta\mathcal{R}^{(\Gamma)})_{\mathcal{U}},\ (R\tau)_{\mathcal{U}} + (Ri)_{\mathcal{U}}], \tag{2.52}$$

where

$$(R\tau)_{\mathcal{U}} = \widehat{\delta}_G T_{\mathcal{U}} + *_G^{-1}\left[\Omega_{\mathcal{U}}, *_G T_{\mathcal{U}}\right], \quad (Ri)_{\mathcal{U}} = *_G^{-1}\left[\chi_{\mathcal{U}}, *_G \mathcal{R}_{\mathcal{U}}^{(\Gamma)}\right].$$

Form $(Ri)_{\mathcal{U}}$ from (2.52) is locally constructed by using components of the Ricci tensor (see (2.24)) as follows from decomposition on the local adapted basis $X^{\overline{\mu}} = \delta u^{\overline{\mu}}$:

$$(Ri)_{\mathcal{U}} = e_{\widehat{\alpha}} \otimes (-1)^{\overline{n}+1} R_{\overline{\lambda}\overline{\nu}} g^{\widehat{\alpha}\overline{\lambda}} \delta u^{\overline{\mu}}.$$

We remark that for isotropic torsionless pseudo-Riemannian spaces the requirement that $\left(\Delta\overline{\mathcal{R}}\right)_{\mathcal{U}} = 0$, i.e., imposing the connection (2.48) to satisfy Yang-Mills equations (2.39) (equivalently (2.44) or (2.45)) we obtain [132, 133] the equivalence of the mentioned gauge gravitational equations with the vacuum Einstein equations $R_{ij} = 0$. In the case of ha–spaces with arbitrary given torsion, even considering vacuum gravitational fields, we have to introduce a source for gauge gravitational equations in order to compensate for the contribution of torsion and to obtain equivalence with the Einstein equations.

Considerations presented in this section constitute the proof of the following result:

Theorem 2.1. *The Einstein equations (2.26) for ha–gravity are equivalent to the Yang-Mills equations*

$$\left(\Delta\overline{\mathcal{R}}\right) = \overline{\mathcal{J}} \tag{2.53}$$

for the induced Cartan connection $\overline{\Theta}$ *(see (2.48) and (2.50)) in the bundle of locally adapted affine frames* $\mathcal{A}a\left(V^{(\overline{n})}\right)$ *with the source* $\overline{\mathcal{J}}_{\mathcal{U}}$ *constructed locally by using the same formulas (2.52) for* $\left(\Delta\overline{\mathcal{R}}\right)$*, but where* $R_{\overline{\alpha}\overline{\beta}}$ *is changed by the matter source* $E_{\overline{\alpha}\overline{\beta}} - \frac{1}{2}g_{\overline{\alpha}\overline{\beta}}E$ *with* $E_{\overline{\alpha}\overline{\beta}} = k\Upsilon_{\overline{\alpha}\overline{\beta}} - \lambda g_{\overline{\alpha}\overline{\beta}}$.

We note that this theorem is an extension for higher order anisotropic spacetimes of the Popov and Daikhin result [133] with respect to a possible gauge like treatment of the Einstein gravity. Similar theorems have been proved for locally anisotropic gauge gravity [186] and in the framework of some variants of locally (and higher order) anisotropic supergravity [172].

2.6 Nonlinear De Sitter Gauge Ha–Gravity

The equivalent reexpression of the Einstein theory as a gauge like theory implies, for both locally isotropic and anisotropic space–times, the nonsemisimplicity of the gauge group, which leads to a nonvariational theory in the total space of the bundle of locally adapted affine frames. A variational gauge gravitational theory can be formulated by using a minimal extension of the affine structural group $\mathcal{A}f_{\overline{n}}(\mathcal{R})$ to the de Sitter gauge group $S_{\overline{n}} = SO(\overline{n})$ acting on distinguished $\mathcal{R}^{\overline{n}+1}$ space.

2.6.1 Nonlinear gauge theories of de Sitter group

Let us consider the de Sitter space $\Sigma^{\overline{n}}$ as a hypersurface given by the equations $\eta_{AB}u^A u^B = -l^2$ in the flat $(\overline{n}+1)$–dimensional space enabled with diagonal metric $\eta_{AB}, \eta_{AA} = \pm 1$ (in this subsection $A, B, C, ... = 1, 2, ..., \overline{n}+1), (\overline{n} = n + m_1 + ... + m_z)$, where $\{u^A\}$ are global Cartesian coordinates in $\mathcal{R}^{\overline{n}+1}; l > 0$ is the curvature of de Sitter space. The de Sitter group $S_{(\eta)} = SO_{(\eta)}(\overline{n}+1)$ is defined as the isometry group of $\Sigma^{\overline{n}}$–space with $\frac{\overline{n}}{2}(\overline{n}+1)$ generators of Lie algebra $so_{(\eta)}(\overline{n}+1)$ satisfying the commutation relations

$$[M_{AB}, M_{CD}] = \eta_{AC}M_{BD} - \eta_{BC}M_{AD} - \eta_{AD}M_{BC} + \eta_{BD}M_{AC}. \qquad (2.54)$$

Decomposing indices $A, B, ...$ as $A = (\widehat{\alpha}, \overline{n}+1), B = \left(\widehat{\beta}, \overline{n}+1\right)$, ..., the metric η_{AB} as $\eta_{AB} = \left(\eta_{\widehat{\alpha}\widehat{\beta}}, \eta_{(\overline{n}+1)(\overline{n}+1)}\right)$, and operators M_{AB} as $M_{\widehat{\alpha}\widehat{\beta}} = \mathcal{F}_{\widehat{\alpha}\widehat{\beta}}$ and $P_{\widehat{\alpha}} = l^{-1}M_{\overline{n}+1,\widehat{\alpha}}$, we can write (2.54) as

$$\left[\mathcal{F}_{\widehat{\alpha}\widehat{\beta}}, \mathcal{F}_{\widehat{\gamma}\widehat{\delta}}\right] = \eta_{\widehat{\alpha}\widehat{\gamma}}\mathcal{F}_{\widehat{\beta}\widehat{\delta}} - \eta_{\widehat{\beta}\widehat{\gamma}}\mathcal{F}_{\widehat{\alpha}\widehat{\delta}} + \eta_{\widehat{\beta}\widehat{\delta}}\mathcal{F}_{\widehat{\alpha}\widehat{\gamma}} - \eta_{\widehat{\alpha}\widehat{\delta}}\mathcal{F}_{\widehat{\beta}\widehat{\gamma}},$$

$$\left[P_{\widehat{\alpha}}, P_{\widehat{\beta}}\right] = -l^{-2}\mathcal{F}_{\widehat{\alpha}\widehat{\beta}}, \quad \left[P_{\widehat{\alpha}}, \mathcal{F}_{\widehat{\beta}\widehat{\gamma}}\right] = \eta_{\widehat{\alpha}\widehat{\beta}}P_{\widehat{\gamma}} - \eta_{\widehat{\alpha}\widehat{\gamma}}P_{\widehat{\beta}},$$

where we have indicated the possibility to decompose $so_{(\eta)}(\overline{n}+1)$ into a direct sum, $so_{(\eta)}(\overline{n}+1) = so_{(\eta)}(\overline{n}) \oplus v_{\overline{n}}$, where $v_{\overline{n}}$ is the vector space stretched on vectors $P_{\widehat{\alpha}}$. We remark that $\Sigma^{\overline{n}} = S_{(\eta)}/L_{(\eta)}$, where $L_{(\eta)} = SO_{(\eta)}(\overline{n})$. For $\eta_{AB} = diag(1, -1, -1, -1)$ and $S_{10} = SO(1,4), L_6 = SO(1,3)$ is the group of Lorentz rotations.

Let $W\left(\mathcal{E}, \mathcal{R}^{\overline{n}+1}, S_{(\eta)}, \mathcal{P}\right)$ be the vector bundle associated with the principal bundle $\mathcal{P}\left(S_{(\eta)}, \mathcal{E}\right)$ on ha-spacetime $v_{\overline{n}}$, where $S_{(\eta)}$ is taken to be the structural group and by $\mathcal{E}$ it is denoted the total space. The action of the structural group $S_{(\eta)}$ on $\mathcal{E}$ can be realized by using $\overline{n} \times \overline{n}$ matrices with a parametrization distinguishing subgroup $L_{(\eta)}$:

$$B = bB_L, \qquad (2.55)$$

where

$$B_L = \begin{pmatrix} L & 0 \\ 0 & 1 \end{pmatrix},$$

$L \in L_{(\eta)}$ is the de Sitter bust matrix transforming the vector $(0,0,...,\rho) \in \mathcal{R}^{\overline{n}+1}$ into the point $(v^1, v^2, ..., v^{\overline{n}+1}) \in \Sigma_\rho^{\overline{n}} \subset \mathcal{R}^{\overline{n}+1}$ for which

$$v_A v^A = -\rho^2, v^A = t^A \rho.$$

Matrix b can be expressed

$$b = \begin{pmatrix} \delta^{\widehat{\alpha}}{}_{\widehat{\beta}} + \frac{t^{\widehat{\alpha}} t_{\widehat{\beta}}}{(1+t^{\overline{n}+1})} & t^{\widehat{\alpha}} \\ t_{\widehat{\beta}} & t^{\overline{n}+1} \end{pmatrix}.$$

The de Sitter gauge field is associated with a linear connection in W, i.e., with a $so_{(\eta)}\,(\overline{n}+1)$-valued connection 1–form on $V^{(\overline{n})}$:

$$\breve{\Theta} = \begin{pmatrix} \omega^{\widehat{\alpha}}{}_{\widehat{\beta}} & \breve{\theta}^{\widehat{\alpha}} \\ \breve{\theta}_{\widehat{\beta}} & 0 \end{pmatrix}, \tag{2.56}$$

where $\omega^{\widehat{\alpha}}{}_{\widehat{\beta}} \in so(\overline{n})_{(\eta)}$, $\breve{\theta}^{\widehat{\alpha}} \in \mathcal{R}^{\overline{n}}, \breve{\theta}_{\widehat{\beta}} \in \eta_{\widehat{\beta}\widehat{\alpha}} \breve{\theta}^{\widehat{\alpha}}$.

Because $S_{(\eta)}$-transforms mix $\omega^{\widehat{\alpha}}{}_{\widehat{\beta}}$ and $\breve{\theta}^{\widehat{\alpha}}$ fields in (2.56) (the introduced parametrization is invariant on action on $SO_{(\eta)}\,(\overline{n})$ group we cannot identify $\omega^{\widehat{\alpha}}{}_{\widehat{\beta}}$ and $\breve{\theta}^{\widehat{\alpha}}$, respectively, with the connection $\Gamma^{\overline{\alpha}}{}_{\overline{\beta}\overline{\gamma}}$ and the fundamental form $\chi^{\overline{\alpha}}$ in $V^{(\overline{n})}$ (as we have for (2.48) and (2.50)). To avoid this difficulty we consider [153, 131] a nonlinear gauge realization of the de Sitter group $S_{(\eta)}$ by introducing the nonlinear gauge field

$$\Theta = b^{-1}\breve{\Theta} b + b^{-1} db = \begin{pmatrix} \Gamma^{\widehat{\alpha}}{}_{\widehat{\beta}} & \theta^{\widehat{\alpha}} \\ \theta_{\widehat{\beta}} & 0 \end{pmatrix}, \tag{2.57}$$

where

$$\Gamma^{\widehat{\alpha}}{}_{\widehat{\beta}} = \omega^{\widehat{\alpha}}{}_{\widehat{\beta}} - \left(t^{\widehat{\alpha}} D t_{\widehat{\beta}} - t_{\widehat{\beta}} D t^{\widehat{\alpha}}\right) / \left(1 + t^{\overline{n}+1}\right),$$

$$\theta^{\widehat{\alpha}} = t^{\overline{n}+1} \breve{\theta}^{\widehat{\alpha}} + D t^{\widehat{\alpha}} - t^{\widehat{\alpha}} \left(d t^{\overline{n}+1} + \breve{\theta}_{\widehat{\gamma}} t^{\widehat{\gamma}}\right) / \left(1 + t^{\overline{n}+1}\right),$$

$$D t^{\widehat{\alpha}} = d t^{\widehat{\alpha}} + \omega^{\widehat{\alpha}}{}_{\widehat{\beta}} t^{\widehat{\beta}}.$$

The action of the group $S(\eta)$ is nonlinear, yielding transforms

$$\Gamma' = L'\Gamma\left(L'\right)^{-1} + L'd\left(L'\right)^{-1}, \theta' = L\theta,$$

where the nonlinear matrix-valued function $L' = L'\left(t^{\overline{\alpha}}, b, B_T\right)$ is defined from $B_b = b'B_{L'}$ (see the parametrization (2.55)).

Now, we can identify components of (2.57) with components of $\Gamma^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}}$ and $\chi^{\widehat{\alpha}}_{\overline{\alpha}}$ on $V^{(\overline{n})}$ and induce in a consistent manner on the base of bundle $W(\mathcal{E}, \mathcal{R}^{\overline{n}+1}, S_{(\eta)}, \mathcal{P})$ the ha–geometry.

2.6.2 Dynamics of the nonlinear de Sitter ha–gravity

Instead of the gravitational potential (2.48), we introduce the gravitational connection (similar to (2.57))

$$\Gamma = \begin{pmatrix} \Gamma^{\widehat{\alpha}}_{\widehat{\beta}} & l_0^{-1}\chi^{\widehat{\alpha}} \\ l_0^{-1}\chi_{\widehat{\beta}} & 0 \end{pmatrix} \qquad (2.58)$$

where

$$\Gamma^{\widehat{\alpha}}_{\widehat{\beta}} = \Gamma^{\widehat{\alpha}}_{\widehat{\beta}\overline{\mu}}\delta u^{\overline{\mu}},$$

$$\Gamma^{\widehat{\alpha}}{}_{\widehat{\beta}\overline{\mu}} = \chi^{\widehat{\alpha}}{}_{\overline{\alpha}}\chi^{\widehat{\beta}}{}_{\overline{\beta}}\Gamma^{\overline{\alpha}}{}_{\overline{\beta}\overline{\gamma}} + \chi^{\widehat{\alpha}}{}_{\overline{\alpha}}\delta_{\overline{\mu}}\chi^{\overline{\alpha}}{}_{\widehat{\beta}},$$

$\chi^{\widehat{\alpha}} = \chi^{\widehat{\alpha}}{}_{\overline{\mu}}\delta u^{\overline{\mu}}$, and $g_{\overline{\alpha}\overline{\beta}} = \chi^{\widehat{\alpha}}{}_{\overline{\alpha}}\chi^{\widehat{\beta}}{}_{\overline{\beta}}\eta_{\widehat{\alpha}\widehat{\beta}}$, and $\eta_{\widehat{\alpha}\widehat{\beta}}$ is parametrized as

$$\eta_{\widehat{\alpha}\widehat{\beta}} = \begin{pmatrix} \eta_{ij} & 0 & ... & 0 \\ 0 & \eta_{a_1b_1} & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \eta_{a_zb_z} \end{pmatrix},$$

$\eta_{ij} = (1, -1, ..., -1), ...\eta_{ij} = (\pm 1, \pm 1, ..., \pm 1), ..., l_0$ is a dimensional constant.

The curvature of (2.58), $\mathcal{R}^{(\Gamma)} = d\Gamma + \Gamma \bigwedge \Gamma$, can be written as

$$\mathcal{R}^{(\Gamma)} = \begin{pmatrix} \mathcal{R}^{\widehat{\alpha}}{}_{\widehat{\beta}} + l_0^{-1}\pi^{\widehat{\alpha}}_{\widehat{\beta}} & l_0^{-1}T^{\widehat{\alpha}} \\ l_0^{-1}T^{\widehat{\beta}} & 0 \end{pmatrix}, \qquad (2.59)$$

where

$$\pi^{\widehat{\alpha}}_{\widehat{\beta}} = \chi^{\widehat{\alpha}} \bigwedge \chi_{\widehat{\beta}}, \mathcal{R}^{\widehat{\alpha}}{}_{\widehat{\beta}} = \frac{1}{2}\mathcal{R}^{\widehat{\alpha}}{}_{\widehat{\beta}\overline{\mu}\overline{\nu}}\delta u^{\overline{\mu}} \bigwedge \delta u^{\overline{\nu}},$$

and

$$\mathcal{R}^{\widehat{\alpha}}{}_{\widehat{\beta}\overline{\mu\nu}} = \chi_{\widehat{\beta}}{}^{\overline{\beta}}\chi_{\overline{\alpha}}{}^{\widehat{\alpha}}R^{\overline{\alpha}}_{\overline{\beta}.\overline{\mu\nu}}$$

(see (2.23) for components of d-curvatures). The de Sitter gauge group is semisimple and we are able to construct a variational gauge gravitational locally anisotropic theory (bundle metric (2.36) is nondegenerate). The Lagrangian of the theory is postulated as

$$L = L_{(G)} + L_{(m)}$$

where the gauge gravitational Lagrangian is defined as

$$L_{(G)} = \frac{1}{4\pi}Tr\left(\mathcal{R}^{(\Gamma)}\bigwedge *_G\mathcal{R}^{(\Gamma)}\right) = \mathcal{L}_{(G)}\left|g\right|^{1/2}\delta^{\overline{n}}u,$$

$$\mathcal{L}_{(G)} = \frac{1}{2l^2}T^{\widehat{\alpha}}{}_{\overline{\mu\nu}}T_{\widehat{\alpha}}{}^{\overline{\mu\nu}} + \frac{1}{8\lambda}\mathcal{R}^{\widehat{\alpha}}{}_{\widehat{\beta}\overline{\mu\nu}}\mathcal{R}^{\widehat{\beta}}{}_{\widehat{\alpha}}{}^{\overline{\mu\nu}} - \frac{1}{l^2}\left(\overleftarrow{R}\,(\Gamma) - 2\lambda_1\right), \qquad (2.60)$$

$T^{\widehat{\alpha}}{}_{\overline{\mu\nu}} = \chi^{\widehat{\alpha}}{}_{\overline{\alpha}}T^{\overline{\alpha}}{}_{\overline{\mu\nu}}$ (the gravitational constant l^2 in (2.60) satisfies the relations $l^2 = 2l_0^2\lambda, \lambda_1 = -3/l_0$], Tr denotes the trace on $\widehat{\alpha},\widehat{\beta}$ indices, and the matter field Lagrangian is defined as

$$L_{(m)} = \frac{1}{2}Tr\left(\Gamma\bigwedge *_G\mathcal{I}\right) = \mathcal{L}_{(m)}\left|g\right|^{1/2}\delta^{\overline{n}}u,$$

$$\mathcal{L}_{(m)} = \frac{1}{2}\Gamma^{\widehat{\alpha}}{}_{\widehat{\beta}\overline{\mu}}S^{\widehat{\beta}}{}_{\widehat{\alpha}}{}^{\overline{\mu}} - t^{\overline{\mu}}{}_{\widehat{\alpha}}l^{\widehat{\alpha}}{}_{\overline{\mu}}. \qquad (2.61)$$

The matter field source $\mathcal{I}$ is obtained as a variational derivation of $\mathcal{L}_{(m)}$ on Γ and is parametrized as

$$\mathcal{I} = \begin{pmatrix} S^{\widehat{\alpha}}{}_{\widehat{\beta}} & -l_0t^{\widehat{\alpha}} \\ -l_0t_{\widehat{\beta}} & 0 \end{pmatrix} \qquad (2.62)$$

with $t^{\widehat{\alpha}} = t^{\widehat{\alpha}}{}_{\overline{\mu}}\delta u^{\overline{\mu}}$ and $S^{\widehat{\alpha}}{}_{\widehat{\beta}} = S^{\widehat{\alpha}}{}_{\widehat{\beta}\overline{\mu}}\delta u^{\overline{\mu}}$ being respectively the canonical tensors of energy-momentum and spin density. Because of the contraction of the "interior" indices $\widehat{\alpha},\widehat{\beta}$ in (2.60) and (2.61) we used the Hodge operator $*_G$ instead of $*_H$ (hereafter we consider $*_G = *$).

Varying the action

$$S = \int \left|g\right|^{1/2}\delta^{\overline{n}}u\left(\mathcal{L}_{(G)} + \mathcal{L}_{(m)}\right)$$

on the Γ-variables (2.58), we obtain the gauge–gravitational field equations:

$$d\left(*\mathcal{R}^{(\Gamma)}\right)+\Gamma\bigwedge\left(*\mathcal{R}^{(\Gamma)}\right)-\left(*\mathcal{R}^{(\Gamma)}\right)\bigwedge\Gamma=-\lambda\left(*\mathcal{I}\right). \qquad (2.63)$$

Specifying the variations on $\Gamma^{\widehat{\alpha}}{}_{\widehat{\beta}}$ and $l^{\widehat{\alpha}}$-variables, we rewrite (2.63) as

$$\widehat{\mathcal{D}}\left(*\mathcal{R}^{(\Gamma)}\right)+\frac{2\lambda}{l^2}\left(\widehat{\mathcal{D}}\left(*\pi\right)+\chi\bigwedge\left(*T^T\right)-\left(*T\right)\bigwedge\chi^T\right)=-\lambda\left(*S\right), \quad (2.64)$$

$$\widehat{\mathcal{D}}\left(*T\right)-\left(*\mathcal{R}^{(\Gamma)}\right)\bigwedge\chi-\frac{2\lambda}{l^2}\left(*\pi\right)\bigwedge\chi=\frac{l^2}{2}\left(*t+\frac{1}{\lambda}*\tau\right), \qquad (2.65)$$

where

$$T^t=\{T_{\widehat{\alpha}}=\eta_{\widehat{\alpha}\widehat{\beta}}T^{\widehat{\beta}},\ T^{\widehat{\beta}}=\frac{1}{2}T^{\widehat{\beta}}{}_{\overline{\mu}\overline{\nu}}\delta u^{\overline{\mu}}\bigwedge\delta u^{\overline{\nu}}\},$$

$$\chi^T=\{\chi_{\widehat{\alpha}}=\eta_{\widehat{\alpha}\widehat{\beta}}\chi^{\widehat{\beta}},\ \chi^{\widehat{\beta}}=\chi^{\widehat{\beta}}{}_{\overline{\mu}}\delta u^{\overline{\mu}}\}, \qquad \widehat{\mathcal{D}}=d+\widehat{\Gamma}$$

($\widehat{\Gamma}$ acts as $\Gamma^{\widehat{\alpha}}{}_{\widehat{\beta}\overline{\mu}}$ on indices $\widehat{\gamma},\widehat{\delta},...$ and as $\Gamma^{\overline{\alpha}}{}_{\overline{\beta}\overline{\mu}}$ on indices $\overline{\gamma},\overline{\delta},...$). In (2.65), τ defines the energy–momentum tensor of the $S_{(\eta)}$–gauge gravitational field $\widehat{\Gamma}$:

$$\tau_{\overline{\mu}\overline{\nu}}\left(\widehat{\Gamma}\right)=\frac{1}{2}Tr\left(\mathcal{R}_{\overline{\mu}\overline{\alpha}}\mathcal{R}^{\overline{\alpha}}{}_{\overline{\nu}}-\frac{1}{4}\mathcal{R}_{\overline{\alpha}\overline{\beta}}\mathcal{R}^{\overline{\alpha}\overline{\beta}}g_{\overline{\mu}\overline{\nu}}\right). \qquad (2.66)$$

Equations (2.63) (or, equivalently, (2.64) and (2.65)) make up the complete system of variational field equations for nonlinear de Sitter gauge gravity with higher order anisotropy. They can be interpreted as a variant of gauge like equations for ha–gravity [186] when the (pseudo) Riemannian base frames and torsions are considered to be induced by an anholonomic frame structure with associated N–connection

A. Tseytlin [153] presented a quantum analysis of the isotropic version of equations (2.64) and (2.65). Of course, the problem of quantizing gravitational interactions is unsolved for both variants of locally anisotropic and isotropic gauge de Sitter gravitational theories, but we think that the generalized Lagrange version of $S_{(\eta)}$-gravity is more adequate for studying quantum radiational and statistical gravitational processes. This is a matter for further investigations.

Finally, we remark that we can obtain a nonvariational Poincare gauge gravitational theory on ha–spaces if we consider the contraction of the gauge potential (2.58) to a potential with values in the Poincare Lie algebra

$$\Gamma = \begin{pmatrix} \Gamma^{\widehat{\alpha}}{}_{\widehat{\beta}} & l_0^{-1}\chi^{\widehat{\alpha}} \\ l_0^{-1}\chi_{\widehat{\beta}} & 0 \end{pmatrix} \to \Gamma = \begin{pmatrix} \Gamma^{\widehat{\alpha}}{}_{\widehat{\beta}} & l_0^{-1}\chi^{\widehat{\alpha}} \\ 0 & 0 \end{pmatrix}.$$

Isotropic Poincare gauge gravitational theories are studied in a number of papers (see, for example, [202, 153, 131]). In a manner similar to considerations presented in this work, we can generalize Poincare gauge models for spaces with local anisotropy.

2.7 An Ansatz for 4D d–Metrics

We consider a 4D space–time $V^{(3+1)}$ provided with a d–metric (1.39) when $g_i = g_i(x^k)$ and $h_a = h_a(x^k, z)$ for $y^a = (z, y^4)$. The N–connection coefficients are some functions on three coordinates (x^i, z),

$$\begin{aligned} N_1^3 &= q_1(x^i, z),\ N_2^3 = q_2(x^i, z), \\ N_1^4 &= n_1(x^i, z),\ N_2^4 = n_2(x^i, z). \end{aligned} \tag{2.67}$$

For simplicity, we shall use brief denotations of partial derivatives, like

$$\begin{aligned} \dot{a} &= \partial a/\partial x^1, a' = \partial a/\partial x^2, \\ a^* &= \partial a/\partial z \dot{a}' = \partial^2 a/\partial x^1 \partial x^2, \\ a^{**} &= \partial^2 a/\partial z \partial z. \end{aligned}$$

The non–trivial components of the Ricci d–tensor (2.10), for the mentioned type of d–metrics depending on three variables, are

$$R_1^1 = R_2^2 = \frac{1}{2g_1 g_2}[-(g_1^{''} + \ddot{g}_2) + \frac{1}{2g_2}\left(\dot{g}_2^2 + g_1' g_2'\right) + \frac{1}{2g_1}\left(g_1'^{\ 2} + \dot{g}_1 \dot{g}_2\right)]; \quad (2.68)$$

$$S_3^3 = S_4^4 = \frac{1}{h_3 h_4}[-h_4^{**} + \frac{1}{2h_4}(h_4^*)^2 + \frac{1}{2h_3} h_3^* h_4^*];$$

$$P_{31} = \frac{q_1}{2}[\left(\frac{h_3^*}{h_3}\right)^2 - \frac{h_3^{**}}{h_3} + \frac{h_4^*}{2h_4^{\ 2}} - \frac{h_3^* h_4^*}{2h_3 h_4}] + \frac{1}{2h_4}[\frac{\dot{h}_4}{2h_4} h_4^* - \dot{h}_4^* + \frac{\dot{h}_3}{2h_3} h_4^*], \quad (2.69)$$

$$P_{32} = \frac{q_2}{2}[\left(\frac{h_3^*}{h_3}\right)^2 - \frac{h_3^{**}}{h_3} + \frac{h_4^*}{2h_4^{\ 2}} - \frac{h_3^* h_4^*}{2h_3 h_4}] + \frac{1}{2h_4}[\frac{h_4'}{2h_4} h_4^* - h_4'^{\ *} + \frac{h_3'}{2h_3} h_4^*];$$

$$P_{41} = -\frac{h_4}{2h_3} n_1^{**} + \frac{1}{4h_3}(\frac{h_4}{h_3} h_3^* - 3h_4^*) n_1^*, \quad (2.70)$$

$$P_{42} = -\frac{h_4}{2h_3} n_2^{**} + \frac{1}{4h_3}(\frac{h_4}{h_3} h_3^* - 3h_4^*) n_2^*.$$

The curvature scalar $\overleftarrow{R}$ (2.11) is defined by the sum of two non-trivial components $\widehat{R} = 2R_1^1$ and $S = 2S_3^3$.

The system of Einstein equations (2.12) transforms into

$$R_1^1 = -\kappa \Upsilon_3^3 = -\kappa \Upsilon_4^4, \quad (2.71)$$

$$S_3^3 = -\kappa \Upsilon_1^1 = -\kappa \Upsilon_2^2, \quad (2.72)$$

$$P_{3i} = \kappa \Upsilon_{3i}, \quad (2.73)$$

$$P_{4i} = \kappa \Upsilon_{4i}, \quad (2.74)$$

where the values of R_1^1, S_3^3, P_{ai}, are taken respectively from (2.68), (2.68), (2.69), (2.70).

By using the equations (2.73) and (2.74) we can define the N–coefficients (2.67), $q_i(x^k, z)$ and $n_i(x^k, z)$, if the functions $g_i(x^k)$ and $h_i(x^k, z)$ are known as respective solutions of the equations (2.71) and (2.72). Let consider an ansatz for a 4D d–metric of type

$$\delta s^2 = g_1(x^k)(dx^1)^2 + (dx^2)^2 + h_3(x^i, t)(\delta t)^2 + h_4(x^i, t)(\delta y^4)^2, \quad (2.75)$$

where the z–parameter is considered to be the time like coordinate and the energy momentum d–tensor is taken

$$\Upsilon_\alpha^\beta = [p_1, p_2, -\varepsilon, p_4 = p].$$

The aim of this section is to analyze the system of partial differential equations following from the Einsteni field equations for these d–metric and energy–momentum d–tensor.

2.7.1 The h–equations

The Einstein equations (2.71), with the Ricci h–tensor (2.68), for the d–metric (2.75) transform into

$$\frac{\partial^2 g_1}{\partial (x^1)^2} - \frac{1}{2g_1}\left(\frac{\partial g_1}{\partial x^1}\right)^2 + 2\kappa\varepsilon g_1 = 0. \tag{2.76}$$

By introducing the coordinates $\chi^i = x^i/\sqrt{\kappa\varepsilon}$ and the variable

$$q = g_1'/g_1, \tag{2.77}$$

where by 'prime' in this Section is considered the partial derivative ∂/χ^2, the equation (2.76) transforms into

$$q' + \frac{q^2}{2} + 2\epsilon = 0, \tag{2.78}$$

where the vacuum case should be parametrized for $\epsilon = 0$ with $\chi^i = x^i$ and $\epsilon = -1$ for a matter state with $\varepsilon = -p$.

The integral curve of (2.78), intersecting a point $\left(\chi^2_{(0)}, q_{(0)}\right)$, considered as a differential equation on χ^2 is defined by the functions [82]

$$q = \frac{q_{(0)}}{1 + \frac{q_{(0)}}{2}\left(\chi^2 - \chi^2_{(0)}\right)}, \qquad \epsilon = 0; \tag{2.79}$$

$$q = \frac{q_{(0)} - 2\tan\left(\chi^2 - \chi^2_{(0)}\right)}{1 + \frac{q_{(0)}}{2}\tan\left(\chi^2 - \chi^2_{(0)}\right)}, \qquad \epsilon < 0. \tag{2.80}$$

Because the function q depends also parametrically on variable χ^1 we can consider functions $\chi^2_{(0)} = \chi^2_{(0)}\left(\chi^1\right)$ and $q_{(0)} = q_{(0)}\left(\chi^1\right)$. We elucidate the non–vacuum case with $\epsilon < 0$. The general formula for the non–trivial component of h–metric is to be obtained after integration on χ^1 of (2.77) by using the solution (2.80)

$$g_1\left(\chi^1, \chi^2\right) = g_{1(0)}\left(\chi^1\right)\left\{\sin[\chi^2 - \chi^2_{(0)}\left(\chi^1\right)] + \arctan\frac{2}{q_{(0)}\left(\chi^1\right)}\right\}^2,$$

for $q_{(0)}\left(\chi^1\right) \neq 0$, and

$$g_1\left(\chi^1,\chi^2\right) = g_{1(0)}\left(\chi^1\right)\ \cos^2[\chi^2 - \chi^2_{(0)}\left(\chi^1\right)] \qquad (2.81)$$

for $q_{(0)}(\chi^1) = 0$, where $g_{1(0)}(\chi^1), \chi^2_{(0)}(\chi^1)$ and $q_{(0)}(\chi^1)$ are some functions of necessary smoothness class on variable χ^1. For simplicity, in our further considerations we shall apply the solution (2.81).

2.7.2 The v–equations

For the ansatz (2.75) the Einstein equations (2.72) with the Ricci h–tensor (2.68) transforms into

$$\frac{\partial^2 h_4}{\partial t^2} - \frac{1}{2h_4}\left(\frac{\partial h_4}{\partial t}\right)^2 - \frac{1}{2h_3}\left(\frac{\partial h_3}{\partial t}\right)\left(\frac{\partial h_4}{\partial t}\right) - \frac{\kappa}{2}\Upsilon_1 h_3 h_4 = 0$$

(here we write down the partial derivatives on t in explicit form) which relates some first and second order partial on z derivatives of diagonal components $h_a(x^i,t)$ of a v–metric with a source

$$\Upsilon_1(x^i,z) = \kappa\Upsilon^1_1 = \kappa\Upsilon^2_2 = p_1 = p_2$$

in the h–subspace. We can consider as unknown the function $h_3(x^i,t)$ (or, inversely, $h_4(x^i,t)$) for some compatible values of $h_4(x^i,t)$ (or $h_3(x^i,t)$) and source $\Upsilon_1(x^i,t)$. By introducing a new variable $\beta = h_4^*/h_4$ the equation (2.7.2) transforms into

$$\beta^* + \frac{1}{2}\beta^2 - \frac{\beta h_3^*}{2h_3} - 2\kappa\Upsilon_1 h_3 = 0 \qquad (2.82)$$

which relates two functions $\beta\left(x^i,t\right)$ and $h_3\left(x^i,t\right)$. There are two possibilities: 1) to define β (i. e. h_4) when $\kappa\Upsilon_1$ and h_3 are prescribed and, inversely 2) to find h_3 for given $\kappa\Upsilon_1$ and h_4 (i. e. β); in both cases one considers only "*" derivatives on t–variable with coordinates x^i treated as parameters.

1. In the first case the explicit solutions of (2.82) have to be constructed by using the integral varieties of the general Riccati equation [82] which by a corresponding redefinition of variables, $t \to t\left(\varsigma\right)$ and $\beta\left(t\right) \to \eta\left(\varsigma\right)$ (for simplicity, we omit dependencies on x^i) could be written in the canonical form

$$\frac{\partial\eta}{\partial\varsigma} + \eta^2 + \Psi\left(\varsigma\right) = 0$$

where Ψ vanishes for vacuum gravitational fields. In vacuum cases the Riccati equation reduces to a Bernoulli equation which (we can use the former variables) for $s(t) = \beta^{-1}$ transforms into a linear differential (on t) equation,

$$s^* + \frac{h_3^*}{2h_3}s - \frac{1}{2} = 0. \tag{2.83}$$

2. In the second (inverse) case when h_3 is to be found for some prescribed $\kappa\Upsilon_1$ and β the equation (2.82) is to be treated as a Bernoulli type equation,

$$h_3^* = -\frac{4\kappa\Upsilon_1}{\beta}(h_3)^2 + \left(\frac{2\beta^*}{\beta} + \beta\right) h_3 \tag{2.84}$$

which can be solved by standard methods. In the vacuum case the squared on h_3 term vanishes and we obtain a linear differential (on t) equation.

Finally, in this Section we conclude that the system of equations (2.72) is satisfied by arbitrary functions

$$h_3 = a_3(\chi^i) \text{ and } h_4 = a_4(\chi^i).$$

If v–metrics depending on three coordinates are introduced, $h_a = h_a(\chi^i, t)$, the v–components of the Einstein equations transforms into (2.7.2) which reduces to (2.82) for prescribed values of $h_3(\chi^i, t)$, and, inversely, to (2.84) if $h_4(\chi^i, t)$ is prescribed.

2.7.3 H–v equations

For the ansatz (2.75) with $h_4 = h_4(x^i)$ and a diagonal energy–momentum d–tensor the h–v–components of Einstein equations (2.73) and (2.74) are written respectively as

$$P_{5i} = \frac{q_i}{2h_3}[(\frac{\partial h_3}{\partial t})^2 - \frac{\partial^2 h_3}{\partial t^2}] = 0, \tag{2.85}$$

and

$$P_{6i} = \frac{h_4}{4(h_3)^2}\frac{\partial n_i}{\partial t}\frac{\partial h_3}{\partial t} - \frac{h_4}{2h_3}\frac{\partial^2 n_i}{\partial t^2} = 0. \tag{2.86}$$

The equations (2.85) are satisfied by arbitrary coefficients $q_i(x^k, t)$ if the d–metric coefficient h_3 is a solution of

$$(\frac{\partial h_3}{\partial t})^2 - \frac{\partial^2 h_3}{\partial t^2} = 0 \qquad (2.87)$$

and the q–coefficients must vanish if this condition is not satisfied. In the last case we obtain a $3+1$ anisotropy. The general solution of equations (2.86) are written in the form

$$n_i = l_i^{(0)}(x^k) \int \sqrt{|h_3(x^k, t)|}dt + n_i^{(0)}(x^k)$$

where $l_i^{(0)}(x^k)$ and $n_i^{(0)}(x^k)$ are arbitrary functions on x^k which have to be defined by some boundary conditions.

2.8 Anisotropic Cosmological Solutions

The aim of this section is to construct two classes of solutions of Einstein equations describing Friedman–Robertson–Walker (FRW) like universes with corresponding symmetries or rotational ellipsoid (elongated and flattened) and torus.

2.8.1 Rotation ellipsoid FRW universes

We proof that there are cosmological solutions constructed as locally anisotropic deformations of the FRW spherical symmetric solution to the rotation ellipsoid configuration. There are two types of rotation ellipsoids, elongated and flattened ones. We examine both cases of such horizon configurations.

Rotation elongated ellipsoid configuration

An elongated rotation ellipsoid hypersurface is given by the formula [89]

$$\frac{x^2+y^2}{\sigma^2-1} + \frac{z^2}{\sigma^2} = \rho^2, \qquad (2.88)$$

where $\sigma \geq 1$, x, y, z are Cartezian coordinates and ρ is similar to the radial coordinate in the spherical symmetric case. The 3D special coordinate system is defined

$$\begin{aligned} x &= \rho \sinh u \sin v \cos \varphi, \ y = \rho \sinh u \sin v \sin \varphi, \\ z &= \rho \ \cosh u \cos v, \end{aligned}$$

where $\sigma = \cosh u, (0 \leq u < \infty,\ 0 \leq v \leq \pi,\ 0 \leq \varphi < 2\pi)$. The hypersurface metric (2.88) is

$$\begin{aligned} g_{uu} &= g_{vv} = \rho^2 \left(\sinh^2 u + \sin^2 v\right), \\ g_{\varphi\varphi} &= \rho^2 \sinh^2 u \sin^2 v. \end{aligned} \tag{2.89}$$

Let us introduce a d–metric of class (2.75)

$$\delta s^2 = g_1(u,v)du^2 + dv^2 + h_3(u,v,\tau)(\delta\tau)^2 + h_4(u,v)(\delta\varphi)^2, \tag{2.90}$$

where $x^1 = u, x^2 = v$, $y^4 = \varphi$, $y^3 = \tau$ is the time like cosmological coordinate and $\delta\tau$ and $\delta\varphi$ are N–elongated differentials. As a particular solution of (2.90) for the h–metric we choose (see (2.81)) the coefficient

$$g_1(u,v) = \cos^2 v \tag{2.91}$$

and set for the v–metric components

$$h_3(u,v,\tau) = -\frac{1}{\rho^2(\tau)(\sinh^2 u + \sin^2 v)} \tag{2.92}$$

and

$$h_4(u,v,\tau) = \frac{\sinh^2 u \sin^2 v}{(\sinh^2 u + \sin^2 v)}. \tag{2.93}$$

The set of coefficients (2.91),(2.92), and (2.93), for the d–metric (2.90, and of $q_i = 0$ and n_i being solutions of (2.87), for the N–connection, defines a solution of the Einstein equations (2.12). The physical treatment of the obtained solutions follows from the locally isotropic limit of a conformal transform of this d–metric: Multiplying (2.90) on

$$\rho^2(\tau)(\sinh^2 u + \sin^2 v),$$

and considering $cos^2 v \simeq 1$ and $n_i \simeq= 0$ for locally isotropic spacetimes we get the interval

$$\begin{aligned} ds^2 &= -d\tau^2 + \rho^2(\tau)[(\sinh^2 u + \sin^2 v)(du^2 + dv^2) + \sinh^2 u \sin^2 v d\varphi^2] \\ &\quad \text{for ellipsoidal coordinates on hypersurface (2.89);} \\ &= -d\tau^2 + \rho^2(\tau)[dx^2 + dy^2 + dz^2] \text{ for Cartezian coordinates,} \end{aligned}$$

which defines just the Robertson–Walker metric. So, the d–metric (2.90), the coefficients of N–connection being solutions of (2.73) and (2.74), describes a 4D cosmological solution of the Einstein equations when instead of a spherical symmetry one has a locally anisotropic deformation to the symmetry of rotation elongated ellipsoid. The explicit dependence on time τ of the cosmological factor ρ must be constructed by using additionally the matter state equations for a cosmological model with local anisotropy.

Flattened rotation ellipsoid coordinates

In a similar fashion we can construct a locally anisotropic deformation of the FRW metric with the symmetry of flattened rotation ellipsoid . The parametric equation for a such hypersurface is [89]

$$\frac{x^2+y^2}{1+\sigma^2}+\frac{z^2}{\sigma^2}=\rho^2,$$

where $\sigma \geq 0$ and $\sigma = \sinh u$. The proper for ellipsoid 3D space coordinate system is defined

$$\begin{aligned} x &= \rho\cosh u\sin v\cos\varphi, y=\rho\cosh u\sin v\sin\varphi \\ z &= \rho\sinh u\cos v, \end{aligned}$$

where $0 \leq u < \infty,\ 0 \leq v \leq \pi,\ 0 \leq \varphi < 2\pi$. The hypersurface metric is

$$\begin{aligned} g_{uu} &= g_{vv}=\rho^2\left(\sinh^2 u+\cos^2 v\right), \\ g_{\varphi\varphi} &= \rho^2\sinh^2 u\cos^2 v. \end{aligned}$$

In the rest the cosmological la–solution is described by the same formulas as in the previous subsection but with respect to new canonical coordinates for flattened rotation ellipsoid.

2.8.2 Toroidal FRW universes

Let us construct a cosmological solution of the Einstein equations with toroidal symmetry. The hypersurface formula of a torus is [89]

$$\left(\sqrt{x^2+y^2}-\rho\ c\tanh\sigma\right)^2+z^2=\frac{\rho^2}{\sinh^2\sigma}.$$

The 3D space coordinate system is defined

$$\begin{aligned} x &= \frac{\rho\sinh\alpha\cos\varphi}{\cosh\alpha-\cos\sigma}, \qquad y=\frac{\rho\sin\sigma\sin\varphi}{\cosh\alpha-\cos\sigma}, \\ z &= \frac{\rho\sinh\sigma}{\cosh\tau-\cos\sigma}, \\ & (-\pi<\sigma<\pi, 0\leq\alpha<\infty, 0\leq\varphi<2\pi)\,. \end{aligned}$$

The hypersurface metric is

$$g_{\sigma\sigma}=g_{\alpha\alpha}=\frac{\rho^2}{\left(\cosh\alpha-\cos\sigma\right)^2}, g_{\varphi\varphi}=\frac{\rho^2\sin^2\sigma}{\left(\cosh\alpha-\cos\sigma\right)^2}. \qquad (2.94)$$

The d–metric of class (2.75) is chosen

$$\delta s^2 = g_1(\alpha)d\sigma^2 + d\alpha^2 + h_3(\sigma,\alpha,\tau)(\delta\tau)^2 + h_4(\sigma)(\delta\varphi)^2, \tag{2.95}$$

where $x^1 = \sigma, x^2 = \alpha$, $y^4 = \varphi$, $y^3 = \tau$ is the time like cosmological coordinate and $\delta\tau$ and $\delta\varphi$ are N–elongated differentials. As a particular solution of (2.94) for the h–metric we choose (see (2.81)) the coefficient

$$g_1(\alpha) = \cos^2\alpha \tag{2.96}$$

and set for the v–metric components

$$\begin{aligned} h_3(\sigma,\alpha,\tau) &= -\frac{(\cosh\alpha - \cos\sigma)^2}{\rho^2(\tau)} \\ h_4(\sigma) &= \sin^2\sigma. \end{aligned} \tag{2.97}$$

Multiplying (2.95) on

$$\frac{\rho^2(\tau)}{(\cosh\alpha - \cos\sigma)^2},$$

and considering $cos\alpha \simeq 1$ and $n_i \simeq= 0$ in the locally isotropic limit we get the interval

$$ds^2 = -d\tau^2 + \frac{\rho^2(\tau)}{(\cosh\alpha - \cos\sigma)^2}[(d\sigma^2 + d\alpha^2 + \sin^2\sigma d\varphi^2]$$

where the space part is just the torus hypersurface metric (2.94). So, the set of coefficients (2.96) and (2.97), for the d–metric (2.95, and of $q_i = 0$ and n_i being solutions of (2.87), for the N–connection, defines a cosmological solution of the Einstein equations (2.12) with the torus symmetry, when the explicit form of the function $\rho(\tau)$ is to be defined by considering some additional equations for the matter state (for instance, with a scalar field defining the torus inflation).

2.9 Concluding Remarks

In this Chapter we have developed the method of anholonomic frames on (pseudo) Riemannian spacetimes by considering associated nonlinear connection (N–connection) strucutres. We provided a rigorous geometric background for description of gravitational systems with mixed holonomic and anholonomic (anisotropic) degrees of freedom by considering first and higher order anisotropies induced by anholonomic constraints and corresponding frame bases.

The first key result of this paper is the proof that generic anisotropic structures of different order are contained in the Einstein theory. We reformulated the tensor and linear connection formalism for (pseudo) Riemannian spaces enables with N–connections and computed the horizonal–vertical splitting, with respect to anholonomic frames with associated N–connections, of the Einstein equations. The (pseudo) Riemannian spaces enabled with compatible anholonomic frame and associated N–connection structures and the metric being a solution of the Einstein equations were called as locally anisotropic spacetimes (in brief, anisotropic spaces). The next step was the definition of gauge field interactions on such spacetimes. We have applied the bundle formalism and extended it to the case of locally anisotropic bases and considered a 'pure' geometric method of deriving the Yang–Mills equations for generic locally anisotropic gauge interactions, by genalizing the absolut differential calculus and dual forms symmetries for anisotropic spaces.

The second key result was the proof by geometric methods that the Yang–Mills equations for a correspondingly defined Cartan connection in the bundle of affine frames on locally anisotropic spacetimes are equivalent to the Einstein equations with anholonomic (N–connection) structures (the original Popov–Daikhin papers [132, 133] were for the locally isotropic spaces). The result was obtained by applying an auxiliary bilinear form on the typical fiber because of degeneration of the Killing form for the affine groups. After projection on base spacetimes the dependence on auxiliar values is eliminated. We analyzed also a variant of variational gauge locally anisotropic gauge theory by considering a minimal extension of the affine structural group to the de Sitter one, with a nonlinear realization for the gauge group as one was performed in a locally isotropic version in Tseytlin's paper [153]. If some former our works [186, 172] where devoted to extensions of some models of gauge gravity to generalized Lagrange and Finsler spaces, in this paper we demonstrated which manner we could manage with anisotropies arrising in locally isotropic, but with anholonomic structures, variants of gauge gravity. Here it should be emphasized that anisotropies of different type (Finsler like, or more general ones) could be induced in all variants of gravity theories dealing with frame (tetrad, vierbiend, in four dimensions) fields and decompisitions of geometrical and physical objects in comonents with respect to such frames and associated N–connections. In a similar fashion anisotropies could arise under nontrivial reductions from higher to lower dimensions in Kaluza–Klein theories; in this case the N–connection should be treated as a splitting field modeling the anholonomic (anisotropic) character of some degrees of freedom.

The third basic result is the construction of a new class of solutions, with generic local anisotropy, of the Einstein equations. For simplicity, we defined these solutions in the framework of general relativity, but they can be removed to various variants of gauge and spinor gravity by using corresponding decompositions of the metric into the frame fields. We note that the obtained class of solutions also holds true for the gauge models of gravity which, in this paper, were constructed to b e equivalent to the Einstein theory. In explicit form we considered the metric ansatz

$$ds^2 = g_{\alpha\beta}\ du^\alpha du^\beta$$

when $g_{\alpha\beta}$ are parametrized by matrices of type

$$\left[\begin{array}{cccc} g_1 + {q_1}^2 h_3 + {n_1}^2 h_4 & q_1 q_2 h_3 + n_1 n_2 h_4 & q_1 h_3 & n_1 h_4 \\ q_1 q_2 h_3 + n_1 n_2 h_4 & g_2 + {q_2}^2 h_3 + {n_2}^2 h_4 & q_2 h_3 & n_2 h_4 \\ q_1 h_3 & q_2 h_3 & h_3 & 0 \\ n_1 h_4 & n_2 h_4 & 0 & h_4 \end{array} \right] \tag{2.98}$$

with coefficients being some functions of necessary smooth class

$$g_i = g_i(x^j), q_i = q_i(x^j, t), n_i = n_i(x^j, t), h_a = h_a(x^j, t).$$

Latin indices run respectively $i, j, k, ... = 1, 2$ and $a, b, c, ... = 3, 4$ and the local coordinates are denoted $u^\alpha = (x^i, y^3 = t, y^4)$, where t is treated as a time like coordinate. A metric (2.98) can be diagonalized,

$$\delta s^2 = g_i(x^j) \left(dx^i\right)^2 + h_a(x^j, t) \left(\delta y^a\right)^2, \tag{2.99}$$

with respect to anholonomic frames (2.3) and (2.4), here we write down only the 'elongated' differentials

$$\delta t = dz + q_i(x^j, t) dx^i,\ \ \delta y^4 = dy^4 + n_i(x^j, t) dx^i.$$

The ansatz (2.98) was formally introduced in [177] in order to construct locally anisotropic black hole solutions; in this paper we applied it to cosmological locally anisotropic space–times. In result, we get new metrics which describe locally anisotropic Friedman–Robertson–Walker like universes with the spherical symmetry deformed to that of rotation (elongated and/or flattened) ellipsoid and torus. Such solutions are contained in general relativity: in the simplest diagonal form they are parametrized by distinguished metrics of type (2.99), given with respect to anholonomic bases, but could be also described equivalently with respect to a coordinate base by matrices of type (2.98). The topic of construction of cosmological models with generic spacetime and matter field distribution and fluctuation anisotropies is under consideration.

Now, we point the item of definition of reference frames in gravity theories: The form of basic field equations and fundamental laws in general relativity do not depend on choosing of coordinate systems and frame bases. Nevertheless, the problem of fixing of an adequate system of reference is also a very important physical task which is not solved by any dynamical equations but following some arguments on measuring of physical observables, imposed symmetry of interactions, types of horizons and singularities, and by taken into consideration the posed Cauchy problem. Having fixed a class of frame variables, the frame coefficients being presented in the Einstein equations, the type of constructed solution depends on the chosen holonomic or anholonomic frame structure. As a result one could model various forms of anisotropies in the framework of the Einsten theory (roughly, on (pseudo) Riemannian spacetimes with corresponding anholonomic frame structures it is possible to model Finsler like metrics, or more general ones with anisotropies). Finally, it should be noted that such questions on stability of obtained solutions, analysis of energy–momentum conditions should be performed in the simplest form with respect to the chosen class of anholonomic frames.

Chapter 3

Anisotropic Taub NUT – Dirac Spaces

The aim of this chapter is to outline the theory of gravity on vector bundles provided with nonlinear connection structures [108, 109] and to proof that anholonomic frames with associated nonlinear connection structures can be introduced in general relativity and in low dimensional and extra dimension models of gravity on (pseudo) Riemannian space–times [177, 179].

3.1 Anholonomic Frames and Nonlinear Connections in General Relativity

The geometry of nonlinear connections on vector and higher order vector bundles can be reformulated for anholonomic frames given on a (pseudo) Riemannian spacetime of dimension $n+m$, or $n+m_1+m_2+...+m_z$, and provided with a d-metric structure which induces on space–time a canonical d–connection structure (1.49). In this case we can consider a formal splitting of indices with respect to some holonomic and anholonomic frame basis vectors. This approach was developed in references [177, 179] with the aim to construct exact solutions with generic local anisotropy in general relativity and its low and extra dimension modifications. For simplicity, in the further sections of this chapter we shall restrict our constructions only to first order anisotropic structures.

Recently one has proposed a new method of construction of exact solutions of the Einstein equations on (pseudo) Riemannian spaces of three, four and extra dimensions (in brief, 3D, 4D,...), by applying the formalism of anholonomic moving frames [195]. There were constructed static solutions for black holes / tori, soliton–dilaton systems and wormhole / flux tube configurations and for anisotropic generalizations of the Taub NUT metric [194]; all such solutions being, in general, with generic local anisotropy. The method was elaborated following the geometry of anholonomic frame (super) bundles and associated nonlinear connections (in brief, N–connection) [180] which has a number of applications in generalized Finsler and Lagrange geometry, anholonomic spinor geometry, (super) gravity and strings with anisotropic (anholonomic) frame structures.

In this chapter we restrict our considerations for the 5D Einstein gravity. In this case the N–connection coefficients are defined by some particular parametrizations of funfbein, or pentadic, coefficients defining a frame structure on (pseudo) Riemannian spacetime and describing a gravitational and matter field dynamics with mixed holonomic (unconstrained) and anholonomic (constrained) variables. We emphasize that the Einstein gravity theory in arbitrary dimensions can be equivalently formulated with respect to both holonomic (coordinate) and anholonomic frames. In the anholonomic cases the rules of partial and covariant derivation are modified by some pentad transforms. The point is to find such values of the anholonomic frame (and associated N–connection) coefficients when the metric is diagonalized and the Einstein equations are written in a simplified form admitting exact solutions.

The class of new exact solutions of vacuum Einstein equations describing anisotropic Taub NUT like spacetimes [199] is defined by off–diagonal metrics if they are given with respect to usual coordinate bases. Such metrics can be anholonomically transformed into diagonal ones with coefficients being very similar to the coefficients of the isotropic Taub NUT solution but having additional dependencies on the 5th coordinate and angular parameters.

We shall use the term locally anisotropic (spacetime) space (in brief, anisotropic space) for a (pseudo) Riemannian space provided with an anholonomic frame structure induced by a procedure of anholonomic diagonalization of a off–diagonal metric.

The Hawking's [62] suggestion that the Euclidean Taub–NUT metric might give rise to the gravitational analogue of the Yang–Mills instanton holds true on anisotropic spaces but in this case both the metric and instanton have some anisotropically renormalized parameters being of higher dimension gravitational vacuum polarization origin. The anisotropic Euclidean Taub-NUT metric also satisfies the vacuum Einstein's equations with zero cosmological constant when the spherical symmetry is deformed, for instance, into ellipsoidal or even toroidal configuration. Such anisotropic Taub-NUT metrics can be used for generation of deformations of the space part of the line element defining an anisotropic modification of the Kaluza-Klein monopole solutions proposed by Gross and Perry [76] and Sorkin [139].

In the long-distance limit, neglecting radiation, the relative motion of two such anisotropic monopoles can be also described by geodesic motions, like in Ref. [94, 95, 22], but these motions are some anholonomic ones with associated nonlinear connection structure and effective torsion induced by the anholonomy of the systems of reference used for modeling anisotropies. The torsion and N–connection corrections vanish if the geometrical objects are transferred with respect to holonomic (coordinate) frames.

From the mathematical point of view, the new anholonomic geometry of anisotropic Taub-NUT spaces is also very interesting. In the locally isotropic Taub-NUT geometry there are four Killing-Yano tensors [72]. Three of them form a complex structure realizing the quaternionic algebra and the Taub-NUT manifold is hyper-Kähler. In addition to such three vector-like Killing-Yano tensors, there is a scalar one which exists by virtue of the metric being of class D, according to Petrov's classification. Anisotropic deformations of metrics to off–diagonal components introduce substantial changes in the geometrical picture. Nevertheless, working with respect to anholonomic frames with associated nonlinear connection structure the basic properties and relations, even being anisotropically modified, are preserved and transformed to similar ones for deformed symmetries [199].

The Schrödinger quantum modes in the Euclidean Taub-NUT geometry were analyzed using algebraic and analytical methods [72, 73, 58, 48, 71, 49, 67]. The Dirac equation was studied in such locally isotropic curved backgrounds [57, 88, 23]. One of the aims of this paper is to prove that this approach can be developed as to include into consideration anisotropic Taub-NUT backgrounds in the context of the standard relativistic gauge-invariant theory [203, 34] of the Dirac field.

The purpose of the present work is to develop a general $SO(4,1)$ gauge-invariant theory of the Dirac fermions [93] which can be considered for locally anisotropic spaces, for instance, in the external field of the Kaluza-Klein monopole [57, 88, 23] which is anisotropically deformed.

Our goal is also to point out new features of the Einstein theory in higher dimension spacetime when the locally anisotropic properties, induced by anholonomic constraints and extra dimension gravity, are emphasized. We shall analyze such effects by constructing new classes of exact solutions of the Einstein–Dirac equations defining 3D soliton–spinor configurations propagating self–consistently in an anisotropic 5D Taub NUT spacetime.

We note that in this paper the 5D spacetime is modeled as a direct time extension of a 4D Riemannian space provided with a corresponding spinor structure, i. e. our spinor constructions are not defined by some Clifford algebra associated to a 5D bilinear form but, for simplicity, they are considered to be extended from a spinor geometry defined for a 4D Riemannian space.

3.1.1 Anholonomic Einstein–Dirac Equations

In this Section we introduce an ansatz for pseudo Riemannian off–diagonal metrics and consider the anholonomic transforms diagonalizing such metrics. The system of field Einstein equations with the spinor matter energy–momentum tensor and of Dirac equations are formulated on 5D pseudo–Riemannian spacetimes constructed as a trivial extension by the time variable of a 4D Riemannian space (an anisotropic deformation of the Taub NUT instanton [199]).

Ansatz for metrics

We consider a 5D pseudo–Riemannian spacetime of signature $(+,-,-,-,-)$, with local coordinates

$$u^{\alpha}=(x^{i},y^{a})=(x^{0}=t,x^{1}=r,x^{2}=\theta,y^{3}=s,y^{4}=p),$$

– or more compactly $u=(x,y)$ – where the Greek indices are conventionally split into two subsets x^i and y^a labeled respectively by Latin indices of type $i,j,k,...=0,1,2$ and $a,b,...=3,4$. The 5D (pseduo) Riemannian metric

$$ds^{2}=g_{\alpha\beta}du^{\alpha}du^{\beta} \tag{3.1}$$

is given by a metric ansatz parametrized in the form

$$g_{\alpha\beta}=\left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & g_{1}+{w_{1}}^{2}h_{3}+{n_{1}}^{2}h_{4} & w_{1}w_{2}h_{3}+n_{1}n_{2}h_{4} & w_{1}h_{3} & n_{1}h_{4} \\ 0 & w_{2}w_{1}h_{3}+n_{1}n_{2}h_{4} & g_{2}+{w_{2}}^{2}h_{3}+{n_{2}}^{2}h_{4} & w_{2}h_{3} & n_{2}h_{4} \\ 0 & w_{1}h_{3} & w_{2}h_{3} & h_{3} & 0 \\ 0 & n_{1}h_{4} & n_{2}h_{4} & 0 & h_{4} \end{array}\right], \tag{3.2}$$

where the coefficients are some functions of type

$$\begin{aligned} g_{1,2} &= g_{1,2}(x^{1},x^{2}),h_{3,4}=h_{3,4}(x^{1},x^{2},s), \\ w_{1,2} &= w_{1,2}(x^{1},x^{2},s),n_{1,2}=n_{1,2}(x^{1},x^{2},s). \end{aligned} \tag{3.3}$$

Both the inverse matrix (metric) as well the metric (3.2) is off–diagonal with respect to the coordinate basis

$$\partial_{\alpha}\equiv\frac{\partial}{du^{\alpha}}=(\partial_{i}=\frac{\partial}{dx^{i}},\partial_{a}=\frac{\partial}{dy^{a}}) \tag{3.4}$$

and, its dual basis,

$$d^{\alpha}\equiv du^{\alpha}=(d^{i}=dx^{i},d^{a}=dy^{a}). \tag{3.5}$$

The metric (3.1) with coefficients (3.2) can be equivalently rewritten in the diagonal form

$$\begin{aligned} \delta s^2 &= dt^2 + g_1(x)(dx^1)^2 + g_2(x)(dx^2)^2 \\ &\quad + h_3(x,s)(\delta y^3)^2 + h_4(x,s)(\delta y^4)^2, \end{aligned} \tag{3.6}$$

if instead the coordinate bases (3.4) and (3.5) we introduce the anholonomic frames (anisotropic bases)

$$\delta_\alpha \equiv \frac{\delta}{du^\alpha} = (\delta_i = \partial_i - N_i^b(u)\ \partial_b, \partial_a = \frac{\partial}{dy^a}) \tag{3.7}$$

and

$$\delta^\alpha \equiv \delta u^\alpha = (\delta^i = dx^i, \delta^a = dy^a + N_k^a(u)\ dx^k) \tag{3.8}$$

where the N–coefficients are parametrized

$$N_0^a = 0,\ N_{1,2}^3 = w_{1,2} \text{ and } N_{1,2}^4 = n_{1,2}$$

and define the associated nonlinear connection (N–connection) structure, see details in Refs [195, 199, 180].

Einstein equations with anholonomic variables

The metric (3.1) with coefficients (3.2) (equivalently, the d–metric (3.6)) is assumed to solve the 5D Einstein equations

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \kappa \Upsilon_{\alpha\beta}, \tag{3.9}$$

where κ and $\Upsilon_{\alpha\beta}$ are respectively the coupling constant and the energy–momentum tensor.

The nontrivial components of the Ricci tensor for the metric (3.1) with coefficients (3.2) (equivalently, the d–metric (3.6)) are

$$
\begin{aligned}
R_1^1 &= R_2^2 = -\frac{1}{2g_1g_2}[g_2^{\bullet\bullet} - \frac{g_1^{\bullet}g_2^{\bullet}}{2g_1} - \frac{(g_2^{\bullet})^2}{2g_2} + g_1^{''} - \frac{g_1^{'}g_2^{'}}{2g_2} - \frac{(g_1^{'})^2}{2g_1}], \quad (3.10)\\
R_3^3 &= R_4^4 = -\frac{\beta}{2h_3h_4}, \quad (3.11)\\
R_{31} &= -w_1\frac{\beta}{2h_4} - \frac{\alpha_1}{2h_4}, \quad (3.12)\\
R_{32} &= -w_2\frac{\beta}{2h_4} - \frac{\alpha_2}{2h_4},\\
R_{41} &= -\frac{h_4}{2h_3}\left[n_1^{**} + \gamma n_1^{*}\right], \quad (3.13)\\
R_{42} &= -\frac{h_4}{2h_3}\left[n_2^{**} + \gamma n_2^{*}\right],
\end{aligned}
$$

where, for simplicity, the partial derivatives are denoted $h^{\bullet} = \partial h/\partial x^1, f' = \partial f/\partial x^2$ and $f^* = \partial f/\partial s$.

The scalar curvature is computed

$$R = 2\left(R_1^1 + R_3^3\right).$$

In result of the obtained equalities for some Ricci and Einstein tensor components, we conclude that for the metric ansatz (3.2) the Einstein equations with matter sources are compatible if the coefficients of the energy–momentum d–tensor give with respect to anholonomic bases satisfy the conditions

$$\Upsilon_0^0 = \Upsilon_1^1 + \Upsilon_3^3, \Upsilon_1^1 = \Upsilon_2^2 = \Upsilon_1, \Upsilon_3^3 = \Upsilon_4^4 = \Upsilon_3, \quad (3.14)$$

and could be written in the form

$$
\begin{aligned}
R_1^1 &= -\kappa\Upsilon_3, \quad (3.15)\\
R_3^3 &= -\kappa\Upsilon_1, \quad (3.16)\\
R_{3\widehat{i}} &= \kappa\Upsilon_{3\widehat{i}}, \quad (3.17)\\
R_{4\widehat{i}} &= \kappa\Upsilon_{4\widehat{i}}, \quad (3.18)
\end{aligned}
$$

where $\widehat{i} = 1, 2$ and the left parts are given by the components of the Ricci tensor (3.10)-(3.13).

The Einstein equations (3.9), equivalently (3.15)–(3.18), reduce to this system of second order partial derivation equations:

$$
\begin{aligned}
g_2^{\bullet\bullet} - \frac{g_1^{\bullet} g_2^{\bullet}}{2g_1} - \frac{(g_2^{\bullet})^2}{2g_2} + g_1^{''} - \frac{g_1^{'} g_2^{'}}{2g_2} - \frac{(g_1^{'})^2}{2g_1} &= -2g_1 g_2 \Upsilon_3, & (3.19)\\
h_4^{**} - \frac{(h_4^*)^2}{2h_4} - \frac{h_4^* h_3^*}{2h_3} &= -2h_3 h_4 \Upsilon_1, & (3.20)\\
\beta w_i + \alpha_i &= -2h_4 \kappa \Upsilon_{3i}, & (3.21)\\
n_i^{**} + \gamma n_i^* &= -\frac{2h_3}{h_4} \kappa \Upsilon_{4i}, & (3.22)
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_1 &= h_4^{*\bullet} - \frac{h_4^*}{2}\left(\frac{h_3^{\bullet}}{h_3} + \frac{h_4^{\bullet}}{h_4}\right), & (3.23)\\
\alpha_2 &= h_4^{*\prime} - \frac{h_4^*}{2}\left(\frac{h_3^{\prime}}{h_3} + \frac{h_4^{\prime}}{h_4}\right), & (3.24)\\
\beta &= h_4^{**} - \frac{(h_4^*)^2}{2h_4} - \frac{h_4^* h_3^*}{2h_3}, & (3.25)\\
\gamma &= \frac{3}{2}\frac{h_4{}^*}{h_4} - \frac{h_3{}^*}{h_3}, & (3.26)
\end{aligned}
$$

and the partial derivatives are denoted, for instance,

$$
\begin{aligned}
g_2^{\bullet} &= \partial g_2/\partial x^1 = \partial g_2/\partial r, g_1^{'} = \partial g_1/\partial x^2 = \partial g_1/\theta,\\
h_3^* &= \partial h_3/\partial s = \partial h_3/\partial \varphi \text{ (or } \partial h_3/\partial y^4, \text{ for } s = y^4).
\end{aligned}
$$

Dirac equations in anisotropic space–times

The problem of definition of spinors in locally anisotropic spaces and in spaces with higher order anisotropy was solved in Refs. [180]. In this paper we consider locally anisotropic Dirac spinors given with respect to anholonomic frames with associated N–connection structure on a 5D (pseudo) Riemannian space $V^{(1,2,2)}$ constructed by a direct time extension of a 4D Riemannian space with two holonomic and two anholonomic variables.

Having an anisotropic d–metric

$$
\begin{aligned}
g_{\alpha\beta}(u) &= (g_{ij}(u), h_{ab}(u)) = (1, g_{\widehat{i}}(u), h_a(u)),\\
\widehat{i} &= 1,2; i = 0,1,2; a = 3,4,
\end{aligned}
$$

defined with respect to an anholonomic basis (1.16) we can easily define the funfbein (pentad) fields

$$
\begin{aligned}
f_{\underline{\mu}} &= f_{\underline{\mu}}^{\mu}\delta_{\mu} = \{f_{\underline{i}} = f_{\underline{i}}^{i}\delta_i, f_{\underline{a}} = f_{\underline{a}}^{a}\partial_a\}, & (3.27)\\
f^{\underline{\mu}} &= f_{\mu}^{\underline{\mu}}\delta^{\mu} = \{f^{\underline{i}} = f_{i}^{\underline{i}}d^i, , f^{\underline{a}} = f_{a}^{\underline{a}}\delta^a\}
\end{aligned}
$$

satisfying the conditions

$$\begin{aligned} g_{ij} &= f_i^{\underline{i}} f_j^{\underline{j}} g_{\underline{ij}} \text{ and } h_{ab} = f_a^{\underline{a}} f_b^{\underline{b}} h_{\underline{ab}}, \\ g_{\underline{ij}} &= diag[1, -1-1] \text{ and } h_{\underline{ab}} = diag[-1, -1]. \end{aligned}$$

For a diagonal d-metric of type (1.39) we have

$$f_i^{\underline{i}} = \sqrt{|g_i|}\delta_i^{\underline{i}} \text{ and } f_a^{\underline{a}} = \sqrt{|h_a|}\delta_a^{\underline{a}},$$

where $\delta_i^{\underline{i}}$ and $\delta_a^{\underline{a}}$ are Kronecker's symbols.

We can also introduce the corresponding funfbiends which are related with the off–diagonal metric ansatz (3.2) for $g_{\alpha\beta}$,

$$e_{\underline{\mu}} = e_{\underline{\mu}}^{\mu} \partial_\mu \text{ and } e^{\underline{\mu}} = e_{\mu}^{\underline{\mu}} \partial^\mu \tag{3.28}$$

satisfying the conditions

$$\begin{aligned} g_{\alpha\beta} &= e_\alpha^{\underline{\alpha}} e_\beta^{\underline{\beta}} g_{\underline{\alpha\beta}} \text{ for } g_{\underline{\alpha\beta}} = diag[1, -1, -1, -1, -1], \\ e_\alpha^{\underline{\alpha}} e_{\underline{\alpha}}^{\mu} &= \delta_\alpha^\mu \text{ and } e_\alpha^{\underline{\alpha}} e_{\underline{\mu}}^{\alpha} = \delta_{\underline{\mu}}^{\underline{\alpha}}. \end{aligned}$$

The Dirac spinor fields on locally anisotropic deformations of Taub NUT spaces,

$$\Psi(u) = [\Psi^{\overline{\alpha}}(u)] = [\psi^{\widehat{I}}(u), \chi_{\widehat{I}}(u)],$$

where $\widehat{I} = 0, 1,$ are defined with respect to the 4D Euclidean tangent subspace belonging the tangent space to $V^{(1,2,2)}$. The 4×4 dimensional gamma matrices $\gamma^{\underline{\alpha}'} = [\gamma^{\underline{1}'}, \gamma^{\underline{2}'}, \gamma^{\underline{3}'}, \gamma^{\underline{4}'}]$ are defined as to satisfy the relation

$$\left\{\gamma^{\underline{\alpha}'}, \gamma^{\underline{\beta}'}\right\} = 2g^{\underline{\alpha}'\underline{\beta}'}, \tag{3.29}$$

where $\left\{\gamma^{\underline{\alpha}'} \gamma^{\underline{\beta}'}\right\}$ is a symmetric commutator, $g^{\underline{\alpha}'\underline{\beta}'} = (-1, -1, -1, -1)$, which generates a Clifford algebra distinguished on two holonomic and two anholonomic directions (hereafter the primed indices will run values on the Euclidean and/or Riemannian, 4D component of the 5D pseudo–Riemannian spacetime). In order to extend the (3.29) relations for unprimed indices $\alpha, \beta...$ we conventionally complete the set of primed gamma matrices with a matrix $\gamma^{\underline{0}}$, i. .e. write $\gamma^{\underline{\alpha}} = [\gamma^{\underline{0}}, \gamma^{\underline{1}}, \gamma^{\underline{2}}, \gamma^{\underline{3}}, \gamma^{\underline{4}}]$ when

$$\left\{\gamma^{\underline{\alpha}}, \gamma^{\underline{\beta}}\right\} = 2g^{\underline{\alpha}\underline{\beta}}.$$

The coefficients of gamma matrices can be computed with respect to coordinate bases (1.2) or with respect to anholonomic bases (1.16) by using respectively the funfbein coefficients (3.27) and (3.28),

$$\gamma^{\alpha}(u) = e^{\alpha}_{\underline{\alpha}}(u)\gamma^{\underline{\alpha}} \text{ and } \widehat{\gamma}^{\beta}(u) = f^{\beta}_{\underline{\beta}}(u)\gamma^{\underline{\beta}},$$

were by $\gamma^{\alpha}(u)$ we denote the curved spacetime gamma matrices and by $\widehat{\gamma}^{\beta}(u)$ we denote the gamma matrices adapted to the N–connection structure.

The covariant derivation of Dirac spinor field $\Psi\left(u\right)$, $\bigtriangledown_{\alpha}\Psi$, can be defined with respect to a pentad decomposition of the off–diagonal metric (3.2)

$$\bigtriangledown_{\alpha}\Psi = \left[\partial_{\alpha} + \frac{1}{4}C_{\underline{\alpha}\underline{\beta}\underline{\gamma}}\left(u\right)\ e^{\underline{\alpha}}_{\alpha}\left(u\right)\gamma^{\underline{\beta}}\gamma^{\underline{\gamma}}\right]\Psi, \tag{3.30}$$

where the coefficients

$$C_{\underline{\alpha}\underline{\beta}\underline{\gamma}}\left(u\right) = \left(D_{\gamma}e^{\alpha}_{\underline{\alpha}}\right)e_{\underline{\beta}\alpha}e^{\gamma}_{\underline{\gamma}}$$

are called the rotation Ricci coefficients; the covariant derivative D_{γ} is defined by the usual Christoffel symbols for the off–diagonal metric.

We can also define an equivalent covariant derivation of the Dirac spinor field, $\overrightarrow{\bigtriangledown}_{\alpha}\Psi$, by using pentad decompositions of the diagonalized d–metric (1.39),

$$\overrightarrow{\bigtriangledown}_{\alpha}\Psi = \left[\delta_{\alpha} + \frac{1}{4}C^{[\delta]}_{\underline{\alpha}\underline{\beta}\underline{\gamma}}\left(u\right)\ f^{\underline{\alpha}}_{\alpha}\left(u\right)\gamma^{\underline{\beta}}\gamma^{\underline{\gamma}}\right]\Psi, \tag{3.31}$$

where there are introduced N–elongated partial derivatives and the coefficients

$$C^{[\delta]}_{\underline{\alpha}\underline{\beta}\underline{\gamma}}\left(u\right) = \left(D^{[\delta]}_{\gamma}f^{\alpha}_{\underline{\alpha}}\right)f_{\underline{\beta}\alpha}f^{\gamma}_{\underline{\gamma}}$$

are transformed into rotation Ricci d–coefficients which together with the d–covariant derivative $D^{[\delta]}_{\gamma}$ are defined by anholonomic pentads and anholonomic transforms of the Christoffel symbols.

For diagonal d–metrics the funfbein coefficients can be taken in their turn in diagonal form and the corresponding gamma matrix $\widehat{\gamma}^{\alpha}\left(u\right)$ for anisotropic curved spaces are proportional to the usual gamma matrix in flat spaces $\gamma^{\underline{\gamma}}$. The Dirac equations for locally anisotropic spacetimes are written in the simplest form with respect to anholonomic frames,

$$(i\widehat{\gamma}^{\alpha}\left(u\right)\overrightarrow{\bigtriangledown_{\alpha}} - \mu)\Psi = 0, \tag{3.32}$$

where μ is the mass constant of the Dirac field. The Dirac equations are the Euler equations for the Lagrangian

$$\begin{aligned}\mathcal{L}^{(1/2)}(u) &= \sqrt{|g|}\{[\Psi^{+}(u)\widehat{\gamma}^{\alpha}(u)\overrightarrow{\bigtriangledown_{\alpha}}\Psi(u) \\ &\quad -(\overrightarrow{\bigtriangledown_{\alpha}}\Psi^{+}(u))\widehat{\gamma}^{\alpha}(u)\Psi(u)] - \mu\Psi^{+}(u)\Psi(u)\},\end{aligned} \tag{3.33}$$

where by $\Psi^{+}(u)$ we denote the complex conjugation and transposition of the column $\Psi(u)$.

Varying $\mathcal{L}^{(1/2)}$ on d–metric (3.33) we obtain the symmetric energy–momentum d–tensor

$$\begin{aligned}\Upsilon_{\alpha\beta}(u) &= \frac{i}{4}[\Psi^{+}(u)\widehat{\gamma}_{\alpha}(u)\overrightarrow{\bigtriangledown_{\beta}}\Psi(u) + \Psi^{+}(u)\widehat{\gamma}_{\beta}(u)\overrightarrow{\bigtriangledown_{\alpha}}\Psi(u) \\ &\quad -(\overrightarrow{\bigtriangledown_{\alpha}}\Psi^{+}(u))\widehat{\gamma}_{\beta}(u)\Psi(u) - (\overrightarrow{\bigtriangledown_{\beta}}\Psi^{+}(u))\widehat{\gamma}_{\alpha}(u)\Psi(u)].\end{aligned} \tag{3.34}$$

We choose such spinor field configurations in curved spacetime as to be satisfied the conditions (3.14).

One can introduce similar formulas to (3.32)–(3.34) for spacetimes provided with off-diagonal metrics with respect to holonomic frames by changing of operators $\widehat{\gamma}_{\alpha}(u) \rightarrow \gamma_{\alpha}(u)$ and $\overrightarrow{\bigtriangledown_{\beta}} \rightarrow \bigtriangledown_{\beta}$.

3.1.2 Anisotropic Taub NUT – Dirac Spinor Solutions

By straightforward calculations we can verify that because the conditions $D_{\gamma}^{[\delta]} f_{\underline{\alpha}}^{\alpha} = 0$ are satisfied the Ricci rotation coefficients vanishes,

$$C_{\underline{\alpha\beta\gamma}}^{[\delta]}(u) = 0 \text{ and } \overrightarrow{\bigtriangledown_{\alpha}}\Psi = \delta_{\alpha}\Psi,$$

and the anisotropic Dirac equations (3.32) transform into

$$(i\widehat{\gamma}^{\alpha}(u)\,\delta_{\alpha} - \mu)\Psi = 0. \tag{3.35}$$

Further simplifications are possible for Dirac fields depending only on coordinates $(t, x^1 = r, x^2 = \theta)$, i. e. $\Psi = \Psi(x^k)$ when the equation (3.35) transforms into

$$(i\underline{\gamma^{0}}\partial_t + i\underline{\gamma^{1}}\frac{1}{\sqrt{|g_1|}}\partial_1 + i\underline{\gamma^{2}}\frac{1}{\sqrt{|g_2|}}\partial_2 - \mu)\Psi = 0.$$

The equation (3.35) simplifies substantially in ζ–coordinates

$$\left(t, \zeta^1 = \zeta^1(r,\theta), \zeta^2 = \zeta^2(r,\theta)\right),$$

defined as to be satisfied the conditions

$$\frac{\partial}{\partial \zeta^1} = \frac{1}{\sqrt{|g_1|}}\partial_1 \text{ and } \frac{\partial}{\partial \zeta^2} = \frac{1}{\sqrt{|g_2|}}\partial_2 \tag{3.36}$$

We get

$$(-i\gamma_{\underline{0}}\frac{\partial}{\partial t} + i\gamma_{\underline{1}}\frac{\partial}{\partial \zeta^1} + i\gamma_{\underline{2}}\frac{\partial}{\partial \zeta^2} - \mu)\Psi(t, \zeta^1, \zeta^2) = 0. \tag{3.37}$$

The equation (3.37) describes the wave function of a Dirac particle of mass μ propagating in a three dimensional Minkowski flat plane which is imbedded as an anisotropic distribution into a 5D pseudo–Riemannian spacetime.

The solution $\Psi = \Psi(t, \zeta^1, \zeta^2)$ of (3.37) can be written

$$\Psi = \left\{ \begin{array}{rcl} \Psi^{(+)}(\zeta) & = & \exp\left[-i(k_0 t + k_1\zeta^1 + k_2\zeta^2)\right]\varphi^0(k) \\ & & \text{for positive energy;} \\ \Psi^{(-)}(\zeta) & = & \exp\left[i(k_0 t + k_1\zeta^1 + k_2\zeta^2)\right]\chi^0(k) \\ & & \text{for negative energy,} \end{array} \right.$$

with the condition that k_0 is identified with the positive energy and $\varphi^0(k)$ and $\chi^0(k)$ are constant bispinors. To satisfy the Klein–Gordon equation we must have

$$k^2 = k_0^2 - k_1^2 - k_2^2 = \mu^2.$$

The Dirac equations implies

$$(\sigma^i k_i - \mu)\varphi^0(k) \text{ and } (\sigma^i k_i + \mu)\chi^0(k),$$

where $\sigma^i(i = 0, 1, 2)$ are Pauli matrices corresponding to a realization of gamma matrices as to a form of splitting to usual Pauli equations for the bispinors $\varphi^0(k)$ and $\chi^0(k)$.

In the rest frame for the horizontal plane parametrized by coordinates $\zeta = \{t, \zeta^1, \zeta^2\}$ there are four independent solutions of the Dirac equations,

$$\varphi^0_{(1)}(\mu, 0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ \varphi^0_{(2)}(\mu, 0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\chi^0_{(1)}(\mu, 0) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \ \chi^0_{(2)}(\mu, 0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

In order to satisfy the conditions (3.14) for compatibility of the equations (3.19)–(3.22) we must consider wave packets of type (for simplicity, we can use only superpositions of positive energy solutions)

$$\Psi^{(+)}(\zeta) = \int \frac{d^3p}{2\pi^3}\frac{\mu}{\sqrt{\mu^2+(k^2)^2}} \times \sum_{[\alpha]=1,2,3} b(p,[\alpha])\varphi^{[\alpha]}(k)\exp\left[-ik_i\zeta^i\right] \tag{3.38}$$

when the coefficients $b(p,[\alpha])$ define a current (the group velocity)

$$J^2 = \sum_{[\alpha]=1,2,3}\int \frac{d^3p}{2\pi^3}\frac{\mu}{\sqrt{\mu^2+(k^2)^2}}|b(p,[\alpha])|^2\frac{p^2}{\sqrt{\mu^2+(k^2)^2}} \equiv < \frac{p^2}{\sqrt{\mu^2+(k^2)^2}} >$$

with $|p^2| \sim \mu$ and the energy–momentum d–tensor (3.34) has the next nontrivial coefficients

$$\Upsilon^0_0 = 2\Upsilon(\zeta^1,\zeta^2) = k_0\Psi^+\gamma_0\Psi, \quad \Upsilon^1_1 = -k_1\Psi^+\gamma_1\Psi, \Upsilon^2_2 = -k_2\Psi^+\gamma_2\Psi \tag{3.39}$$

where the holonomic coordinates can be reexpressed $\zeta^i = \zeta^i(x^i)$. We must take two or more waves in the packet and choose such coefficients $b(p,[\alpha])$, satisfying corresponding algebraic equations, as to have in (3.39) the equalities

$$\Upsilon^1_1 = \Upsilon^2_2 = \Upsilon(\zeta^1,\zeta^2) = \Upsilon(x^1,x^2), \tag{3.40}$$

required by the conditions (3.34).

3.2 Taub NUT Solutions with Generic Local Anisotropy

The Kaluza-Klein monopole [76, 139] was obtained by embedding the Taub-NUT gravitational instanton into five-dimensional theory, adding the time coordinate in a trivial way. There are anisotropic variants of such solutions [199] when anisotropies are modelled by effective polarizations of the induced magnetic field. The aim of this Section is to analyze such Taub–NUT solutions for both cases of locally isotropic and locally anisotropic configurations.

3.2.1 A conformal transform of the Taub NUT metric

We consider the Taub NUT solutions and introduce a conformal transformation and a such redefinition of variables which will be useful for further generalizations to anisotropic vacuum solutions.

The Taub NUT solution

This locally isotropic solution of the 5D vacuum Einstein equations is expressed by the line element

$$\begin{aligned} ds^2_{(5D)} &= dt^2 + ds^2_{(4D)}; \\ ds^2_{(4D)} &= -V^{-1}(dr^2 + r^2 d\theta^2 + \sin^2\theta d\varphi^2) - V(dx^4 + A_i dx^i)^2 \end{aligned} \quad (3.41)$$

where

$$V^{-1} = 1 + \frac{m_0}{r}, m_0 = const.$$

The functions A_i are static ones associated to the electromagnetic potential,

$$A_r = 0, A_\theta = 0, A_\varphi = 4m_0(1 - \cos\theta)$$

resulting into "pure" magnetic field

$$\vec{B} = \text{rot}\,\vec{A} = m_0 \frac{\overrightarrow{r}}{r^3}\,. \quad (3.42)$$

of a Euclidean instanton; $\overrightarrow{r}$ is the spherical coordinate's unity vector. The spacetime defined by (3.41) has the global symmetry of the group $G_s = SO(3) \otimes U_4(1) \otimes T_t(1)$ since the line element is invariant under the global rotations of the Cartesian space coordinates and y^4 and t translations of the Abelian groups $U_4(1)$ and $T_t(1)$ respectively. We note that the $U_4(1)$ symmetry eliminates the so called NUT singularity if y^4 has the period $4\pi m_0$.

Conformally transformed Taub NUT metrics

With the aim to construct anisotropic generalizations it is more convenient to introduce a new 5th coordinate,

$$y^4 \to \varsigma = y^4 - \int \mu^{-1}(\theta,\varphi) d\xi(\theta,\varphi), \quad (3.43)$$

with the property that

$$d\varsigma + 4m_0(1 - \cos\theta)d\theta = dy^4 + 4m_0(1 - \cos\theta)d\varphi,$$

which holds for

$$d\xi = \mu(\theta,\varphi)d(\varsigma - y^4) = \frac{\partial \xi}{\partial \theta}d\theta + \frac{\partial \xi}{\partial \varphi}d\varphi,$$

when

$$\frac{\partial \xi}{\partial \theta} = 4m_0(1-\cos\theta)\mu, \quad \frac{\partial \xi}{\partial \varphi} = -4m_0(1-\cos\theta)\mu,$$

and, for instance,

$$\mu = (1-\cos\theta)^{-2}\exp[\theta - \varphi].$$

The changing of coordinate (3.43) describe a re–orientation of the 5th coordinate in a such way as we could have only one nonvanishing component of the electromagnetic potential

$$A_\theta = 4m_0\left(1-\cos\theta\right).$$

The next step is to perform a conformal transform,

$$ds^2_{(4D)} \rightarrow d\widehat{s}^2_{(4D)} = V ds^2_{(4D)}$$

and to consider the 5D metric

$$\begin{aligned} ds^2_{(5D)} &= dt^2 + d\widehat{s}^2_{(4D)}; \\ d\widehat{s}^2_{(4D)} &= -(dr^2 + r^2 d\theta^2) - r^2\sin^2\theta d\varphi^2 - V^2(d\zeta + A_\theta d\theta)^2, \end{aligned} \tag{3.44}$$

(not being an exact solution of the Einstein equations) which will transform into some exact solutions after corresponding anholonomic transforms.

Here, we emphasize that we chose the variant of transformation of a locally isotropic non–Einsteinian metrics into an anisotropic one solving the vacuum Einstein equations in order to illustrate a more simple procedure of construction of 5D vacuum metrics with generic local anisotropy. As a metter of principle we could remove vacuum isotropic solutions into vacuum anisotropic ones, but the formula in this case would became very combersome.. The fact of selection as an isotropic 4D Riemannian background just the metric from the linear interval $d\widehat{s}^2_{(4D)}$ can be treated as a conformal transformation of an instanton solution which is anisotropically deformed and put trivially (by extension to the time like coordinate) into a 5D metric as to generate a locally isotropic vacuum gravitational field.

3.2.2 Anisotropic Taub NUT solutions with magnetic polarization

We outline two classes of exact solutions of 5D vacuum Einstein equations with generic anisotropies (see details in Ref. [199]) which will be extended to configurations with spinor matter field source.

Solutions with angular polarization

The ansatz for a d–metric (1.39), with a distinguished anisotropic dependence on the angular coordinate φ, when $s=\varphi$, is taken in the form

$$\begin{aligned} \delta s^2 &= dt^2 - \delta s^2_{(4D)}, \\ \delta s^2_{(4D)} &= -(dr^2 + r^2 d\theta^2) - r^2 \sin^2\theta d\varphi^2 - V^2(r)\eta_4^2(\theta,\varphi)\delta\varsigma^2, \\ \delta\varsigma &= d\varsigma + n_2(\theta,\varphi)d\theta, \end{aligned}$$

where the values $\eta_4^2(\theta,\varphi)$ (we use non–negative values η_4^2 not changing the signature of metrics) and $n_2(\theta,\varphi)$ must be found as to satisfy the vacuum Einstein equations in the form (3.19)–(3.22). We can verify that the data

$$\begin{aligned} x^0 &= t, x^1 = r, x^2 = \theta, y^3 = s = \varphi, y^4 = \varsigma, \\ g_0 &= 1, g_1 = -1, g_2 = -r^2, h_3 = -r^2\sin^2\theta, \\ h_4 &= V^2(r)\,\eta^2_{(\varphi)}, \eta^2_{(\varphi)} = [1+\varpi(r,\theta)\varphi]^2, w_i = 0; \\ n_{0,1} &= 0; n_2 = n_{2[0]}(r,\theta) + n_{2[1]}(r,\theta)/[1+\varpi(r,\theta)\varphi]^2. \end{aligned} \quad (3.45)$$

give an exact solution. If we impose the condition to obtain in the locally isotropic limit just the metric (3.44), we have to choose the arbitrary functions from the general solution of (3.20) as to have

$$\eta^2_{(\varphi)} = [1+\varpi(r,\theta)\varphi]^2 \to 1 \text{ for } \varpi(r,\theta)\varphi \to 0.$$

For simplicity, we can analyze only angular anisotropies with $\varpi = \varpi(\theta)$, when

$$\eta^2_{(\varphi)} = \eta^2_{(\varphi)}(\theta,\varphi) = [1+\varpi(\theta)\varphi]^2.$$

In the locally isotropic limit of the solution for $n_2(r,\theta,\varphi)$, when $\varpi\varphi \to 0$, we could obtain the particular magnetic configuration contained in the metric (3.44) if we impose the condition that

$$n_{2[0]}(r,\theta) + n_{2[1]}(r,\theta) = A_\theta = 4m_0(1-\cos\theta),$$

which defines only one function from two unknown values $n_{2[0]}(r,\theta)$ and $n_{2[1]}(r,\theta)$. This could have a corresponding physical motivation. From the usual Kaluza–Klein procedure we induce the 4D gravitational field (metric) and 4D electro–magnetic field (potentials A_i), which satisfy the Maxwell equations in 4D pseudo–Riemannian space–time. For the case of spherical, locally isotropic, symmetries the Maxwell equations can be written for vacuum magnetic fields without any polarizations. When we introduce into consideration anholonomic constraints and locally anisotropic gravitational configurations the effective magnetic field could be effectively renormalized by higher dimension gravitational field. This effect, for some classes of anisotropies, can be modeled by considering that the constant m_0 is polarized,

$$m_0 \to m(r,\theta,\varphi) = m_0 \eta_m(r,\theta,\varphi)$$

for the electro–magnetic potential and resulting magnetic field. For "pure" angular anisotropies we write that

$$\begin{aligned} n_2(\theta,\varphi) &= n_{2[0]}(\theta) + n_{2[1]}(\theta)/[1+\varpi(\theta)\varphi]^2 \\ &= 4m_0\eta_m(\theta,\varphi)(1-\cos\theta), \end{aligned}$$

for

$$\eta^2_{(\varphi)}(\theta,\varphi) = \eta^2_{(\varphi)[0]}(\theta) + \eta^2_{(\varphi)[1]}(\theta)/[1+\varpi(\theta)\varphi]^2.$$

This could result in a constant angular renormalization even $\varpi(\theta)\varphi \to 0$.

Solutions with extra–dimension induced polarization

Another class of solutions is constructed if we consider a d–metric of the type (1.39), when $s=\varsigma$, with anisotropic dependence on the 5th coordinate ς,

$$\begin{aligned} \delta s^2 &= dt^2 - \delta s^2_{(4D)}, \\ \delta s^2_{(4D)} &= -(dr^2 + r^2 d\theta^2) - r^2\sin^2\theta d\varphi^2 - V^2(r)\eta^2_{(\varsigma)}(\theta,\varsigma)\delta\varsigma^2, \\ \delta\varsigma &= d\varsigma + w_3(\theta,\varsigma)d\theta, \end{aligned}$$

where, for simplicity, we omit possible anisotropies on variable r, i. e. we state that $\eta_{(\varsigma)}$ and w_2 are not functions on r.

The data for a such solution are

$$\begin{aligned} x^0 &= t, x^1 = r, x^2 = \theta, y^3 = s = \varsigma, y^4 = \varphi, \\ g_0 &= 1, g_1 = -1, g_2 = -r^2, h_4 = -r^2\sin^2\theta, \\ h_3 &= V^2(r)\eta^2_{(\varsigma)}, \eta^2_{(\varsigma)} = \eta^2_{(\varsigma)}(r,\theta,\varsigma), n_{0,1} = 0; \\ w_{0,1} &= 0, w_2 = 4m_0\eta_m(\theta,\varsigma)(1-\cos\theta), n_0 = 0, \\ n_{1,2} &= n_{1,2[0]}(r,\theta) + n_{1,2[1]}(r,\theta)\int \eta^{-3}_{(\varsigma)}(r,\theta,\varsigma)d\varsigma, \end{aligned} \quad (3.46)$$

where the function $\eta_{(\varsigma)} = \eta_{(\varsigma)}(r, \theta, \varsigma)$ is an arbitrary one as follow for the case $h_4^* = 0$, for angular polarizations we state, for simplicity, that $\eta_{(\varsigma)}$ does not depend on r, i. e. $\eta_{(\varsigma)} = \eta_{(\varsigma)}(\theta, \varsigma)$. We chose the coefficient

$$w_4 = 4m_0\eta_m\left(\theta, \varsigma\right)\left(1 - \cos\theta\right)$$

as to have compatibility with the locally isotropic limit when $w_2 \simeq A_\theta$ with a "polarization" effect modeled by $\eta_m\left(\theta, \varsigma\right)$, which could have a constant component $\eta_m \simeq \eta_{m[0]} = const$ for small anisotropies. In the simplest cases we can fix the conditions $n_{1,2[0,1]}\left(r, \theta\right) = 0$. All functions $\eta^2_{(\varsigma)}, \eta_m$ and $n_{1,2[0,1]}$ can be treated as some possible induced higher dimensional polarizations.

3.3 Anisotropic Taub NUT–Dirac Fields

In this Section we construct two new classes of solutions of the 5D Einstein–Dirac fields in a manner as to extend the locally anisotropic Taub NUT metrics defined by data (3.45) and (3.46) as to be solutions of the Einstein equations (3.19)–(3.22) with a nonvanishing diagonal energy momentum d–tensor

$$\Upsilon^\alpha_\beta = \{2\Upsilon(r,\theta), \Upsilon(r,\theta), \Upsilon(r,\theta), 0, 0\}$$

for a Dirac wave packet satisfying the conditions (3.39) and (3.40).

3.3.1 Dirac fields and angular polarizations

In order to generate from the data (3.45) a new solution with Dirac spinor matter field we consider instead of a linear dependence of polarization,

$$\eta_{(\varphi)} \sim [1 + \varpi\left(r, \theta\right)\varphi],$$

an arbitrary function $\eta_{(\varphi)}\left(r, \theta, \varphi\right)$ for which

$$h_4 = V^2(r)\eta^2_{(\varphi)}\left(r, \theta, \varphi\right)$$

is an exact solution of the equation (3.20) with $\Upsilon_1 = \Upsilon\left(r, \theta\right)$. With respect to the variable $\eta^2_{(\varphi)}\left(r, \theta, \varphi\right)$ this component of the Einstein equations becomes linear

$$\eta^{**}_{(\varphi)} + r^2\sin^2\theta\Upsilon\eta_{(\varphi)} = 0 \qquad (3.47)$$

which is a second order linear differential equation on variable φ with parametric dependencies of the coefficient $r^2\sin^2\theta\Upsilon$ on coordinates (r, θ). The solution of equation (3.47) is to be found following the method outlined in Ref. [82]:

$$\begin{aligned} \eta_{(\varphi)} &= C_1(r,\theta)\cosh[\varphi r\sin\theta\sqrt{|\Upsilon(r,\theta)|}+C_2(r,\theta)], \\ & \quad \Upsilon(r,\theta)<0; \qquad (3.48) \\ &= C_1(r,\theta)+C_2(r,\theta)\varphi, \Upsilon(r,\theta)=0; \qquad (3.49) \\ &= C_1(r,\theta)\cos[\varphi r\sin\theta\sqrt{\Upsilon(r,\theta)}+C_2(r,\theta)], \\ & \quad \Upsilon(r,\theta)>0, \qquad (3.50) \end{aligned}$$

where $C_{1,2}(r,\theta)$ are some functions to be defined from some boundary conditions. The first solution (3.48), for negative densities of energy should be excluded as unphysical, the second solution (3.49) is just that from (3.45) for the vacuum case. A new interesting physical situation is described by the solution (3.50) when we obtain a Taub NUT anisotropic metric with periodic anisotropic dependencies on the angle φ where the periodicity could variate on coordinates (r,θ) as it is defined by the energy density $\Upsilon(r,\theta)$. For simplicity, we can consider a package of spinor waves with constant value of $\Upsilon=\Upsilon_0$ and fix some boundary and coordinate conditions when $C_{1,2}=C_{1,2[0]}$ are constant. This type of anisotropic Taub NUT solutions are described by a d–metric coefficient

$$h_4=V^2(r)C_{1[0]}^2\cos^2[\varphi r\sin\theta\sqrt{\Upsilon_0}+C_{2[0]}]. \qquad (3.51)$$

Putting this value into the formulas (3.23), (3.24) and (3.25) for coefficients in equations (3.21) we can express $\alpha_{1,2}=\alpha_{1,2}[h_3,h_4,\Upsilon_0]$ and $\beta=\beta[h_3,h_4,\Upsilon_0]$ (we omit these rather simple but cumbersome formulas) and in consequence we can define the values $w_{1,2}$ by solving linear algebraic equations:

$$w_{1,2}(r,\theta,\varphi)=\alpha_{1,2}(r,\theta,\varphi)/\beta(r,\theta,\varphi).$$

Having defined the values (3.51) it is a simple task of two integrations on φ in order to define

$$\begin{aligned} n_2 &= n_{2[0]}(r,\theta)\left[\ln\frac{1+\cos\tilde{\kappa}}{1-\cos\tilde{\kappa}}+\frac{1}{1-\cos\tilde{\kappa}}+\frac{1}{1-\sin\tilde{\kappa}}\right] \\ & \quad +n_{2[1]}(r,\theta), \qquad (3.52) \end{aligned}$$

were

$$\tilde{\kappa}=\varphi r\sin\theta\sqrt{\Upsilon_0}+C_{2[0]},$$

$n_{2[0,1]}(r,\theta)$ are some arbitrary functions to be defined by boundary conditions. We put $n_{0,1}=0$ to obtain in the vacuum limit the solution (3.45).

Finally, we can summarize the data defining an exact solution for an anisotropic (on angle φ) Dirac wave packet – Taub NUT configuration:

$$\begin{aligned} x^0 &= t, x^1 = r, x^2 = \theta, y^3 = s = \varphi, y^4 = \varsigma, \\ g_0 &= 1, g_1 = -1, g_2 = -r^2, h_3 = -r^2 \sin^2\theta, \\ h_4 &= V^2(r)\,\eta^2_{(\varphi)}, \eta_{(\varphi)} = C_1(r,\theta)\cos\tilde{\kappa}(r,\theta,\varphi), \\ w_i &= 0, n_{0,1} = 0, n_2 = n_2(r,\theta,\tilde{\kappa}(r,\theta,\varphi)) \text{ see (3.52)}, \\ \Psi &= \Psi^{(+)}\left(\zeta^{1,2}(x^1,x^2)\right) \text{ see (3.38)}, \\ \Upsilon &= \Upsilon\left(\zeta^{1,2}(x^1,x^2)\right) \text{ see (3.39)}. \end{aligned} \tag{3.53}$$

This solution will be extended to additional soliton anisotropic configurations in the next Section.

3.3.2 Dirac fields and extra dimension polarizations

Now we consider a generalization of the data (3.46) for generation of a new solution, with generic local anisotropy on extra dimension 5th coordinate, of the Einstein – Dirac equations. Following the equation (3.21) we conclude that there are not nonvacuum solutions of the Einstein equations (with $\Upsilon \neq 0$) if $h_4^* = 0$ which impose the condition $\Upsilon = 0$ for $h_3, h_4 \neq 0$. So, we have to consider that the d–metric component $h_4 = -r^2\sin^2\theta$ from the data (3.46) is generalized to a function $h_4(r,\theta,\varsigma)$ satisfying a second order nonlinear differential equation on variable ς with coefficients depending parametrically on coordinates (r,θ). The equation (i. e. (3.21)) can be linearized (see Ref. [82]) if we introduce a new variable $h_4 = h^2$,

$$h^{**} - \frac{h_3^*}{2h_3}h^* + h_3\Upsilon h = 0,$$

which, in its turn, can be transformed :

a) to a Riccati form if we introduce a new variable v, for which $h = v^*/v$,

$$v^* + v^2 - \frac{h_3^*}{2h_3}v + h_3\Upsilon = 0; \tag{3.54}$$

b) to the so–called normal form [82],

$$\lambda^{**} + I\lambda = 0, \tag{3.55}$$

obtained by a redefinition of variables like

$$\lambda = h\exp\left[-\frac{1}{4}\int\frac{h_3^*}{h_3}d\varsigma\right] = h\ h_3^{-1/4}$$

where

$$I = h_3 \Upsilon - \frac{1}{16} \frac{h_3^*}{h_3} + \frac{1}{4} \left(\frac{h_3^*}{h_3} \right)^* .$$

We can construct explicit series and/or numeric solutions (for instance, by using Mathematica or Maple programs) of both type of equations (3.54) and normal (3.55) for some stated boundary conditions and type of polarization of the coefficient $h_3(r,\theta,\varsigma) = V^2(r)\,\eta^2_{(\varsigma)}(r,\theta,\varsigma)$ and, in consequence, to construct different classes of solutions for $h_4(r,\theta,\varsigma)$. In order to have compatibility with the data (3.46) we must take h_4 in the form

$$h_4(r,\theta,\varsigma) = -r^2 \sin^2\theta + h_{4(\varsigma)}(r,\theta,\varsigma),$$

where $h_{4(\varsigma)}(r,\theta,\varsigma)$ vanishes for $\Upsilon \to 0$.

Having defined a value of $h_4(r,\theta,\varsigma)$ we can compute the coefficients (3.23), (3.24) and (3.25) and find from the equations (3.21)

$$w_{1,2}(r,\theta,\varsigma) = \alpha_{1,2}(r,\theta,\varsigma) / \beta(r,\theta,\varsigma).$$

From the equations (3.22), after two integrations on variable ς one obtains the values of $n_{1,2}(r,\theta,\varsigma)$. Two integrations of equations (3.22) define

$$n_i(r,\theta,\varsigma) = n_{i[0]}(r,\theta) \int_0^\varsigma dz \int_0^z ds P(r,\theta,s) + n_{i[1]}(r,\theta),$$

where

$$P \equiv \frac{1}{2}\left(\frac{h_3^*}{h_3} - 3\frac{h_4^*}{h_4}\right)$$

and the functions $n_{i[0]}(r,\theta)$ and $n_{i[1]}(r,\theta)$ on (r,θ) have to be defined by solving the Cauchy problem. The boundary conditions of both type of coefficients $w_{1,2}$ and $n_{1,2}$ should be expressed in some forms transforming into corresponding values for the data (3.46) if the source $\Upsilon \to 0$. We omit explicit formulas for exact Einstein–Dirac solutions with ς–polarizations because their forms depend very strongly on the type of polarizations and vacuum solutions.

3.4 Anholonomic Dirac–Taub NUT Solitons

In the next subsections we analyze two explicit examples when the spinor field induces two dimensional, depending on three variables, solitonic anisotropies.

3.4.1 Kadomtsev–Petviashvili type solitons

By straightforward verification we conclude that the d–metric component $h_4(r,\theta,s)$ could be a solution of Kadomtsev–Petviashvili (KdP) equation [81] (the first methods of integration of 2+1 dimensional soliton equations where developed by Dryuma [55] and Zakharov and Shabat [212])

$$h_4^{**}+\epsilon\left(\dot{h}_4+6h_4h_4^{\prime}+h_4^{\prime\prime\prime}\right)^{\prime}=0,\epsilon=\pm 1, \tag{3.56}$$

if the component $h_3(r,\theta,s)$ satisfies the Bernoulli equations [82]

$$h_3^{*}+Y\left(r,\theta,s\right)\left(h_3\right)^2+F_{\epsilon}\left(r,\theta,s\right)h_3=0, \tag{3.57}$$

where, for $h_4^{*}\neq 0$,

$$Y\left(r,\theta,s\right)=\kappa\Upsilon\frac{h_4}{h_4^{*}}, \tag{3.58}$$

and

$$F_{\epsilon}\left(r,\theta,s\right)=\frac{h_4^{*}}{h_4}+\frac{2\epsilon}{h_4^{*}}\left(\dot{h}_4+6h_4h_4^{\prime}+h_4^{\prime\prime\prime}\right)^{\prime}.$$

The three dimensional integral variety of (3.57) is defined by formulas

$$h_3^{-1}\left(r,\theta,s\right)=h_{3(x)}^{-1}\left(r,\theta\right)E_{\epsilon}\left(x^{i},s\right)\times\int\frac{Y\left(r,\theta,s\right)}{E_{\epsilon}\left(r,\theta,s\right)}ds,$$

where

$$E_{\epsilon}\left(r,\theta,s\right)=\exp\int F_{\epsilon}\left(r,\theta,s\right)ds$$

and $h_{3(x)}\left(r,\theta\right)$ is a nonvanishing function.

In the vacuum case $Y\left(r,\theta,s\right)=0$ and we can write the integral variety of (3.57)

$$h_3^{(vac)}\left(r,\theta,s\right)=h_{3(x)}^{(vac)}\left(r,\theta\right)\exp\left[-\int F_{\epsilon}\left(r,\theta,s\right)ds\right].$$

We conclude that a solution of KdP equation (3.57) could be generated by a non–perturbative component $h_4(r,\theta,s)$ of a diagonal h–metric if the second component $h_3\left(r,\theta,s\right)$ is a solution of Bernoulli equations (3.57) with coefficients determined both by h_4 and its partial derivatives and by the Υ_1^1 component of the energy–momentum d–tensor (see (3.40)). The parameters (coefficients) of (2+1) dimensional KdV solitons are induced by gravity and spinor constants and spinor field configuration defining locally anisotropic interactions of packets of Dirac's spinor waves.

3.4.2 (2+1) sine–Gordon type solitons

In a similar manner we can prove that solutions $h_4(r,\theta,s)$ of (2+1) sine–Gordon equation (see, for instance, [61, 90, 207])

$$h_4^{**} + h_4^{''} - \ddot{h}_4 = \sin(h_4)$$

also induce solutions for $h_3\,(r,\theta,s)$ following from the Bernoulli equation

$$h_3^{*} + \kappa E(r,\theta)\frac{h_4}{h_4^{*}}(h_3)^2 + F\,(r,\theta,s)\,h_3 = 0, h_4^{*} \neq 0,$$

where

$$F\,(r,\theta,s) = \frac{h_4^{*}}{h_4} + \frac{2}{h_4^{*}}\left[h_4^{''} - \ddot{h}_4 - \sin(h_4)\right].$$

The general solutions (with energy–momentum sources and in vacuum cases) are constructed by a corresponding redefinition of coefficients in the formulas from the previous subsection. We note that we can consider both type of anisotropic solitonic polarizations, depending on angular variable φ or on extra dimension coordinate ς. Such classes of solutions of the Einstein–Dirac equations describe three dimensional spinor wave packets induced and moving self–consistently on solitonic gravitational locally anisotropic configurations. In a similar manner, we can consider Dirac wave packets generating and propagating on locally anisotropic black hole (with rotation ellipsoid horizons), black tori, anisotropic disk and two or three dimensional black hole anisotropic gravitational structures [195]. Finally, we note that such gravitational solitons are induced by Dirac field matter sources and are different from those soliton solutions of vacuum Einstein equations originally considered by Belinski and Zakharov [31].

Finally, we conclude that we have argued that the anholonomic frame method can be applied for construction on new classes of Einstein–Dirac equations in five dimensional (5D) space–times. Subject to a form of metric ansatz with dependencies of coefficients on two holonomic and one anholonomic variables we obtained a very simplified form of field equations which admit exact solutions. We have identified two classes of solutions describing Taub NUT like metrics with anisotropic dependencies on angular parameter or on the fifth coordinate. We have shown that both classes of anisotropic vacuum solutions can be generalized to matter sources with the energy–momentum tensor defined by some wave packets of Dirac fields. Although the Dirac equation is a quantum one, in the quasi–classical approximation we can consider such spinor fields as some spinor waves propagating in a three dimensional Minkowski plane which is imbedded in a self–consistent manner in a Taub–NUT anisotropic space–time. At the classical level it should be emphasized that the results of this paper are very general in nature, depending in a crucial way only on the locally Lorentzian nature of 5D space–time and on the supposition that this space–time is constructed as a trivial time extension of 4D space–times. We have proved that the new classes of solutions admit generalizations to nontrivial topological configurations of 3D dimensional solitons (induced by anisotropic spinor matter) defined as solutions Kadomtsev–Petviashvili or sine–Gordon equations.

Part II

Anisotropic Spinors

Spinor variables and interactions of spinor fields on Finsler spaces were used in a heuristic manner, for instance, in works [19, 124], where the problem of a rigorous definition of spinors for locally anisotropic spaces was not considered. Here we note that, in general, the nontrivial nonlinear connection and torsion structures and possible incompatibility of metric and connections makes the solution of the mentioned problem very sophisticate. The geometric definition of locally anisotropic spinors and a detailed study of the relationship between Clifford, spinor and nonlinear and distinguished connections structures in vector bundles, generalized Lagrange and Finsler spaces are presented in Refs. [163, 162, 165].

The purpose of this Part is to summarize our investigations [163, 162, 165, 186, 169] on formulation of the theory of classical and quantum field interactions on locally anisotropic spaces. We receive primary attention to the development of the necessary geometric framework: to propose an abstract spinor formalism and formulate the differential geometry of locally anisotropic spaces (the second step after the definition of locally anisotropic spinors in [163, 162]). The next step is the investigation of locally anisotropic interactions of fundamental fields on generic locally anisotropic spaces [165].

For our considerations on the locally anisotropic spinor theory it will be convenient to extend the Penrose and Rindler abstract index formalism [127, 128, 129] (see also the Luehr and Rosenbaum index free methods [91]) proposed for spinors on locally isotropic spaces. We note that in order to formulate the locally anisotropic physics usually we have dimensions $d > 4$ for the fundamental locally anisotropic space-time. In this case the 2-spinor calculus does not play a preferential role.

Chapter 4

Anisotropic Clifford Structures

If a nonlinear connection structure is defined on a vector (covector, or higher order vector–covector) bundle, or on a pseudo–Riemannian spacetime, the geometrical objects on this space are distinguished into some "horizontal" and "vertical" (co-vertical, or higher order vertical–covertical) invariant components. Our idea on definition of Clifford and spinor structure on such locally anisotropic spaces is to consider distinguished Cliffor algebras, which consists from blocks of usual Clifford algebras for every horizontal and vertical subspace (for every "shall" of higher order anisotropies). For symplicity, we restrict our constructions only to vector bundles (the covector bundles with respective Clifford co-algebras are similar dual constructions [198], we can for instance to develop a respective theory fo Clifford co–structures on Hamilton and Cartan spaces).

4.1 Distinguished Clifford Algebras

The typical fiber of a vector bunde (v-bundle) ξ_d , $\pi_d :\ HE \oplus VE \rightarrow E$ is a d-vector space, $\mathcal{F} = h\mathcal{F} \oplus v\mathcal{F}$, split into horizontal $h\mathcal{F}$ and vertical $v\mathcal{F}$ subspaces, with metric $G(g, h)$ induced by v-bundle metric (1.39). Clifford algebras (see, for example, Refs. [83, 154, 129]) formulated for d-vector spaces will be called Clifford d-algebras [163, 162] . In this section we shall consider the main properties of Clifford d-algebras. The proof of theorems will be based on the technique developed in Ref. [83] correspondingly adapted to the distinguished character of spaces in consideration.

Let k be a number field (for our purposes $k = \mathcal{R}$ or $k = \mathcal{C}, \mathcal{R}$ and $\mathcal{C}$, are, respectively real and complex number fields) and define $\mathcal{F}$, as a d-vector space on k provided with nondegenerate symmetric quadratic form (metric) G. Let C be an algebra on k (not necessarily commutative) and $j\ :\ \mathcal{F} \rightarrow C$

a homomorphism of underlying vector spaces such that $j(u)^2 = G(u) \cdot 1$ (1 is the unity in algebra C and d-vector $u \in \mathcal{F}$). We are interested in definition of the pair (C, j) satisfying the next universitality conditions. For every k-algebra A and arbitrary homomorphism $\varphi : \mathcal{F} \to A$ of the underlying d-vector spaces, such that $(\varphi(u))^2 \to G(u) \cdot 1$, there is a unique homomorphism of algebras $\psi : C \to A$ transforming the diagram 1 into a commutative one.

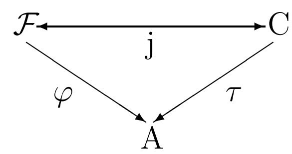

Figure 4.1: Diagram 1

The algebra solving this problem will be denoted as $C(\mathcal{F}, A)$ [equivalently as $C(G)$ or $C(\mathcal{F})$] and called as Clifford d-algebra associated with pair $(\mathcal{F}, G)$.

Theorem 4.1. *The above-presented diagram has a unique solution* (C, j) *up to isomorphism.*

Proof: (We adapt for d-algebras that of Ref. [83], p. 127.) For a universal problem the uniqueness is obvious if we prove the existence of solution $C(G)$. To do this we use tensor algebra $\mathcal{L}^{(F)} = \oplus \mathcal{L}^{pr}_{qs}(\mathcal{F}) = \oplus_{i=0}^{\infty} T^i(\mathcal{F})$, where $T^0(\mathcal{F}) = k$ and $T^i(\mathcal{F}) = k$ and $T^i(\mathcal{F}) = \mathcal{F} \otimes ... \otimes \mathcal{F}$ for $i > 0$. Let $I(G)$ be the bilateral ideal generated by elements of form $\epsilon(u) = u \otimes u - G(u) \cdot 1$ where $u \in \mathcal{F}$ and 1 is the unity element of algebra $\mathcal{L}(\mathcal{F})$. Every element from $I(G)$ can be written as $\sum_i \lambda_i \epsilon(u_i) \mu_i$, where $\lambda_i, \mu_i \in \mathcal{L}(\mathcal{F})$ and $u_i \in \mathcal{F}$. Let $C(G) = \mathcal{L}(\mathcal{F})/I(G)$ and define $j : \mathcal{F} \to C(G)$ as the composition of monomorphism $i : \mathcal{F} \to L^1(\mathcal{F}) \subset \mathcal{L}(\mathcal{F})$ and projection $p : \mathcal{L}(\mathcal{F}) \to C(G)$. In this case pair $(C(G), j)$ is the solution of our problem. From the general properties of tensor algebras the homomorphism $\varphi : \mathcal{F} \to A$ can be extended to $\mathcal{L}(\mathcal{F})$, i.e., the diagram 2 is commutative, where ρ is a monomorphism

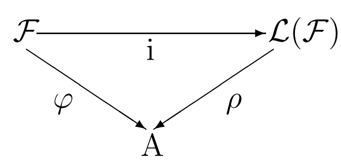

Figure 4.2: Diagram 2

of algebras. Because $(\varphi(u))^2 = G(u) \cdot 1$, then ρ vanishes on ideal $I(G)$ and in this case the necessary homomorphism τ is defined. As a consequence of uniqueness of ρ, the homomorphism τ is unique.

Tensor d-algebra $\mathcal{L}(\mathcal{F})$ can be considered as a $\mathcal{Z}/2$ graded algebra. Really, let us introduce $\mathcal{L}^{(0)}(\mathcal{F}) = \sum_{i=1}^{\infty} T^{2i}(\mathcal{F})$ and $\mathcal{L}^{(1)}(\mathcal{F}) = \sum_{i=1}^{\infty} T^{2i+1}(\mathcal{F})$. Setting $I^{(\alpha)}(G) = I(G) \cap \mathcal{L}^{(\alpha)}(\mathcal{F})$. Define $C^{(\alpha)}(G)$ as $p\left(\mathcal{L}^{(\alpha)}(\mathcal{F})\right)$, where $p : \mathcal{L}(\mathcal{F}) \rightarrow C(G)$ is the canonical projection. Then $C(G) = C^{(0)}(G) \oplus C^{(1)}(G)$ and in consequence we obtain that the Clifford d-algebra is $\mathcal{Z}/2$ graded.

It is obvious that Clifford d-algebra functorially depends on pair $(\mathcal{F}, G)$. If $f : \mathcal{F} \rightarrow \mathcal{F}'$ is a homomorphism of k-vector spaces, such that $G'(f(u)) = G(u)$, where G and G' are, respectively, metrics on $\mathcal{F}$ and $\mathcal{F}'$, then f induces an homomorphism of d-algebras

$$C(f) : C(G) \rightarrow C(G')$$

with identities $C(\varphi \cdot f) = C(\varphi) C(f)$ and $C(Id_{\mathcal{F}}) = Id_{C(\mathcal{F})}$.

If $\mathcal{A}^{\alpha}$ and $\mathcal{B}^{\beta}$ are $\mathcal{Z}/2$–graded d–algebras, then their graded tensorial product $\mathcal{A}^{\alpha} \otimes \mathcal{B}^{\beta}$ is defined as a d-algebra for k-vector d-space $\mathcal{A}^{\alpha} \otimes \mathcal{B}^{\beta}$ with the graded product induced as $(a \otimes b)(c \otimes d) = (-1)^{\alpha\beta} ac \otimes bd$, where $b \in \mathcal{B}^{\alpha}$ and $c \in \mathcal{A}^{\alpha}$ $\quad (\alpha, \beta = 0, 1)$.

Now we reformulate for d–algebras the Chevalley theorem [45]:

Theorem 4.2. *The Clifford d-algebra $C(h\mathcal{F} \oplus v\mathcal{F}, g + h)$ is naturally isomorphic to $C(g) \otimes C(h)$.*

Proof. Let $n : h\mathcal{F} \rightarrow C(g)$ and $n' : v\mathcal{F} \rightarrow C(h)$ be canonical maps and map
$m : h\mathcal{F} \oplus v\mathcal{F} \rightarrow C(g) \otimes C(h)$ is defined as $m(x, y) = n(x) \otimes 1 + 1 \otimes n'(y)$, $x \in h\mathcal{F}, y \in v\mathcal{F}$. We have $(m(x, y))^2 = \left[(n(x))^2 + (n'(y))^2\right] \cdot 1 = [g(x) + h(y)]$. Taking into account the universality property of Clifford d-algebras we conclude that m induces the homomorphism

$$\zeta : C(h\mathcal{F} \oplus v\mathcal{F}, g + h) \rightarrow C(h\mathcal{F}, g) \widehat{\otimes} C(v\mathcal{F}, h).$$

We also can define a homomorphism

$$\upsilon : C(h\mathcal{F}, g) \widehat{\otimes} C(v\mathcal{F}, h) \rightarrow C(h\mathcal{F} \oplus v\mathcal{F}, g + h)$$

by using formula $\upsilon(x \otimes y) = \delta(x) \delta'(y)$, where homomorphysms δ and δ' are, respectively, induced by imbeddings of $h\mathcal{F}$ and $v\mathcal{F}$ into $h\mathcal{F} \oplus v\mathcal{F}$:

$$\delta : C(h\mathcal{F}, g) \rightarrow C(h\mathcal{F} \oplus v\mathcal{F}, g + h),$$

$$\delta' : C(v\mathcal{F}, h) \rightarrow C(h\mathcal{F} \oplus v\mathcal{F}, g + h).$$

Because $x\in C^{(\alpha)}(g)$ and $y\in C^{(\alpha)}(g)$, we have

$$\delta(x)\,\delta'(y)=(-1)^{(\alpha)}\,\delta'(y)\,\delta(x).$$

Superpositions of homomorphisms ζ and υ lead to identities

$$\upsilon\zeta=Id_{C(h\mathcal{F},g)\widehat{\otimes}C(v\mathcal{F},h)},\ \zeta\upsilon=Id_{C(h\mathcal{F},g)\widehat{\otimes}C(v\mathcal{F},h)}. \tag{4.1}$$

Really, the d-dalgebra $C(h\mathcal{F}\oplus v\mathcal{F},g+h)$ is generated by elements of type $m(x,y)$. Calculating

$$\begin{aligned}\upsilon\zeta(m(x,y)) &= \upsilon(n(x)\otimes 1+1\otimes n'(y))\\ &= \delta(n(x))\,\delta(n'(y))=m(x,0)+m(0,y)=m(x,y),\end{aligned}$$

we prove the first identity in (4.1).

On the other hand, d-algebra $C(h\mathcal{F},g)\widehat{\otimes}C(v\mathcal{F},h)$ is generated by elements of type $n(x)\otimes 1$ and $1\otimes n'(y)$, we prove the second identity in (4.1).

Following from the above -mentioned properties of homomorphisms ζ and υ we can assert that the natural isomorphism is explicitly constructed.□

In consequence of theorem 4.2 we conclude that all operations with Clifford d-algebras can be reduced to calculations for $C(h\mathcal{F},g)$ and $C(v\mathcal{F},h)$ which are usual Clifford algebras of dimension 2^n and, respectively, 2^m [83, 21].

Of special interest is the case when $k=\mathcal{R}$ and $\mathcal{F}$ is isomorphic to vector space $\mathcal{R}^{p+q,a+b}$ provided with quadratic form $-x_1^2-...-x_p^2+x_{p+q}^2-y_1^2-...-y_a^2+...+y_{a+b}^2$. In this case, the Clifford algebra, denoted as $\left(C^{p,q},C^{a,b}\right)$, is generated by symbols $e_1^{(x)},e_2^{(x)},...,e_{p+q}^{(x)},e_1^{(y)},e_2^{(y)},...,e_{a+b}^{(y)}$ satisfying properties $(e_i)^2=-1\ (1\leq i\leq p),(e_j)^2=-1\ (1\leq j\leq a),(e_k)^2=1\ (p{+}1\leq k\leq p{+}q)$,

$(e_j)^2=1\ (n+1\leq s\leq a+b),\ e_ie_j=-e_je_i,\ i\neq j.$ Explicit calculations of $C^{p,q}$ and $C^{a,b}$ are possible by using isomorphisms [83, 129]

$$C^{p+n,q+n}\simeq C^{p,q}\otimes M_2(\mathcal{R})\otimes...\otimes M_2(\mathcal{R})\cong C^{p,q}\otimes M_{2^n}(\mathcal{R})\cong M_{2^n}(C^{p,q}),$$

where $M_s(A)$ denotes the ring of quadratic matrices of order s with coefficients in ring A. Here we write the simplest isomorphisms $C^{1,0}\simeq\mathcal{C},C^{0,1}\simeq\mathcal{R}\oplus\mathcal{R}$, and $C^{2,0}=\mathcal{H}$, where by $\mathcal{H}$ is denoted the body of quaternions. We summarize this calculus as (as in Ref. [21])

$$\begin{aligned}
C^{0,0} &= \mathcal{R}, C^{1,0}=\mathcal{C}, C^{0,1}=\mathcal{R}\oplus\mathcal{R}, C^{2,0}=\mathcal{H}, C^{0,2}=M_2\left(\mathcal{R}\right),\\
C^{3,0} &= \mathcal{H}\oplus\mathcal{H}, C^{0,3}=M_2\left(\mathcal{R}\right),\\
C^{4,0} &= M_2\left(\mathcal{H}\right), C^{0,4}=M_2\left(\mathcal{H}\right), C^{5,0}=M_4\left(\mathcal{C}\right),\\
C^{0,5} &= M_2\left(\mathcal{H}\right)\oplus M_2\left(\mathcal{H}\right), C^{6,0}=M_8\left(\mathcal{R}\right), C^{0,6}=M_4\left(\mathcal{H}\right),\\
C^{7,0} &= M_8\left(\mathcal{R}\right)\oplus M_8\left(\mathcal{R}\right),\ C^{0,7}=M_8\left(\mathcal{C}\right),\\
C^{8,0} &= M_{16}\left(\mathcal{R}\right),\ C^{0,8}=M_{16}\left(\mathcal{R}\right).
\end{aligned}$$

One of the most important properties of real algebras $C^{0,p}\ \left(C^{0,a}\right)$ and $C^{p,0}\ \left(C^{a,0}\right)$ is eightfold periodicity of $p(a)$.

Now, we emphasize that H^{2n}-spaces admit locally a structure of Clifford algebra on complex vector spaces. Really, by using almost Hermitian structure $J_{\alpha}{}^{\beta}$ and considering complex space $\mathcal{C}^n$ with nondegenarate quadratic form

$$\sum_{a=1}^{n}\left|z_a\right|^2,\ z_a\in\mathcal{C}^2$$

induced locally by metric (1.39) (rewritten in complex coordinates $z_a=x_a+iy_a$) we define Clifford algebra

$$\overleftarrow{C}^n=\underbrace{\overleftarrow{C}^1\otimes...\otimes\overleftarrow{C}^1}_{n},$$

where $\overleftarrow{C}^1=\mathcal{C}\otimes_R\mathcal{C}=\mathcal{C}\oplus\mathcal{C}$ or in consequence, $\overleftarrow{C}^n\simeq C^{n,0}\otimes_{\mathcal{R}}\mathcal{C}\approx C^{0,n}\otimes_{\mathcal{R}}\mathcal{C}$. Explicit calculations lead to isomorphisms $\overleftarrow{C}^2=C^{0,2}\otimes_{\mathcal{R}}\mathcal{C}\approx M_2\left(\mathcal{R}\right)\otimes_{\mathcal{R}}\mathcal{C}\approx M_2\left(\overleftarrow{C}^n\right),\ C^{2p}\approx M_{2^p}\left(\mathcal{C}\right)$ and $\overleftarrow{C}^{2p+1}\approx M_{2^p}\left(\mathcal{C}\right)\oplus M_{2^p}\left(\mathcal{C}\right)$, which show that complex Clifford algebras, defined locally for H^{2n}-spaces, have periodicity 2 on p.

Considerations presented in the proof of theorem 4.2 show that map $j:\mathcal{F}\to C\left(\mathcal{F}\right)$ is monomorphic, so we can identify space $\mathcal{F}$ with its image in $C\left(\mathcal{F},G\right)$, denoted as $u\to\overline{u}$, if $u\in C^{(0)}\left(\mathcal{F},G\right)\ \left(u\in C^{(1)}\left(\mathcal{F},G\right)\right)$; then $u=\overline{u}$ (respectively, $\overline{u}=-u$).

Definition 4.1. *The set of elements* $u\in C\left(G\right)^*$, *where* $C\left(G\right)^*$ *denotes the multiplicative group of invertible elements of* $C\left(\mathcal{F},G\right)$ *satisfying* $\overline{u}\mathcal{F}u^{-1}\in\mathcal{F}$, *is called the twisted Clifford d-group, denoted as* $\widetilde{\Gamma}\left(\mathcal{F}\right)$.

Let $\widetilde{\rho}:\widetilde{\Gamma}\left(\mathcal{F}\right)\to GL\left(\mathcal{F}\right)$ be the homorphism given by $u\to\rho\widetilde{u}$, where $\widetilde{\rho}_u\left(w\right)=\overline{u}wu^{-1}$. We can verify that $\ker\widetilde{\rho}=\mathcal{R}^*$is a subgroup in $\widetilde{\Gamma}\left(\mathcal{F}\right)$.

Canonical map $j : \mathcal{F} \to C(\mathcal{F})$ can be interpreted as the linear map $\mathcal{F} \to C(\mathcal{F})^0$ satisfying the universal property of Clifford d-algebras. This leads to a homomorphism of algebras, $C(\mathcal{F}) \to C(\mathcal{F})^t$, considered by an anti-involution of $C(\mathcal{F})$ and denoted as $u \to {}^t u$. More exactly, if $u_1...u_n \in \mathcal{F}$, then $t_u = u_n...u_1$ and ${}^t\overline{u} = \overline{{}^t u} = (-1)^n u_n...u_1$.

Definition 4.2. *The spinor norm of arbitrary $u \in C(\mathcal{F})$ is defined as* $S(u) = {}^t\overline{u} \cdot u \in C(\mathcal{F})$.

It is obvious that if $u, u', u'' \in \widetilde{\Gamma}(\mathcal{F})$, then $S(u, u') = S(u)S(u')$ and $S(uu'u'') = S(u)S(u')S(u'')$. For $u, u' \in \mathcal{F}S(u) = -G(u)$ and $S(u, u') = S(u)S(u') = S(uu')$.

Let us introduce the orthogonal group $O(G) \subset GL(G)$ defined by metric G on $\mathcal{F}$ and denote sets $SO(G) = \{u \in O(G), \det|u| = 1\}$, $Pin(G) = \{u \in \widetilde{\Gamma}(\mathcal{F}), S(u) = 1\}$ and $Spin(G) = Pin(G) \cap C^0(\mathcal{F})$. For $\mathcal{F} \cong \mathcal{R}^{n+m}$ we write $Spin(n+m)$. By straightforward calculations (see similar considerations in Ref. [83]) we can verify the exactness of these sequences:

$$
\begin{aligned}
1 &\to \mathcal{Z}/2 \to Pin(G) \to O(G) \to 1, \\
1 &\to \mathcal{Z}/2 \to Spin(G) \to SO(G) \to 0, \\
1 &\to \mathcal{Z}/2 \to Spin(n+m) \to SO(n+m) \to 1.
\end{aligned}
$$

We conclude this section by emphasizing that the spinor norm was defined with respect to a quadratic form induced by a metric in v-bundle ξ_d (or by an H^{2n}-metric in the case of generalized Lagrange spaces). This approach differs from those presented in Refs. [19] and [124].

4.2 Anisotropic Clifford Bundles and Spinor Structures

There are two possibilities for generalizing our spinor constructions defined for d-vector spaces to the case of vector bundle spaces enabled with the structure of N-connection. The first is to use the extension to the category of vector bundles. The second is to define the Clifford fibration associated with compatible linear d-connection and metric G on a vector bundle (or with an H^{2n}-metric on GL-space). Let us consider both variants.

4.2.1 Clifford d-module structure

Because functor $\mathcal{F} \to C(\mathcal{F})$ is smooth we can extend it to the category of vector bundles of type $\xi_d = \{\pi_d : HE \oplus VE \to E\}$. Recall that by $\mathcal{F}$

we denote the typical fiber of such bundles. For ξ_d we obtain a bundle of algebras, denoted as $C(\xi_d)$, such that $C(\xi_d)_u = C(\mathcal{F}_u)$. Multiplication in every fibre defines a continuous map $C(\xi_d) \times C(\xi_d) \to C(\xi_d)$. If ξ_d is a vector bundle on number field k, the structure of the $C(\xi_d)$-module, the d-module, the d-module, on ξ_d is given by the continuous map $C(\xi_d) \times_E \xi_d \to \xi_d$ with every fiber $\mathcal{F}_u$ provided with the structure of the $C(\mathcal{F}_u)$ −module, correlated with its k-module structure, Because $\mathcal{F} \subset C(\mathcal{F})$, we have a fiber to fiber map $\mathcal{F} \times_E \xi_d \to \xi_d$, inducing on every fiber the map $\mathcal{F}_u \times_E \xi_{d(u)} \to \xi_{d(u)}$ ($\mathcal{R}$-linear on the first factor and k-linear on the second one). Inversely, every such bilinear map defines on ξ_d the structure of the $C(\xi_d)$-module by virtue of universal properties of Clifford d-algebras. Equivalently, the above-mentioned bilinear map defines a morphism of v-bundles $m : \xi_d \to HOM(\xi_d, \xi_d)$ [$HOM(\xi_d, \xi_d)$ denotes the bundles of homomorphisms] when $(m(u))^2 = G(u)$ on every point.

Vector bundles ξ_d provided with $C(\xi_d)$-structures are objects of the category with morphisms being morphisms of v-bundles, which induce on every point $u \in \xi$ morphisms of $C(\mathcal{F}_u)$ −modules. This is a Banach category contained in the category of finite-dimensional d-vector spaces on filed k. We shall not use category formalism in this work, but point to its advantages in further formulation of new directions of K-theory (see , for example, an introduction in Ref. [83]) concerned with locally anisotorpic spaces.

Let us denote by $H^s(\xi, GL_{n+m}(\mathcal{R}))$ the s-dimensional cohomology group of the algebraic sheaf of germs of continuous maps of v-bundle ξ with group $GL_{n+m}(\mathcal{R})$ the group of automorphisms of $\mathcal{R}^{n+m}$ (for the language of algebraic topology see, for example, Refs. [83] and [74]). We shall also use the group $SL_{n+m}(\mathcal{R}) = \{A \subset GL_{n+m}(\mathcal{R}), \det A = 1\}$. Here we point out that cohomologies $H^s(M, Gr)$ characterize the class of a principal bundle $\pi : P \to M$ on M with structural group Gr. Taking into account that we deal with bundles distinguished by an N-connection we introduce into consideration cohomologies $H^s(\xi, GL_{n+m}(\mathcal{R}))$ as distinguished classes (d-classes) of bundles ξ provided with a global N-connection structure.

For a real vector bundle ξ_d on compact base ξ we can define the orientation on ξ_d as an element $\alpha_d \in H^1(\xi, GL_{n+m}(\mathcal{R}))$ whose image on map

$$H^1(\xi, SL_{n+m}(\mathcal{R})) \to H^1(\xi, GL_{n+m}(\mathcal{R}))$$

is the d-class of bundle ξ.

Definition 4.3. *The spinor structure on ξ_d is defined as an element $\beta_d \in H^1(\xi, Spin(n+m))$ whose image in the composition*

$$H^1(\xi, Spin(n+m)) \to H^1(\xi, SO(n+m)) \to H^1(\xi, GL_{n+m}(\mathcal{R}))$$

is the d-class of ξ.

The above definition of spinor structures can be reformulated in terms of principal bundles. Let ξ_d be a real vector bundle of rank n+m on a compact base ξ. If there is a principal bundle P_d with structural group $SO\left(n+m\right)$ [or $Spin\left(n+m\right)$], this bundle ξ_d can be provided with orientation (or spinor) structure. The bundle P_d is associated with element $\alpha_d \in H^1\left(\xi, SO(n+m)\right)$ [or $\beta_d \in H^1\left(\xi, Spin\left(n+m\right)\right).$

We remark that a real bundle is oriented if and only if its first Stiefel-Whitney d-class vanishes,

$$w_1\left(\xi_d\right) \in H^1\left(\xi, \mathcal{Z}/2\right) = 0,$$

where $H^1\left(\xi, \mathcal{Z}/2\right)$ is the first group of Chech cohomology with coefficients in $\mathcal{Z}/2$, Considering the second Stiefel-Whitney class $w_2\left(\xi_d\right) \in H^{21}\left(\xi, \mathcal{Z}/2\right)$ it is well known that vector bundle ξ_d admits the spinor structure if and only if $w_2\left(\xi_d\right) = 0$. Finally, in this subsection, we emphasize that taking into account that base space ξ is also a v-bundle, $p : E \to M$, we have to make explicit calculations in order to express cohomologies $H^s\left(\xi, GL_{n+m}\right)$ and $H^s\left(\xi, SO\left(n+m\right)\right)$ through cohomologies $H^s\left(M, GL_n\right), H^s\left(M, SO\left(m\right)\right)$, which depends on global topological structures of spaces M and ξ. For general bundle and base spaces this requires a cumbersome cohomological calculus.

4.2.2 Anisotropic Clifford fibration

Another way of defining the spinor structure is to use Clifford fibrations. Consider the principal bundle with the structural group Gr being a subgroup of orthogonal group $O\left(G\right)$, where G is a quadratic nondegenerate form (see(1.39)) defined on the base (also being a bundle space) space ξ. The fibration associated to principal fibration $P\left(\xi, Gr\right)$ [or $P\left(H^{2n}, Gr\right)$] with a typical fiber having Clifford algebra $C\left(G\right)$ is, by definition, the Clifford fibration $PC\left(\xi, Gr\right)$. We can always define a metric on the Clifford fibration if every fiber is isometric to $PC\left(\xi, G\right)$ (this result is proved for arbitrary quadratic forms G on pseudo-Riemannian bases [154]). If, additionally, $Gr \subset SO\left(G\right)$ a global section can be defined on $PC\left(G\right)$.

Let $\mathcal{P}\left(\xi, Gr\right)$ be the set of principal bundles with differentiable base ξ and structural group Gr. If $g : Gr \to Gr'$ is an homomorphism of Lie groups and $P\left(\xi, Gr\right) \subset \mathcal{P}\left(\xi, Gr\right)$ (for simplicity in this section we shall denote mentioned bundles and sets of bundles as P, P' and respectively, $\mathcal{P}, \mathcal{P}'$), we can always construct a principal bundle with the property that there is as homomorphism $f : P' \to P$ of principal bundles which can be projected to the identity map of ξ and corresponds to isomorphism $g : Gr \to Gr'$. If the inverse statement also holds, the bundle P' is called as the extension of P associated to g and f is called the extension homomorphism denoted as $\widetilde{g}$.

Now we can define distinguished spinor structures on bundle spaces (compare with definition 2.3).

Definition 4.4. *Let $P \in \mathcal{P}(\xi, O(G))$ be a principal bundle. A distinguished spinor structure of P, equivalently a ds-structure of ξ is an extension $\widetilde{P}$ of P associated to homomorphism $h : PinG \to O(G)$ where $O(G)$ is the group of orthogonal rotations, generated by metric G, in bundle ξ.*

So, if $\widetilde{P}$ is a spinor structure of the space ξ, then $\widetilde{P} \in \mathcal{P}(\xi, PinG)$.

The definition of spinor structures on varieties was given in Ref.[50]. In Refs. [51] and [51] it is proved that a necessary and sufficient condition for a space time to be orientable is to admit a global field of orthonormalized frames. We mention that spinor structures can be also defined on varieties modeled on Banach spaces [1]. As we have shown in this subsection, similar constructions are possible for the cases when space time has the structure of a v-bundle with an N-connection.

Definition 4.5. *A special distinguished spinor structure, ds-structure, of principal bundle $P = P(\xi, SO(G))$ is a principal bundle $\widetilde{P} = \widetilde{P}(\xi, SpinG)$ for which a homomorphism of principal bundles $\widetilde{p} : \widetilde{P} \to P$, projected on the identity map of ξ (or of H^{2n}) and corresponding to representation*

$$R : SpinG \to SO(G),$$

is defined.

In the case when the base space variety is oriented, there is a natural bijection between tangent spinor structures with a common base. For special ds-structures we can define, as for any spinor structure, the concepts of spin tensors, spinor connections, and spinor covariant derivations (see Refs. [162, 189, 165]).

4.3 Almost Complex Anisotropic Spinor Structures

Almost complex structures are an important characteristic of H^{2n}-spaces. We can rewrite the almost Hermitian metric [108, 109], in complex form [163]:

$$G = H_{ab}(z,\xi)\, dz^a \otimes dz^b, \tag{4.2}$$

where

$$z^a = x^a + iy^a,\ \overline{z^a} = x^a - iy^a,\ H_{ab}(z,\overline{z}) = g_{ab}(x,y)\mid^{x=x(z,\overline{z})}_{y=y(z,\overline{z})},$$

and define almost complex spinor structures. For given metric (4.2) on H^{2n}-space there is always a principal bundle P^U with unitary structural group U(n) which allows us to transform H^{2n}-space into v-bundle $\xi^U \approx P^U \times_{U(n)} \mathcal{R}^{2n}$. This statement will be proved after we introduce complex spinor structures on oriented real vector bundles [83].

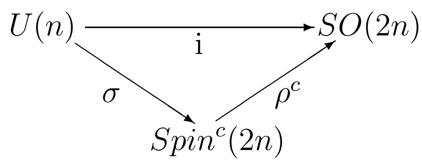

Figure 4.3: Diagram 3

Let us consider momentarily $k = \mathcal{C}$ and introduce into consideration [instead of the group $Spin(n)$] the group $Spin^c \times_{\mathcal{Z}/2} U(1)$ being the factor group of the product $Spin(n) \times U(1)$ with the respect to equivalence

$$(y,z) \sim (-y,-a), \quad y \in Spin(m).$$

This way we define the short exact sequence

$$1 \to U(1) \to Spin^c(n) \overset{S^c}{\to} SO(n) \to 1,$$

where $\rho^c(y,a) = \rho^c(y)$. If λ is oriented , real, and rank n, γ-bundle $\pi : E_\lambda \to M^n$, with base M^n, the complex spinor structure, spin structure, on λ is given by the principal bundle P with structural group $Spin^c(m)$ and isomorphism $\lambda \approx P \times_{Spin^c(n)} \mathcal{R}^n$. For such bundles the categorial equivalence can be defined as

$$\epsilon^c : \mathcal{E}^T_{\mathcal{C}}(M^n) \to \mathcal{E}^\lambda_{\mathcal{C}}(M^n), \tag{4.3}$$

where $\epsilon^c(E^c) = P \triangle_{Spin^c(n)} E^c$ is the category of trivial complex bundles on $M^n, \mathcal{E}^\lambda_{\mathcal{C}}(M^n)$ is the category of complex v-bundles on M^n with action of Clifford bundle $C(\lambda), P\triangle_{Spin^c(n)}$ and E^c is the factor space of the bundle product $P \times_M E^c$ with respect to the equivalence $(p,e) \sim (p\widehat{g}^{-1}, \widehat{g}e), p \in$

$P, e \in E^c$, where $\widehat{g} \in Spin^c(n)$ acts on E by via the imbedding $Spin(n) \subset C^{0,n}$ and the natural action $U(1) \subset \mathcal{C}$ on complex v-bundle $\xi^c, E^c = tot\xi^c$, for bundle $\pi^c : E^c \to M^n$.

Now we return to the bundle ξ. A real v-bundle (not being a spinor bundle) admits a complex spinor structure if and only if there exist a homomorphism $\sigma : U(n) \to Spin^c(2n)$ making the diagram 3 commutative. The explicit construction of σ for arbitrary γ-bundle is given in Refs. [83] and [21]. For H^{2n}-spaces it is obvious that a diagram similar to (4.3) can be defined for the tangent bundle TM^n, which directly points to the possibility of defining the ${}^cSpin$-structure on H^{2n}-spaces.

Let λ be a complex, rank n, spinor bundle with

$$\tau : Spin^c(n) \times_{\mathcal{Z}/2} U(1) \to U(1) \tag{4.4}$$

the homomorphism defined by formula $\tau(\lambda, \delta) = \delta^2$. For P_s being the principal bundle with fiber $Spin^c(n)$ we introduce the complex linear bundle $L(\lambda^c) = P_S \times_{Spin^c(n)} \mathcal{C}$ defined as the factor space of $P_S \times \mathcal{C}$ on equivalence relation

$$(pt, z) \sim \left(p, l(t)^{-1} z\right),$$

where $t \in Spin^c(n)$. This linear bundle is associated to complex spinor structure on λ^c.

If λ^c and $\lambda^{c'}$ are complex spinor bundles, the Whitney sum $\lambda^c \oplus \lambda^{c'}$ is naturally provided with the structure of the complex spinor bundle. This follows from the holomorphism

$$\omega' : Spin^c(n) \times Spin^c(n') \to Spin^c(n + n'), \tag{4.5}$$

given by formula $[(\beta, z), (\beta', z')] \to [\omega(\beta, \beta'), zz']$, where ω is the homomorphism making the following diagram 4 commutative. Here, $z, z' \in U(1)$. It

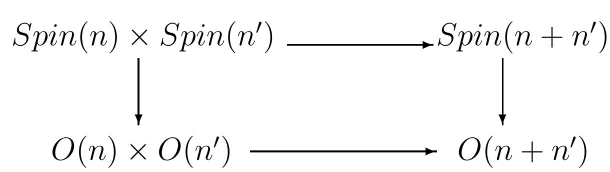

Figure 4.4: Diagram 4

is obvious that $L\left(\lambda^c \oplus \lambda^{c'}\right)$ is isomorphic to $L(\lambda^c) \otimes L\left(\lambda^{c'}\right)$.

We conclude this section by formulating our main result on complex spinor structures for H^{2n}-spaces:

Theorem 4.3. *Let λ^c be a complex spinor bundle of rank n and H^{2n}-space considered as a real vector bundle $\lambda^c \oplus \lambda^{c'}$ provided with almost complex structure $J^{\alpha}{}_{\beta}$; multiplication on i is given by $\begin{pmatrix} 0 & -\delta^i_j \\ \delta^i_j & 0 \end{pmatrix}$. Then, the diagram 5 is commutative up to isomorphisms ϵ^c and $\widetilde{\epsilon}^c$ defined as in (2.49), $\mathcal{H}$ is functor $E^c \to E^c \otimes L(\lambda^c)$ and $\mathcal{E}^{0,2n}_{\mathcal{C}}(M^n)$ is defined by functor $\mathcal{E}_{\mathcal{C}}(M^n) \to \mathcal{E}^{0,2n}_{\mathcal{C}}(M^n)$ given as correspondence $E^c \to \Lambda(\mathcal{C}^n) \otimes E^c$ (which is a categorial equivalence), $\Lambda(\mathcal{C}^n)$ is the exterior algebra on $\mathcal{C}^n$. W is the real bundle $\lambda^c \oplus \lambda^{c'}$ provided with complex structure.*

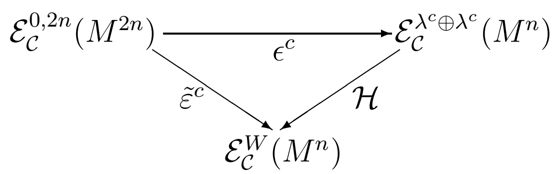

Figure 4.5: Diagram 5

Proof: We use composition of homomorphisms

$$\mu : Spin^c(2n) \xrightarrow{\pi} SO(n) \xrightarrow{r} U(n) \xrightarrow{\sigma} Spin^c(2n) \times_{\mathcal{Z}/2} U(1),$$

commutative diagram 6 and introduce composition of homomorphisms

$$\mu : Spin^c(n) \xrightarrow{\Delta} Spin^c(n) \times Spin^c(n) \xrightarrow{\omega^c} Spin^c(n),$$

where Δ is the diagonal homomorphism and ω^c is defined as in (4.5). Using homomorphisms (4.4) and (4.5) we obtain formula $\mu(t) = \mu(t)\, r(t)$.

Now consider bundle $P \times_{Spin^c(n)} Spin^c(2n)$ as the principal $Spin^c(2n)$-bundle, associated to $M \oplus M$ being the factor space of the product $P \times Spin^c(2n)$ on the equivalence relation $(p,t,h) \sim \left(p, \mu(t)^{-1} h\right)$. In this case the categorial equivalence (4.3) can be rewritten as

$$\epsilon^c(E^c) = P \times_{Spin^c(n)} Spin^c(2n)\, \Delta_{Spin^c(2n)} E^c$$

and seen as factor space of $P \times Spin^c(2n) \times_M E^c$ on equivalence relation

$$(pt, h, e) \sim \left(p, \mu(t)^{-1} h, e\right) \text{ and } (p, h_1, h_2, e) \sim \left(p, h_1, h_2^{-1} e\right)$$

(projections of elements p and e coincides on base M). Every element of $\epsilon^c(E^c)$ can be represented as $P \Delta_{Spin^c(n)} E^c$, i.e., as a factor space $P \Delta E^c$ on equivalence relation $(pt, e) \sim (p, \mu^c(t), e)$, when $t \in Spin^c(n)$. The complex

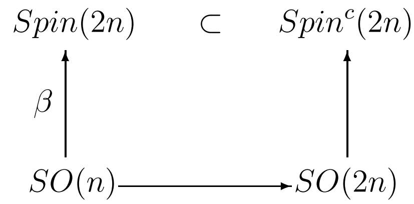

Figure 4.6: Diagram 6

line bundle $L\left(\lambda^{c}\right)$ can be interpreted as the factor space of $P\times_{Spin^{c}(n)}\mathcal{C}$ on equivalence relation $(pt,\delta)\sim\left(p,r\left(t\right)^{-1}\delta\right)$.

Putting $(p,e)\otimes(p,\delta)\,(p,\delta e)$ we introduce morphism

$$\epsilon^{c}\left(E\right)\times L\left(\lambda^{c}\right)\rightarrow\epsilon^{c}\left(\lambda^{c}\right)$$

with properties $(pt,e)\otimes(pt,\delta)\rightarrow(pt,\delta e)=\left(p,\mu^{c}\left(t\right)^{-1}\delta e\right)$,

$\left(p,\mu^{c}\left(t\right)^{-1}e\right)\otimes\left(p,l\left(t\right)^{-1}e\right)\rightarrow\left(p,\mu^{c}\left(t\right)r\left(t\right)^{-1}\delta e\right)$ pointing to the fact that we have defined the isomorphism correctly and that it is an isomorphism on every fiber. □

Chapter 5

Spinors and Anisotropic Spaces

The purpose of this Chapter is to show how a corresponding abstract spinor technique entailing notational and calculations advantages can be developed for arbitrary splits of dimensions of a d-vector space $\mathcal{F} = h\mathcal{F} \oplus v\mathcal{F}$, where $\dim h\mathcal{F} = n$ and $\dim v\mathcal{F} = m$. For convenience we shall also present some necessary coordinate expressions.

The problem of a rigorous definition of spinors on locally anisotropic spaces (anisotropic spinors, d–spinors) was posed and solved [163, 162, 189] in the framework of the formalism of Clifford and spinor structures on v–bundles provided with compatible nonlinear and distinguished connections and metric . We introduced d–spinors as corresponding objects of the Clifford d–algebra $\mathcal{C}(\mathcal{F}, G)$, defined for a d–vector space $\mathcal{F}$ in a standard manner (see, for instance, [83]) and proved that operations with $\mathcal{C}(\mathcal{F}, G)$ can be reduced to calculations for $\mathcal{C}(h\mathcal{F}, g)$ and $\mathcal{C}(v\mathcal{F}, h)$, which are usual Clifford algebras of respective dimensions 2^n and 2^m (if it is necessary we can use quadratic forms g and h correspondingly induced on $h\mathcal{F}$ and $v\mathcal{F}$ by a metric $\mathbf{G}$ (1.39)). Considering the orthogonal subgroup $O(\mathbf{G}) \subset GL(\mathbf{G})$ defined by a metric $\mathbf{G}$ we can define the d-spinor norm and parametrize d-spinors by ordered pairs of elements of Clifford algebras $\mathcal{C}(h\mathcal{F}, g)$ and $\mathcal{C}(v\mathcal{F}, h)$. We emphasize that the splitting of a Clifford d-algebra associated to a v-bundle $\mathcal{E}$ is a straightforward consequence of the global decomposition defining a N-connection structure in $\mathcal{E}$.

In this Chapter, as a rule, we shall omit proofs which in most cases are mechanical but rather tedious. We can apply the methods developed in [127, 128, 129, 91] in a straightforward manner on h- and v-subbundles in order to verify the correctness of affirmations.

5.1 Anisotropic Clifford Algebras, Spinors and Twistors

In order to relate the succeeding constructions with Clifford d-algebras [163, 162] we consider a locally anisotropic frame decomposition of the metric (1.39):

$$G_{\alpha\beta}\left(u\right)=l_{\alpha}^{\widehat{\alpha}}\left(u\right)l_{\beta}^{\widehat{\beta}}\left(u\right)G_{\widehat{\alpha}\widehat{\beta}}, \tag{5.1}$$

where the frame d-vectors and constant metric matrices are distinguished as

$$l_{\alpha}^{\widehat{\alpha}}\left(u\right)=\left(\begin{array}{cc} l_{j}^{\widehat{j}}\left(u\right) & 0\\ 0 & l_{a}^{\widehat{a}}\left(u\right)\end{array}\right),G_{\widehat{\alpha}\widehat{\beta}}\left(\begin{array}{cc} g_{\widehat{i}\widehat{j}} & 0\\ 0 & h_{\widehat{a}\widehat{b}}\end{array}\right), \tag{5.2}$$

$g_{\widehat{i}\widehat{j}}$ and $h_{\widehat{a}\widehat{b}}$ are diagonal matrices with $g_{\widehat{i}\widehat{i}}=h_{\widehat{a}\widehat{a}}=\pm 1.$

To generate Clifford d-algebras we start with matrix equations

$$\sigma_{\widehat{\alpha}}\sigma_{\widehat{\beta}}+\sigma_{\widehat{\beta}}\sigma_{\widehat{\alpha}}=-G_{\widehat{\alpha}\widehat{\beta}}I, \tag{5.3}$$

where I is the identity matrix, matrices $\sigma_{\widehat{\alpha}}$ (σ-objects) act on a d-vector space $\mathcal{F}=h\mathcal{F}\oplus v\mathcal{F}$ and theirs components are distinguished as

$$\sigma_{\widehat{\alpha}}\ =\left\{ (\sigma_{\widehat{\alpha}})_{\underline{\beta}}^{\cdot\underline{\gamma}}=\left(\begin{array}{cc} (\sigma_{\widehat{i}})_{\underline{j}}^{\cdot\underline{k}} & 0\\ 0 & (\sigma_{\widehat{a}})_{\underline{b}}^{\cdot\underline{c}}\end{array}\right)\right\}, \tag{5.4}$$

indices $\underline{\beta}$,$\underline{\gamma}$,... refer to spin spaces of type $\mathcal{S}=S_{(h)}\oplus S_{(v)}$ and underlined Latin indices $\underline{j}$,$\underline{k}$, ... and $\underline{b},\underline{c}$, ... refer respectively to a h-spin space $\mathcal{S}_{(h)}$ and a v-spin space $\mathcal{S}_{(v)}$, which are correspondingly associated to a h- and v-decomposition of a v-bundle $\mathcal{E}_{(d)}$. The irreducible algebra of matrices $\sigma_{\widehat{\alpha}}$ of minimal dimension $N\times N$, where $N=N_{(n)}+N_{(m)}$, $\dim\mathcal{S}_{(h)}$=$N_{(n)}$ and $\dim\mathcal{S}_{(v)}$=$N_{(m)}$, has these dimensions

$$N_{(n)} = \left\{ \begin{array}{cc} 2^{(n-1)/2}, & n=2k+1\\ 2^{n/2}, & n=2k;\end{array}\right.$$
$$N_{(m)} = \left\{ \begin{array}{cc} 2^{(m-1)/2}, & m=2k+1\\ 2^{m/2}, & m=2k,\end{array}\right.$$

where $k=1,2,...$.

The Clifford d-algebra is generated by sums on $n+1$ elements of form

$$A_1I+B^{\widehat{i}}\sigma_{\widehat{i}}+C^{\widehat{i}\widehat{j}}\sigma_{\widehat{i}\widehat{j}}+D^{\widehat{i}\widehat{j}\widehat{k}}\sigma_{\widehat{i}\widehat{j}\widehat{k}}+...$$

and sums of $m+1$ elements of form

$$A_2 I + B^{\widehat{a}}\sigma_{\widehat{a}} + C^{\widehat{a}\widehat{b}}\sigma_{\widehat{a}\widehat{b}} + D^{\widehat{a}\widehat{b}\widehat{c}}\sigma_{\widehat{a}\widehat{b}\widehat{c}} + ...$$

with antisymmetric coefficients $C^{\widehat{i}\widehat{j}} = C^{[\widehat{i}\widehat{j}]}, C^{\widehat{a}\widehat{b}} = C^{[\widehat{a}\widehat{b}]}, D^{\widehat{i}\widehat{j}\widehat{k}} = D^{[\widehat{i}\widehat{j}\widehat{k}]}, D^{\widehat{a}\widehat{b}\widehat{c}} = D^{[\widehat{a}\widehat{b}\widehat{c}]}$, ... and matrices $\sigma_{\widehat{i}\widehat{j}} = \sigma_{[\widehat{i}}\sigma_{\widehat{j}]}, \sigma_{\widehat{a}\widehat{b}} = \sigma_{[\widehat{a}}\sigma_{\widehat{b}]}, \sigma_{\widehat{i}\widehat{j}\widehat{k}} = \sigma_{[\widehat{i}}\sigma_{\widehat{j}}\sigma_{\widehat{k}]}, ...$. Really, we have 2^{n+1} coefficients $\left(A_1, C^{\widehat{i}\widehat{j}}, D^{\widehat{i}\widehat{j}\widehat{k}}, ...\right)$ and 2^{m+1} coefficients $\left(A_2, C^{\widehat{a}\widehat{b}}, D^{\widehat{a}\widehat{b}\widehat{c}}, ...\right)$ of the Clifford algebra on $\mathcal{F}$.

For simplicity, in this subsection, we shall present the necessary geometric constructions only for h-spin spaces $\mathcal{S}_{(h)}$ of dimension $N_{(n)}$. Considerations for a v-spin space $\mathcal{S}_{(v)}$ are similar but with proper characteristics for a dimension $N_{(m)}$.

In order to define the scalar (spinor) product on $\mathcal{S}_{(h)}$ we introduce into consideration this finite sum (because of a finite number of elements $\sigma_{[\widehat{i}\widehat{j}...\widehat{k}]}$) :

$$^{(\pm)}E_{\underline{km}}^{\underline{ij}} = \delta_{\underline{k}}^{\underline{i}}\delta_{\underline{m}}^{\underline{j}} + \frac{2}{1!}(\sigma_{\widehat{i}})_{\underline{k}}^{.\underline{i}}(\sigma^{\widehat{i}})_{\underline{m}}^{.\underline{j}} + \frac{2^2}{2!}(\sigma_{\widehat{i}\widehat{j}})_{\underline{k}}^{.\underline{i}}(\sigma^{\widehat{i}\widehat{j}})_{\underline{m}}^{.\underline{j}} + \frac{2^3}{3!}(\sigma_{\widehat{i}\widehat{j}\widehat{k}})_{\underline{k}}^{.\underline{i}}(\sigma^{\widehat{i}\widehat{j}\widehat{k}})_{\underline{m}}^{.\underline{j}} + ... \quad (5.5)$$

which can be factorized as

$$^{(\pm)}E_{\underline{km}}^{\underline{ij}} = N_{(n)}\ {}^{(\pm)}\epsilon_{\underline{km}}\ {}^{(\pm)}\epsilon^{\underline{ij}} \text{ for } n = 2k \quad (5.6)$$

and

$$\begin{aligned} ^{(+)}E_{\underline{km}}^{\underline{ij}} &= 2N_{(n)}\epsilon_{\underline{km}}\epsilon^{\underline{ij}},\ ^{(-)}E_{\underline{km}}^{\underline{ij}} = 0 \text{ for } n = 3(mod4), \\ ^{(+)}E_{\underline{km}}^{\underline{ij}} &= 0,\ ^{(-)}E_{\underline{km}}^{\underline{ij}} = 2N_{(n)}\epsilon_{\underline{km}}\epsilon^{\underline{ij}} \text{ for } n = 1(mod4). \end{aligned} \quad (5.7)$$

Antisymmetry of $\sigma_{\widehat{i}\widehat{j}\widehat{k}...}$ and the construction of the objects (5.5),(5.6) and (5.7) define the properties of ϵ-objects $^{(\pm)}\epsilon_{\underline{km}}$ and $\epsilon_{\underline{km}}$ which have an eight-fold periodicity on n (see details in [129] and, with respect to locally anisotropic spaces, [163]).

For even values of n it is possible the decomposition of every h-spin space $\mathcal{S}_{(h)}$into irreducible h-spin spaces $\mathbf{S}_{(h)}$ and $\mathbf{S}'_{(h)}$ (one considers splitting of h-indices, for instance, $\underline{l} = L \oplus L', \underline{m} = M \oplus M', ...$; for v-indices we shall write $\underline{a} = A \oplus A', \underline{b} = B \oplus B', ...$) and defines new ϵ-objects

$$\epsilon^{\underline{lm}} = \frac{1}{2}\left(^{(+)}\epsilon^{\underline{lm}} +^{(-)} \epsilon^{\underline{lm}}\right) \text{ and } \widetilde{\epsilon}^{\underline{lm}} = \frac{1}{2}\left(^{(+)}\epsilon^{\underline{lm}} -^{(-)} \epsilon^{\underline{lm}}\right). \quad (5.8)$$

We shall omit similar formulas for ϵ-objects with lower indices.

We can verify, by using expressions (5.7) and straightforward calculations, these parametrizations on symmetry properties of ϵ-objects (5.8)

$$\epsilon^{\underline{lm}} = \begin{pmatrix} \epsilon^{LM} = \epsilon^{ML} & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \widetilde{\epsilon}^{\underline{lm}} = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{\epsilon}^{LM} = \widetilde{\epsilon}^{ML} \end{pmatrix}$$
$$\text{for } n = 0(mod8);$$
$$\epsilon^{\underline{lm}} = -\frac{1}{2}{}^{(-)}\epsilon^{\underline{lm}} = \epsilon^{\underline{ml}}, \text{where } {}^{(+)}\epsilon^{\underline{lm}} = 0, \text{ and} \quad (5.9)$$
$$\widetilde{\epsilon}^{\underline{lm}} = -\frac{1}{2}{}^{(-)}\epsilon^{\underline{lm}} = \widetilde{\epsilon}^{\underline{ml}} \text{for } n = 1(mod8);$$
$$\epsilon^{\underline{lm}} = \begin{pmatrix} 0 & 0 \\ \epsilon^{L'M} & 0 \end{pmatrix} \text{ and } \widetilde{\epsilon}^{\underline{lm}} = \begin{pmatrix} 0 & \widetilde{\epsilon}^{LM'} = -\epsilon^{M'L} \\ 0 & 0 \end{pmatrix}$$
$$\text{for } n = 2(mod8);$$
$$\epsilon^{\underline{lm}} = -\frac{1}{2}{}^{(+)}\epsilon^{\underline{lm}} = -\epsilon^{\underline{ml}}, \text{ where } {}^{(-)}\epsilon^{\underline{lm}} = 0, \text{ and}$$
$$\widetilde{\epsilon}^{\underline{lm}} = \frac{1}{2}{}^{(+)}\epsilon^{\underline{lm}} = -\widetilde{\epsilon}^{\underline{ml}} \text{for } n = 3(mod8);$$
$$\epsilon^{\underline{lm}} = \begin{pmatrix} \epsilon^{LM} = -\epsilon^{ML} & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \widetilde{\epsilon}^{\underline{lm}} = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{\epsilon}^{LM} = -\widetilde{\epsilon}^{ML} \end{pmatrix}$$
$$\text{for } n = 4(mod8);$$
$$\epsilon^{\underline{lm}} = -\frac{1}{2}{}^{(-)}\epsilon^{\underline{lm}} = -\epsilon^{\underline{ml}}, \text{ where } {}^{(+)}\epsilon^{\underline{lm}} = 0, \text{ and}$$
$$\widetilde{\epsilon}^{\underline{lm}} = -\frac{1}{2}{}^{(-)}\epsilon^{\underline{lm}} = -\widetilde{\epsilon}^{\underline{ml}} \text{for } n = 5(mod8);$$
$$\epsilon^{\underline{lm}} = \begin{pmatrix} 0 & 0 \\ \epsilon^{L'M} & 0 \end{pmatrix} \text{ and } \widetilde{\epsilon}^{\underline{lm}} = \begin{pmatrix} 0 & \widetilde{\epsilon}^{LM'} = \epsilon^{M'L} \\ 0 & 0 \end{pmatrix}$$
$$\text{for } n = 6(mod8);$$
$$\epsilon^{\underline{lm}} = \frac{1}{2}{}^{(-)}\epsilon^{\underline{lm}} = \epsilon^{\underline{ml}}, \text{ where } {}^{(+)}\epsilon^{\underline{lm}} = 0, \text{ and}$$
$$\widetilde{\epsilon}^{\underline{lm}} = -\frac{1}{2}{}^{(-)}\epsilon^{\underline{lm}} = \widetilde{\epsilon}^{\underline{ml}} \text{ for } n = 7(mod8).$$

Let denote reduced and irreducible h-spinor spaces in a form pointing to the symmetry of spinor inner products in dependence of values $n = 8k + l$ $(k = 0, 1, 2, ...; l = 1, 2, ...7)$ of the dimension of the horizontal subbundle (we shall write respectively $\triangle$ and $\circ$ for antisymmetric and symmetric inner products of reduced spinors and $\diamondsuit = (\triangle, \circ)$ and $\widetilde{\diamondsuit} = (\circ, \triangle)$ for corresponding parametrizations of inner products, in brief *i.p.*, of irreducible spinors; properties of scalar products of spinors are defined by ϵ-objects (5.9); we shall use $\diamond$ for a general *i.p.* when the symmetry is not pointed out):

$$
\begin{aligned}
\mathcal{S}_{(h)}\ (8k) &= \mathbf{S}_{\circ}\oplus\mathbf{S}'_{\circ}; \\
\mathcal{S}_{(h)}\ (8k+1) &= \mathcal{S}^{(-)}_{\circ}\ (i.p.\text{ is defined by an }{}^{(-)}\epsilon\text{-object}); \\
\mathcal{S}_{(h)}\ (8k+2) &= \left\{ \begin{array}{c} \mathcal{S}_{\diamond}=(\mathbf{S}_{\diamond},\mathbf{S}_{\diamond}),\ \text{or} \\ \mathcal{S}'_{\diamond}=(\mathbf{S}'_{\widetilde{\diamond}},\mathbf{S}'_{\widetilde{\diamond}}); \end{array} \right. \\
\mathcal{S}_{(h)}\ (8k+3) &= \mathcal{S}^{(+)}_{\triangle}\ (i.p.\text{ is defined by an }{}^{(+)}\epsilon\text{-object}); \\
\mathcal{S}_{(h)}\ (8k+4) &= \mathbf{S}_{\triangle}\oplus\mathbf{S}'_{\triangle}; \\
\mathcal{S}_{(h)}\ (8k+5) &= \mathcal{S}^{(-)}_{\triangle}\ (i.p.\text{ is defined by an }{}^{(-)}\epsilon\text{-object}), \\
\mathcal{S}_{(h)}\ (8k+6) &= \left\{ \begin{array}{c} \mathcal{S}_{\diamond}=(\mathbf{S}_{\diamond},\mathbf{S}_{\diamond}),\ \text{or} \\ \mathcal{S}'_{\diamond}=(\mathbf{S}'_{\widetilde{\diamond}},\mathbf{S}'_{\widetilde{\diamond}}); \end{array} \right. \\
\mathcal{S}_{(h)}\ (8k+7) &= \mathcal{S}^{(+)}_{\circ}\ (i.p.\text{ is defined by an }{}^{(+)}\epsilon\text{-object}).
\end{aligned} \tag{5.10}
$$

We note that by using corresponding ϵ-objects we can lower and rise indices of reduced and irreducible spinors (for $n = 2, 6(mod4)$ we can exclude primed indices, or inversely, see details in [127, 128, 129]).

The similar v-spinor spaces are denoted by the same symbols as in (5.10) provided with a left lower mark "|" and parametrized with respect to the values $m = 8k' + l$ (k'=0,1,...; l=1,2,...,7) of the dimension of the vertical subbundle, for example, as

$$\mathcal{S}_{(v)}(8k') = \mathbf{S}_{|\circ}\oplus\mathbf{S}'_{|\circ}, \mathcal{S}_{(v)}\ (8k+1) = \mathcal{S}^{(-)}_{|\circ}, ... \tag{5.11}$$

We use "$\widetilde{\ }$"-overlined symbols,

$$\widetilde{\mathcal{S}}_{(h)}\ (8k) = \widetilde{\mathbf{S}}_{\circ}\oplus\widetilde{S}'_{\circ}, \widetilde{\mathcal{S}}_{(h)}\ (8k+1) = \widetilde{\mathcal{S}}^{(-)}_{\circ}, ... \tag{5.12}$$

and

$$\widetilde{\mathcal{S}}_{(v)}(8k') = \widetilde{\mathbf{S}}_{|\circ}\oplus\widetilde{S}'_{|\circ}, \widetilde{\mathcal{S}}_{(v)}\ (8k'+1) = \widetilde{\mathcal{S}}^{(-)}_{|\circ}, ... \tag{5.13}$$

respectively for the dual to (5.10) and (5.11) spinor spaces.

The spinor spaces (5.10)-(5.13) are called the prime spinor spaces, in brief p-spinors . They are considered as building blocks of distinguished (n,m)–spinor spaces constructed in this manner:

$$ \tag{2.65}$$

$$\begin{array}{rcl}\mathcal{S}(_{\circ\circ,\circ\circ}) & = & \mathbf{S}_{\circ}\oplus\mathbf{S}'_{\circ}\oplus\mathbf{S}_{|\circ}\oplus\mathbf{S}'_{|\circ},\mathcal{S}(_{\circ\circ,\circ}|^{\circ})=\mathbf{S}_{\circ}\oplus\mathbf{S}'_{\circ}\oplus\mathbf{S}_{|\circ}\oplus\widetilde{\mathbf{S}}'_{|\circ}, \\ \mathcal{S}(_{\circ\circ,} & | & ^{\circ\circ})=\mathbf{S}_{\circ}\oplus\mathbf{S}'_{\circ}\oplus\widetilde{\mathbf{S}}_{|\circ}\oplus\widetilde{\mathbf{S}}'_{|\circ},\mathcal{S}(_{\circ}|^{\circ\circ\circ})=\mathbf{S}_{\circ}\oplus\widetilde{\mathbf{S}}'_{\circ}\oplus\widetilde{\mathbf{S}}_{|\circ}\oplus\widetilde{\mathbf{S}}'_{|\circ}, \\ & & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \mathcal{S}(_{\triangle,\triangle}) & = & \mathcal{S}^{(+)}_{\triangle}\oplus S^{(+)}_{|\triangle},S(_{\triangle},^{\triangle})=\mathcal{S}^{(+)}_{\triangle}\oplus\widetilde{S}^{(+)}_{|\triangle}, \\ & & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \mathcal{S}(_{\triangle}|^{\circ},_{\diamondsuit}) & = & \mathbf{S}_{\triangle}\oplus\widetilde{S}_{\circ}{}'\oplus\mathcal{S}_{|\diamondsuit},\mathcal{S}(_{\triangle}|^{\circ},^{\diamondsuit})=\mathbf{S}_{\triangle}\oplus\widetilde{S}_{\circ}{}'\oplus\widetilde{\mathcal{S}}^{\diamondsuit}_{|}, \\ & & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{array} \qquad (5.14)$$

Considering the operation of dualisation of prime components in (5.14) we can generate different isomorphic variants of distinguished (n,m)–spinor spaces.

We define a d–spinor space $\mathcal{S}_{(n,m)}$ as a direct sum of a horizontal and a vertical spinor spaces of type (5.14), for instance,

$$\begin{array}{rcl}\mathcal{S}_{(8k,8k')} & = & \mathbf{S}_{\circ}\oplus\mathbf{S}'_{\circ}\oplus\mathbf{S}_{|\circ}\oplus\mathbf{S}'_{|\circ},\mathcal{S}_{(8k,8k'+1)}=\mathbf{S}_{\circ}\oplus\mathbf{S}'_{\circ}\oplus\mathcal{S}^{(-)}_{|\circ},..., \\ \mathcal{S}_{(8k+4,8k'+5)} & = & \mathbf{S}_{\triangle}\oplus\mathbf{S}'_{\triangle}\oplus\mathcal{S}^{(-)}_{|\triangle},...\end{array}$$

The scalar product on a $\mathcal{S}_{(n,m)}$ is induced by (corresponding to fixed values of n and m) ϵ-objects (5.9) considered for h- and v-components.

Having introduced d-spinors for dimensions (n,m) we can write out the generalization for locally anisotropic spaces of twistor equations [128] by using the distinguished σ-objects (5.4):

$$(\sigma_{(\widehat{\alpha}})^{\cdot\cdot\underline{\gamma}}_{|\underline{\beta}|}\quad\frac{\delta\omega^{\underline{\beta}}}{\delta u^{\widehat{\beta})}}=\frac{1}{n+m}\quad G_{\widehat{\alpha}\widehat{\beta}}(\sigma^{\widehat{\epsilon}})^{\cdot\cdot\underline{\gamma}}_{\underline{\beta}}\quad\frac{\delta\omega^{\underline{\beta}}}{\delta u^{\widehat{\epsilon}}}, \qquad (5.15)$$

where $\left|\underline{\beta}\right|$ denotes that we do not consider symmetrization on this index. The general solution of (5.15) on the d-vector space $\mathcal{F}$ looks like as

$$\omega^{\underline{\beta}}=\Omega^{\underline{\beta}}+u^{\widehat{\alpha}}(\sigma_{\widehat{\alpha}})^{\cdot\cdot\underline{\beta}}_{\underline{\epsilon}}\Pi^{\underline{\epsilon}}, \qquad (5.16)$$

where $\Omega^{\underline{\beta}}$ and $\Pi^{\underline{\epsilon}}$ are constant d-spinors. For fixed values of dimensions n and m we mast analyze the reduced and irreducible components of h- and v-parts of equations (5.15) and their solutions (5.16) in order to find the symmetry properties of a d-twistor $\mathbf{Z}^{\alpha}$ defined as a pair of d–spinors

$$\mathbf{Z}^{\alpha}=(\omega^{\underline{\alpha}},\pi'_{\underline{\beta}}),$$

where $\pi_{\underline{\beta}'}=\pi^{(0)}_{\underline{\beta}'}\in\widetilde{\mathcal{S}}_{(n,m)}$ is a constant dual d-spinor. The problem of definition of spinors and twistors on locally anisotropic spaces was firstly considered in [189] (see also [156]) in connection with the possibility to extend the equations (5.15) and their solutions (5.16), by using nearly autoparallel maps, on curved, locally isotropic or anisotropic, spaces.

5.2 Mutual Transforms of Tensors and Spinors

The spinor algebra for spaces of higher dimensions can not be considered as a real alternative to the tensor algebra as for locally isotropic spaces of dimensions $n = 3, 4$ [127, 128, 129]. The same holds true for locally anisotropic spaces and we emphasize that it is not quite convenient to perform a spinor calculus for dimensions $n, m >> 4$. Nevertheless, the concept of spinors is important for every type of spaces, we can deeply understand the fundamental properties of geometical objects on locally anisotropic spaces, and we shall consider in this subsection some questions concerning transforms of d-tensor objects into d-spinor ones.

5.2.1 Transformation of d-tensors into d-spinors

In order to pass from d-tensors to d-spinors we must use σ-objects (5.4) written in reduced or irreduced form (in dependence of fixed values of dimensions n and m):

$$(\sigma_{\widehat{\alpha}})^{\cdot\gamma}_{\underline{\beta}},\ (\sigma^{\widehat{\alpha}})^{\underline{\beta\gamma}},\ (\sigma^{\widehat{\alpha}})_{\underline{\beta\gamma}}, ..., (\sigma_{\widehat{a}})^{\underline{bc}}, ..., (\sigma_{\widehat{i}})_{\underline{jk}}, ..., (\sigma_{\widehat{a}})^{AA'}, ..., (\sigma^{\widehat{i}})_{II'}, ... \quad (5.17)$$

It is obvious that contracting with corresponding σ-objects (5.17) we can introduce instead of d-tensors indices the d-spinor ones, for instance,

$$\omega^{\underline{\beta\gamma}} = (\sigma^{\widehat{\alpha}})^{\underline{\beta\gamma}}\omega_{\widehat{\alpha}}, \quad \omega_{AB'} = (\sigma^{\widehat{a}})_{AB'}\omega_{\widehat{a}}, \quad ..., \zeta^{\underline{i}}_{\cdot\underline{j}} = (\sigma^{\widehat{k}})^{\underline{i}}_{\cdot\underline{j}}\zeta_{\widehat{k}}, ...$$

For d-tensors containing groups of antisymmetric indices there is a more simple procedure of theirs transforming into d-spinors because the objects

$$(\sigma_{\widehat{\alpha}\widehat{\beta}...\widehat{\gamma}})^{\underline{\delta\nu}}, \quad (\sigma^{\widehat{a}\widehat{b}...\widehat{c}})^{\underline{de}}, \quad ..., (\sigma^{\widehat{i}\widehat{j}...\widehat{k}})_{II'}, \quad ... \quad (5.18)$$

can be used for sets of such indices into pairs of d-spinor indices. Let us enumerate some properties of σ-objects of type (5.18) (for simplicity we consider only h-components having q indices $\widehat{i}, \widehat{j}, \widehat{k}, ...$ taking values from 1 to n; the properties of v-components can be written in a similar manner with respect to indices $\widehat{a}, \widehat{b}, \widehat{c}...$ taking values from 1 to m):

$$(\sigma_{\widehat{i}...\widehat{j}})^{\underline{kl}} \text{ is } \left\{ \begin{array}{c} \text{symmetric on } \underline{k}, \underline{l} \text{ for } n-2q \equiv 1,7\ (mod\ 8); \\ \text{antisymmetric on } \underline{k}, \underline{l} \text{ for } n-2q \equiv 3,5\ (mod\ 8) \end{array} \right\} \quad (5.19)$$

for odd values of n, and an object

$$(\sigma_{\widehat{i}...\widehat{j}})^{IJ} \ \left((\sigma_{\widehat{i}...\widehat{j}})^{I'J'}\right)$$

$$\text{is} \left\{ \begin{array}{c} \text{symmetric on } I,J \ (I',J') \text{ for } n-2q \equiv 0 \ (mod\ 8); \\ \text{antisymmetric on } I,J \ (I',J') \text{ for } n-2q \equiv 4 \ (mod\ 8) \end{array} \right\} \quad (5.20)$$

or

$$(\sigma_{\widehat{i}...\widehat{j}})^{IJ'} = \pm (\sigma_{\widehat{i}...\widehat{j}})^{J'I} \{ \begin{array}{c} n+2q \equiv 6(mod 8); \\ n+2q \equiv 2(mod 8), \end{array} \quad (5.21)$$

with vanishing of the rest of reduced components of the d-tensor $(\sigma_{\widehat{i}...\widehat{j}})^{\underline{kl}}$ with prime/unprime sets of indices.

5.2.2 Fundamental d–spinors

We can transform every d-spinor $\xi^{\underline{\alpha}} = (\xi^{\underline{i}}, \xi^{\underline{a}})$ into a corresponding d-tensor. For simplicity, we consider this construction only for a h-component $\xi^{\underline{i}}$ on a h-space being of dimension n. The values

$$\xi^{\underline{\alpha}} \xi^{\underline{\beta}} (\sigma^{\widehat{i}...\widehat{j}})_{\underline{\alpha\beta}} \quad (n \text{ is odd}) \quad (5.22)$$

or

$$\xi^{I} \xi^{J} (\sigma^{\widehat{i}...\widehat{j}})_{IJ} \ \left(\text{or } \xi^{I'} \xi^{J'} (\sigma^{\widehat{i}...\widehat{j}})_{I'J'}\right) \ (n \text{ is even}) \quad (5.23)$$

with a different number of indices $\widehat{i}...\widehat{j}$, taken together, defines the h-spinor $\xi^{\underline{i}}$ to an accuracy to the sign. We emphasize that it is necessary to choose only those h-components of d-tensors (5.22) (or (5.23)) which are symmetric on pairs of indices $\underline{\alpha\beta}$ (or IJ (or $I'J'$)) and the number q of indices $\widehat{i}...\widehat{j}$ satisfies the condition (as a respective consequence of the properties (5.19) and/or (5.20), (5.21))

$$n - 2q \equiv 0, 1, 7 \ (mod\ 8). \quad (5.24)$$

Of special interest is the case when

$$q = \frac{1}{2}(n \pm 1) \ \ (n \text{ is odd}) \quad (5.25)$$

or

$$q = \frac{1}{2} n \ \ (n \text{ is even}). \quad (5.26)$$

If all expressions (5.22) and/or (5.23) are zero for all values of q with the exception of one or two ones defined by the condition (5.25) (or (5.26)), the value $\xi^{\widehat{i}}$ (or ξ^{I} $(\xi^{I'})$) is called a fundamental h-spinor. Defining in a similar manner the fundamental v-spinors we can introduce fundamental d-spinors as pairs of fundamental h- and v-spinors. Here we remark that a h(v)-spinor $\xi^{\widehat{i}}$ $(\xi^{\widehat{a}})$ (we can also consider reduced components) is always a fundamental one for $n(m) < 7$, which is a consequence of (5.24)).

Finally, in this section, we note that the geometry of fundamental h- and v-spinors is similar to that of usual fundamental spinors (see Appendix to the monograph [129]). We omit such details in this work, but emphasize that constructions with fundamental d-spinors, for a locally anisotropic space, must be adapted to the corresponding global splitting by N-connection of the space.

5.3 Anisotropic Spinor Differential Geometry

The goal of the section is to formulate the differential geometry of d-spinors for locally anisotropic spaces.

We shall use denotations of type

$$v^{\alpha} = (v^{i}, v^{a}) \in \sigma^{\alpha} = (\sigma^{i}, \sigma^{a}) \text{ and } \zeta^{\underline{\alpha}} = (\zeta^{\underline{i}}, \zeta^{\underline{a}}) \in \sigma^{\underline{\alpha}} = (\sigma^{\underline{i}}, \sigma^{\underline{a}})$$

for, respectively, elements of modules of d-vector and irreduced d-spinor fields (see details in [163]). D-tensors and d-spinor tensors (irreduced or reduced) will be interpreted as elements of corresponding σ -modules, for instance,

$$q^{\alpha}{}_{\beta\ldots} \in \sigma^{\alpha}{}_{\beta}, \psi^{\underline{\alpha}}{}_{\underline{\beta}}{}^{\underline{\gamma}}{}_{\ldots} \in \sigma^{\underline{\alpha}}{}_{\underline{\beta}}{}^{\underline{\gamma}}{}_{\ldots}, \xi^{II'}{}_{JK'N'} \in \sigma^{II'}{}_{JK'N'}, \ldots$$

We can establish a correspondence between the d–metric $g_{\alpha\beta}$ (1.39) and d-spinor metric $\epsilon_{\underline{\alpha\beta}}$ (ϵ-objects (5.9) for both h- and v-subspaces of $\mathcal{E}$) of a locally anisotropic space $\mathcal{E}$ by using the relation

$$g_{\alpha\beta} = -\frac{1}{N(n)+N(m)}((\sigma_{(\alpha}(u))^{\underline{\alpha}_1\underline{\beta}_1}(\sigma_{\beta)}(u))^{\underline{\beta}_2\underline{\alpha}_2})\epsilon_{\underline{\alpha}_1\underline{\alpha}_2}\epsilon_{\underline{\beta}_1\underline{\beta}_2}, \tag{5.27}$$

where

$$(\sigma_{\alpha}(u))^{\underline{\nu\gamma}} = l_{\alpha}^{\widehat{\alpha}}(u)(\sigma_{\widehat{\alpha}})^{\underline{\nu\gamma}}, \tag{5.28}$$

which is a consequence of formulas (5.1)-(5.6). In brief we can write (5.27) as

$$g_{\alpha\beta} = \epsilon_{\underline{\alpha}_1\underline{\alpha}_2}\epsilon_{\underline{\beta}_1\underline{\beta}_2} \tag{5.29}$$

if the σ-objects are considered as a fixed structure, whereas ϵ-objects are treated as caring the metric "dynamics " , on locally anisotropic space–times. This variant is used, for instance, in the so-called 2-spinor geometry [128, 129] and should be preferred if we have to make explicit the algebraic symmetry properties of d-spinor objects. An alternative way is to consider as fixed the algebraic structure of ϵ-objects and to use variable components of σ-objects of type (5.28) for developing a variational d-spinor approach to gravitational and matter field interactions on locally anisotropic spaces (the spinor Ashtekar variables [20] are introduced in this manner).

We note that a d–spinor metric

$$\epsilon_{\underline{\nu\tau}} = \begin{pmatrix} \epsilon_{\underline{ij}} & 0 \\ 0 & \epsilon_{\underline{ab}} \end{pmatrix}$$

on the d-spinor space $\mathcal{S} = (\mathcal{S}_{(h)}, \mathcal{S}_{(v)})$ can have symmetric or antisymmetric h (v) -components $\epsilon_{\underline{ij}}$ $(\epsilon_{\underline{ab}})$, see ϵ-objects (5.9). For simplicity, in this section (in order to avoid cumbersome calculations connected with eight-fold periodicity on dimensions n and m of a locally anisotropic space $\mathcal{E}$) we shall develop a general d-spinor formalism only by using irreduced spinor spaces $\mathcal{S}_{(h)}$ and $\mathcal{S}_{(v)}$.

5.4 D-covariant derivation

Let $\mathcal{E}$ be a locally anisotropic space. We define the action on a d-spinor of a d-covariant operator

$$\bigtriangledown_\alpha = (\bigtriangledown_i, \bigtriangledown_a) = (\sigma_\alpha)^{\underline{\alpha}_1\underline{\alpha}_2}\bigtriangledown_{\underline{\alpha}_1\underline{\alpha}_2} = \left((\sigma_i)^{\underline{i}_1\underline{i}_2}\bigtriangledown_{\underline{i}_1\underline{i}_2},\ (\sigma_a)^{\underline{a}_1\underline{a}_2}\bigtriangledown_{\underline{a}_1\underline{a}_2}\right)$$

(in brief, we shall write

$$\bigtriangledown_\alpha = \bigtriangledown_{\underline{\alpha}_1\underline{\alpha}_2} = (\bigtriangledown_{\underline{i}_1\underline{i}_2},\ \bigtriangledown_{\underline{a}_1\underline{a}_2}))$$

as a map

$$\bigtriangledown_{\underline{\alpha}_1\underline{\alpha}_2}\ :\ \sigma^{\underline{\beta}} \to \sigma_\alpha^{\underline{\beta}} = \sigma_{\underline{\alpha}_1\underline{\alpha}_2}^{\underline{\beta}}$$

satisfying conditions

$$\bigtriangledown_\alpha(\xi^{\underline{\beta}} + \eta^{\underline{\beta}}) = \bigtriangledown_\alpha\xi^{\underline{\beta}} + \bigtriangledown_\alpha\eta^{\underline{\beta}},$$

and

$$\bigtriangledown_\alpha(f\xi^{\underline{\beta}}) = f\ \bigtriangledown_\alpha \xi^{\underline{\beta}} + \xi^{\underline{\beta}}\ \bigtriangledown_\alpha f$$

for every $\xi^{\underline{\beta}}, \eta^{\underline{\beta}} \in \sigma^{\underline{\beta}}$ and f being a scalar field on $\mathcal{E}$. It is also required that one holds the Leibnitz rule

$$(\bigtriangledown_\alpha \zeta_{\underline{\beta}})\eta^{\underline{\beta}} = \bigtriangledown_\alpha(\zeta_{\underline{\beta}}\eta^{\underline{\beta}}) - \zeta_{\underline{\beta}} \bigtriangledown_\alpha \eta^{\underline{\beta}}$$

and that $\bigtriangledown_\alpha$ is a real operator, i.e. it commuters with the operation of complex conjugation:

$$\overline{\bigtriangledown_\alpha \psi_{\underline{\alpha}\underline{\beta}\underline{\gamma}\ldots}} = \bigtriangledown_\alpha(\overline{\psi}_{\underline{\alpha}\underline{\beta}\underline{\gamma}\ldots}).$$

Let now analyze the question on uniqueness of action on d-spinors of an operator $\bigtriangledown_\alpha$ satisfying necessary conditions . Denoting by $\bigtriangledown_\alpha^{(1)}$ and $\bigtriangledown_\alpha$ two such d-covariant operators we consider the map

$$(\bigtriangledown_\alpha^{(1)} - \bigtriangledown_\alpha) : \sigma^{\underline{\beta}} \to \sigma_{\underline{\alpha}_\infty \underline{\alpha}_\in}^{\underline{\beta}}. \tag{5.30}$$

Because the action on a scalar f of both operators $\bigtriangledown_\alpha^{(1)}$ and $\bigtriangledown_\alpha$ must be identical, i.e.

$$\bigtriangledown_\alpha^{(1)} f = \bigtriangledown_\alpha f, \tag{5.31}$$

the action (5.30) on $f = \omega_{\underline{\beta}}\xi^{\underline{\beta}}$ must be written as

$$(\bigtriangledown_\alpha^{(1)} - \bigtriangledown_\alpha)(\omega_{\underline{\beta}}\xi^{\underline{\beta}}) = 0.$$

In consequence we conclude that there is an element $\Theta_{\underline{\alpha}_1\underline{\alpha}_2\underline{\beta}}{}^{\underline{\gamma}} \in \sigma_{\underline{\alpha}_1\underline{\alpha}_2\underline{\beta}}{}^{\underline{\gamma}}$ for which

$$\bigtriangledown_{\underline{\alpha}_1\underline{\alpha}_2}^{(1)} \xi^{\underline{\gamma}} = \bigtriangledown_{\underline{\alpha}_1\underline{\alpha}_2} \xi^{\underline{\gamma}} + \Theta_{\underline{\alpha}_1\underline{\alpha}_2\underline{\beta}}{}^{\underline{\gamma}} \xi^{\underline{\beta}} \tag{5.32}$$

and

$$\bigtriangledown_{\underline{\alpha}_1\underline{\alpha}_2}^{(1)} \omega_{\underline{\beta}} = \bigtriangledown_{\underline{\alpha}_1\underline{\alpha}_2} \omega_{\underline{\beta}} - \Theta_{\underline{\alpha}_1\underline{\alpha}_2\underline{\beta}}{}^{\underline{\gamma}} \omega_{\underline{\gamma}}\ .$$

The action of the operator (5.30) on a d-vector $v^\beta = v^{\underline{\beta}_1\underline{\beta}_2}$ can be written by using formula (5.32) for both indices $\underline{\beta}_1$ and $\underline{\beta}_2$:

$$\begin{aligned}(\bigtriangledown_\alpha^{(1)} - \bigtriangledown_\alpha)v^{\underline{\beta}_1\underline{\beta}_2} &= \Theta_{\alpha\underline{\gamma}}{}^{\underline{\beta}_1} v^{\underline{\gamma}\underline{\beta}_2} + \Theta_{\alpha\underline{\gamma}}{}^{\underline{\beta}_2} v^{\underline{\beta}_1\underline{\gamma}} \\ &= (\Theta_{\alpha\underline{\gamma}_1}{}^{\underline{\beta}_1} \delta_{\underline{\gamma}_2}{}^{\underline{\beta}_2} + \Theta_{\alpha\underline{\gamma}_1}{}^{\underline{\beta}_2} \delta_{\underline{\gamma}_2}{}^{\underline{\beta}_1}) v^{\underline{\gamma}_1\underline{\gamma}_2} = Q^\beta{}_{\alpha\gamma} v^\gamma,\end{aligned}$$

where

$$Q^\beta{}_{\alpha\gamma} = Q^{\underline{\beta}_1\underline{\beta}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\ \underline{\gamma}_1\underline{\gamma}_2} = \Theta_{\alpha\underline{\gamma}_1}{}^{\underline{\beta}_1} \delta_{\underline{\gamma}_2}{}^{\underline{\beta}_2} + \Theta_{\alpha\underline{\gamma}_1}{}^{\underline{\beta}_2} \delta_{\underline{\gamma}_2}{}^{\underline{\beta}_1}. \tag{5.33}$$

The d-commutator $\bigtriangledown_{[\alpha}\bigtriangledown_{\beta]}$ defines the d–torsion. So, applying operators $\bigtriangledown^{(1)}_{[\alpha}\bigtriangledown^{(1)}_{\beta]}$ and $\bigtriangledown_{[\alpha}\bigtriangledown_{\beta]}$ on $f=\omega_{\underline{\beta}}\xi^{\underline{\beta}}$ we can write

$$T^{(1)\gamma}{}_{\alpha\beta}-T^{\gamma}{}_{\alpha\beta}=Q^{\gamma}{}_{\beta\alpha}-Q^{\gamma}{}_{\alpha\beta}$$

with $Q^{\gamma}{}_{\alpha\beta}$ from (5.33).

The action of operator $\bigtriangledown^{(1)}_{\alpha}$ on d-spinor tensors of type $\chi_{\underline{\alpha}_1\underline{\alpha}_2\underline{\alpha}_3\ldots}{}^{\underline{\beta}_1\underline{\beta}_2\ldots}$ must be constructed by using formula (5.32) for every upper index $\underline{\beta}_1\underline{\beta}_2\ldots$ and formula (5.33) for every lower index $\underline{\alpha}_1\underline{\alpha}_2\underline{\alpha}_3\ldots$.

5.5 Infeld - van der Waerden coefficients

Let

$$\delta_{\underline{\alpha}}{}^{\underline{\alpha}}=\left(\delta_{\underline{\mathbf{1}}}{}^{\underline{i}},\delta_{\underline{\mathbf{2}}}{}^{\underline{i}},\ldots,\delta_{\underline{\mathbf{N(n)}}}{}^{\underline{i}},\delta_{\underline{\mathbf{1}}}{}^{\underline{a}},\delta_{\underline{\mathbf{2}}}{}^{\underline{a}},\ldots,\delta_{\underline{\mathbf{N(m)}}}{}^{\underline{i}}\right)$$

be a d-spinor basis. The dual to it basis is denoted as

$$\delta_{\underline{\alpha}}{}^{\underline{\alpha}}=\left(\delta_{\underline{i}}{}^{\underline{\mathbf{1}}},\delta_{\underline{i}}{}^{\underline{\mathbf{2}}},\ldots,\delta_{\underline{i}}{}^{\underline{\mathbf{N(n)}}},\delta_{\underline{i}}{}^{\underline{\mathbf{1}}},\delta_{\underline{i}}{}^{\underline{\mathbf{2}}},\ldots,\delta_{\underline{i}}{}^{\underline{\mathbf{N(m)}}}\right).$$

A d-spinor $\kappa^{\underline{\alpha}}\in\sigma^{\ \underline{\alpha}}$ has components $\kappa^{\underline{\alpha}}=\kappa^{\underline{\alpha}}\delta_{\underline{\alpha}}{}^{\underline{\alpha}}$. Taking into account that

$$\delta_{\underline{\alpha}}{}^{\underline{\alpha}}\delta_{\underline{\beta}}{}^{\underline{\beta}}\bigtriangledown_{\underline{\alpha}\underline{\beta}}=\bigtriangledown_{\underline{\alpha}\underline{\beta}},$$

we write out the components $\bigtriangledown_{\underline{\alpha}\underline{\beta}}\ \kappa^{\underline{\gamma}}$ as

$$\begin{aligned}\delta_{\underline{\alpha}}{}^{\underline{\alpha}}\ \delta_{\underline{\beta}}{}^{\underline{\beta}}\ \delta_{\underline{\gamma}}{}^{\underline{\gamma}}\ \bigtriangledown_{\underline{\alpha}\underline{\beta}}\ \kappa^{\underline{\gamma}} &= \delta_{\underline{\epsilon}}{}^{\underline{\tau}}\ \delta_{\underline{\tau}}{}^{\underline{\gamma}}\ \bigtriangledown_{\underline{\alpha}\underline{\beta}}\ \kappa^{\underline{\epsilon}}+\kappa^{\underline{\epsilon}}\ \delta_{\underline{\epsilon}}{}^{\underline{\gamma}}\ \bigtriangledown_{\underline{\alpha}\underline{\beta}}\ \delta_{\underline{\epsilon}}{}^{\underline{\epsilon}}\\ &= \bigtriangledown_{\underline{\alpha}\underline{\beta}}\kappa^{\underline{\gamma}}+\kappa^{\underline{\epsilon}}\gamma^{\underline{\gamma}}{}_{\underline{\alpha}\underline{\beta}\underline{\epsilon}},\end{aligned} \tag{5.34}$$

where the coordinate components of the d-spinor connection $\gamma^{\underline{\gamma}}{}_{\underline{\alpha}\underline{\beta}\underline{\epsilon}}$ are defined as

$$\gamma^{\underline{\gamma}}{}_{\underline{\alpha}\underline{\beta}\underline{\epsilon}}\doteq\delta_{\underline{\tau}}{}^{\underline{\gamma}}\ \bigtriangledown_{\underline{\alpha}\underline{\beta}}\ \delta_{\underline{\epsilon}}{}^{\underline{\tau}}. \tag{5.35}$$

We call the Infeld - van der Waerden d-symbols a set of σ-objects $(\sigma_{\alpha})^{\underline{\alpha}\underline{\beta}}$ parametrized with respect to a coordinate d-spinor basis. Defining

$$\bigtriangledown_{\alpha}=(\sigma_{\alpha})^{\underline{\alpha}\underline{\beta}}\ \bigtriangledown_{\underline{\alpha}\underline{\beta}},$$

introducing denotations

$$\gamma^{\underline{\gamma}}{}_{\alpha\underline{\tau}}\doteq\gamma^{\underline{\gamma}}{}_{\underline{\alpha}\underline{\beta}\underline{\tau}}\ (\sigma_{\alpha})^{\underline{\alpha}\underline{\beta}}$$

and using properties (5.34) we can write relations

$$l_{\alpha}^{\alpha}\ \delta_{\underline{\beta}}\ ^{\underline{\beta}}\ \bigtriangledown_{\alpha}\ \kappa^{\underline{\beta}} = \bigtriangledown_{\alpha}\kappa^{\underline{\beta}} + \kappa^{\underline{\delta}}\ \gamma^{\underline{\beta}}_{\ \alpha\underline{\delta}} \tag{5.36}$$

and

$$l_{\alpha}^{\alpha}\ \delta_{\underline{\beta}}\ ^{\underline{\beta}}\ \bigtriangledown_{\alpha}\ \mu_{\underline{\beta}} = \bigtriangledown_{\alpha}\ \mu_{\underline{\beta}} - \mu_{\underline{\delta}}\gamma^{\underline{\delta}}_{\ \alpha\underline{\beta}} \tag{5.37}$$

for d-covariant derivations $\bigtriangledown_{\underline{\alpha}}\ \kappa^{\underline{\beta}}$ and $\bigtriangledown_{\underline{\alpha}}\ \mu_{\underline{\beta}}$.

We can consider expressions similar to (5.36) and (5.37) for values having both types of d-spinor and d-tensor indices, for instance,

$$l_{\alpha}^{\alpha}\ l_{\gamma}^{\gamma}\ \delta_{\underline{\delta}}\ ^{\underline{\delta}}\ \bigtriangledown_{\alpha}\ \theta_{\underline{\delta}}^{\ \gamma} = \bigtriangledown_{\alpha}\theta_{\underline{\delta}}^{\ \gamma} - \theta_{\underline{\epsilon}}^{\ \gamma}\gamma^{\underline{\epsilon}}_{\ \alpha\underline{\delta}} + \theta_{\underline{\delta}}^{\ \tau}\ \Gamma^{\ \gamma}_{\ \ \alpha\tau}$$

(we can prove this by a straightforward calculation of the derivation $\bigtriangledown_{\alpha}(\theta_{\underline{\epsilon}}^{\ \tau}\ \ \delta_{\underline{\delta}}\ ^{\underline{\epsilon}}\ l_{\tau}^{\gamma}))$.

Now we shall consider some possible relations between components of d-connections $\gamma^{\underline{\epsilon}}_{\ \alpha\underline{\delta}}$ and $\Gamma^{\ \gamma}_{\ \ \alpha\tau}$ and derivations of $(\sigma_{\alpha})^{\underline{\alpha}\underline{\beta}}$. We can write

$$\begin{aligned}
\Gamma^{\alpha}_{\ \beta\gamma} &= l_{\alpha}^{\alpha}\bigtriangledown_{\gamma} l_{\beta}^{\alpha} = l_{\alpha}^{\alpha}\bigtriangledown_{\gamma}(\sigma_{\beta})^{\underline{\epsilon\tau}} = l_{\alpha}^{\alpha}\bigtriangledown_{\gamma}((\sigma_{\beta})^{\underline{\epsilon\tau}}\delta_{\underline{\epsilon}}^{\ \underline{\epsilon}}\delta_{\underline{\tau}}^{\ \underline{\tau}}) \\
&= l_{\alpha}^{\alpha}\delta_{\underline{\alpha}}^{\ \underline{\alpha}}\delta_{\underline{\epsilon}}^{\ \underline{\epsilon}}\bigtriangledown_{\gamma}(\sigma_{\beta})^{\underline{\alpha\epsilon}} + l_{\alpha}^{\alpha}(\sigma_{\beta})^{\underline{\epsilon\tau}}(\delta_{\underline{\tau}}^{\ \underline{\tau}}\bigtriangledown_{\gamma}\delta_{\underline{\epsilon}}^{\ \underline{\epsilon}} + \delta_{\underline{\epsilon}}^{\ \underline{\epsilon}}\bigtriangledown_{\gamma}\delta_{\underline{\tau}}^{\ \underline{\tau}}) \\
&= l_{\underline{\epsilon\tau}}^{\alpha}\ \bigtriangledown_{\gamma}(\sigma_{\beta})^{\underline{\epsilon\tau}} + l_{\underline{\mu\nu}}^{\alpha}\delta_{\underline{\epsilon}}^{\ \underline{\mu}}\delta_{\underline{\tau}}^{\ \underline{\nu}}(\sigma_{\beta})^{\underline{\epsilon\tau}}(\delta_{\underline{\tau}}^{\ \underline{\tau}}\bigtriangledown_{\gamma}\delta_{\underline{\epsilon}}^{\ \underline{\epsilon}} + \delta_{\underline{\epsilon}}^{\ \underline{\epsilon}}\bigtriangledown_{\gamma}\delta_{\underline{\tau}}^{\ \underline{\tau}}),
\end{aligned}$$

where $l_{\alpha}^{\alpha} = (\sigma_{\underline{\epsilon\tau}})^{\alpha}$, from which it follows

$$(\sigma_{\alpha})^{\underline{\mu\nu}}(\sigma_{\underline{\alpha\beta}})^{\beta}\Gamma^{\alpha}_{\ \gamma\beta} = (\sigma_{\underline{\alpha\beta}})^{\beta}\bigtriangledown_{\gamma}(\sigma_{\alpha})^{\underline{\mu\nu}} + \delta_{\underline{\beta}}^{\ \underline{\nu}}\gamma^{\underline{\mu}}_{\ \gamma\underline{\alpha}} + \delta_{\underline{\alpha}}^{\ \underline{\mu}}\gamma^{\underline{\nu}}_{\ \gamma\underline{\beta}}.$$

Connecting the last expression on $\underline{\beta}$ and $\underline{\nu}$ and using an orthonormalized d-spinor basis when $\gamma^{\underline{\beta}}_{\ \gamma\underline{\beta}} = 0$ (a consequence from (5.35)) we have

$$\gamma^{\underline{\mu}}_{\ \gamma\underline{\alpha}} = \frac{1}{N(n)+N(m)}(\Gamma^{\underline{\mu\beta}}_{\ \ \ \gamma\ \underline{\alpha\beta}} - (\sigma_{\underline{\alpha\beta}})^{\beta}\bigtriangledown_{\gamma}(\sigma_{\beta})^{\underline{\mu\beta}}), \tag{5.38}$$

where

$$\Gamma^{\underline{\mu\beta}}_{\ \ \ \gamma\ \underline{\alpha\beta}} = (\sigma_{\alpha})^{\underline{\mu\beta}}(\sigma_{\underline{\alpha\beta}})^{\beta}\Gamma^{\alpha}_{\ \gamma\beta}. \tag{5.39}$$

We also note here that, for instance, for the canonical and Berwald connections, Christoffel d-symbols we can express d-spinor connection (5.39) through corresponding locally adapted derivations of components of metric and N-connection by introducing respectively the coefficients of the Barwald, canonical or another type d–connections.

5.6 D-spinors of Anisotropic Curvature and Torsion

The d-tensor indices of the commutator $\Delta_{\alpha\beta}$ can be transformed into d-spinor ones:

$$\square_{\underline{\alpha}\underline{\beta}} = (\sigma^{\alpha\beta})_{\underline{\alpha}\underline{\beta}}\Delta_{\alpha\beta} = (\square_{\underline{i}\underline{j}}, \square_{\underline{a}\underline{b}}), \tag{5.40}$$

with h- and v-components,

$$\square_{\underline{i}\underline{j}} = (\sigma^{\alpha\beta})_{\underline{i}\underline{j}}\Delta_{\alpha\beta} \text{ and } \square_{\underline{a}\underline{b}} = (\sigma^{\alpha\beta})_{\underline{a}\underline{b}}\Delta_{\alpha\beta},$$

being symmetric or antisymmetric in dependence of corresponding values of dimensions n and m (see eight-fold parametizations (5.18)–(5.20)). Considering the actions of operator (5.40) on d-spinors $\pi^{\underline{\gamma}}$ and $\mu_{\underline{\gamma}}$ we introduce the d-spinor curvature $X_{\underline{\delta}}{}^{\underline{\gamma}}{}_{\underline{\alpha}\underline{\beta}}$ as to satisfy equations

$$\square_{\underline{\alpha}\underline{\beta}}\ \pi^{\underline{\gamma}} = X_{\underline{\delta}}{}^{\underline{\gamma}}{}_{\underline{\alpha}\underline{\beta}}\pi^{\underline{\delta}} \tag{5.41}$$

and

$$\square_{\underline{\alpha}\underline{\beta}}\ \mu_{\underline{\gamma}} = X_{\underline{\gamma}}{}^{\underline{\delta}}{}_{\underline{\alpha}\underline{\beta}}\mu_{\underline{\delta}}.$$

The gravitational d-spinor $\Psi_{\underline{\alpha}\underline{\beta}\underline{\gamma}\underline{\delta}}$ is defined by a corresponding symmetrization of d-spinor indices:

$$\Psi_{\underline{\alpha}\underline{\beta}\underline{\gamma}\underline{\delta}} = X_{(\underline{\alpha}|\underline{\beta}|\underline{\gamma}\underline{\delta})}. \tag{5.42}$$

We note that d-spinor tensors $X_{\underline{\delta}}{}^{\underline{\gamma}}{}_{\underline{\alpha}\underline{\beta}}$ and $\Psi_{\underline{\alpha}\underline{\beta}\underline{\gamma}\underline{\delta}}$ are transformed into similar 2-spinor objects on locally isotropic spaces [128, 129] if we consider vanishing of the N-connection structure and a limit to a locally isotropic space.

Putting $\delta_{\underline{\gamma}}{}^{\underline{\gamma}}$ instead of $\mu_{\underline{\gamma}}$ in (5.41) and using (5.42) we can express respectively the curvature and gravitational d-spinors as

$$X_{\underline{\gamma}\underline{\delta}\underline{\alpha}\underline{\beta}} = \delta_{\underline{\delta}\underline{\tau}}\square_{\underline{\alpha}\underline{\beta}}\delta_{\underline{\gamma}}{}^{\underline{\tau}}$$

and

$$\Psi_{\underline{\gamma}\underline{\delta}\underline{\alpha}\underline{\beta}} = \delta_{\underline{\delta}\underline{\tau}}\square_{(\underline{\alpha}\underline{\beta}}\delta_{\underline{\gamma})}{}^{\underline{\tau}}.$$

The d-spinor torsion $T^{\underline{\gamma}_1\underline{\gamma}_2}{}_{\underline{\alpha}\underline{\beta}}$ is defined similarly as for d-tensors) by using the d-spinor commutator (5.40) and equations

$$\square_{\underline{\alpha}\underline{\beta}} f = T^{\underline{\gamma}_1\underline{\gamma}_2}{}_{\underline{\alpha}\underline{\beta}}\nabla_{\underline{\gamma}_1\underline{\gamma}_2} f.$$

The d-spinor components $R_{\underline{\gamma}_1\underline{\gamma}_2}{}^{\underline{\delta}_1\underline{\delta}_2}{}_{\underline{\alpha\beta}}$ of the curvature d-tensor $R_{\gamma}{}^{\delta}{}_{\alpha\beta}$ can be computed by using the relations (5.39), (5.40) and (5.42) as to satisfy the equations (the d-spinor analogous of equations (1.79))

$$(\Box_{\underline{\alpha\beta}} - T^{\underline{\gamma}_1\underline{\gamma}_2}{}_{\underline{\alpha\beta}}\nabla_{\underline{\gamma}_1\underline{\gamma}_2})V^{\underline{\delta}_1\underline{\delta}_2} = R_{\underline{\gamma}_1\underline{\gamma}_2}{}^{\underline{\delta}_1\underline{\delta}_2}{}_{\underline{\alpha\beta}}V^{\underline{\gamma}_1\underline{\gamma}_2}, \tag{5.43}$$

here d-vector $V^{\underline{\gamma}_1\underline{\gamma}_2}$ is considered as a product of d-spinors, i.e. $V^{\underline{\gamma}_1\underline{\gamma}_2} = \nu^{\underline{\gamma}_1}\mu^{\underline{\gamma}_2}$. We find

$$\begin{aligned} R_{\underline{\gamma}_1\underline{\gamma}_2}{}^{\underline{\delta}_1\underline{\delta}_2}{}_{\underline{\alpha\beta}} &= \left(X_{\underline{\gamma}_1}{}^{\underline{\delta}_1}{}_{\underline{\alpha\beta}} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha\beta}}\ \gamma^{\underline{\delta}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_1}\right)\delta_{\underline{\gamma}_2}{}^{\underline{\delta}_2} \\ &+ \left(X_{\underline{\gamma}_2}{}^{\underline{\delta}_2}{}_{\underline{\alpha\beta}} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha\beta}}\ \gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_2}\right)\delta_{\underline{\gamma}_1}{}^{\underline{\delta}_1}. \end{aligned} \tag{5.44}$$

It is convenient to use this d-spinor expression for the curvature d-tensor

$$\begin{aligned} R_{\underline{\gamma}_1\underline{\gamma}_2}{}^{\underline{\delta}_1\underline{\delta}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\beta}_1\underline{\beta}_2} &= \left(X_{\underline{\gamma}_1}{}^{\underline{\delta}_1}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\beta}_1\underline{\beta}_2} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\beta}_1\underline{\beta}_2}\ \gamma^{\underline{\delta}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_1}\right)\delta_{\underline{\gamma}_2}{}^{\underline{\delta}_2} \\ &+ \left(X_{\underline{\gamma}_2}{}^{\underline{\delta}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\beta}_1\underline{\beta}_2} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\beta}_1\underline{\beta}_2}\ \gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_2}\right)\delta_{\underline{\gamma}_1}{}^{\underline{\delta}_1} \end{aligned}$$

in order to get the d-spinor components of the Ricci d-tensor

$$\begin{aligned} R_{\underline{\gamma}_1\underline{\gamma}_2\underline{\alpha}_1\underline{\alpha}_2} &= R_{\underline{\gamma}_1\underline{\gamma}_2}{}^{\underline{\delta}_1\underline{\delta}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\delta}_2} = X_{\underline{\gamma}_1}{}^{\underline{\delta}_1}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2}\ \gamma^{\underline{\delta}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_1} \\ &+ X_{\underline{\gamma}_2}{}^{\underline{\delta}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\gamma}_1\underline{\delta}_2}\ \gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_2} \end{aligned} \tag{5.45}$$

and this d-spinor decomposition of the scalar curvature:

$$\begin{aligned} q\overleftarrow{R} &= R^{\underline{\alpha}_1\underline{\alpha}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2} = X^{\underline{\alpha}_1\underline{\delta}_1}{}_{\underline{\alpha}_1}{}^{\underline{\alpha}_2}{}_{\underline{\delta}_1\underline{\alpha}_2} + T^{\underline{\tau}_1\underline{\tau}_2\underline{\alpha}_1}{}_{\underline{\alpha}_2\underline{\delta}_1}{}^{\underline{\alpha}_2}\ \gamma^{\underline{\delta}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\alpha}_1} \\ &+ X^{\underline{\alpha}_2\underline{\delta}_2\underline{\alpha}_1}{}_{\underline{\alpha}_2\underline{\delta}_2\underline{\alpha}_1} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1}{}^{\underline{\alpha}_2\underline{\alpha}_1}{}_{\underline{\delta}_2}\ \gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\alpha}_2}. \end{aligned} \tag{5.46}$$

Putting (2.96) and (2.97) into (2.43) and, correspondingly, (2.41) we find the d-spinor components of the Einstein and $\Phi_{\alpha\beta}$ d-tensors:

$$\begin{aligned} \overleftarrow{G}_{\gamma\alpha} &= \overleftarrow{G}_{\underline{\gamma}_1\underline{\gamma}_2\underline{\alpha}_1\underline{\alpha}_2} = X_{\underline{\gamma}_1}{}^{\underline{\delta}_1}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2}\ \gamma^{\underline{\delta}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_1} \\ &+ X_{\underline{\gamma}_2}{}^{\underline{\delta}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\gamma}_1\underline{\delta}_2}\ \gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_2} \\ &- \frac{1}{2}\varepsilon_{\underline{\gamma}_1\underline{\alpha}_1}\varepsilon_{\underline{\gamma}_2\underline{\alpha}_2}[X^{\underline{\beta}_1\underline{\mu}_1}{}_{\underline{\beta}_1}{}^{\underline{\beta}_2}{}_{\underline{\mu}_1\underline{\beta}_2} + T^{\underline{\tau}_1\underline{\tau}_2\underline{\beta}_1}{}_{\underline{\beta}_2\underline{\mu}_1}{}^{\underline{\beta}_2}\ \gamma^{\underline{\mu}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\beta}_1} \\ &+ X^{\underline{\beta}_2\underline{\mu}_2\underline{\nu}_1}{}_{\underline{\beta}_2\underline{\mu}_2\underline{\nu}_1} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\beta}_1}{}^{\underline{\beta}_2\underline{\beta}_1}{}_{\underline{\delta}_2}\ \gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\beta}_2}] \end{aligned} \tag{5.47}$$

and

$$
\begin{aligned}
\Phi_{\gamma\alpha} \quad = \quad & \Phi_{\underline{\gamma}_1\underline{\gamma}_2\underline{\alpha}_1\underline{\alpha}_2} = \frac{1}{2(n+m)} \varepsilon_{\underline{\gamma}_1\underline{\alpha}_1} \varepsilon_{\underline{\gamma}_2\underline{\alpha}_2} [X^{\underline{\beta}_1\underline{\mu}_1}{}_{\underline{\beta}_1}{}^{\underline{\beta}_2}{}_{\underline{\mu}_1\underline{\beta}_2} \qquad (5.48) \\
& + T^{\underline{\tau}_1\underline{\tau}_2\underline{\beta}_1}{}_{\underline{\beta}_2\underline{\mu}_1}{}^{\underline{\beta}_2} \gamma^{\underline{\mu}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\beta}_1} + X^{\underline{\beta}_2\underline{\mu}_2\underline{\nu}_1}{}_{\underline{\beta}_2\underline{\mu}_2\underline{\nu}_1} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\beta}_1}{}^{\underline{\beta}_2\underline{\beta}_1}{}_{\underline{\delta}_2} \gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\beta}_2}] \\
& - \frac{1}{2}[X_{\underline{\gamma}_1}{}^{\underline{\delta}_1}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2} \gamma^{\underline{\delta}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_1} \\
& + X_{\underline{\gamma}_2}{}^{\underline{\delta}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\gamma}_1\underline{\delta}_2} \gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_2}].
\end{aligned}
$$

The components of the conformal Weyl d-spinor can be computed by putting d-spinor values of the curvature (5.44) and Ricci (5.44) d-tensors into corresponding expression for the d-tensor (1.82). We omit this calculus in this work.

Chapter 6

Anisotropic Spinors and Field Equations

The problem of formulation gravitational and gauge field equations on different types of locally anisotropic spaces is considered, for instance, in [109, 27, 19] and [186]. In this section we shall introduce the basic field equations for gravitational and matter field locally anisotropic interactions in a generalized form for generic locally anisotropic spaces.

6.1 Anisotropic Scalar Field Interactions

Let $\varphi(u) = (\varphi_1(u), \varphi_2(u), ..., \varphi_k(u))$ be a complex k-component scalar field of mass μ on locally anisotropic space $\mathcal{E}$. The d-covariant generalization of the conformally invariant (in the massless case) scalar field equation [128, 129] can be defined by using the d'Alambert locally anisotropic operator [4, 168] $\square = D^\alpha D_\alpha$, where D_α is a d-covariant derivation on $\mathcal{E}$ constructed, for simplicity, by using Christoffel d–simbols (all formulas for field equations and conservation values can be deformed by using corresponding defrormation d–tensors $P^\alpha_{\beta\gamma}$ from the Cristoffel d–simbols, or the canonical d–connection to a general d-connection into consideration):

$$(\square + \frac{n+m-2}{4(n+m-1)} \overleftarrow{R} + \mu^2)\varphi(u) = 0. \tag{6.1}$$

We must change d-covariant derivation D_α into ${}^\diamond D_\alpha = D_\alpha + ieA_\alpha$ and take into account the d-vector current

$$J^{(0)}_\alpha(u) = i((\overline{\varphi}(u) D_\alpha \varphi(u) - D_\alpha \overline{\varphi}(u))\varphi(u))$$

if interactions between locally anisotropic electromagnetic field (d-vector potential A_α), where e is the electromagnetic constant, and charged scalar field φ are considered. The equations (6.1) are (locally adapted to the N-connection structure) Euler equations for the Lagrangian

$$\mathcal{L}^{(0)}(u) = \sqrt{|g|}\left[g^{\alpha\beta}\delta_\alpha\overline{\varphi}(u)\,\delta_\beta\varphi(u) - \left(\mu^2 + \frac{n+m-2}{4(n+m-1)}\right)\overline{\varphi}(u)\,\varphi(u)\right], \tag{6.2}$$

where $|g| = det g_{\alpha\beta}$.

The locally adapted variations of the action with Lagrangian (6.2) on variables $\varphi(u)$ and $\overline{\varphi}(u)$ leads to the locally anisotropic generalization of the energy-momentum tensor,

$$E^{(0,can)}_{\alpha\beta}(u) = \delta_\alpha\overline{\varphi}(u)\,\delta_\beta\varphi(u) + \delta_\beta\overline{\varphi}(u)\,\delta_\alpha\varphi(u) - \frac{1}{\sqrt{|g|}}g_{\alpha\beta}\mathcal{L}^{(0)}(u), \tag{6.3}$$

and a similar variation on the components of a d-metric (1.39) leads to a symmetric metric energy-momentum d-tensor,

$$E^{(0)}_{\alpha\beta}(u) = E^{(0,can)}_{(\alpha\beta)}(u) - \frac{n+m-2}{2(n+m-1)}\left[R_{(\alpha\beta)} + D_{(\alpha}D_{\beta)} - g_{\alpha\beta}\Box\right]\overline{\varphi}(u)\,\varphi(u). \tag{6.4}$$

Here we note that we can obtain a nonsymmetric energy-momentum d-tensor if we firstly vary on $G_{\alpha\beta}$ and than impose the constraints of compatibility with the N-connection structure. We also conclude that the existence of a N-connection in v-bundle $\mathcal{E}$ results in a nonequivalence of energy-momentum d-tensors (6.3) and (6.4), nonsymmetry of the Ricci tensor (see (1.77)), nonvanishing of the d-covariant derivation of the Einstein d-tensor (1.85), $D_\alpha \overleftarrow{G}^{\alpha\beta} \neq 0$ and, in consequence, a corresponding breaking of conservation laws on locally anisotropic spaces when $D_\alpha E^{\alpha\beta} \neq 0$ [108, 109]. The problem of formulation of conservation laws on locally anisotropic spaces is discussed in details and two variants of its solution (by using nearly autoparallel maps and tensor integral formalism on locally anisotropic multispaces) are proposed in [168]. In this section we shall present only straightforward generalizations of field equations and necessary formulas for energy-momentum d-tensors of matter fields on $\mathcal{E}$ considering that it is naturally that the conservation laws (usually being consequences of global, local and/or intrinsic symmetries of the fundamental space-time and of the type of field interactions) have to be broken on locally anisotropic spaces.

6.2 Anisotropic Proca equations

Let consider a d-vector field $\varphi_\alpha(u)$ with mass μ^2 (locally anisotropic Proca field) interacting with exterior locally anisotropic gravitational field. From the Lagrangian

$$\mathcal{L}^{(1)}(u) = \sqrt{|g|}\left[-\frac{1}{2}\overline{f}_{\alpha\beta}(u)\, f^{\alpha\beta}(u) + \mu^2\overline{\varphi}_\alpha(u)\,\varphi^\alpha(u)\right], \tag{6.5}$$

where $f_{\alpha\beta} = D_\alpha\varphi_\beta - D_\beta\varphi_\alpha$, one follows the Proca equations on locally anisotropic spaces

$$D_\alpha f^{\alpha\beta}(u) + \mu^2\varphi^\beta(u) = 0. \tag{6.6}$$

Equations (6.6) are a first type constraints for $\beta = 0$. Acting with D_α on (6.6), for $\mu \neq 0$ we obtain second type constraints

$$D_\alpha\varphi^\alpha(u) = 0. \tag{6.7}$$

Putting (6.7) into (6.6) we obtain second order field equations with respect to φ_α :

$$\Box\varphi_\alpha(u) + R_{\alpha\beta}\varphi^\beta(u) + \mu^2\varphi_\alpha(u) = 0. \tag{6.8}$$

The energy-momentum d-tensor and d-vector current following from the (6.8) can be written as

$$E_{\alpha\beta}^{(1)}(u) = -g^{\varepsilon\tau}\left(\overline{f}_{\beta\tau}f_{\alpha\varepsilon} + \overline{f}_{\alpha\varepsilon}f_{\beta\tau}\right) + \mu^2\left(\overline{\varphi}_\alpha\varphi_\beta + \overline{\varphi}_\beta\varphi_\alpha\right) - \frac{g_{\alpha\beta}}{\sqrt{|g|}}\mathcal{L}^{(1)}(u).$$

and

$$J_\alpha^{(1)}(u) = i\left(\overline{f}_{\alpha\beta}(u)\,\varphi^\beta(u) - \overline{\varphi}^\beta(u)\, f_{\alpha\beta}(u)\right).$$

For $\mu = 0$ the d-tensor $f_{\alpha\beta}$ and the Lagrangian (6.5) are invariant with respect to locally anisotropic gauge transforms of type

$$\varphi_\alpha(u) \to \varphi_\alpha(u) + \delta_\alpha\Lambda(u),$$

where $\Lambda(u)$ is a d-differentiable scalar function, and we obtain a locally anisotropic variant of Maxwell theory.

6.3 Anisotropic Gravitons and Backgrounds

Let a massless d-tensor field $h_{\alpha\beta}(u)$ is interpreted as a small perturbation of the locally anisotropic background metric d-field $g_{\alpha\beta}(u)$. Considering, for simplicity, a torsionless background we have locally anisotropic Fierz-Pauli equations

$$\Box h_{\alpha\beta}(u) + 2R_{\tau\alpha\beta\nu}(u)\ h^{\tau\nu}(u) = 0 \tag{6.9}$$

and d-gauge conditions

$$D_\alpha h^\alpha_\beta(u) = 0, \quad h(u) \equiv h^\alpha_\beta(u) = 0, \tag{6.10}$$

where $R_{\tau\alpha\beta\nu}(u)$ is curvature d-tensor of the locally anisotropic background space (these formulae can be obtained by using a perturbation formalism with respect to $h_{\alpha\beta}(u)$ developed in [75]; in our case we must take into account the distinguishing of geometrical objects and operators on locally anisotropic spaces).

We note that we can rewrite d-tensor formulas (6.1)-(6.10) into similar d-spinor ones by using formulas (5.27)–(5.29), (5.39), (5.41) and (5.45)–(6.6) (for simplicity, we omit these considerations in this work).

6.4 Anisotropic Dirac Equations

Let denote the Dirac d-spinor field on $\mathcal{E}$ as $\psi(u) = (\psi^{\underline{\alpha}}(u))$ and consider as the generalized Lorentz transforms the group of automorphysm of the metric $G_{\widehat{\alpha}\widehat{\beta}}$ (see the locally anisotropic frame decomposition of d-metric (5.3)). The d-covariant derivation of field $\psi(u)$ is written as

$$\overrightarrow{\bigtriangledown_\alpha}\psi = \left[\delta_\alpha + \frac{1}{4}C_{\widehat{\alpha}\widehat{\beta}\widehat{\gamma}}(u)\ l^{\widehat{\alpha}}_\alpha(u)\,\sigma^{\widehat{\beta}}\sigma^{\widehat{\gamma}}\right]\psi, \tag{6.11}$$

where coefficients $C_{\widehat{\alpha}\widehat{\beta}\widehat{\gamma}} = (D_\gamma l^\alpha_{\widehat{\alpha}})\, l_{\widehat{\beta}\alpha} l^\gamma_{\widehat{\gamma}}$ generalize for locally anisotropic spaces the corresponding Ricci coefficients on Riemannian spaces [60]. Using σ-objects $\sigma^\alpha(u)$ (see (5.28) and (5.4)) we define the Dirac equations on locally anisotropic spaces:

$$(i\sigma^\alpha(u)\overrightarrow{\bigtriangledown_\alpha} - \mu)\psi = 0, \tag{6.12}$$

which are the Euler equations for the Lagrangian

$$\begin{aligned}\mathcal{L}^{(1/2)}(u) \quad = \quad & \sqrt{|g|}\{[\psi^+(u)\,\sigma^\alpha(u)\,\overrightarrow{\bigtriangledown_\alpha}\psi(u) \\ & -(\overrightarrow{\bigtriangledown_\alpha}\psi^+(u))\sigma^\alpha(u)\,\psi(u)] - \mu\psi^+(u)\,\psi(u)\},\end{aligned} \tag{6.13}$$

where $\psi^{+}(u)$ is the complex conjugation and transposition of the column $\psi(u)$.

From (6.13) we obtain the d-metric energy-momentum d-tensor

$$\begin{aligned} E_{\alpha\beta}^{(1/2)}(u) &= \frac{i}{4}[\psi^{+}(u)\sigma_{\alpha}(u)\overrightarrow{\bigtriangledown_{\beta}}\psi(u)+\psi^{+}(u)\sigma_{\beta}(u)\overrightarrow{\bigtriangledown_{\alpha}}\psi(u) \\ &-(\overrightarrow{\bigtriangledown_{\alpha}}\psi^{+}(u))\sigma_{\beta}(u)\psi(u)-(\overrightarrow{\bigtriangledown_{\beta}}\psi^{+}(u))\sigma_{\alpha}(u)\psi(u)] \end{aligned}$$

and the d-vector source

$$J_{\alpha}^{(1/2)}(u)=\psi^{+}(u)\sigma_{\alpha}(u)\psi(u).$$

We emphasize that locally anisotropic interactions with exterior gauge fields can be introduced by changing the locally anisotropic partial derivation from (6.11) in this manner:

$$\delta_{\alpha}\rightarrow\delta_{\alpha}+ie^{\star}B_{\alpha}, \tag{6.14}$$

where $e^{\star}$ and B_{α} are respectively the constant d-vector potential of locally anisotropic gauge interactions on locally anisotropic spaces (see [186] and the next section).

6.5 Yang-Mills Equations in Anisotropic Spinor Form

We consider a v-bundle $\mathcal{B}_{E}$, $\pi_{B}:\mathcal{B}\rightarrow\mathcal{E}$, on locally anisotropic space $\mathcal{E}$. Additionally to d-tensor and d-spinor indices we shall use capital Greek letters, $\Phi,\Upsilon,\Xi,\Psi,...$ for fibre (of this bundle) indices (see details in [128, 129] for the case when the base space of the v-bundle π_{B} is a locally isotropic space-time). Let $\underline{\bigtriangledown}_{\alpha}$ be, for simplicity, a torsionless, linear connection in $\mathcal{B}_{E}$ satisfying conditions:

$$\begin{aligned} \underline{\bigtriangledown}_{\alpha} &: \quad \Upsilon^{\Theta}\rightarrow\ \Upsilon_{\alpha}^{\Theta}\quad\left[\text{or }\ \Xi^{\Theta}\rightarrow\ \Xi_{\alpha}^{\Theta}\right], \\ \underline{\bigtriangledown}_{\alpha}\left(\lambda^{\Theta}+\nu^{\Theta}\right) &= \underline{\bigtriangledown}_{\alpha}\lambda^{\Theta}+\underline{\bigtriangledown}_{\alpha}\nu^{\Theta}, \\ \underline{\bigtriangledown}_{\alpha}\left(f\lambda^{\Theta}\right) &= \lambda^{\Theta}\underline{\bigtriangledown}_{\alpha}f+f\underline{\bigtriangledown}_{\alpha}\lambda^{\Theta},\quad f\in\ \Upsilon^{\Theta}\ [\text{or }\ \Xi^{\Theta}], \end{aligned}$$

where by Υ^{Θ} $\left(\Xi^{\Theta}\right)$ we denote the module of sections of the real (complex) v-bundle $\mathcal{B}_{E}$ provided with the abstract index Θ. The curvature of connection $\underline{\bigtriangledown}_{\alpha}$ is defined as

$$K_{\alpha\beta\Omega}\ {}^{\Theta}\lambda^{\Omega}=\left(\underline{\bigtriangledown}_{\alpha}\underline{\bigtriangledown}_{\beta}-\underline{\bigtriangledown}_{\beta}\underline{\bigtriangledown}_{\alpha}\right)\lambda^{\Theta}.$$

For Yang-Mills fields as a rule one considers that $\mathcal{B}_E$ is enabled with a unitary (complex) structure (complex conjugation changes mutually the upper and lower Greek indices). It is useful to introduce instead of $K_{\alpha\beta\Omega}{}^{\Theta}$ a Hermitian matrix $F_{\alpha\beta\Omega}{}^{\Theta} = i\ K_{\alpha\beta\Omega}{}^{\Theta}$ connected with components of the Yang-Mills d-vector potential $B_{\alpha\Xi}{}^{\Phi}$ according the formula:

$$\frac{1}{2}F_{\alpha\beta\Xi}{}^{\Phi} = \underline{\bigtriangledown}_{[\alpha}B_{\beta]\Xi}{}^{\Phi} - iB_{[\alpha|\Lambda|}{}^{\Phi}B_{\beta]\Xi}{}^{\Lambda}, \tag{6.15}$$

where the locally anisotropic space indices commute with capital Greek indices. The gauge transforms are written in the form:

$$\begin{aligned} B_{\alpha\Theta}{}^{\Phi} &\mapsto B_{\alpha\widehat{\Theta}}{}^{\widehat{\Phi}} = B_{\alpha\Theta}{}^{\Phi}\, s_{\Phi}{}^{\widehat{\Phi}}\, q_{\widehat{\Theta}}{}^{\Theta} + is_{\Theta}{}^{\widehat{\Phi}}\underline{\bigtriangledown}_{\alpha}\, q_{\widehat{\Theta}}{}^{\Theta}, \\ F_{\alpha\beta\Xi}{}^{\Phi} &\mapsto F_{\alpha\beta\widehat{\Xi}}{}^{\widehat{\Phi}} = F_{\alpha\beta\Xi}{}^{\Phi}s_{\Phi}{}^{\widehat{\Phi}}q_{\widehat{\Xi}}{}^{\Xi}, \end{aligned}$$

where matrices $s_{\Phi}{}^{\widehat{\Phi}}$ and $q_{\widehat{\Xi}}{}^{\Xi}$ are mutually inverse (Hermitian conjugated in the unitary case). The Yang-Mills equations on torsionless locally anisotropic spaces [186] (see details in the next Chapter) are written in this form:

$$\underline{\bigtriangledown}^{\alpha}F_{\alpha\beta\Theta}{}^{\Psi} = J_{\beta\ \Theta}{}^{\Psi}, \tag{6.16}$$

$$\underline{\bigtriangledown}_{[\alpha}F_{\beta\gamma]\Theta}{}^{\Xi} = 0. \tag{6.17}$$

We must introduce deformations of connection of type, $\underline{\bigtriangledown}^{\star}_{\alpha} \longrightarrow \underline{\bigtriangledown}_{\alpha} + P_{\alpha}$, (the deformation d-tensor P_{α} is induced by the torsion in v-bundle $\mathcal{B}_E$) into the definition of the curvature of locally anisotropic gauge fields (6.15) and motion equations (6.16) and (6.17) if interactions are modeled on a generic locally anisotropic space.

Now we can write out the field equations of the Einstein-Cartan theory in the d-spinor form. So, for the Einstein equations (1.84) we have

$$\overleftarrow{G}_{\underline{\gamma}_1\underline{\gamma}_2\underline{\alpha}_1\underline{\alpha}_2} + \lambda\varepsilon_{\underline{\gamma}_1\underline{\alpha}_1}\varepsilon_{\underline{\gamma}_2\underline{\alpha}_2} = \kappa E_{\underline{\gamma}_1\underline{\gamma}_2\underline{\alpha}_1\underline{\alpha}_2},$$

with $\overleftarrow{G}_{\underline{\gamma}_1\underline{\gamma}_2\underline{\alpha}_1\underline{\alpha}_2}$ from (5.47), or, by using the d-tensor (5.48),

$$\Phi_{\underline{\gamma}_1\underline{\gamma}_2\underline{\alpha}_1\underline{\alpha}_2} + (\frac{\overleftarrow{R}}{4} - \frac{\lambda}{2})\varepsilon_{\underline{\gamma}_1\underline{\alpha}_1}\varepsilon_{\underline{\gamma}_2\underline{\alpha}_2} = -\frac{\kappa}{2}E_{\underline{\gamma}_1\underline{\gamma}_2\underline{\alpha}_1\underline{\alpha}_2},$$

which are the d-spinor equivalent of the equations (1.86). These equations must be completed by the algebraic equations (1.87) for the d-torsion and d-spin density with d-tensor indices changed into corresponding d-spinor ones.

Part III

Higher Order Anisotropic Spinors

The theory of anisotropic spinors formulated in the Part II is extended for higher order anisotropic (ha) spaces. In brief, such spinors will be called ha–spinors which are defined as some Clifford ha–structures defined with respect to a distinguished quadratic form (1.43) on a hvc–bundle. For simplicity, the bulk of formulas will be given with respect to higher order vector bundles. To rewrite such formulas for hvc–bundles is to consider for the "dual" shells of higher order anisotropy some dual vector spaces and associated dual spinors.

Chapter 7

Clifford Ha–Structures

7.1 Distinguished Clifford Algebras

The typical fiber of dv–bundle ξ_d , π_d : $HE \oplus V_1E \oplus ... \oplus V_zE \rightarrow E$ is a d-vector space, $\mathcal{F} = h\mathcal{F} \oplus v_1\mathcal{F} \oplus ... \oplus v_z\mathcal{F}$, split into horizontal $h\mathcal{F}$ and verticals $v_p\mathcal{F}, p = 1, ..., z$ subspaces, with a bilinear quadratic form $G(g, h)$ induced by a hvc–bundle metric (1.43). Clifford algebras (see, for example, Refs. [83, 154, 129]) formulated for d-vector spaces will be called Clifford d-algebras [163, 162, 189]. We shall consider the main properties of Clifford d–algebras. The proof of theorems will be based on the technique developed in Ref. [83] correspondingly adapted to the distinguished character of spaces in consideration.

Let k be a number field (for our purposes $k = \mathcal{R}$ or $k = \mathcal{C}, \mathcal{R}$ and $\mathcal{C}$, are, respectively real and complex number fields) and define $\mathcal{F}$, as a d-vector space on k provided with nondegenerate symmetric quadratic form (metric) G. Let C be an algebra on k (not necessarily commutative) and $j : \mathcal{F} \rightarrow C$ a homomorphism of underlying vector spaces such that $j(u)^2 = G(u) \cdot 1$ (1 is the unity in algebra C and d-vector $u \in \mathcal{F}$). We are interested in definition of the pair (C, j) satisfying the next universitality conditions. For every k-algebra A and arbitrary homomorphism $\varphi : \mathcal{F} \rightarrow A$ of the underlying d-vector spaces, such that $(\varphi(u))^2 \rightarrow G(u) \cdot 1$, there is a unique homomorphism of algebras $\psi : C \rightarrow A$ transforming the diagram 1 into a commutative one.

The algebra solving this problem will be denoted as $C(\mathcal{F}, A)$ [equivalently as $C(G)$ or $C(\mathcal{F})$] and called as Clifford d–algebra associated with pair $(\mathcal{F}, G)$.

Theorem 7.1. *The above-presented diagram has a unique solution (C, j) up to isomorphism.*

Proof: (We adapt for d-algebras that of Ref. [83], p. 127 and extend for higher order anisotropies a similar proof presented in the Part II). For a universal problem the uniqueness is obvious if we prove the existence of solution $C(G)$. To do this we use tensor algebra $\mathcal{L}^{(F)} = \oplus\mathcal{L}^{pr}_{qs}(\mathcal{F}) = \oplus^{\infty}_{i=0}T^i(\mathcal{F})$, where $T^0(\mathcal{F}) = k$ and $T^i(\mathcal{F}) = k$ and $T^i(\mathcal{F}) = \mathcal{F}\otimes...\otimes\mathcal{F}$ for $i > 0$. Let $I(G)$ be the bilateral ideal generated by elements of form $\epsilon(u) = u \otimes u - G(u) \cdot 1$ where $u \in \mathcal{F}$ and 1 is the unity element of algebra $\mathcal{L}(\mathcal{F})$. Every element from $I(G)$ can be written as $\sum_i \lambda_i \epsilon(u_i)\mu_i$, where $\lambda_i, \mu_i \in \mathcal{L}(\mathcal{F})$ and $u_i \in \mathcal{F}$. Let $C(G) = \mathcal{L}(\mathcal{F})/I(G)$ and define $j : \mathcal{F} \to C(G)$ as the composition of monomorphism $i : \mathcal{F} \to L^1(\mathcal{F}) \subset \mathcal{L}(\mathcal{F})$ and projection $p : \mathcal{L}(\mathcal{F}) \to C(G)$. In this case pair $(C(G), j)$ is the solution of our problem. From the general properties of tensor algebras the homomorphism $\varphi : \mathcal{F} \to A$ can be extended to $\mathcal{L}(\mathcal{F})$, i.e., the diagram 2 is commutative, where ρ is a monomorphism of algebras. Because $(\varphi(u))^2 = G(u) \cdot 1$, then ρ vanishes on ideal $I(G)$ and in this case the necessary homomorphism τ is defined. As a consequence of uniqueness of ρ, the homomorphism τ is unique.

Tensor d–algebra $\mathcal{L}(\mathcal{F})$ can be considered as a $\mathcal{Z}/2$ graded algebra. Really, let us introduce $\mathcal{L}^{(0)}(\mathcal{F}) = \sum^{\infty}_{i=1} T^{2i}(\mathcal{F})$ and $\mathcal{L}^{(1)}(\mathcal{F}) = \sum^{\infty}_{i=1} T^{2i+1}(\mathcal{F})$. Setting $I^{(\alpha)}(G) = I(G) \cap \mathcal{L}^{(\alpha)}(\mathcal{F})$. Define $C^{(\alpha)}(G)$ as $p\left(\mathcal{L}^{(\alpha)}(\mathcal{F})\right)$, where $p : \mathcal{L}(\mathcal{F}) \to C(G)$ is the canonical projection. Then $C(G) = C^{(0)}(G) \oplus C^{(1)}(G)$ and in consequence we obtain that the Clifford d–algebra is $\mathcal{Z}/2$ graded.

It is obvious that Clifford d-algebra functorially depends on pair $(\mathcal{F}, G)$. If $f : \mathcal{F} \to \mathcal{F}'$ is a homomorphism of k-vector spaces, such that $G'(f(u)) = G(u)$, where G and G' are, respectively, metrics on $\mathcal{F}$ and $\mathcal{F}'$, then f induces an homomorphism of d-algebras

$$C(f) : C(G) \to C(G')$$

with identities $C(\varphi \cdot f) = C(\varphi) C(f)$ and $C(Id_{\mathcal{F}}) = Id_{C(\mathcal{F})}$.

If $\mathcal{A}^{\alpha}$ and $\mathcal{B}^{\beta}$ are $\mathcal{Z}/2$–graded d–algebras, then their graded tensorial product $\mathcal{A}^{\alpha} \otimes \mathcal{B}^{\beta}$ is defined as a d-algebra for k-vector d-space $\mathcal{A}^{\alpha} \otimes \mathcal{B}^{\beta}$ with the graded product induced as $(a \otimes b)(c \otimes d) = (-1)^{\alpha\beta} ac \otimes bd$, where $b \in \mathcal{B}^{\alpha}$ and $c \in \mathcal{A}^{\alpha} \quad (\alpha, \beta = 0, 1)$.

Now we re–formulate for d–algebras the Chevalley theorem [45]:

Theorem 7.2. *The Clifford d-algebra*

$$C(h\mathcal{F} \oplus v_1\mathcal{F} \oplus ... \oplus v_z\mathcal{F}, g + h_1 + ... + h_z)$$

is naturally isomorphic to $C(g) \otimes C(h_1) \otimes ... \otimes C(h_z)$.

Proof. Let $n : h\mathcal{F} \to C(g)$ and $n'_{(p)} : v_{(p)}\mathcal{F} \to C\left(h_{(p)}\right)$ be canonical maps and map

$$m : h\mathcal{F} \oplus v_1\mathcal{F} \oplus ... \oplus v_z\mathcal{F} \to C(g) \otimes C(h_1) \otimes ... \otimes C(h_z)$$

is defined as

$$m(x,y_{(1)},...,y_{(z)}) =$$
$$n(x)\otimes 1\otimes ...\otimes 1+1\otimes n^{\prime }(y_{(1)})\otimes ...\otimes 1+1\otimes ...\otimes 1\otimes n^{\prime }(y_{(z)}),$$

$x\in h\mathcal{F},y_{(1)}\in v_{(1)}\mathcal{F},...,y_{(z)}\in v_{(z)}\mathcal{F}$. We have

$$\begin{aligned} \left(m(x,y_{(1)},...,y_{(z)})\right) ^{2} &= \left[\left(n\left(x\right) \right) ^{2}+\left(n^{\prime }\left(y_{(1)}\right) \right) ^{2}+...+\left(n^{\prime }\left(y_{(z)}\right) \right) ^{2}\right] \cdot 1 \\ &= [g\left(x\right) +h\left(y_{(1)}\right) +...+h\left(y_{(z)}\right)]. \end{aligned}$$

Taking into account the universality property of Clifford d-algebras we conclude that $m_{1}+...+m_{z}$ induces the homomorphism

$$\zeta :C\left(h\mathcal{F}\oplus v_{1}\mathcal{F}\oplus ...\oplus v_{z}\mathcal{F},g+h_{1}+...+h_{z}\right) \rightarrow$$
$$C\left(h\mathcal{F},g\right) \widehat{\otimes }C\left(v_{1}\mathcal{F},h_{1}\right) \widehat{\otimes }...C\left(v_{z}\mathcal{F},h_{z}\right) .$$

We also can define a homomorphism

$$\upsilon :C\left(h\mathcal{F},g\right) \widehat{\otimes }C\left(v_{1}\mathcal{F},h_{(1)}\right) \widehat{\otimes }...\widehat{\otimes }C\left(v_{z}\mathcal{F},h_{(z)}\right) \rightarrow$$
$$C\left(h\mathcal{F}\oplus v_{1}\mathcal{F}\oplus ...\oplus v_{z}\mathcal{F},g+h_{(1)}+...+h_{(z)}\right)$$

by using formula $\upsilon \left(x\otimes y_{(1)}\otimes ...\otimes y_{(z)}\right) =\delta \left(x\right) \delta _{(1)}^{\prime }\left(y_{(1)}\right) ...\delta _{(z)}^{\prime }\left(y_{(z)}\right)$, where homomorphysms δ and $\delta _{(1)}^{\prime },...,\delta _{(z)}^{\prime }$ are, respectively, induced by imbeddings of $h\mathcal{F}$ and $v_{1}\mathcal{F}$ into $h\mathcal{F}\oplus v_{1}\mathcal{F}\oplus ...\oplus v_{z}\mathcal{F}$:

$$\begin{aligned} \delta &: C\left(h\mathcal{F},g\right) \rightarrow C\left(h\mathcal{F}\oplus v_{1}\mathcal{F}\oplus ...\oplus v_{z}\mathcal{F},g+h_{(1)}+...+h_{(z)}\right) , \\ \delta _{(1)}^{\prime } &: C\left(v_{1}\mathcal{F},h_{(1)}\right) \rightarrow C\left(h\mathcal{F}\oplus v_{1}\mathcal{F}\oplus ...\oplus v_{z}\mathcal{F},g+h_{(1)}+...+h_{(z)}\right) , \\ & .... \\ \delta _{(z)}^{\prime } &: C\left(v_{z}\mathcal{F},h_{(z)}\right) \rightarrow C\left(h\mathcal{F}\oplus v_{1}\mathcal{F}\oplus ...\oplus v_{z}\mathcal{F},g+h_{(1)}+...+h_{(z)}\right) . \end{aligned}$$

Superpositions of homomorphisms ζ and υ lead to identities

$$\begin{aligned} \upsilon \zeta &= Id_{C\left(h\mathcal{F},g\right) \widehat{\otimes }C\left(v_{1}\mathcal{F},h_{(1)}\right) \widehat{\otimes }...\widehat{\otimes }C\left(v_{z}\mathcal{F},h_{(z)}\right) }, \\ \zeta \upsilon &= Id_{C\left(h\mathcal{F},g\right) \widehat{\otimes }C\left(v_{1}\mathcal{F},h_{(1)}\right) \widehat{\otimes }...\widehat{\otimes }C\left(v_{z}\mathcal{F},h_{(z)}\right) }. \end{aligned} \tag{7.1}$$

Really, d-algebra $C\left(h\mathcal{F}\oplus v_{1}\mathcal{F}\oplus ...\oplus v_{z}\mathcal{F},g+h_{(1)}+...+h_{(z)}\right)$ is generated by elements of type $m(x,y_{(1)},...y_{(z)})$. Calculating

$$\begin{aligned} & \upsilon \zeta \left(m\left(x,y_{(1)},...y_{(z)}\right) \right) =\upsilon (n\left(x\right) \otimes 1\otimes ...\otimes 1+1\otimes n_{(1)}^{\prime }\left(y_{(1)}\right) \otimes ...\otimes 1 \\ & +...+1\otimes ....\otimes n_{(z)}^{\prime }\left(y_{(z)}\right))=\delta \left(n\left(x\right) \right) \delta \left(n_{(1)}^{\prime }\left(y_{(1)}\right) \right) ...\delta \left(n_{(z)}^{\prime }\left(y_{(z)}\right) \right) \\ & =m\left(x,0,...,0\right) +m(0,y_{(1)},...,0)+...+m(0,0,...,y_{(z)}) \\ & =m\left(x,y_{(1)},...,y_{(z)}\right) , \end{aligned}$$

we prove the first identity in (7.1).

On the other hand, d-algebra

$$C(h\mathcal{F},g)\widehat{\otimes}C\left(v_1\mathcal{F},h_{(1)}\right)\widehat{\otimes}...\widehat{\otimes}C\left(v_z\mathcal{F},h_{(z)}\right)$$

is generated by elements of type

$$n(x)\otimes 1\otimes ...\otimes,1\otimes n'_{(1)}\left(y_{(1)}\right)\otimes ...\otimes 1,...1\otimes ....\otimes n'_{(z)}\left(y_{(z)}\right),$$

we prove the second identity in (7.1).

Following from the above–mentioned properties of homomorphisms ζ and υ we can assert that the natural isomorphism is explicitly constructed.□

In consequence of the presented in this section Theorems we conclude that all operations with Clifford d-algebras can be reduced to calculations for $C(h\mathcal{F},g)$ and $C\left(v_{(p)}\mathcal{F},h_{(p)}\right)$ which are usual Clifford algebras of dimension 2^n and, respectively, 2^{m_p} [83, 21].

Of special interest is the case when $k=\mathcal{R}$ and $\mathcal{F}$ is isomorphic to vector space $\mathcal{R}^{p+q,a+b}$ provided with quadratic form

$$-x_1^2-...-x_p^2+x_{p+q}^2-y_1^2-...-y_a^2+...+y_{a+b}^2.$$

In this case, the Clifford algebra, denoted as $\left(C^{p,q},C^{a,b}\right)$, is generated by symbols $e_1^{(x)},e_2^{(x)},...,e_{p+q}^{(x)},e_1^{(y)},e_2^{(y)},...,e_{a+b}^{(y)}$ satisfying properties

$$\begin{aligned}(e_i)^2 &= -1\ (1\leq i\leq p),(e_j)^2=-1\ (1\leq j\leq a),\\ (e_k)^2 &= 1\ (p+1\leq k\leq p+q),\\ (e_j)^2 &= 1\ (n+1\leq s\leq a+b),\ e_ie_j=-e_je_i,\ i\neq j.\end{aligned}$$

Explicit calculations of $C^{p,q}$ and $C^{a,b}$ are possible by using isomorphisms [83, 129]

$$\begin{aligned}C^{p+n,q+n} &\simeq C^{p,q}\otimes M_2(\mathcal{R})\otimes ...\otimes M_2(\mathcal{R})\\ &\cong C^{p,q}\otimes M_{2^n}(\mathcal{R})\cong M_{2^n}(C^{p,q}),\end{aligned}$$

where $M_s(A)$ denotes the ring of quadratic matrices of order s with coefficients in ring A. Here we write the simplest isomorphisms $C^{1,0}\simeq\mathcal{C},C^{0,1}\simeq\mathcal{R}\oplus\mathcal{R}$ and $C^{2,0}=\mathcal{H}$, where by $\mathcal{H}$ is denoted the body of quaternions.

Now, we emphasize that higher order Lagrange and Finsler spaces, denoted H^{2n}–spaces, admit locally a structure of Clifford algebra on complex vector spaces. Really, by using almost Hermitian structure $J_\alpha^{\ \ \beta}$ and considering complex space $\mathcal{C}^n$ with nondegenarate quadratic form $\sum_{a=1}^n|z_a|^2,\ z_a\in\mathcal{C}^2$ induced locally by metric (1.43) (rewritten in complex coordinates $z_a=$

$x_a + iy_a)$ we define Clifford algebra $\overleftarrow{C}^n = \underbrace{\overleftarrow{C}^1 \otimes ... \otimes \overleftarrow{C}^1}_{n}$, where $\overleftarrow{C}^1 = \mathcal{C}\otimes_R\mathcal{C} = \mathcal{C} \oplus \mathcal{C}$ or in consequence, $\overleftarrow{C}^n \simeq C^{n,0} \otimes_{\mathcal{R}} \mathcal{C} \approx C^{0,n} \otimes_{\mathcal{R}} \mathcal{C}$. Explicit calculations lead to isomorphisms

$$\overleftarrow{C}^2 = C^{0,2} \otimes_{\mathcal{R}} \mathcal{C} \approx M_2(\mathcal{R}) \otimes_{\mathcal{R}} \mathcal{C} \approx M_2\left(\overleftarrow{C}^n\right),\ C^{2p} \approx M_{2^p}(\mathcal{C})$$

and

$$\overleftarrow{C}^{2p+1} \approx M_{2^p}(\mathcal{C}) \oplus M_{2^p}(\mathcal{C}),$$

which show that complex Clifford algebras, defined locally for H^{2n}-spaces, have periodicity 2 on p.

Considerations presented in the proof of theorem 2.2 show that map $j : \mathcal{F} \to C(\mathcal{F})$ is monomorphic, so we can identify space $\mathcal{F}$ with its image in $C(\mathcal{F}, G)$, denoted as $u \to \overline{u}$, if $u \in C^{(0)}(\mathcal{F}, G)$ $\left(u \in C^{(1)}(\mathcal{F}, G)\right)$; then $u = \overline{u}$ (respectively, $\overline{u} = -u$).

Definition 7.1. *The set of elements $u \in C(G)^*$, where $C(G)^*$ denotes the multiplicative group of invertible elements of $C(\mathcal{F}, G)$ satisfying $\overline{u}\mathcal{F}u^{-1} \in \mathcal{F}$, is called the twisted Clifford d-group, denoted as $\widetilde{\Gamma}(\mathcal{F})$.*

Let $\widetilde{\rho} : \widetilde{\Gamma}(\mathcal{F}) \to GL(\mathcal{F})$ be the homorphism given by $u \to \rho\widetilde{u}$, where $\widetilde{\rho}_u(w) = \overline{u}wu^{-1}$. We can verify that $\ker\widetilde{\rho} = \mathcal{R}^*$is a subgroup in $\widetilde{\Gamma}(\mathcal{F})$.

The canonical map $j : \mathcal{F} \to C(\mathcal{F})$ can be interpreted as the linear map $\mathcal{F} \to C(\mathcal{F})^0$ satisfying the universal property of Clifford d-algebras. This leads to a homomorphism of algebras, $C(\mathcal{F}) \to C(\mathcal{F})^t$, considered by an anti-involution of $C(\mathcal{F})$ and denoted as $u \to {}^tu$. More exactly, if $u_1...u_n \in \mathcal{F}$, then $t_u = u_n...u_1$ and ${}^t\overline{u} = \overline{{}^tu} = (-1)^n u_n...u_1$.

Definition 7.2. *The spinor norm of arbitrary $u \in C(\mathcal{F})$ is defined as* $S(u) = {}^t\overline{u} \cdot u \in C(\mathcal{F})$.

It is obvious that if $u, u', u'' \in \widetilde{\Gamma}(\mathcal{F})$, then $S(u, u') = S(u)S(u')$ and $S(uu'u'') = S(u)S(u')S(u'')$. For $u, u' \in \mathcal{F}S(u) = -G(u)$ and $S(u, u') = S(u)S(u') = S(uu')$.

Let us introduce the orthogonal group $O(G) \subset GL(G)$ defined by metric G on $\mathcal{F}$ and denote sets

$$SO(G) = \{u \in O(G), \det|u| = 1\},\ Pin(G) = \{u \in \widetilde{\Gamma}(\mathcal{F}), S(u) = 1\}$$

and $Spin(G) = Pin(G) \cap C^0(\mathcal{F})$. For $\mathcal{F} \cong \mathcal{R}^{n+m}$ we write $Spin(n_E)$. By straightforward calculations (see similar considerations in Ref. [83]) we can verify the exactness of these sequences:

$$\begin{aligned} 1 &\rightarrow \mathcal{Z}/2 \rightarrow Pin\,(G) \rightarrow O\,(G) \rightarrow 1, \\ 1 &\rightarrow \mathcal{Z}/2 \rightarrow Spin\,(G) \rightarrow SO\,(G) \rightarrow 0, \\ 1 &\rightarrow \mathcal{Z}/2 \rightarrow Spin\,(n_E) \rightarrow SO\,(n_E) \rightarrow 1. \end{aligned}$$

We conclude this subsection by emphasizing that the spinor norm was defined with respect to a quadratic form induced by a metric in dv-bundle $\mathcal{E}^{<z>}$. This approach differs from those presented in Refs. [19] and [124].

7.2 Clifford Ha–Bundles

We shall consider two variants of generalization of spinor constructions defined for d-vector spaces to the case of distinguished vector bundle spaces enabled with the structure of N-connection. The first is to use the extension to the category of vector bundles. The second is to define the Clifford fibration associated with compatible linear d-connection and metric G on a dv–bundle. We shall analyze both variants.

7.2.1 Clifford d–module structure in dv–bundles

Because functor $\mathcal{F} \rightarrow C(\mathcal{F})$ is smooth we can extend it to the category of vector bundles of type

$$\xi^{<z>} = \{\pi_d : HE^{<z>} \oplus V_1E^{<z>} \oplus ... \oplus V_zE^{<z>} \rightarrow E^{<z>}\}.$$

Recall that by $\mathcal{F}$ we denote the typical fiber of such bundles. For $\xi^{<z>}$ we obtain a bundle of algebras, denoted as $C\,(\xi^{<z>})$, such that $C\,(\xi^{<z>})_u = C\,(\mathcal{F}_u)$. Multiplication in every fibre defines a continuous map

$$C\,(\xi^{<z>}) \times C\,(\xi^{<z>}) \rightarrow C\,(\xi^{<z>})\,.$$

If $\xi^{<z>}$ is a distinguished vector bundle on number field k, the structure of the $C\,(\xi^{<z>})$-module, the d-module, the d-module, on $\xi^{<z>}$ is given by the continuous map $C\,(\xi^{<z>}) \times_E \xi^{<z>} \rightarrow \xi^{<z>}$ with every fiber $\mathcal{F}_u$ provided with the structure of the $C\,(\mathcal{F}_u)$ −module, correlated with its k-module structure, Because $\mathcal{F} \subset C\,(\mathcal{F})$, we have a fiber to fiber map $\mathcal{F} \times_E \xi^{<z>} \rightarrow \xi^{<z>}$, inducing on every fiber the map $\mathcal{F}_u \times_E \xi^{<z>}_{(u)} \rightarrow \xi^{<z>}_{(u)}$ ($\mathcal{R}$-linear on the first factor and k-linear on the second one). Inversely, every such bilinear map defines on $\xi^{<z>}$ the structure of the $C\,(\xi^{<z>})$-module by virtue of universal properties of Clifford d–algebras. Equivalently, the above–mentioned bilinear map defines a morphism of v–bundles

$$m:\xi^{<z>}\rightarrow HOM\left(\xi^{<z>},\xi^{<z>}\right)\quad\left[HOM\left(\xi^{<z>},\xi^{<z>}\right)\right.$$

denotes the bundles of homomorphisms] when $\left(m\left(u\right)\right)^{2}=G\left(u\right)$ on every point.

Vector bundles $\xi^{<z>}$ provided with $C\left(\xi^{<z>}\right)$–structures are objects of the category with morphisms being morphisms of dv-bundles, which induce on every point $u\in\xi^{<z>}$ morphisms of $C\left(\mathcal{F}_{u}\right)-$modules. This is a Banach category contained in the category of finite-dimensional d-vector spaces on filed k.

Let us denote by $H^{s}\left(\mathcal{E}^{<z>},GL_{n_{E}}\left(\mathcal{R}\right)\right),$ where $n_{E}=n+m_{1}+...+m_{z},$ the s-dimensional cohomology group of the algebraic sheaf of germs of continuous maps of dv-bundle $\mathcal{E}^{<z>}$ with group $GL_{n_{E}}\left(\mathcal{R}\right)$ the group of automorphisms of $\mathcal{R}^{n_{E}}$ (for the language of algebraic topology see, for example, Refs. [83] and [74]). We shall also use the group $SL_{n_{E}}\left(\mathcal{R}\right)=\left\{A\subset GL_{n_{E}}\left(\mathcal{R}\right),\det A=1\right\}.$ Here we point out that cohomologies $H^{s}(M,Gr)$ characterize the class of a principal bundle $\pi:P\rightarrow M$ on M with structural group $Gr.$ Taking into account that we deal with bundles distinguished by an N-connection we introduce into consideration cohomologies $H^{s}\left(\mathcal{E}^{<z>},GL_{n_{E}}\left(\mathcal{R}\right)\right)$ as distinguished classes (d-classes) of bundles $\mathcal{E}^{<z>}$ provided with a global N-connection structure.

For a real vector bundle $\xi^{<z>}$ on compact base $\mathcal{E}^{<z>}$ we can define the orientation on $\xi^{<z>}$ as an element $\alpha_{d}\in H^{1}\left(\mathcal{E}^{<z>},GL_{n_{E}}\left(\mathcal{R}\right)\right)$ whose image on map

$$H^{1}\left(\mathcal{E}^{<z>},SL_{n_{E}}\left(\mathcal{R}\right)\right)\rightarrow H^{1}\left(\mathcal{E}^{<z>},GL_{n_{E}}\left(\mathcal{R}\right)\right)$$

is the d-class of bundle $\mathcal{E}^{<z>}$.

Definition 7.3. *The spinor structure on $\xi^{<z>}$ is defined as an element $\beta_{d}\in H^{1}\left(\mathcal{E}^{<z>},Spin\left(n_{E}\right)\right)$ whose image in the composition*

$$H^{1}\left(\mathcal{E}^{<z>},Spin\left(n_{E}\right)\right)\rightarrow H^{1}\left(\mathcal{E}^{<z>},SO\left(n_{E}\right)\right)\rightarrow H^{1}\left(\mathcal{E}^{<z>},GL_{n_{E}}\left(\mathcal{R}\right)\right)$$

is the d-class of $\mathcal{E}^{<z>}$.

The above definition of spinor structures can be re–formulated in terms of principal bundles. Let $\xi^{<z>}$ be a real vector bundle of rank n+m on a compact base $\mathcal{E}^{<z>}$. If there is a principal bundle P_{d} with structural group $SO(n_{E})$ or $Spin(n_{E})$], this bundle $\xi^{<z>}$ can be provided with orientation (or spinor) structure. The bundle P_d is associated with element $\alpha_d \in H^1(\mathcal{E}^{<z>}, SO(n_{<z>}))$ [or $\beta_d \in H^1(\mathcal{E}^{<z>}, Spin(n_E))$.

We remark that a real bundle is oriented if and only if its first Stiefel–Whitney d–class vanishes,

$$w_1(\xi_d) \in H^1(\xi, \mathcal{Z}/2) = 0,$$

where $H^1(\mathcal{E}^{<z>}, \mathcal{Z}/2)$ is the first group of Chech cohomology with coefficients in $\mathcal{Z}/2$, Considering the second Stiefel–Whitney class $w_2(\xi^{<z>}) \in H^2(\mathcal{E}^{<z>}, \mathcal{Z}/2)$ it is well known that vector bundle $\xi^{<z>}$ admits the spinor structure if and only if $w_2(\xi^{<z>}) = 0$. Finally, we emphasize that taking into account that base space $\mathcal{E}^{<z>}$ is also a v-bundle, $p : E^{<z>} \to M$, we have to make explicit calculations in order to express cohomologies $H^s(\mathcal{E}^{<z>}, GL_{n+m})$ and $H^s(\mathcal{E}^{<z>}, SO(n+m))$ through cohomologies

$$H^s(M, GL_n), H^s(M, SO(m_1)), ...H^s(M, SO(m_z)),$$

which depends on global topological structures of spaces M and $\mathcal{E}^{<z>}$. For general bundle and base spaces this requires a cumbersome cohomological calculus.

7.2.2 Clifford fibration

Another way of defining the spinor structure is to use Clifford fibrations. Consider the principal bundle with the structural group Gr being a subgroup of orthogonal group $O(G)$, where G is a quadratic nondegenerate form) defined on the base (also being a bundle space) space $\mathcal{E}^{<z>}$. The fibration associated to principal fibration $P(\mathcal{E}^{<z>}, Gr)$ with a typical fiber having Clifford algebra $C(G)$ is, by definition, the Clifford fibration $PC(\mathcal{E}^{<z>}, Gr)$. We can always define a metric on the Clifford fibration if every fiber is isometric to $PC(\mathcal{E}^{<z>}, G)$ (this result is proved for arbitrary quadratic forms G on pseudo–Riemannian bases [154]). If, additionally, $Gr \subset SO(G)$ a global section can be defined on $PC(G)$.

Let $\mathcal{P}(\mathcal{E}^{<z>}, Gr)$ be the set of principal bundles with differentiable base $\mathcal{E}^{<z>}$ and structural group Gr. If $g : Gr \to Gr'$ is an homomorphism of Lie groups and $P(\mathcal{E}^{<z>}, Gr) \subset \mathcal{P}(\mathcal{E}^{<z>}, Gr)$ (for simplicity in this subsection we shall denote mentioned bundles and sets of bundles as P, P' and respectively, $\mathcal{P}, \mathcal{P}'$), we can always construct a principal bundle with the property that there is an homomorphism $f : P' \to P$ of principal bundles which can be projected to the identity map of $\mathcal{E}^{<z>}$ and corresponds to isomorphism $g : Gr \to Gr'$. If the inverse statement also holds, the bundle P' is called as the extension of P associated to g and f is called the extension homomorphism denoted as $\widetilde{g}$.

Now we can define distinguished spinor structures on bundle spaces .

Definition 7.4. *Let $P \in \mathcal{P}(\mathcal{E}^{<z>}, O(G))$ be a principal bundle. A distinguished spinor structure of P, equivalently a ds-structure of $\mathcal{E}^{<z>}$ is an extension $\widetilde{P}$ of P associated to homomorphism $h : PinG \to O(G)$ where $O(G)$ is the group of orthogonal rotations, generated by metric G, in bundle $\mathcal{E}^{<z>}$.*

So, if $\widetilde{P}$ is a spinor structure of the space $\mathcal{E}^{<z>}$, then $\widetilde{P} \in \mathcal{P}(\mathcal{E}^{<z>}, PinG)$.

The definition of spinor structures on varieties was given in Ref.[50]. In Refs. [51] and [51] it is proved that a necessary and sufficient condition for a space time to be orientable is to admit a global field of orthonormalized frames. We mention that spinor structures can be also defined on varieties modeled on Banach spaces [1]. As we have shown similar constructions are possible for the cases when space time has the structure of a v-bundle with an N-connection.

Definition 7.5. *A special distinguished spinor structure, ds-structure, of principal bundle $P = P(\mathcal{E}^{<z>}, SO(G))$ is a principal bundle $\widetilde{P} = \widetilde{P}(\mathcal{E}^{<z>}, SpinG)$ for which a homomorphism of principal bundles $\widetilde{p} : \widetilde{P} \to P$, projected on the identity map of $\mathcal{E}^{<z>}$ and corresponding to representation*

$$R : SpinG \to SO(G),$$

is defined.

In the case when the base space variety is oriented, there is a natural bijection between tangent spinor structures with a common base. For special ds–structures we can define, as for any spinor structure, the concepts of spin tensors, spinor connections, and spinor covariant derivations (see Refs. [162, 189, 165]).

7.3 Almost Complex Spinor Structures

Almost complex structures are an important characteristic of H^{2n}-spaces and of osculator bundles $Osc^{k=2k_1}(M)$, where $k_1 = 1, 2, ...$. For simplicity in this subsection we restrict our analysis to the case of H^{2n}-spaces. We can rewrite the almost Hermitian metric [108, 109], H^{2n}-metric in complex form [163]:

$$G = H_{ab}(z, \xi)\, dz^a \otimes dz^b, \tag{7.2}$$

where

$$z^a = x^a + iy^a,\ \overline{z^a} = x^a - iy^a,\ H_{ab}(z, \overline{z}) = g_{ab}(x, y)\mid^{x=x(z,\overline{z})}_{y=y(z,\overline{z})},$$

and define almost complex spinor structures. For given metric (7.2) on H^{2n}-space there is always a principal bundle P^U with unitary structural group U(n) which allows us to transform H^{2n}-space into v-bundle $\xi^U \approx P^U \times_{U(n)} \mathcal{R}^{2n}$. This statement will be proved after we introduce complex spinor structures on oriented real vector bundles [83].

Let us consider momentarily $k = \mathcal{C}$ and introduce into consideration [instead of the group $Spin(n)$] the group $Spin^c \times_{\mathcal{Z}/2} U(1)$ being the factor group of the product $Spin(n) \times U(1)$ with the respect to equivalence

$$(y, z) \sim (-y, -a), \quad y \in Spin(m).$$

This way we define the short exact sequence

$$1 \rightarrow U(1) \rightarrow Spin^c(n) \overset{S^c}{\rightarrow} SO(n) \rightarrow 1, \tag{7.3}$$

where $\rho^c(y, a) = \rho^c(y)$. If λ is oriented , real, and rank n, γ-bundle $\pi : E_\lambda \rightarrow M^n$, with base M^n, the complex spinor structure, spin structure, on λ is given by the principal bundle P with structural group $Spin^c(m)$ and isomorphism $\lambda \approx P \times_{Spin^c(n)} \mathcal{R}^n$ (see (7.3)). For such bundles the categorial equivalence can be defined as

$$\epsilon^c : \mathcal{E}^T_{\mathcal{C}}(M^n) \rightarrow \mathcal{E}^\lambda_{\mathcal{C}}(M^n), \tag{7.4}$$

where $\epsilon^c(E^c) = P \triangle_{Spin^c(n)} E^c$ is the category of trivial complex bundles on $M^n, \mathcal{E}^\lambda_{\mathcal{C}}(M^n)$ is the category of complex v-bundles on M^n with action of Clifford bundle $C(\lambda), P\triangle_{Spin^c(n)}$ and E^c is the factor space of the bundle product $P \times_M E^c$ with respect to the equivalence $(p, e) \sim (p\widehat{g}^{-1}, \widehat{g}e), p \in P, e \in E^c$, where $\widehat{g} \in Spin^c(n)$ acts on E by via the imbedding $Spin(n) \subset C^{0,n}$ and the natural action $U(1) \subset \mathcal{C}$ on complex v-bundle $\xi^c, E^c = tot\xi^c$, for bundle $\pi^c : E^c \rightarrow M^n$.

Now we return to the bundle $\xi = \mathcal{E}^{<1>}$. A real v-bundle (not being a spinor bundle) admits a complex spinor structure if and only if there exist a homomorphism $\sigma : U(n) \rightarrow Spin^c(2n)$ making the diagram 3 commutative. The explicit construction of σ for arbitrary γ-bundle is given in Refs. [83] and [21]. For H^{2n}-spaces it is obvious that a diagram similar to (7.4) can be defined for the tangent bundle TM^n, which directly points to the possibility of defining the ${}^cSpin$-structure on H^{2n}–spaces.

Let λ be a complex, rank n, spinor bundle with

$$\tau : Spin^c(n) \times_{\mathcal{Z}/2} U(1) \rightarrow U(1) \tag{7.5}$$

the homomorphism defined by formula $\tau(\lambda,\delta)=\delta^2$. For P_s being the principal bundle with fiber $Spin^c(n)$ we introduce the complex linear bundle $L(\lambda^c)=P_S\times_{Spin^c(n)}\mathcal{C}$ defined as the factor space of $P_S\times\mathcal{C}$ on equivalence relation

$$(pt,z)\sim\left(p,l(t)^{-1}z\right),$$

where $t\in Spin^c(n)$. This linear bundle is associated to complex spinor structure on λ^c.

If λ^c and $\lambda^{c'}$ are complex spinor bundles, the Whitney sum $\lambda^c\oplus\lambda^{c'}$ is naturally provided with the structure of the complex spinor bundle. This follows from the holomorphism

$$\omega':Spin^c(n)\times Spin^c(n')\to Spin^c(n+n'), \tag{7.6}$$

given by formula $[(\beta,z),(\beta',z')]\to[\omega(\beta,\beta'),zz']$, where ω is the homomorphism making the diagram 4 commutative. Here, $z,z'\in U(1)$. It is obvious that $L\left(\lambda^c\oplus\lambda^{c'}\right)$ is isomorphic to $L(\lambda^c)\otimes L\left(\lambda^{c'}\right)$.

We conclude this subsection by formulating our main result on complex spinor structures for H^{2n}-spaces:

Theorem 7.3. *Let λ^c be a complex spinor bundle of rank n and H^{2n}-space considered as a real vector bundle $\lambda^c\oplus\lambda^{c'}$ provided with almost complex structure $J^{\alpha}{}_{\beta}$; multiplication on i is given by $\begin{pmatrix}0 & -\delta^i_j\\ \delta^i_j & 0\end{pmatrix}$. Then, the diagram 5 is commutative up to isomorphisms ϵ^c and $\widetilde{\epsilon}^c$ defined as in (7.4), $\mathcal{H}$ is functor $E^c\to E^c\otimes L(\lambda^c)$ and $\mathcal{E}^{0,2n}_{\mathcal{C}}(M^n)$ is defined by functor $\mathcal{E}_{\mathcal{C}}(M^n)\to\mathcal{E}^{0,2n}_{\mathcal{C}}(M^n)$ given as correspondence $E^c\to\Lambda(\mathcal{C}^n)\otimes E^c$ (which is a categorial equivalence), $\Lambda(\mathcal{C}^n)$ is the exterior algebra on $\mathcal{C}^n$. W is the real bundle $\lambda^c\oplus\lambda^{c'}$ provided with complex structure.*

Proof: We use composition of homomorphisms

$$\mu:Spin^c(2n)\xrightarrow{\pi}SO(n)\xrightarrow{r}U(n)\xrightarrow{\sigma}Spin^c(2n)\times_{\mathcal{Z}/2}U(1),$$

commutative diagram 6 and introduce composition of homomorphisms

$$\mu:Spin^c(n)\xrightarrow{\Delta}Spin^c(n)\times Spin^c(n)\xrightarrow{\omega^c}Spin^c(n),$$

where Δ is the diagonal homomorphism and ω^c is defined as in (7.6). Using homomorphisms (7.5) and ((7.6)) we obtain formula $\mu(t)=\mu(t)\,r(t)$.

Now consider bundle $P\times_{Spin^c(n)}Spin^c(2n)$ as the principal $Spin^c(2n)$-bundle, associated to $M\oplus M$ being the factor space of the product $P\times$

$Spin^c(2n)$ on the equivalence relation $(p,t,h) \sim \left(p, \mu(t)^{-1} h\right)$. In this case the categorial equivalence (7.4) can be rewritten as

$$\epsilon^c(E^c) = P \times_{Spin^c(n)} Spin^c(2n) \Delta_{Spin^c(2n)} E^c$$

and seen as factor space of $P \times Spin^c(2n) \times_M E^c$ on equivalence relation

$$(pt,h,e) \sim \left(p, \mu(t)^{-1} h, e\right) \text{and} (p,h_1,h_2,e) \sim \left(p,h_1,h_2^{-1}e\right)$$

(projections of elements p and e coincides on base M). Every element of $\epsilon^c(E^c)$ can be represented as $P\Delta_{Spin^c(n)}E^c$, i.e., as a factor space $P\Delta E^c$ on equivalence relation $(pt,e) \sim (p, \mu^c(t), e)$, when $t \in Spin^c(n)$. The complex line bundle $L(\lambda^c)$ can be interpreted as the factor space of $P \times_{Spin^c(n)} \mathcal{C}$ on equivalence relation $(pt,\delta) \sim \left(p, r(t)^{-1} \delta\right)$.

Putting $(p,e) \otimes (p,\delta)(p,\delta e)$ we introduce morphism

$$\epsilon^c(E) \times L(\lambda^c) \to \epsilon^c(\lambda^c)$$

with properties

$$\begin{aligned} (pt,e) \otimes (pt,\delta) &\to (pt,\delta e) = \left(p, \mu^c(t)^{-1} \delta e\right), \\ \left(p, \mu^c(t)^{-1} e\right) \otimes \left(p, l(t)^{-1} e\right) &\to \left(p, \mu^c(t)\, r(t)^{-1} \delta e\right) \end{aligned}$$

pointing to the fact that we have defined the isomorphism correctly and that it is an isomorphism on every fiber. □

Chapter 8

Spinors and Ha–Spaces

8.1 D–Spinor Techniques

The purpose of this section is to show how a corresponding abstract spinor technique entailing notational and calculations advantages can be developed for arbitrary splits of dimensions of a d-vector space $\mathcal{F} = h\mathcal{F} \oplus v_1\mathcal{F} \oplus ... \oplus v_z\mathcal{F}$, where $\dim h\mathcal{F} = n$ and $\dim v_p\mathcal{F} = m_p$. For convenience we shall also present some necessary coordinate expressions.

The problem of a rigorous definition of spinors on la-spaces (la-spinors, d-spinors) was posed and solved [163, 162, 165] in the framework of the formalism of Clifford and spinor structures on v-bundles provided with compatible nonlinear and distinguished connections and metric. We introduced d-spinors as corresponding objects of the Clifford d-algebra $\mathcal{C}(\mathcal{F}, G)$, defined for a d-vector space $\mathcal{F}$ in a standard manner (see, for instance, [83]) and proved that operations with $\mathcal{C}(\mathcal{F}, G)$ can be reduced to calculations for $\mathcal{C}(h\mathcal{F}, g), \mathcal{C}(v_1\mathcal{F}, h_1), ...$ and $\mathcal{C}(v_z\mathcal{F}, h_z)$, which are usual Clifford algebras of respective dimensions $2^n, 2^{m_1}, ...$ and 2^{m_z} (if it is necessary we can use quadratic forms g and h_p correspondingly induced on $h\mathcal{F}$ and $v_p\mathcal{F}$ by a metric $\mathbf{G}$ (1.43)). Considering the orthogonal subgroup $O(\mathbf{G}) \subset GL(\mathbf{G})$ defined by a metric $\mathbf{G}$ we can define the d-spinor norm and parametrize d-spinors by ordered pairs of elements of Clifford algebras $\mathcal{C}(h\mathcal{F}, g)$ and $\mathcal{C}(v_p\mathcal{F}, h_p), p = 1, 2, ...z$. We emphasize that the splitting of a Clifford d-algebra associated to a dv-bundle $\mathcal{E}^{<z>}$ is a straightforward consequence of the global decomposition defining a N-connection structure in $\mathcal{E}^{<z>}$.

In this subsection we shall omit detailed proofs which in most cases are mechanical but rather tedious. We can apply the methods developed in [127, 128, 129, 91] in a straightforward manner on h- and v-subbundles in order to verify the correctness of affirmations.

8.1.1 Clifford d–algebra, d–spinors and d–twistors

In order to relate the succeeding constructions with Clifford d-algebras [163, 162] we consider a la-frame decomposition of the metric (1.43):

$$G_{<\alpha><\beta>}(u) = l_{<\alpha>}^{<\widehat{\alpha}>}(u)\, l_{<\beta>}^{<\widehat{\beta}>}(u)\, G_{<\widehat{\alpha}><\widehat{\beta}>},$$

where the frame d-vectors and constant metric matrices are distinguished as

$$l_{<\alpha>}^{<\widehat{\alpha}>}(u) = \begin{pmatrix} l_j^{\widehat{j}}(u) & 0 & ... & 0 \\ 0 & l_{a_1}^{\widehat{a}_1}(u) & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & .. & l_{a_z}^{\widehat{a}_z}(u) \end{pmatrix},$$

$$G_{<\widehat{\alpha}><\widehat{\beta}>} = \begin{pmatrix} g_{\widehat{i}\widehat{j}} & 0 & ... & 0 \\ 0 & h_{\widehat{a}_1\widehat{b}_1} & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & 0 & h_{\widehat{a}_z\widehat{b}_z} \end{pmatrix},$$

$g_{\widehat{i}\widehat{j}}$ and $h_{\widehat{a}_1\widehat{b}_1}, ..., h_{\widehat{a}_z\widehat{b}_z}$ are diagonal matrices with $g_{\widehat{i}\widehat{i}} = h_{\widehat{a}_1\widehat{a}_1} = ... = h_{\widehat{a}_z\widehat{b}_z} = \pm 1$.

To generate Clifford d-algebras we start with matrix equations

$$\sigma_{<\widehat{\alpha}>}\sigma_{<\widehat{\beta}>} + \sigma_{<\widehat{\beta}>}\sigma_{<\widehat{\alpha}>} = -G_{<\widehat{\alpha}><\widehat{\beta}>}I, \tag{8.1}$$

where I is the identity matrix, matrices $\sigma_{<\widehat{\alpha}>}$ (σ-objects) act on a d-vector space $\mathcal{F} = h\mathcal{F} \oplus v_1\mathcal{F} \oplus ... \oplus v_z\mathcal{F}$ and theirs components are distinguished as

$$\sigma_{<\widehat{\alpha}>} = \left\{ (\sigma_{<\widehat{\alpha}>})_{\underline{\beta}}^{\cdot\underline{\gamma}} = \begin{pmatrix} (\sigma_{\widehat{i}})_{\underline{j}}^{\cdot\underline{k}} & 0 & ... & 0 \\ 0 & (\sigma_{\widehat{a}_1})_{\underline{b}_1}^{\cdot\underline{c}_1} & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & (\sigma_{\widehat{a}_z})_{\underline{b}_z}^{\cdot\underline{c}_z} \end{pmatrix} \right\}, \tag{8.2}$$

indices $\underline{\beta},\underline{\gamma},...$ refer to spin spaces of type $\mathcal{S} = S_{(h)} \oplus S_{(v_1)} \oplus ... \oplus S_{(v_z)}$ and underlined Latin indices $\underline{j},\underline{k},...$ and $\underline{b}_1, \underline{c}_1, ..., \underline{b}_z, \underline{c}_z...$ refer respectively to h-spin space $\mathcal{S}_{(h)}$ and v$_p$-spin space $\mathcal{S}_{(v_p)}, (p = 1, 2, ..., z)$ which are correspondingly associated to a h- and v$_p$-decomposition of a dv-bundle $\mathcal{E}^{<z>}$. The irreducible algebra of matrices $\sigma_{<\widehat{\alpha}>}$ of minimal dimension $N \times N$, where $N = N_{(n)} + N_{(m_1)} + ... + N_{(m_z)}$, $\dim \mathcal{S}_{(h)}{=}N_{(n)}$ and $\dim \mathcal{S}_{(v_p)}{=}N_{(m_p)}$, has these dimensions

$$N_{(n)} = \left\{ \begin{array}{cc} 2^{(n-1)/2}, & n = 2k+1 \\ 2^{n/2}, & n = 2k; \end{array} \right. ,$$

$$N_{(m_p)} = \left| \begin{array}{cc} 2^{(m_p-1)/2}, & m_p = 2k_p+1 \\ 2^{m_p}, & m_p = 2k_p \end{array} \right|,$$

where $k = 1, 2, ..., k_p = 1, 2, ...$

The Clifford d-algebra is generated by sums on $n+1$ elements of form

$$A_1 I + B^{\widehat{i}}\sigma_{\widehat{i}} + C^{\widehat{ij}}\sigma_{\widehat{ij}} + D^{\widehat{ijk}}\sigma_{\widehat{ijk}} + ... \tag{8.3}$$

and sums of $m_p + 1$ elements of form

$$A_{2(p)} I + B^{\widehat{a}_p}\sigma_{\widehat{a}_p} + C^{\widehat{a}_p\widehat{b}_p}\sigma_{\widehat{a}_p\widehat{b}_p} + D^{\widehat{a}_p\widehat{b}_p\widehat{c}_p}\sigma_{\widehat{a}_p\widehat{b}_p\widehat{c}_p} + ...$$

with antisymmetric coefficients

$$C^{\widehat{ij}} = C^{[\widehat{ij}]}, C^{\widehat{a}_p\widehat{b}_p} = C^{[\widehat{a}_p\widehat{b}_p]}, D^{\widehat{ijk}} = D^{[\widehat{ijk}]}, D^{\widehat{a}_p\widehat{b}_p\widehat{c}_p} = D^{[\widehat{a}_p\widehat{b}_p\widehat{c}_p]}, ...$$

and matrices

$$\sigma_{\widehat{ij}} = \sigma_{[\widehat{i}}\sigma_{\widehat{j}]}, \sigma_{\widehat{a}_p\widehat{b}_p} = \sigma_{[\widehat{a}_p}\sigma_{\widehat{b}_p]}, \sigma_{\widehat{ijk}} = \sigma_{[\widehat{i}}\sigma_{\widehat{j}}\sigma_{\widehat{k}]}, ...$$

Really, we have 2^{n+1} coefficients $\left(A_1, C^{\widehat{ij}}, D^{\widehat{ijk}}, ...\right)$ and 2^{m_p+1} coefficients $(A_{2(p)}, C^{\widehat{a}_p\widehat{b}_p}, D^{\widehat{a}_p\widehat{b}_p\widehat{c}_p}, ...)$ of the Clifford algebra on $\mathcal{F}$.

For simplicity, we shall present the necessary geometric constructions only for h-spin spaces $\mathcal{S}_{(h)}$ of dimension $N_{(n)}$. Considerations for a v-spin space $\mathcal{S}_{(v)}$ are similar but with proper characteristics for a dimension $N_{(m)}$.

In order to define the scalar (spinor) product on $\mathcal{S}_{(h)}$ we introduce into consideration this finite sum (because of a finite number of elements $\sigma_{[\widehat{ij}...\widehat{k}]}$):

$$\begin{aligned} {}^{(\pm)}E^{\underline{ij}}_{\underline{km}} &= \delta^{\underline{i}}_{\underline{k}}\delta^{\underline{j}}_{\underline{m}} + \frac{2}{1!}(\sigma_{\widehat{i}})^{.\underline{i}}_{\underline{k}}(\sigma^{\widehat{i}})^{.\underline{j}}_{\underline{m}} + \frac{2^2}{2!}(\sigma_{\widehat{ij}})^{.\underline{i}}_{\underline{k}}(\sigma^{\widehat{ij}})^{.\underline{j}}_{\underline{m}} \\ &+ \frac{2^3}{3!}(\sigma_{\widehat{ijk}})^{.\underline{i}}_{\underline{k}}(\sigma^{\widehat{ijk}})^{.\underline{j}}_{\underline{m}} + ... \end{aligned} \tag{8.4}$$

which can be factorized as

$${}^{(\pm)}E^{\underline{ij}}_{\underline{km}} = N_{(n)}\ {}^{(\pm)}\epsilon_{\underline{km}}\ {}^{(\pm)}\epsilon^{\underline{ij}} \text{ for } n = 2k \tag{8.5}$$

and

$$\begin{aligned} {}^{(+)}E^{\underline{ij}}_{\underline{km}} &= 2N_{(n)}\epsilon_{\underline{km}}\epsilon^{\underline{ij}},\ {}^{(-)}E^{\underline{ij}}_{\underline{km}} = 0 \text{ for } n = 3(mod4), \\ {}^{(+)}E^{\underline{ij}}_{\underline{km}} &= 0,\ {}^{(-)}E^{\underline{ij}}_{\underline{km}} = 2N_{(n)}\epsilon_{\underline{km}}\epsilon^{\underline{ij}} \text{ for } n = 1(mod4). \end{aligned} \tag{8.6}$$

Antisymmetry of $\sigma_{\widehat{ijk}...}$ and the construction of the objects (8.3)–(8.6) define the properties of ϵ-objects ${}^{(\pm)}\epsilon_{\underline{km}}$ and $\epsilon_{\underline{km}}$ which have an eight-fold periodicity on n (see details in [129] and, with respect to locally anisotropic spaces, [163]).

For even values of n it is possible the decomposition of every h-spin space $\mathcal{S}_{(h)}$ into irreducible h-spin spaces $\mathbf{S}_{(h)}$ and $\mathbf{S}'_{(h)}$ (one considers splitting of h-indices, for instance, $\underline{l}= L \oplus L', \underline{m} = M \oplus M', ...$; for v_p-indices we shall write $\underline{a}_p = A_p \oplus A'_p, \underline{b}_p = B_p \oplus B'_p, ...$) and defines new ϵ-objects

$$\epsilon^{\underline{lm}} = \frac{1}{2}\left({}^{(+)}\epsilon^{\underline{lm}} +^{(-)} \epsilon^{\underline{lm}}\right) \text{ and } \tilde{\epsilon}^{\underline{lm}} = \frac{1}{2}\left({}^{(+)}\epsilon^{\underline{lm}} -^{(-)} \epsilon^{\underline{lm}}\right) \quad (8.7)$$

We shall omit similar formulas for ϵ-objects with lower indices.

In general, the spinor ϵ-objects should be defined for every shell of anisotropy according the formulas (5.9) where instead of dimension n we shall consider the dimensions m_p, $1 \le p \le z$, of shells.

We define a d-spinor space $\mathcal{S}_{(n,m_1)}$ as a direct sum of a horizontal and a vertical spinor spaces of type (5.4), for instance,

$$\mathcal{S}_{(8k,8k')} = \mathbf{S}_\circ \oplus \mathbf{S}'_\circ \oplus \mathbf{S}_{|\circ} \oplus \mathbf{S}'_{|\circ}, \mathcal{S}_{(8k,8k'+1)} = \mathbf{S}_\circ \oplus \mathbf{S}'_\circ \oplus \mathcal{S}^{(-)}_{|\circ}, ...,$$

$$\mathcal{S}_{(8k+4,8k'+5)} = \mathbf{S}_\triangle \oplus \mathbf{S}'_\triangle \oplus \mathcal{S}^{(-)}_{|\triangle}, ...$$

The scalar product on a $\mathcal{S}_{(n,m_1)}$ is induced by (corresponding to fixed values of n and m_1) ϵ-objects (5.9) considered for h- and v_1-components. We present also an example for $\mathcal{S}_{(n,m_1+...+m_z)}$:

$$\mathcal{S}_{(8k+4,8k_{(1)}+5,...,8k_{(p)}+4,...8k_{(z)})} =$$
$$[\mathbf{S}_\triangle \oplus \mathbf{S}'_\triangle \oplus \mathcal{S}^{(-)}_{|(1)\triangle} \oplus ... \oplus \mathbf{S}_{|(p)\triangle} \oplus \mathbf{S}'_{|(p)\triangle} \oplus ... \oplus \mathbf{S}_{|(z)\circ} \oplus \mathbf{S}'_{|(z)\circ}.$$

Having introduced d-spinors for dimensions $(n, m_1 + ... + m_z)$ we can write out the generalization for ha–spaces of twistor equations [128] by using the distinguished σ-objects (8.2):

$$(\sigma_{(<\widehat{\alpha}>})^{..\underline{\gamma}}_{|\underline{\beta}|} \quad \frac{\delta\omega^{\underline{\beta}}}{\delta u^{<\widehat{\beta}>)}} = \frac{1}{n + m_1 + ... + m_z} \quad G_{<\widehat{\alpha}><\widehat{\beta}>}(\sigma^{\widehat{\epsilon}})^{..\underline{\gamma}}_{\underline{\beta}} \quad \frac{\delta\omega^{\underline{\beta}}}{\delta u^{\widehat{\epsilon}}}, \quad (8.8)$$

where $\left|\underline{\beta}\right|$ denotes that we do not consider symmetrization on this index. The general solution of (8.8) on the d-vector space $\mathcal{F}$ looks like as

$$\omega^{\underline{\beta}} = \Omega^{\underline{\beta}} + u^{<\widehat{\alpha}>}(\sigma_{<\widehat{\alpha}>})^{..\underline{\beta}}_{\underline{\epsilon}}\Pi^{\underline{\epsilon}}, \quad (8.9)$$

where $\Omega^{\underline{\beta}}$ and $\Pi^{\underline{\epsilon}}$ are constant d-spinors. For fixed values of dimensions n and $m = m_1 + ... m_z$ we mast analyze the reduced and irreducible components of h- and v_p-parts of equations (8.8) and their solutions (8.9) in order to find the symmetry properties of a d-twistor $\mathbf{Z}^{\alpha}$ defined as a pair of d-spinors

$$\mathbf{Z}^{\alpha} = (\omega^{\underline{\alpha}}, \pi'_{\underline{\beta}}),$$

where $\pi_{\underline{\beta}'} = \pi^{(0)}_{\underline{\beta}'} \in \widetilde{\mathcal{S}}_{(n,m_1,...,m_z)}$ is a constant dual d-spinor. The problem of definition of spinors and twistors on ha-spaces was firstly considered in [189] (see also [156]) in connection with the possibility to extend the equations (8.9) and theirs solutions (8.10), by using nearly autoparallel maps, on curved, locally isotropic or anisotropic, spaces. We note that the definition of twistors have been extended to higher order anisotropic spaces with trivial N– and d–connections.

8.1.2 Mutual transforms of d-tensors and d-spinors

The spinor algebra for spaces of higher dimensions can not be considered as a real alternative to the tensor algebra as for locally isotropic spaces of dimensions $n = 3, 4$ [127, 128, 129]. The same holds true for ha–spaces and we emphasize that it is not quite convenient to perform a spinor calculus for dimensions $n, m >> 4$. Nevertheless, the concept of spinors is important for every type of spaces, we can deeply understand the fundamental properties of geometical objects on ha–spaces, and we shall consider in this subsection some questions concerning transforms of d-tensor objects into d-spinor ones.

8.1.3 Transformation of d-tensors into d-spinors

In order to pass from d-tensors to d-spinors we must use σ-objects (8.2) written in reduced or irreduced form (in dependence of fixed values of dimensions n and m):

$$(\sigma_{<\widehat{\alpha}>})^{\cdot\gamma}_{\underline{\beta}},\ (\sigma^{<\widehat{\alpha}>})^{\underline{\beta\gamma}},\ (\sigma^{<\widehat{\alpha}>})_{\underline{\beta\gamma}}, ..., (\sigma_{<\widehat{a}>})^{\underline{bc}}, ..., \quad (8.10)$$
$$(\sigma_{\widehat{i}})_{\underline{jk}}, ..., (\sigma_{<\widehat{a}>})^{AA'}, ..., (\sigma^{\widehat{i}})_{II'}, ...$$

It is obvious that contracting with corresponding σ-objects (8.10) we can introduce instead of d-tensors indices the d-spinor ones, for instance,

$$\omega^{\underline{\beta\gamma}} = (\sigma^{<\widehat{\alpha}>})^{\underline{\beta\gamma}}\omega_{<\widehat{\alpha}>}, \quad \omega_{AB'} = (\sigma^{<\widehat{a}>})_{AB'}\omega_{<\widehat{a}>}, \quad ..., \zeta^{\underline{i}}_{\cdot\underline{j}} = (\sigma^{\widehat{k}})^{\underline{i}}_{\cdot\underline{j}}\zeta_{\widehat{k}}, ...$$

For d-tensors containing groups of antisymmetric indices there is a more simple procedure of theirs transforming into d-spinors because the objects

$$(\sigma_{\widehat{\alpha}\widehat{\beta}...\widehat{\gamma}})^{\underline{\delta\nu}}, \quad (\sigma^{\widehat{a}\widehat{b}...\widehat{c}})^{\underline{de}}, \quad ..., (\sigma^{\widehat{i}\widehat{j}...\widehat{k}})_{II'}, \quad ... \quad (8.11)$$

can be used for sets of such indices into pairs of d-spinor indices. Let us enumerate some properties of σ-objects of type (8.11) (for simplicity we consider only h-components having q indices $\widehat{i}, \widehat{j}, \widehat{k}, ...$ taking values from 1 to n; the properties of v_p-components can be written in a similar manner with respect to indices $\widehat{a}_p, \widehat{b}_p, \widehat{c}_p ...$ taking values from 1 to m):

$$(\sigma_{\widehat{i}...\widehat{j}})^{\underline{kl}} \text{ is } \left\{ \begin{array}{c} \text{symmetric on } \underline{k}, \underline{l} \text{ for } n - 2q \equiv 1, 7 \ (mod\ 8); \\ \text{antisymmetric on } \underline{k}, \underline{l} \text{ for } n - 2q \equiv 3, 5 \ (mod\ 8) \end{array} \right\} \quad (8.12)$$

for odd values of n, and an object

$$(\sigma_{\widehat{i}...\widehat{j}})^{IJ} \ \left((\sigma_{\widehat{i}...\widehat{j}})^{I'J'} \right)$$

$$\text{is } \left\{ \begin{array}{c} \text{symmetric on } I, J \ (I', J') \text{ for } n - 2q \equiv 0 \ (mod\ 8); \\ \text{antisymmetric on } I, J \ (I', J') \text{ for } n - 2q \equiv 4 \ (mod\ 8) \end{array} \right\} \quad (8.13)$$

or

$$(\sigma_{\widehat{i}...\widehat{j}})^{IJ'} = \pm (\sigma_{\widehat{i}...\widehat{j}})^{J'I} \{ \begin{array}{c} n + 2q \equiv 6 (mod 8); \\ n + 2q \equiv 2 (mod 8), \end{array} \quad (8.14)$$

with vanishing of the rest of reduced components of the d-tensor $(\sigma_{\widehat{i}...\widehat{j}})^{\underline{kl}}$ with prime/ unprime sets of indices.

8.1.4 Fundamental d–spinors

We can transform every d–spinor $\xi^{\underline{\alpha}} = (\xi^{\underline{i}}, \xi^{\underline{a}_1}, ..., \xi^{\underline{a}_z})$ into a corresponding d-tensor. For simplicity, we consider this construction only for a h-component $\xi^{\underline{i}}$ on a h-space being of dimension n. The values

$$\xi^{\underline{\alpha}} \xi^{\underline{\beta}} (\sigma^{\widehat{i}...\widehat{j}})_{\underline{\alpha\beta}} \quad (n \text{ is odd}) \quad (8.15)$$

or

$$\xi^{I} \xi^{J} (\sigma^{\widehat{i}...\widehat{j}})_{IJ} \ \left(\text{or } \xi^{I'} \xi^{J'} (\sigma^{\widehat{i}...\widehat{j}})_{I'J'} \right) \ (n \text{ is even}) \quad (8.16)$$

with a different number of indices $\widehat{i}...\widehat{j}$, taken together, defines the h-spinor $\xi^{\underline{i}}$ to an accuracy to the sign. We emphasize that it is necessary to choose only those h-components of d-tensors (8.15) (or (8.16)) which are symmetric on pairs of indices $\underline{\alpha\beta}$ (or IJ (or $I'J'$)) and the number q of indices $\widehat{i}...\widehat{j}$ satisfies the condition (as a respective consequence of the properties (8.12) and/ or (8.13), (8.14))

$$n - 2q \equiv 0, 1, 7 \ (mod\ 8). \quad (8.17)$$

Of special interest is the case when

$$q = \frac{1}{2}(n \pm 1) \quad (n \text{ is odd}) \tag{8.18}$$

or

$$q = \frac{1}{2}n \quad (n \text{ is even}). \tag{8.19}$$

If all expressions (8.15) and/or (8.16) are zero for all values of q with the exception of one or two ones defined by the conditions (8.17), (8.18) (or (8.19)), the value $\xi^{\widehat{i}}$ (or ξ^{I} $(\xi^{I'})$) is called a fundamental h-spinor. Defining in a similar manner the fundamental v-spinors we can introduce fundamental d-spinors as pairs of fundamental h- and v-spinors. Here we remark that a h(v_p)-spinor $\xi^{\widehat{i}}$ $(\xi^{\widehat{a}_p})$ (we can also consider reduced components) is always a fundamental one for $n(m) < 7$, which is a consequence of (8.19)).

8.2 Differential Geometry of Ha–Spinors

This subsection is devoted to the differential geometry of d–spinors in higher order anisotropic spaces. We shall use denotations of type

$$v^{<\alpha>} = (v^{i}, v^{<a>}) \in \sigma^{<\alpha>} = (\sigma^{i}, \sigma^{<a>})$$

and

$$\zeta^{\underline{\alpha}_p} = (\zeta^{\underline{i}_p}, \zeta^{\underline{a}_p}) \in \sigma^{\alpha_p} = (\sigma^{i_p}, \sigma^{a_p})$$

for, respectively, elements of modules of d-vector and irreduced d-spinor fields (see details in [163]). D-tensors and d-spinor tensors (irreduced or reduced) will be interpreted as elements of corresponding σ–modules, for instance,

$$q^{<\alpha>}_{\ \ <\beta>...} \in \sigma^{<\alpha>}/'; [-0_{<\beta>}, \psi^{\underline{\alpha}_p \quad \underline{\gamma}_p}_{\quad \underline{\beta}_p \quad ...} \in \sigma^{\underline{\alpha}_p \quad \underline{\gamma}_p}_{\quad \underline{\beta_p} \quad ...}, \xi^{I_p I'_p}_{\quad J_p K'_p N'_p} \in \sigma^{I_p I'_p}_{\quad J_p K'_p N'_p}, ...$$

We can establish a correspondence between the higher order anisotropic adapted to the N–connection metric $g_{\alpha\beta}$ (1.43) and d-spinor metric $\epsilon_{\underline{\alpha\beta}}$ (ϵ-objects (5.9) for both h- and v_p-subspaces of $\mathcal{E}^{<z>}$) of a ha–space $\mathcal{E}^{<z>}$ by using the relation

$$g_{<\alpha><\beta>} = -\frac{1}{N(n) + N(m_1) + ... + N(m_z)} \times \tag{8.20}$$

$$((\sigma_{(<\alpha>}(u))^{\underline{\alpha\beta}}(\sigma_{<\beta>)}(u))^{\underline{\delta\gamma}})\epsilon_{\underline{\alpha\gamma}}\epsilon_{\underline{\beta\delta}},$$

where

$$(\sigma_{<\alpha>}(u))^{\underline{\nu}\underline{\gamma}} = l^{<\widehat{\alpha}>}_{<\alpha>}(u)(\sigma_{<\widehat{\alpha}>})^{<\underline{\nu}><\underline{\gamma}>}, \tag{8.21}$$

which is a consequence of formulas (8.1)–(8.7). In brief we can write (8.20) as

$$g_{<\alpha><\beta>} = \epsilon_{\underline{\alpha}_1\underline{\alpha}_2}\epsilon_{\underline{\beta}_1\underline{\beta}_2} \tag{8.22}$$

if the σ-objects are considered as a fixed structure, whereas ϵ-objects are treated as caring the metric "dynamics " , on higher order anisotropic space. This variant is used, for instance, in the so-called 2-spinor geometry [128, 129] and should be preferred if we have to make explicit the algebraic symmetry properties of d-spinor objects by using metric decomposition (8.22). An alternative way is to consider as fixed the algebraic structure of ϵ-objects and to use variable components of σ-objects of type (8.21) for developing a variational d-spinor approach to gravitational and matter field interactions on ha-spaces (the spinor Ashtekar variables [20] are introduced in this manner).

We note that a d–spinor metric

$$\epsilon_{\underline{\nu}\underline{\tau}} = \begin{pmatrix} \epsilon_{\underline{i}\underline{j}} & 0 & ... & 0 \\ 0 & \epsilon_{\underline{a}_1\underline{b}_1} & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \epsilon_{\underline{a}_z\underline{b}_z} \end{pmatrix}$$

on the d-spinor space $\mathcal{S} = (\mathcal{S}_{(h)}, \mathcal{S}_{(v_1)}, ..., \mathcal{S}_{(v_z)})$ can have symmetric or antisymmetric h (v$_p$) -components $\epsilon_{\underline{i}\underline{j}}$ ($\epsilon_{\underline{a}_p\underline{b}_p}$) , see ϵ-objects (5.9). For simplicity, in order to avoid cumbersome calculations connected with eight-fold periodicity on dimensions n and m_p of a ha-space $\mathcal{E}^{<z>}$, we shall develop a general d-spinor formalism only by using irreduced spinor spaces $\mathcal{S}_{(h)}$ and $\mathcal{S}_{(v_p)}$.

8.2.1 D-covariant derivation on ha–spaces

Let $\mathcal{E}^{<z>}$ be a ha-space. We define the action on a d-spinor of a d-covariant operator

$$\begin{aligned} \bigtriangledown_{<\alpha>} &= (\bigtriangledown_i, \bigtriangledown_{<a>}) \\ &= (\sigma_{<\alpha>})^{\underline{\alpha}_1\underline{\alpha}_2}\bigtriangledown_{\underline{\alpha}_1\underline{\alpha}_2} = \left((\sigma_i)^{\underline{i}_1\underline{i}_2}\bigtriangledown_{\underline{i}_1\underline{i}_2},\ (\sigma_{<a>})^{\underline{a}_1\underline{a}_2}\bigtriangledown_{\underline{a}_1\underline{a}_2}\right) \\ &= \left((\sigma_i)^{\underline{i}_1\underline{i}_2}\bigtriangledown_{\underline{i}_1\underline{i}_2},\ (\sigma_{a_1})^{\underline{a}_1\underline{a}_2}\bigtriangledown_{(1)\underline{a}_1\underline{a}_2}, ..., \right. \\ &\quad \left. (\sigma_{a_p})^{\underline{a}_1\underline{a}_2}\bigtriangledown_{(p)\underline{a}_1\underline{a}_2}, ..., (\sigma_{a_z})^{\underline{a}_1\underline{a}_2}\bigtriangledown_{(z)\underline{a}_1\underline{a}_2}\right) \end{aligned}$$

(in brief, we shall write

$$\bigtriangledown_{<\alpha>} = \bigtriangledown_{\underline{\alpha}_1\underline{\alpha}_2} = \left(\bigtriangledown_{\underline{i}_1\underline{i}_2},\ \bigtriangledown_{(1)\underline{a}_1\underline{a}_2}, ..., \bigtriangledown_{(p)\underline{a}_1\underline{a}_2}, ..., \bigtriangledown_{(z)\underline{a}_1\underline{a}_2}\right))$$

as maps

$$\bigtriangledown_{\underline{\alpha}_1\underline{\alpha}_2} \ : \ \sigma^{\underline{\beta}} \rightarrow \sigma^{\underline{\beta}}_{<\alpha>} = \sigma^{\underline{\beta}}_{\underline{\alpha}_1\underline{\alpha}_2} =$$

$$\left(\sigma^{\underline{\beta}}_{i} = \sigma^{\underline{\beta}}_{\underline{i}_1\underline{i}_2}, \sigma^{\underline{\beta}}_{(1)a_1} = \sigma^{\underline{\beta}}_{(1)\underline{\alpha}_1\underline{\alpha}_2}, ..., \sigma^{\underline{\beta}}_{(p)a_p} = \sigma^{\underline{\beta}}_{(p)\underline{\alpha}_1\underline{\alpha}_2}, ..., \sigma^{\underline{\beta}}_{(z)a_z} = \sigma^{\underline{\beta}}_{(z)\underline{\alpha}_1\underline{\alpha}_2}\right)$$

satisfying conditions

$$\bigtriangledown_{<\alpha>}(\xi^{\underline{\beta}} + \eta^{\underline{\beta}}) = \bigtriangledown_{<\alpha>}\xi^{\underline{\beta}} + \bigtriangledown_{<\alpha>}\eta^{\underline{\beta}},$$

and

$$\bigtriangledown_{<\alpha>}(f\xi^{\underline{\beta}}) = f \bigtriangledown_{<\alpha>} \xi^{\underline{\beta}} + \xi^{\underline{\beta}} \bigtriangledown_{<\alpha>} f$$

for every $\xi^{\underline{\beta}}, \eta^{\underline{\beta}} \in \sigma^{\underline{\beta}}$ and f being a scalar field on $\mathcal{E}^{<z>}$. It is also required that one holds the Leibnitz rule

$$(\bigtriangledown_{<\alpha>}\zeta_{\underline{\beta}})\eta^{\underline{\beta}} = \bigtriangledown_{<\alpha>}(\zeta_{\underline{\beta}}\eta^{\underline{\beta}}) - \zeta_{\underline{\beta}} \bigtriangledown_{<\alpha>} \eta^{\underline{\beta}}$$

and that $\bigtriangledown_{<\alpha>}$ is a real operator, i.e. it commuters with the operation of complex conjugation:

$$\overline{\bigtriangledown_{<\alpha>}\psi_{\underline{\alpha}\underline{\beta}\gamma...}} = \bigtriangledown_{<\alpha>}(\overline{\psi}_{\underline{\alpha}\underline{\beta}\gamma...}).$$

Let now analyze the question on uniqueness of action on d-spinors of an operator $\bigtriangledown_{<\alpha>}$ satisfying necessary conditions . Denoting by $\bigtriangledown^{(1)}_{<\alpha>}$ and $\bigtriangledown_{<\alpha>}$ two such d-covariant operators we consider the map

$$(\bigtriangledown^{(1)}_{<\alpha>} - \bigtriangledown_{<\alpha>}) : \sigma^{\underline{\beta}} \rightarrow \sigma^{\underline{\beta}}_{\underline{\alpha}_\infty\underline{\alpha}_\in}. \tag{8.23}$$

Because the action on a scalar f of both operators $\bigtriangledown^{(1)}_{\alpha}$ and $\bigtriangledown_{\alpha}$ must be identical, i.e.

$$\bigtriangledown^{(1)}_{<\alpha>}f = \bigtriangledown_{<\alpha>}f,$$

the action (8.23) on $f = \omega_{\underline{\beta}}\xi^{\underline{\beta}}$ must be written as

$$(\bigtriangledown^{(1)}_{<\alpha>} - \bigtriangledown_{<\alpha>})(\omega_{\underline{\beta}}\xi^{\underline{\beta}}) = 0.$$

In consequence we conclude that there is an element $\Theta_{\underline{\alpha}_1\underline{\alpha}_2\underline{\beta}}{}^{\underline{\gamma}} \in \sigma_{\underline{\alpha}_1\underline{\alpha}_2\underline{\beta}}{}^{\underline{\gamma}}$ for which

$$\begin{aligned} \bigtriangledown^{(1)}_{\underline{\alpha}_1\underline{\alpha}_2}\xi^{\underline{\gamma}} &= \bigtriangledown_{\underline{\alpha}_1\underline{\alpha}_2}\xi^{\underline{\gamma}} + \Theta_{\underline{\alpha}_1\underline{\alpha}_2\underline{\beta}}{}^{\underline{\gamma}}\xi^{\underline{\beta}}, \\ \bigtriangledown^{(1)}_{\underline{\alpha}_1\underline{\alpha}_2}\omega_{\underline{\beta}} &= \bigtriangledown_{\underline{\alpha}_1\underline{\alpha}_2}\omega_{\underline{\beta}} - \Theta_{\underline{\alpha}_1\underline{\alpha}_2\underline{\beta}}{}^{\underline{\gamma}}\omega_{\underline{\gamma}} \ . \end{aligned} \tag{8.24}$$

The action of the operator (8.23) on a d-vector $v^{<\beta>} = v^{\underline{\beta}_1\underline{\beta}_2}$ can be written by using formula (8.24) for both indices $\underline{\beta}_1$ and $\underline{\beta}_2$:

$$\begin{aligned}(\nabla^{(1)}_{<\alpha>} - \nabla_{<\alpha>})v^{\underline{\beta}_1\underline{\beta}_2} &= \Theta_{<\alpha>\underline{\gamma}}^{\quad\underline{\beta}_1} v^{\underline{\gamma}\underline{\beta}_2} + \Theta_{<\alpha>\underline{\gamma}}^{\quad\underline{\beta}_2} v^{\underline{\beta}_1\underline{\gamma}} \\ &= (\Theta_{<\alpha>\underline{\gamma}_1}^{\quad\underline{\beta}_1}\delta_{\underline{\gamma}_2}^{\ \ \underline{\beta}_2} + \Theta_{<\alpha>\underline{\gamma}_1}^{\quad\underline{\beta}_2}\delta_{\underline{\gamma}_2}^{\ \ \underline{\beta}_1})v^{\underline{\gamma}_1\underline{\gamma}_2} \\ &= Q^{<\beta>}_{\ \ <\alpha><\gamma>} v^{<\gamma>},\end{aligned}$$

where

$$Q^{<\beta>}_{\ \ <\alpha><\gamma>} = Q^{\underline{\beta}_1\underline{\beta}_2}_{\qquad \underline{\alpha}_1\underline{\alpha}_2\ \underline{\gamma}_1\underline{\gamma}_2} = \Theta_{<\alpha>\underline{\gamma}_1}^{\quad\underline{\beta}_1}\delta_{\underline{\gamma}_2}^{\ \ \underline{\beta}_2} + \Theta_{<\alpha>\underline{\gamma}_1}^{\quad\underline{\beta}_2}\delta_{\underline{\gamma}_2}^{\ \ \underline{\beta}_1}. \tag{8.25}$$

The d-commutator $\nabla_{[<\alpha>}\nabla_{<\beta>]}$ defines the d-torsion. So, applying operators $\nabla^{(1)}_{[<\alpha>}\nabla^{(1)}_{<\beta>]}$ and $\nabla_{[<\alpha>}\nabla_{<\beta>]}$ on $f = \omega_{\underline{\beta}}\xi^{\underline{\beta}}$ we can write

$$T^{(1)<\gamma>}_{\quad\ <\alpha><\beta>} - T^{<\gamma>}_{\ \ <\alpha><\beta>} = Q^{<\gamma>}_{\ \ <\beta><\alpha>} - Q^{<\gamma>}_{\ \ <\alpha><\beta>}$$

with $Q^{<\gamma>}_{\ \ <\alpha><\beta>}$ from (8.25).

The action of operator $\nabla^{(1)}_{<\alpha>}$ on d-spinor tensors of type $\chi_{\underline{\alpha}_1\underline{\alpha}_2\underline{\alpha}_3...}^{\qquad\quad \underline{\beta}_1\underline{\beta}_2...}$ must be constructed by using formula (8.24) for every upper index $\underline{\beta}_1\underline{\beta}_2...$ and formula (8.25) for every lower index $\underline{\alpha}_1\underline{\alpha}_2\underline{\alpha}_3...$.

8.2.2 Infeld–van der Waerden coefficients

Let

$$\delta_{\underline{\alpha}}^{\ \ \alpha} = \left(\delta_{\underline{\mathbf{1}}}^{\ \ i}, \delta_{\underline{\mathbf{2}}}^{\ \ i}, ..., \delta_{\underline{\mathbf{N(n)}}}^{\ \ i}, \delta_{\underline{\mathbf{1}}}^{\ \ a}, \delta_{\underline{\mathbf{2}}}^{\ \ a}, ..., \delta_{\underline{\mathbf{N(m)}}}^{\ \ i}\right)$$

be a d–spinor basis. The dual to it basis is denoted as

$$\delta_{\alpha}^{\ \ \underline{\alpha}} = \left(\delta_{\underline{i}}^{\ \ \mathbf{1}}, \delta_{\underline{i}}^{\ \ \mathbf{2}}, ..., \delta_{\underline{i}}^{\ \ \mathbf{N(n)}}, \delta_{\underline{i}}^{\ \ \mathbf{1}}, \delta_{\underline{i}}^{\ \ \mathbf{2}}, ..., \delta_{\underline{i}}^{\ \ \mathbf{N(m)}}\right).$$

A d-spinor $\kappa^{\underline{\alpha}} \in \sigma^{\ \underline{\alpha}}$ has components $\kappa^{\underline{\alpha}} = \kappa^{\alpha}\delta_{\alpha}^{\ \ \underline{\alpha}}$. Taking into account that

$$\delta_{\underline{\alpha}}^{\ \ \alpha}\delta_{\underline{\beta}}^{\ \ \beta}\nabla_{\alpha\beta} = \nabla_{\underline{\alpha}\underline{\beta}},$$

we write out the components $\nabla_{\underline{\alpha}\underline{\beta}}\ \kappa^{\underline{\gamma}}$ as

$$\begin{aligned}\delta_{\underline{\alpha}}^{\ \ \alpha}\ \delta_{\underline{\beta}}^{\ \ \beta}\ \delta_{\underline{\gamma}}^{\ \ \gamma}\ \nabla_{\alpha\beta}\ \kappa^{\underline{\gamma}} &= \delta_{\underline{\epsilon}}^{\ \ \tau}\ \delta_{\underline{\tau}}^{\ \ \gamma}\ \nabla_{\alpha\beta}\ \kappa^{\underline{\epsilon}} + \kappa^{\underline{\epsilon}}\ \delta_{\underline{\epsilon}}^{\ \ \gamma}\ \nabla_{\alpha\beta}\ \delta_{\underline{\epsilon}}^{\ \ \epsilon} \\ &= \nabla_{\underline{\alpha}\underline{\beta}}\kappa^{\underline{\gamma}} + \kappa^{\underline{\epsilon}}\gamma^{\underline{\gamma}}_{\ \underline{\alpha}\underline{\beta}\underline{\epsilon}},\end{aligned} \tag{8.26}$$

where the coordinate components of the d–spinor connection $\gamma^{\underline{\gamma}}_{\ \underline{\alpha\beta\epsilon}}$ are defined as

$$\gamma^{\underline{\gamma}}_{\ \underline{\alpha\beta\epsilon}} \doteq \delta_{\underline{\tau}}^{\ \ \gamma} \ \bigtriangledown_{\underline{\alpha\beta}} \, \delta_{\underline{\epsilon}}^{\ \ \tau}. \tag{8.27}$$

We call the Infeld - van der Waerden d-symbols a set of σ-objects $(\sigma_\alpha)^{\underline{\alpha\beta}}$ parametrized with respect to a coordinate d-spinor basis. Defining

$$\bigtriangledown_{<\alpha>} = (\sigma_{<\alpha>})^{\underline{\alpha\beta}} \ \bigtriangledown_{\underline{\alpha\beta}},$$

introducing denotations

$$\gamma^{\underline{\gamma}}_{\ <\alpha>\underline{\tau}} \doteq \gamma^{\underline{\gamma}}_{\ \underline{\alpha\beta\tau}} \ (\sigma_{<\alpha>})^{\underline{\alpha\beta}}$$

and using properties (8.26) we can write relations

$$\begin{aligned} l^{<\alpha>}_{<\alpha>} \ \delta_{\underline{\beta}}^{\ \ \beta} \ \bigtriangledown_{<\alpha>} \kappa^{\underline{\beta}} &= \bigtriangledown_{<\alpha>} \kappa^{\underline{\beta}} + \kappa^{\underline{\delta}} \ \gamma^{\underline{\beta}}_{\ <\alpha>\underline{\delta}}, \\ l^{<\alpha>}_{<\alpha>} \ \delta_{\underline{\beta}}^{\ \ \beta} \ \bigtriangledown_{<\alpha>} \ \mu_{\underline{\beta}} &= \bigtriangledown_{<\alpha>} \ \mu_{\underline{\beta}} - \mu_{\underline{\delta}} \gamma^{\underline{\delta}}_{\ <\alpha>\underline{\beta}}. \end{aligned} \tag{8.28}$$

for d-covariant derivations $\bigtriangledown_{\underline{\alpha}} \kappa^{\underline{\beta}}$ and $\bigtriangledown_{\underline{\alpha}} \ \mu_{\underline{\beta}}$.

We can consider expressions similar to (8.28) for values having both types of d-spinor and d-tensor indices, for instance,

$$\begin{aligned} & l^{<\alpha>}_{<\alpha>} \ l^{<\gamma>}_{<\gamma>} \ \delta_{\underline{\delta}}^{\ \ \delta} \ \bigtriangledown_{<\alpha>} \ \theta_{\underline{\delta}}^{\ <\gamma>} = \\ & \bigtriangledown_{<\alpha>} \theta_{\underline{\delta}}^{\ <\gamma>} - \theta_{\underline{\epsilon}}^{\ <\gamma>} \gamma^{\underline{\epsilon}}_{\ <\alpha>\underline{\delta}} + \theta_{\underline{\delta}}^{\ <\tau>} \ \Gamma^{\ <\gamma>}_{\ \ <\alpha><\tau>} \end{aligned}$$

(we can prove this by a straightforward calculation).

Now we shall consider some possible relations between components of d-connections $\gamma^{\underline{\epsilon}}_{\ <\alpha>\underline{\delta}}$ and $\Gamma^{\ <\gamma>}_{\ \ <\alpha><\tau>}$ and derivations of $(\sigma_{<\alpha>})^{\underline{\alpha\beta}}$. We can write

$$\begin{aligned} \Gamma^{<\alpha>}_{\ \ <\beta><\gamma>} &= l^{<\alpha>}_{<\alpha>} \ \bigtriangledown_{<\gamma>} \ l^{<\alpha>}_{<\beta>} \\ &= l^{<\alpha>}_{<\alpha>} \ \bigtriangledown_{<\gamma>} \ (\sigma_{<\beta>})^{\underline{\epsilon\tau}} l^{<\alpha>}_{<\alpha>} \ \bigtriangledown_{<\gamma>} \ ((\sigma_{<\beta>})^{\underline{\epsilon\tau}} \delta_{\underline{\epsilon}}^{\ \epsilon} \delta_{\underline{\tau}}^{\ \tau}) \\ &= l^{<\alpha>}_{<\alpha>} \delta_{\underline{\alpha}}^{\ \alpha} \delta_{\underline{\epsilon}}^{\ \epsilon} \ \bigtriangledown_{<\gamma>} \ (\sigma_{<\beta>})^{\underline{\alpha\epsilon}} \\ &+ l^{<\alpha>}_{<\alpha>} (\sigma_{<\beta>})^{\underline{\epsilon\tau}} (\delta_{\underline{\tau}}^{\ \tau} \ \bigtriangledown_{<\gamma>} \ \delta_{\underline{\epsilon}}^{\ \epsilon} + \delta_{\underline{\epsilon}}^{\ \epsilon} \ \bigtriangledown_{<\gamma>} \ \delta_{\underline{\tau}}^{\ \tau}) \\ &= l^{<\alpha>}_{\underline{\epsilon\tau}} \ \ \bigtriangledown_{<\gamma>} \ (\sigma_{<\beta>})^{\underline{\epsilon\tau}} \\ &+ l^{<\alpha>}_{\underline{\mu\nu}} \delta_{\underline{\epsilon}}^{\ \mu} \delta_{\underline{\tau}}^{\ \nu} (\sigma_{<\beta>})^{\underline{\epsilon\tau}} (\delta_{\underline{\tau}}^{\ \tau} \ \bigtriangledown_{<\gamma>} \ \delta_{\underline{\epsilon}}^{\ \epsilon} + \delta_{\underline{\epsilon}}^{\ \epsilon} \ \bigtriangledown_{<\gamma>} \ \delta_{\underline{\tau}}^{\ \tau}), \end{aligned}$$

where $l^{<\alpha>}_{<\alpha>} = (\sigma_{\underline{\epsilon\tau}})^{<\alpha>}$, from which one follows

$$\begin{aligned} & (\sigma_{<\alpha>})^{\underline{\mu\nu}} (\sigma_{\underline{\alpha\beta}})^{<\beta>} \Gamma^{<\alpha>}_{\ \ <\gamma><\beta>} = \\ & (\sigma_{\underline{\alpha\beta}})^{<\beta>} \ \bigtriangledown_{<\gamma>} \ (\sigma_{<\alpha>})^{\underline{\mu\nu}} + \delta_{\underline{\beta}}^{\ \nu} \gamma^{\underline{\mu}}_{\ <\gamma>\underline{\alpha}} + \delta_{\underline{\alpha}}^{\ \mu} \gamma^{\underline{\nu}}_{\ <\gamma>\underline{\beta}}. \end{aligned}$$

Connecting the last expression on $\underline{\beta}$ and $\underline{\nu}$ and using an orthonormalized d-spinor basis when $\gamma^{\underline{\beta}}_{\ <\gamma>\underline{\beta}} = 0$ (a consequence from (8.27)) we have

$$\gamma^{\underline{\mu}}_{\ <\gamma>\underline{\alpha}} = \frac{1}{N(n)+N(m_1)+...+N(m_z)}(\Gamma^{\underline{\mu\beta}}_{\ \ <\gamma>\ \underline{\alpha\beta}} \quad (8.29)$$
$$-(\sigma_{\underline{\alpha\beta}})^{<\beta>} \bigtriangledown_{<\gamma>} (\sigma_{<\beta>})^{\underline{\mu\beta}}),$$

where

$$\Gamma^{\underline{\mu\beta}}_{\ \ <\gamma>\ \underline{\alpha\beta}} = (\sigma_{<\alpha>})^{\underline{\mu\beta}}(\sigma_{\underline{\alpha\beta}})^{\beta}\Gamma^{<\alpha>}_{\ <\gamma><\beta>}. \quad (8.30)$$

We also note here that, for instance, for the canonical and Berwald connections and Christoffel d-symbols we can express d-spinor connection (8.30) through corresponding locally adapted derivations of components of metric and N-connection by introducing corresponding coefficients instead of $\Gamma^{<\alpha>}_{\ <\gamma><\beta>}$ in (8.30) and than in (8.29).

8.2.3 D–spinors of ha–space curvature and torsion

The d-tensor indices of the commutator $\Delta_{<\alpha><\beta>}$ can be transformed into d-spinor ones:

$$\begin{aligned} \square_{\underline{\alpha\beta}} &= (\sigma^{<\alpha><\beta>})_{\underline{\alpha\beta}}\Delta_{\alpha\beta} = (\square_{\underline{ij}}, \square_{\underline{ab}}) \\ &= (\square_{\underline{ij}}, \square_{\underline{a}_1\underline{b}_1}, ..., \square_{\underline{a}_p\underline{b}_p}, ..., \square_{\underline{a}_z\underline{b}_z}), \end{aligned} \quad (8.31)$$

with h- and v_p-components,

$$\square_{\underline{ij}} = (\sigma^{<\alpha><\beta>})_{\underline{ij}}\Delta_{<\alpha><\beta>} \text{ and } \square_{\underline{ab}} = (\sigma^{<\alpha><\beta>})_{\underline{ab}}\Delta_{<\alpha><\beta>},$$

being symmetric or antisymmetric in dependence of corresponding values of dimensions n and m_p (see eight-fold parametizations (5.9) and (5.10)). Considering the actions of operator (8.31) on d-spinors $\pi^{\underline{\gamma}}$ and $\mu_{\underline{\gamma}}$ we introduce the d-spinor curvature $X_{\underline{\delta}\ \ \underline{\alpha\beta}}^{\ \ \underline{\gamma}}$ as to satisfy equations

$$\square_{\underline{\alpha\beta}}\ \pi^{\underline{\gamma}} = X_{\underline{\delta}\ \ \underline{\alpha\beta}}^{\ \ \underline{\gamma}}\pi^{\underline{\delta}} \text{ and } \square_{\underline{\alpha\beta}}\ \mu_{\underline{\gamma}} = X_{\underline{\gamma}\ \ \underline{\alpha\beta}}^{\ \ \underline{\delta}}\mu_{\underline{\delta}}. \quad (8.32)$$

The gravitational d-spinor $\Psi_{\underline{\alpha\beta\gamma\delta}}$ is defined by a corresponding symmetrization of d-spinor indices:

$$\Psi_{\underline{\alpha\beta\gamma\delta}} = X_{(\underline{\alpha}|\underline{\beta}|\underline{\gamma\delta})}. \quad (8.33)$$

We note that d-spinor tensors $X_{\underline{\delta}}\ {}^{\underline{\gamma}}{}_{\underline{\alpha\beta}}$ and $\Psi_{\underline{\alpha\beta\gamma\delta}}$ are transformed into similar 2-spinor objects on locally isotropic spaces [128, 129] if we consider vanishing of the N-connection structure and a limit to a locally isotropic space.

Putting $\delta_{\underline{\gamma}}\ {}^{\underline{\gamma}}$ instead of $\mu_{\underline{\gamma}}$ in (8.32) and using (8.33) we can express respectively the curvature and gravitational d-spinors as

$$X_{\underline{\gamma\delta\alpha\beta}} = \delta_{\underline{\delta\tau}}\square_{\underline{\alpha\beta}}\delta_{\underline{\gamma}}\ {}^{\underline{\tau}} \text{ and } \Psi_{\underline{\gamma\delta\alpha\beta}} = \delta_{\underline{\delta\tau}}\square_{(\underline{\alpha\beta}}\delta_{\underline{\gamma})}\ {}^{\underline{\tau}}.$$

The d-spinor torsion $T^{\underline{\gamma}_1\underline{\gamma}_2}{}_{\underline{\alpha\beta}}$ is defined similarly as for d-tensors by using the d-spinor commutator (8.31) and equations

$$\square_{\underline{\alpha\beta}}f = T^{\underline{\gamma}_1\underline{\gamma}_2}{}_{\underline{\alpha\beta}}\ \bigtriangledown_{\underline{\gamma}_1\underline{\gamma}_2}\ f.$$

The d-spinor components $R_{\underline{\gamma}_1\underline{\gamma}_2}\ {}^{\underline{\delta}_1\underline{\delta}_2}{}_{\underline{\alpha\beta}}$ of the curvature d-tensor $R_{\gamma}\ {}^{\delta}{}_{\alpha\beta}$ can be computed by using relations (8.30), and (8.31) and (8.33) as to satisfy the equations

$$(\square_{\underline{\alpha\beta}} - T^{\underline{\gamma}_1\underline{\gamma}_2}{}_{\underline{\alpha\beta}}\bigtriangledown_{\underline{\gamma}_1\underline{\gamma}_2})V^{\underline{\delta}_1\underline{\delta}_2} = R_{\underline{\gamma}_1\underline{\gamma}_2}\ {}^{\underline{\delta}_1\underline{\delta}_2}{}_{\underline{\alpha\beta}}V^{\underline{\gamma}_1\underline{\gamma}_2},$$

here d-vector $V^{\underline{\gamma}_1\underline{\gamma}_2}$ is considered as a product of d-spinors, i.e. $V^{\underline{\gamma}_1\underline{\gamma}_2} = \nu^{\underline{\gamma}_1}\mu^{\underline{\gamma}_2}$. We find

$$\begin{aligned} R_{\underline{\gamma}_1\underline{\gamma}_2}\ {}^{\underline{\delta}_1\underline{\delta}_2}{}_{\underline{\alpha\beta}} \quad = \quad & \left(X_{\underline{\gamma}_1}\ {}^{\underline{\delta}_1}{}_{\underline{\alpha\beta}} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha\beta}}\quad \gamma^{\underline{\delta}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_1}\right)\delta_{\underline{\gamma}_2}\ {}^{\underline{\delta}_2} \\ & + \left(X_{\underline{\gamma}_2}\ {}^{\underline{\delta}_2}{}_{\underline{\alpha\beta}} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha\beta}}\quad \gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_2}\right)\delta_{\underline{\gamma}_1}\ {}^{\underline{\delta}_1}. \end{aligned} \tag{8.34}$$

It is convenient to use this d-spinor expression for the curvature d-tensor

$$\begin{aligned} R_{\underline{\gamma}_1\underline{\gamma}_2}\ {}^{\underline{\delta}_1\underline{\delta}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\beta}_1\underline{\beta}_2} \quad = \quad & \left(X_{\underline{\gamma}_1}\ {}^{\underline{\delta}_1}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\beta}_1\underline{\beta}_2} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\beta}_1\underline{\beta}_2}\ \gamma^{\underline{\delta}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_1}\right)\delta_{\underline{\gamma}_2}\ {}^{\underline{\delta}_2} \\ & + \left(X_{\underline{\gamma}_2}\ {}^{\underline{\delta}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\beta}_1\underline{\beta}_2} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\beta}_1\underline{\beta}_2}\ \gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_2}\right)\delta_{\underline{\gamma}_1}\ {}^{\underline{\delta}_1} \end{aligned}$$

in order to get the d–spinor components of the Ricci d-tensor

$$\begin{aligned} & R_{\underline{\gamma}_1\underline{\gamma}_2\underline{\alpha}_1\underline{\alpha}_2} = R_{\underline{\gamma}_1\underline{\gamma}_2}\ {}^{\underline{\delta}_1\underline{\delta}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\delta}_2} = X_{\underline{\gamma}_1}\ {}^{\underline{\delta}_1}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2} + \\ & T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2}\ \gamma^{\underline{\delta}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_1} + X_{\underline{\gamma}_2}\ {}^{\underline{\delta}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\gamma}_1\underline{\delta}_2}\ \gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_2} \end{aligned} \tag{8.35}$$

and this d-spinor decomposition of the scalar curvature:

$$\begin{aligned} q\overleftarrow{R} \quad = \quad & R^{\underline{\alpha}_1\underline{\alpha}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2} = X^{\underline{\alpha}_1\underline{\delta}_1}{}_{\underline{\alpha}_1}{}^{\underline{\alpha}_2}{}_{\underline{\delta}_1\underline{\alpha}_2} + T^{\underline{\tau}_1\underline{\tau}_2\underline{\alpha}_1}{}_{\underline{\alpha}_2\underline{\delta}_1}{}^{\underline{\alpha}_2}\ \gamma^{\underline{\delta}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\alpha}_1} \\ & + X^{\underline{\alpha}_2\underline{\delta}_2\underline{\alpha}_1}{}_{\underline{\alpha}_2\underline{\delta}_2\underline{\alpha}_1} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1}{}^{\underline{\alpha}_2\underline{\alpha}_1}{}_{\underline{\delta}_2}\ \gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\alpha}_2}. \end{aligned} \tag{8.36}$$

Putting (8.35) and (8.36) into (1.78) and, correspondingly, (9.14) we find the d–spinor components of the Einstein and $\Phi_{<\alpha><\beta>}$ d–tensors:

$$\overleftarrow{G}_{<\gamma><\alpha>} = \overleftarrow{G}_{\underline{\gamma}_1\underline{\gamma}_2\underline{\alpha}_1\underline{\alpha}_2} = X_{\underline{\gamma}_1}{}^{\underline{\delta}_1}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2}\,\gamma^{\underline{\delta}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_1}$$
$$+X_{\underline{\gamma}_2}{}^{\underline{\delta}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\gamma}_1\underline{\delta}_2}\,\gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_2} -$$
$$\frac{1}{2}\varepsilon_{\underline{\gamma}_1\underline{\alpha}_1}\varepsilon_{\underline{\gamma}_2\underline{\alpha}_2}[X^{\underline{\beta}_1\underline{\mu}_1}{}_{\underline{\beta}_1}{}^{\underline{\beta}_2}{}_{\underline{\mu}_1\underline{\beta}_2} + T^{\underline{\tau}_1\underline{\tau}_2\underline{\beta}_1}{}_{\underline{\beta}_2\underline{\mu}_1}{}^{\underline{\beta}_2}\,\gamma^{\underline{\mu}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\beta}_1} +$$
$$X^{\underline{\beta}_2\underline{\mu}_2\underline{\nu}_1}{}_{\underline{\beta}_2\underline{\mu}_2\underline{\nu}_1} + T^{\underline{\tau}_1\underline{\tau}_2}{}^{\underline{\beta}_2\underline{\beta}_1}_{\underline{\beta}_1}{}_{\underline{\delta}_2}\,\gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\beta}_2}] \qquad (8.37)$$

and

$$\Phi_{<\gamma><\alpha>} = \Phi_{\underline{\gamma}_1\underline{\gamma}_2\underline{\alpha}_1\underline{\alpha}_2} = \frac{1}{2(n+m_1+...+m_z)}\varepsilon_{\underline{\gamma}_1\underline{\alpha}_1}\varepsilon_{\underline{\gamma}_2\underline{\alpha}_2}[X^{\underline{\beta}_1\underline{\mu}_1}{}_{\underline{\beta}_1}{}^{\underline{\beta}_2}{}_{\underline{\mu}_1\underline{\beta}_2} +$$
$$T^{\underline{\tau}_1\underline{\tau}_2\underline{\beta}_1}{}_{\underline{\beta}_2\underline{\mu}_1}{}^{\underline{\beta}_2}\,\gamma^{\underline{\mu}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\beta}_1} + X^{\underline{\beta}_2\underline{\mu}_2\underline{\nu}_1}{}_{\underline{\beta}_2\underline{\mu}_2\underline{\nu}_1} + T^{\underline{\tau}_1\underline{\tau}_2}{}^{\underline{\beta}_2\underline{\beta}_1}_{\underline{\beta}_1}{}_{\underline{\delta}_2}\,\gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\beta}_2}] -$$
$$\frac{1}{2}[X_{\underline{\gamma}_1}{}^{\underline{\delta}_1}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2}\,\gamma^{\underline{\delta}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_1} +$$
$$X_{\underline{\gamma}_2}{}^{\underline{\delta}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\gamma}_1\underline{\delta}_2}\,\gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_2}]. \qquad (8.38)$$

The components of the conformal Weyl d-spinor can be computed by putting d-spinor values of the curvature (8.34) and Ricci (8.35) d-tensors into corresponding expression for the d-tensor (1.77). We omit this calculus in this work.

Chapter 9

Ha-Spinors and Field Interactions

The problem of formulation gravitational and gauge field equations on different types of locally anisotropic spaces is considered, for instance, in [109, 27, 19] and [186]. In this Chapter we shall introduce the basic field equations for gravitational and matter field la-interactions in a generalized form for generic higher order anisotropic spaces.

9.1 Scalar field ha–interactions

Let $\varphi(u) = (\varphi_1(u), \varphi_2(u), ..., \varphi_k(u))$ be a complex k-component scalar field of mass μ on ha-space $\mathcal{E}^{<z>}$. The d-covariant generalization of the conformally invariant (in the massless case) scalar field equation [128, 129] can be defined by using the d'Alambert locally anisotropic operator [4, 168] $\square = D^{<\alpha>}D_{<\alpha>}$, where $D_{<\alpha>}$ is a d-covariant derivation on $\mathcal{E}^{<z>}$ and constructed, for simplicity, by using Christoffel d–symbols (all formulas for field equations and conservation values can be deformed by using corresponding deformations d–tensors $P^{<\alpha>}_{<\beta><\gamma>}$ from the Cristoffel d–symbols, or the canonical d–connection to a general d-connection into consideration):

$$(\square + \frac{n_E - 2}{4(n_E - 1)}\overleftarrow{R} + \mu^2)\varphi(u) = 0, \tag{9.1}$$

where $n_E = n + m_1 + ... + m_z$.We must change d-covariant derivation $D_{<\alpha>}$ into ${}^{\diamond}D_{<\alpha>} = D_{<\alpha>} + ieA_{<\alpha>}$ and take into account the d-vector current

$$J^{(0)}_{<\alpha>}(u) = i((\overline{\varphi}(u) D_{<\alpha>}\varphi(u) - D_{<\alpha>}\overline{\varphi}(u))\varphi(u))$$

if interactions between locally anisotropic electromagnetic field (d-vector potential $A_{<\alpha>}$), where e is the electromagnetic constant, and charged scalar field φ are considered. The equations (9.1) are (locally adapted to the N-connection structure) Euler equations for the Lagrangian

$$\mathcal{L}^{(0)}(u) = \tag{9.2}$$
$$\sqrt{|g|}\left[g^{<\alpha><\beta>}\delta_{<\alpha>}\overline{\varphi}(u)\,\delta_{<\beta>}\varphi(u) - \left(\mu^2 + \frac{n_E-2}{4(n_E-1)}\right)\overline{\varphi}(u)\,\varphi(u)\right],$$

where $|g| = detg_{<\alpha><\beta>}$.

The locally adapted variations of the action with Lagrangian (9.2) on variables $\varphi(u)$ and $\overline{\varphi}(u)$ leads to the locally anisotropic generalization of the energy-momentum tensor,

$$E^{(0,can)}_{<\alpha><\beta>}(u) = \delta_{<\alpha>}\overline{\varphi}(u)\,\delta_{<\beta>}\varphi(u) + \delta_{<\beta>}\overline{\varphi}(u)\,\delta_{<\alpha>}\varphi(u) - \frac{1}{\sqrt{|g|}}g_{<\alpha><\beta>}\mathcal{L}^{(0)}(u), \tag{9.3}$$

and a similar variation on the components of a d-metric (1.43) leads to a symmetric metric energy-momentum d-tensor,

$$E^{(0)}_{<\alpha><\beta>}(u) = E^{(0,can)}_{(<\alpha><\beta>)}(u) - \frac{n_E-2}{2(n_E-1)}\left[R_{(<\alpha><\beta>)} + D_{(<\alpha>}D_{<\beta>)} - g_{<\alpha><\beta>}\Box\right]\overline{\varphi}(u)\,\varphi(u). \tag{9.4}$$

Here we note that we can obtain a nonsymmetric energy-momentum d-tensor if we firstly vary on $G_{<\alpha><\beta>}$ and than impose the constraint of compatibility with the N-connection structure. We also conclude that the existence of a N-connection in dv-bundle $\mathcal{E}^{<z>}$ results in a nonequivalence of energy-momentum d-tensors (9.3) and (9.4), nonsymmetry of the Ricci tensor, nonvanishing of the d-covariant derivation of the Einstein d-tensor, $D_{<\alpha>}\overleftarrow{G}^{<\alpha><\beta>} \neq 0$ and, in consequence, a corresponding breaking of conservation laws on higher order anisotropic spaces when $D_{<\alpha>}E^{<\alpha><\beta>} \neq 0$. The problem of formulation of conservation laws on locally anisotropic spaces is discussed in details and two variants of its solution (by using nearly autoparallel maps and tensor integral formalism on locally anisotropic and higher order multispaces) are proposed in [168].

In this Chapter we present only straightforward generalizations of field equations and necessary formulas for energy-momentum d-tensors of matter fields on $\mathcal{E}^{<z>}$ considering that it is naturally that the conservation laws (usually being consequences of global, local and/or intrinsic symmetries of the fundamental space-time and of the type of field interactions) have to be broken on locally anisotropic spaces.

9.2 Proca equations on ha–spaces

Let consider a d-vector field $\varphi_{<\alpha>}(u)$ with mass μ^2 (locally anisotropic Proca field) interacting with exterior la-gravitational field. From the Lagrangian

$$\mathcal{L}^{(1)}(u) = \sqrt{|g|}\left[-\frac{1}{2}\overline{f}_{<\alpha><\beta>}(u)\, f^{<\alpha><\beta>}(u) + \mu^2\overline{\varphi}_{<\alpha>}(u)\,\varphi^{<\alpha>}(u)\right], \tag{9.5}$$

where $f_{<\alpha><\beta>} = D_{<\alpha>}\varphi_{<\beta>} - D_{<\beta>}\varphi_{<\alpha>}$, one follows the Proca equations on higher order anisotropic spaces

$$D_{<\alpha>}f^{<\alpha><\beta>}(u) + \mu^2\varphi^{<\beta>}(u) = 0. \tag{9.6}$$

Equations (9.6) are a first type constraints for $\beta = 0$. Acting with $D_{<\alpha>}$ on (9.6), for $\mu \neq 0$ we obtain second type constraints

$$D_{<\alpha>}\varphi^{<\alpha>}(u) = 0. \tag{9.7}$$

Putting (9.7) into (9.6) we obtain second order field equations with respect to $\varphi_{<\alpha>}$:

$$\Box\varphi_{<\alpha>}(u) + R_{<\alpha><\beta>}\varphi^{<\beta>}(u) + \mu^2\varphi_{<\alpha>}(u) = 0. \tag{9.8}$$

The energy-momentum d-tensor and d-vector current following from the (9.8) can be written as

$$\begin{aligned} E^{(1)}_{<\alpha><\beta>}(u) &= -g^{<\varepsilon><\tau>}\left(\overline{f}_{<\beta><\tau>}f_{<\alpha><\varepsilon>} + \overline{f}_{<\alpha><\varepsilon>}f_{<\beta><\tau>}\right) \\ &\quad +\mu^2\left(\overline{\varphi}_{<\alpha>}\varphi_{<\beta>} + \overline{\varphi}_{<\beta>}\varphi_{<\alpha>}\right) - \frac{g_{<\alpha><\beta>}}{\sqrt{|g|}}\mathcal{L}^{(1)}(u) \end{aligned}$$

and

$$J^{(1)}_{<\alpha>}(u) = i\left(\overline{f}_{<\alpha><\beta>}(u)\,\varphi^{<\beta>}(u) - \overline{\varphi}^{<\beta>}(u)\, f_{<\alpha><\beta>}(u)\right).$$

For $\mu = 0$ the d-tensor $f_{<\alpha><\beta>}$ and the Lagrangian (9.5) are invariant with respect to locally anisotropic gauge transforms of type

$$\varphi_{<\alpha>}(u) \to \varphi_{<\alpha>}(u) + \delta_{<\alpha>}\Lambda(u),$$

where $\Lambda(u)$ is a d-differentiable scalar function, and we obtain a locally anisotropic variant of Maxwell theory.

9.3 Higher order anisotropic Dirac equations

Let denote the Dirac d–spinor field on $\mathcal{E}^{<z>}$ as $\psi(u) = (\psi^{\underline{\alpha}}(u))$ and consider as the generalized Lorentz transforms the group of automorphysm of the metric $G_{<\widehat{\alpha}><\widehat{\beta}>}$ (see (1.43)).The d–covariant derivation of field $\psi(u)$ is written as

$$\overrightarrow{\nabla_{<\alpha>}}\psi = \left[\delta_{<\alpha>} + \frac{1}{4}C_{\widehat{\alpha}\widehat{\beta}\widehat{\gamma}}(u)\ l^{\widehat{\alpha}}_{<\alpha>}(u)\,\sigma^{\widehat{\beta}}\sigma^{\widehat{\gamma}}\right]\psi, \tag{9.9}$$

where coefficients $C_{\widehat{\alpha}\widehat{\beta}\widehat{\gamma}} = (D_{<\gamma>}l^{<\alpha>}_{\widehat{\alpha}})\,l_{\widehat{\beta}<\alpha>}l^{<\gamma>}_{\widehat{\gamma}}$ generalize for ha-spaces the corresponding Ricci coefficients on Riemannian spaces [60]. Using σ-objects $\sigma^{<\alpha>}(u)$ (see (8.2) and (8.12)–(8.14)) we define the Dirac equations on ha–spaces:

$$(i\sigma^{<\alpha>}(u)\,\overrightarrow{\nabla_{<\alpha>}} - \mu)\psi = 0,$$

which are the Euler equations for the Lagrangian

$$\begin{aligned}\mathcal{L}^{(1/2)}(u) \quad = \quad & \sqrt{|g|}\{[\psi^{+}(u)\,\sigma^{<\alpha>}(u)\,\overrightarrow{\nabla_{<\alpha>}}\psi(u) - \\ & (\overrightarrow{\nabla_{<\alpha>}}\psi^{+}(u))\sigma^{<\alpha>}(u)\,\psi(u)] - \mu\psi^{+}(u)\,\psi(u)\},\end{aligned} \tag{9.10}$$

where $\psi^{+}(u)$ is the complex conjugation and transposition of the column $\psi(u)$.

From (9.10) we obtain the d–metric energy-momentum d-tensor

$$\begin{aligned}&E^{(1/2)}_{<\alpha><\beta>} = \frac{i}{4}[\psi^{+}(u)\,\sigma_{<\alpha>}(u)\,\overrightarrow{\nabla_{<\beta>}}\psi(u) + \psi^{+}(u)\,\sigma_{<\beta>}(u)\,\overrightarrow{\nabla_{<\alpha>}}\psi(u)\\ &-(\overrightarrow{\nabla_{<\alpha>}}\psi^{+}(u))\sigma_{<\beta>}(u)\,\psi(u) - (\overrightarrow{\nabla_{<\beta>}}\psi^{+}(u))\sigma_{<\alpha>}(u)\,\psi(u)]\end{aligned}$$

and the d-vector source

$$J^{(1/2)}_{<\alpha>}(u) = \psi^{+}(u)\,\sigma_{<\alpha>}(u)\,\psi(u).$$

We emphasize that locally anisotropic interactions with exterior gauge fields can be introduced by changing the higher order anisotropic partial derivation from (9.9) in this manner:

$$\delta_{\alpha} \rightarrow \delta_{\alpha} + ie^{\star}B_{\alpha},$$

where $e^{\star}$ and B_{α} are respectively the constant d-vector potential of locally anisotropic gauge interactions on higher order anisotropic spaces (see [186] and the next section).

9.4 D–spinor Yang–Mills fields

We consider a dv–bundle $\mathcal{B}_E,\ \pi_B : \mathcal{B} \to \mathcal{E}^{<z>}$ on ha–space $\mathcal{E}^{<z>}$. Additionally to d-tensor and d-spinor indices we shall use capital Greek letters, $\Phi, \Upsilon, \Xi, \Psi, ...$ for fibre (of this bundle) indices (see details in [128, 129] for the case when the base space of the v-bundle π_B is a locally isotropic space-time). Let $\underline{\bigtriangledown}_{<\alpha>}$ be, for simplicity, a torsionless, linear connection in $\mathcal{B}_E$ satisfying conditions:

$$\begin{aligned}
&\underline{\bigtriangledown}_{<\alpha>} : \Upsilon^{\Theta} \to \Upsilon^{\Theta}_{<\alpha>} \quad \left[\text{or } \Xi^{\Theta} \to \Xi^{\Theta}_{<\alpha>}\right],\\
&\underline{\bigtriangledown}_{<\alpha>} \left(\lambda^{\Theta} + \nu^{\Theta}\right) = \underline{\bigtriangledown}_{<\alpha>} \lambda^{\Theta} + \underline{\bigtriangledown}_{<\alpha>} \nu^{\Theta},\\
&\underline{\bigtriangledown}_{<\alpha>} \left(f\lambda^{\Theta}\right) = \lambda^{\Theta} \underline{\bigtriangledown}_{<\alpha>} f + f \underline{\bigtriangledown}_{<\alpha>} \lambda^{\Theta}, \quad f \in \Upsilon^{\Theta} \ [\text{or } \Xi^{\Theta}],
\end{aligned}$$

where by $\Upsilon^{\Theta}\ \left(\Xi^{\Theta}\right)$ we denote the module of sections of the real (complex) v–bundle $\mathcal{B}_E$ provided with the abstract index Θ. The curvature of connection $\underline{\bigtriangledown}_{<\alpha>}$ is defined as

$$K_{<\alpha><\beta>\Omega}^{\qquad\ \ \Theta} \lambda^{\Omega} = \left(\underline{\bigtriangledown}_{<\alpha>} \underline{\bigtriangledown}_{<\beta>} - \underline{\bigtriangledown}_{<\beta>} \underline{\bigtriangledown}_{<\alpha>}\right) \lambda^{\Theta}.$$

For Yang-Mills fields as a rule one considers that $\mathcal{B}_E$ is enabled with a unitary (complex) structure (complex conjugation changes mutually the upper and lower Greek indices). It is useful to introduce instead of $K_{<\alpha><\beta>\Omega}^{\qquad\ \ \Theta}$ a Hermitian matrix $F_{<\alpha><\beta>\Omega}^{\qquad\ \ \Theta} = i\ K_{<\alpha><\beta>\Omega}^{\qquad\ \ \Theta}$ connected with components of the Yang-Mills d-vector potential $B_{<\alpha>\Xi}^{\quad\ \Phi}$ according the formula:

$$\frac{1}{2} F_{<\alpha><\beta>\Xi}^{\qquad\ \ \Phi} = \underline{\bigtriangledown}_{[<\alpha>} B_{<\beta>]\Xi}^{\quad\ \Phi} - i B_{[<\alpha>|\Lambda|}^{\quad\ \Phi} B_{<\beta>]\Xi}^{\quad\ \Lambda}, \tag{9.11}$$

where the locally anisotropic space indices commute with capital Greek indices. The gauge transforms are written in the form:

$$\begin{aligned}
B_{<\alpha>\Theta}^{\quad\ \Phi} &\mapsto B_{<\alpha>\widehat{\Theta}}^{\quad\ \widehat{\Phi}} = B_{<\alpha>\Theta}^{\quad\ \Phi}\, s_{\Phi}^{\ \ \widehat{\Phi}}\, q_{\widehat{\Theta}}^{\ \ \Theta} + i s_{\Theta}^{\ \ \widehat{\Phi}}\, \underline{\bigtriangledown}_{<\alpha>}\, q_{\widehat{\Theta}}^{\ \ \Theta},\\
F_{<\alpha><\beta>\Xi}^{\qquad\ \ \Phi} &\mapsto F_{<\alpha><\beta>\widehat{\Xi}}^{\qquad\ \ \widehat{\Phi}} = F_{<\alpha><\beta>\Xi}^{\qquad\ \ \Phi} s_{\Phi}^{\ \ \widehat{\Phi}} q_{\widehat{\Xi}}^{\ \ \Xi},
\end{aligned}$$

where matrices $s_{\Phi}^{\ \ \widehat{\Phi}}$ and $q_{\widehat{\Xi}}^{\ \ \Xi}$ are mutually inverse (Hermitian conjugated in the unitary case). The Yang-Mills equations on torsionless locally anisotropic spaces [186] (see details in the next Section) are written in this form:

$$\begin{aligned}
\underline{\bigtriangledown}^{<\alpha>} F_{<\alpha><\beta>\Theta}^{\qquad\ \ \Psi} &= J_{<\beta>\ \Theta}^{\qquad \Psi}, \\
\underline{\bigtriangledown}_{[<\alpha>} F_{<\beta><\gamma>]\Theta}^{\qquad\ \ \Xi} &= 0.
\end{aligned} \tag{9.12}$$

We must introduce deformations of connection of type $\underline{\bigtriangledown}_{\alpha}^{\star} \longrightarrow \underline{\bigtriangledown}_{\alpha} + P_{\alpha}$, (the deformation d-tensor P_{α} is induced by the torsion in dv-bundle $\mathcal{B}_E$) into the definition of the curvature of gauge ha–fields (9.11) and motion equations (9.12) if interactions are modeled on a generic higher order anisotropic space.

9.5 D–spinor Einstein–Cartan Theory

The Einstein equations in some models of higher order anisotropic supergravity have been considered in [169, 172]. Here we note that the Einstein equations and conservation laws on v–bundles provided with N-connection structures were studied in detail in [108, 109, 2, 3, 193, 191, 164]. In Ref. [186] we proved that the locally anisotropic gravity can be formulated in a gauge like manner and analyzed the conditions when the Einstein gravitational locally anisotropic field equations are equivalent to a corresponding form of Yang-Mills equations. Our aim here is to write the higher order anisotropic gravitational field equations in a form more convenient for theirs equivalent reformulation in higher order anisotropic spinor variables.

9.5.1 Einstein ha–equations

We define d-tensor $\Phi_{<\alpha><\beta>}$ as to satisfy conditions

$$-2\Phi_{<\alpha><\beta>} \doteq R_{<\alpha><\beta>} - \frac{1}{n+m_1+...+m_z} \overleftarrow{R} g_{<\alpha><\beta>}$$

which is the torsionless part of the Ricci tensor for locally isotropic spaces [128, 129], i.e. $\Phi_{<\alpha>}^{<\alpha>} \doteq 0$. The Einstein equations on higher order anisotropic spaces

$$\overleftarrow{G}_{<\alpha><\beta>} + \lambda g_{<\alpha><\beta>} = \kappa E_{<\alpha><\beta>}, \tag{9.13}$$

where

$$\overleftarrow{G}_{<\alpha><\beta>} = R_{<\alpha><\beta>} - \frac{1}{2} \overleftarrow{R} g_{<\alpha><\beta>}$$

is the Einstein d–tensor, λ and κ are correspondingly the cosmological and gravitational constants and by $E_{<\alpha><\beta>}$ is denoted the locally anisotropic energy–momentum d–tensor, can be rewritten in equivalent form:

$$\Phi_{<\alpha><\beta>} = -\frac{\kappa}{2}(E_{<\alpha><\beta>} - \frac{1}{n+m_1+...+m_z} E_{<\tau>}^{<\tau>}\ g_{<\alpha><\beta>}). \tag{9.14}$$

Because ha–spaces generally have nonzero torsions we shall add to (9.14) (equivalently to (9.13)) a system of algebraic d–field equations with the source $S^{<\alpha>}_{\ <\beta><\gamma>}$ being the locally anisotropic spin density of matter (if we consider a variant of higher order anisotropic Einstein–Cartan theory):

$$T^{<\gamma>}_{\ <\alpha><\beta>} + 2\delta^{<\gamma>}_{\ [<\alpha>} T^{<\delta>}_{\ <\beta>]<\delta>} = \kappa S^{<\gamma>}_{\ <\alpha><\beta>}. \tag{9.15}$$

From (9.15) one follows the conservation law of higher order anisotropic spin matter:

$$\bigtriangledown_{<\gamma>} S^{<\gamma>}_{\ <\alpha><\beta>} - T^{<\delta>}_{\ <\delta><\gamma>} S^{<\gamma>}_{\ <\alpha><\beta>} = E_{<\beta><\alpha>} - E_{<\alpha><\beta>}.$$

9.5.2 Einstein–Cartan d–equations

Now we can write out the field equations of the Einstein–Cartan theory in the d-spinor form. So, for the Einstein equations (1.78) we have

$$\overleftarrow{G}_{\underline{\gamma}_1\underline{\gamma}_2\underline{\alpha}_1\underline{\alpha}_2} + \lambda \varepsilon_{\underline{\gamma}_1\underline{\alpha}_1} \varepsilon_{\underline{\gamma}_2\underline{\alpha}_2} = \kappa E_{\underline{\gamma}_1\underline{\gamma}_2\underline{\alpha}_1\underline{\alpha}_2},$$

with $\overleftarrow{G}_{\underline{\gamma}_1\underline{\gamma}_2\underline{\alpha}_1\underline{\alpha}_2}$ from (8.37), or, by using the d-tensor (8.38),

$$\Phi_{\underline{\gamma}_1\underline{\gamma}_2\underline{\alpha}_1\underline{\alpha}_2} + (\frac{\overleftarrow{R}}{4} - \frac{\lambda}{2}) \varepsilon_{\underline{\gamma}_1\underline{\alpha}_1} \varepsilon_{\underline{\gamma}_2\underline{\alpha}_2} = -\frac{\kappa}{2} E_{\underline{\gamma}_1\underline{\gamma}_2\underline{\alpha}_1\underline{\alpha}_2},$$

which are the d-spinor equivalent of the equations (9.14). These equations must be completed by the algebraic equations (9.15) for the d-torsion and d-spin density with d-tensor indices changed into corresponding d–spinor ones.

9.5.3 Higher order anisotropic gravitons

Let a massless d-tensor field $h_{<\alpha><\beta>}(u)$ is interpreted as a small perturbation of the locally anisotropic background metric d-field $g_{<\alpha><\beta>}(u)$. Considering, for simplicity, a torsionless background we have locally anisotropic Fierz–Pauli equations

$$\Box h_{<\alpha><\beta>}(u) + 2R_{<\tau><\alpha><\beta><\nu>}(u) \ h^{<\tau><\nu>}(u) = 0$$

and d–gauge conditions

$$D_{<\alpha>} h^{<\alpha>}_{<\beta>}(u) = 0, \quad h(u) \equiv h^{<\alpha>}_{<\beta>}(u) = 0,$$

where $R_{<\tau><\alpha><\beta><\nu>}(u)$ is curvature d-tensor of the locally anisotropic background space (these formulae can be obtained by using a perturbation formalism with respect to $h_{<\alpha><\beta>}(u)$ developed in [75]; in our case we must take into account the distinguishing of geometrical objects and operators on higher order anisotropic spaces).

Finally, we remark that all presented geometric constructions contain those elaborated for generalized Lagrange spaces [108, 109] (for which a tangent bundle TM is considered instead of a v-bundle $\mathcal{E}^{<z>}$) and for constructions on the so called osculator bundles with different prolongations and extensions of Finsler and Lagrange metrics [110]. We also note that the higher order Lagrange (Finsler) geometry is characterized by a metric of type (dmetrichcv) with components parametized as $g_{ij} = \frac{1}{2}\frac{\partial^2 \mathcal{L}}{\partial y^i \partial y^j}$ $\left(g_{ij} = \frac{1}{2}\frac{\partial^2 \Lambda^2}{\partial y^i \partial y^j}\right)$ and $h_{a_p b_p} = g_{ij}$, where $\mathcal{L} = \mathcal{L}\ (x, y_{(1)}, y_{(2)}, ..., y_{(z)})$ $\left(\Lambda = \Lambda\left(x, y_{(1)}, y_{(2)}, ..., y_{(z)}\right)\right)$ is a Lagrangian (Finsler metric) on $TM^{(z)}$ (see details in [108, 109, 96, 27]).

Part IV

Finsler Geometry and Spinor Variables

Chapter 10

Metrics Depending on Spinor Variables

10.1 Lorentz Transformation

We present the transformation character of the connection, the nonlinear connection and the spin connection coefficients with respect to local Lorentz transformations which depend on spinor variables, vector variables as well as coordinates.

For any quantities which transform as

$$f\left(x, y, \xi, \overline{\xi}\right) \to f'\left(x, y, \xi', \overline{\xi}'\right) = U\left(x, y, \xi, \overline{\xi}\right) \tag{10.1}$$

their derivatives with respect to x^i, y^i, ξ_α and $\overline{\xi}^\alpha$ under Lorentz transformations

$$x^{i'} = x^i, y^{i'} = y^i, \xi'_\alpha = \Lambda^\beta{}_\alpha \xi_\beta, \overline{\xi}'^\alpha = \Lambda^{-1\alpha}_{\ \beta}\overline{\xi}^\beta \tag{10.2}$$

will be given as follows

$$\begin{aligned}
a)\quad \frac{\partial U}{\partial x^\lambda} &= \frac{\partial f'}{\partial x^\lambda} + \frac{\partial f'}{\partial \xi'_\alpha}\frac{\partial \Lambda^\beta{}_\alpha}{\partial x^\lambda}\xi_\beta + \frac{\partial f'}{\partial \overline{\xi}'^\alpha}\frac{\partial \Lambda^{-1\alpha}_{\ \beta}}{\partial x^\lambda}\xi_\beta,\\
b)\quad \frac{\partial U}{\partial \xi_\alpha} &= \frac{\partial f'}{\partial \xi'_\beta}\Lambda^\alpha{}_\beta + \frac{\partial f'}{\partial \xi'_\beta}\frac{\partial \Lambda^\gamma{}_\beta}{\partial \xi_\alpha}\xi_\gamma + \frac{\partial f'}{\partial \overline{\xi}'^\beta}\frac{\partial \Lambda^{-1\beta}_{\ \gamma}}{\partial \xi_\alpha}\overline{\xi}^\gamma,\\
c)\quad \frac{\partial U}{\partial \overline{\xi}^\alpha} &= \frac{\partial f'}{\partial \overline{\xi}'^\alpha}\Lambda^{-1\beta}_{\ \alpha} + \frac{\partial f'}{\partial \xi'_\beta}\frac{\partial \Lambda^\gamma{}_\beta}{\partial \overline{\xi}^\alpha}\xi_\gamma + \frac{\partial f'}{\partial \overline{\xi}'^\beta}\frac{\partial \Lambda^{-1\beta}_{\ \gamma}}{\partial \overline{\xi}^\alpha}\overline{\xi}^\gamma,\\
c)\quad \frac{\partial U}{\partial y^\lambda} &= \frac{\partial f'}{\partial y^\lambda} + \frac{\partial f'}{\partial \xi'_\beta}\frac{\partial \Lambda^\gamma{}_\beta}{\partial y^\lambda}\xi_\gamma + \frac{\partial f'}{\partial \overline{\xi}'^\beta}\frac{\partial \Lambda^{-1\beta}_{\ \gamma}}{\partial y^\lambda}\overline{\xi}^\gamma.
\end{aligned} \tag{10.3}$$

Taking into account that (2.23) of [123], namely:

$$\begin{aligned}\frac{\partial^{[*]}}{\partial x^{\lambda}} &= \left(\frac{\partial}{\partial x^{\lambda}} + N_{\alpha\lambda}\frac{\partial}{\partial \xi_{\alpha}} + \overline{N}^{\alpha}_{\lambda}\frac{\partial}{\partial \overline{\xi}^{\alpha}}\right) \\ &\quad - \left(\Gamma^{k}_{\tau\lambda} + \overline{C}^{\kappa\alpha}_{\tau} N_{\alpha\lambda} + \overline{N}^{\alpha}_{\lambda} C^{\kappa}_{\tau\alpha}\right) y^{\tau}\frac{\partial}{\partial y^{\kappa}} \\ &= \frac{\partial^{[*]}}{\partial x^{\lambda}} - \left(\Gamma^{(*)k}_{\tau\lambda} y^{\tau}\right)\frac{\partial}{\partial y^{\kappa}},\end{aligned} \tag{10.4}$$

(where the nonlinear connection coefficients $N_{\alpha\lambda}$ and $\overline{N}^{\alpha}_{\lambda}$ are given in [121]), we substitute (10.3) in (10.4), then the nonlinear connection coefficients have to be transformed for Lorentz scalar quantities as

$$\begin{aligned} a)\quad N'_{\alpha\lambda} &= N_{\beta\lambda}\Lambda^{\beta}_{\ \alpha} + \frac{\partial^{[*]}\Lambda^{\beta}_{\ \alpha}}{\partial x^{\lambda}}\xi_{\beta}, \\ a')\quad \overline{N}'^{\alpha}_{\lambda} &= \overline{N}^{\beta}_{\lambda}\Lambda^{-1\alpha}_{\ \ \beta} + \frac{\partial^{[*]}\Lambda^{-1\alpha}_{\ \ \beta}}{\partial x^{\lambda}}\overline{\xi}^{\beta}. \end{aligned} \tag{10.5}$$

In the above mentioned (10.5) a), a') the relation $\partial^{[*]}/\partial x^{\lambda} = \partial^{[*]\prime}/\partial x^{\lambda}$ was used for $[*]$–differential operators. For the calculation of the transformation character of nonlinear connection coefficients $n_{\alpha\lambda}, \widetilde{n}^{\alpha}_{\lambda}, \widetilde{n}^{0\alpha}_{\lambda}, \widetilde{n}^{\beta\alpha}_{0}, n^{0}_{\beta\alpha}, n^{\beta}_{0\alpha}$ are used the relations

$$\frac{\partial^{[*]}}{\partial \xi_{\alpha}} = \Lambda^{\alpha}_{\ \beta}\frac{\partial^{[*]\prime}}{\partial \xi_{\beta}}, \frac{\partial^{[*]}}{\partial \xi_{\alpha}} = \Lambda^{-1\alpha}_{\ \ \beta}\frac{\partial^{[*]\prime}}{\partial \xi^{\alpha}}, \frac{\partial^{[*]}}{\partial y^{\lambda}} = \frac{\partial^{[*]\prime}}{\partial y'^{\lambda}}.$$

Also by means of (2.23) b), c), d) of [123] and (10.3) we obtain

$$\begin{aligned} b)\quad n'_{\beta\lambda} &= \Lambda^{\alpha}_{\beta} n_{\alpha\lambda} + \frac{\partial^{[*]}\Lambda^{\gamma}_{\ \beta}}{\partial y^{\lambda}}\xi_{\gamma}, \\ b')\quad \widetilde{n}'^{\beta}_{\lambda} &= \Lambda^{-1\beta}_{\ \ \alpha}\widetilde{n}^{\alpha}_{\lambda} + \frac{\partial^{[*]}\Lambda^{-1\beta}_{\ \ \gamma}}{\partial y^{\lambda}}\overline{\xi}^{\gamma}, \\ c)\quad \widetilde{n}'^{0\beta}_{\delta} &= \Lambda^{-1\beta}_{\ \ \alpha}\left(\Lambda^{\varepsilon}_{\ \delta}\widetilde{n}^{0\beta}_{\varepsilon} + \frac{\partial^{[*]}\Lambda^{\gamma}_{\ \delta}}{\partial \xi_{\alpha}}\xi_{\gamma}\right), \\ c')\quad \widetilde{n}'^{\delta\beta}_{0} &= \Lambda^{-1\beta}_{\ \ \alpha}\left(\Lambda^{-1\delta}_{\ \ \varepsilon}\widetilde{n}^{\varepsilon\alpha}_{0} + \frac{\partial^{[*]}\Lambda^{-1\delta}_{\ \ \gamma}}{\partial \xi_{\alpha}}\overline{\xi}^{\gamma}\right), \\ d)\quad n'^{0}_{\beta\alpha} &= \Lambda^{\delta}_{\ \alpha}\left(\Lambda^{\gamma}_{\ \beta} n^{0}_{\gamma\alpha} + \frac{\partial^{[*]}\Lambda^{\gamma}_{\ \beta}}{\partial \overline{\xi}^{\delta}}\xi_{\gamma}\right), \\ d')\quad n'^{\beta}_{0\alpha} &= \Lambda^{\delta}_{\ \alpha}\left(n^{\gamma}_{0\delta}\Lambda^{-1\beta}_{\ \ \gamma} + \overline{\xi}^{\gamma}\frac{\partial^{[*]}\Lambda^{-1\beta}_{\ \ \gamma}}{\partial \overline{\xi}^{\delta}}{}_{\gamma}\right). \end{aligned}$$

Consequently, [∗]–derivatives of the quantities 10.1) will satisfy the following relations:

$$\begin{aligned} a)\quad \frac{\partial^{[*]}U}{\partial x^{\lambda}} &= \frac{\partial f'}{\partial x^{\lambda}} + N'_{\alpha\lambda}\frac{\partial f'}{\partial \xi'_{\alpha}} + \overline{N}'^{\alpha}_{\lambda}\frac{\partial f'}{\partial \overline{\xi}'^{\alpha}} - \Gamma^{(*)\kappa}_{\tau\lambda} y'^{\tau}\frac{\partial f'}{\partial y'^{\kappa}}, \qquad (10.6)\\ b)\quad \frac{\partial^{[*]}U}{\partial y^{\lambda}} &= \frac{\partial f'}{\partial y'^{\lambda}} + n'_{\alpha\lambda}\frac{\partial f'}{\partial \xi'_{\alpha}} + \overline{n}'^{\alpha}_{\lambda}\frac{\partial f'}{\partial \overline{\xi}'^{\alpha}} - C^{(*)\kappa}_{\tau\lambda} y'^{\tau}\frac{\partial f'}{\partial y'^{\kappa}},\\ c)\quad \Lambda^{-1\beta}_{\ \ \alpha}\frac{\partial^{[*]}U}{\partial \xi_{\beta}} &= \frac{\partial f'}{\partial \xi'_{\alpha}} + \widetilde{n}'^{0\alpha}_{\beta}\frac{\partial f'}{\partial \xi'_{\beta}} + \widetilde{n}'^{\beta\alpha}_{0}\frac{\partial f'}{\partial \overline{\xi}'^{\beta}} - \overline{C}'^{(*)\kappa\alpha}_{\tau} y'^{\tau}\frac{\partial f'}{\partial y'^{\kappa}},\\ d)\quad \Lambda^{\beta}_{\ \alpha}\frac{\partial^{[*]}U}{\partial \overline{\xi}^{\beta}} &= \frac{\partial f'}{\partial \overline{\xi}'^{\alpha}} + \widetilde{n}'^{0}_{\beta\alpha}\frac{\partial f'}{\partial \xi'_{\beta}} + \widetilde{n}'^{\beta}_{0\alpha}\frac{\partial f'}{\partial \overline{\xi}^{\beta}} - C'^{(*)\kappa\alpha}_{\tau} y'^{\tau}\frac{\partial f'}{\partial y'^{\kappa}}. \end{aligned}$$

We have Lorentz–scalar quantities

$$f'\left(x, y, \xi', \overline{\xi}'\right) = f\left(x, y, \xi, \overline{\xi}\right), \qquad (10.7)$$

then, the

$$\frac{\partial^{[*]}f}{\partial x^{\lambda}}, \frac{\partial^{[*]}f}{\partial y^{\lambda}}, \frac{\partial^{[*]}f}{\partial \xi_{\alpha}}, \frac{\partial^{[*]}f}{\partial \overline{\xi}_{\alpha}}$$

are transformed as Lorentz–scalar and spinors adjoint to each other, respectively. Consequently [∗]–differentiation are covariant differential operators for Lorentz–scalar quantities. The spin connection coefficients $\omega^{[*]}_{ab\lambda}, \theta^{[*]}_{ab\lambda}, \theta^{[*]\beta}_{ab}$, $\theta^{[*]}_{ab\beta}$ will be transformed by Lorentz transformations as follows:

We consider the relation (3.23) a) of [123], namely:

$$\begin{aligned} \omega^{[*]}_{ab\lambda} &= \left(\frac{\partial^{[*]}h^{\mu}_{a}}{\partial x^{\lambda}} + \Gamma^{(*)\mu}_{\nu\lambda}h^{\nu}_{a}\right)h_{\mu b}, \qquad (10.8)\\ \omega^{[*]\prime}_{ab\lambda} &= \left(\frac{\partial^{[*]}h'^{\mu}_{a}}{\partial x^{\lambda}} + \Gamma^{(*)\prime\mu}_{\nu\lambda}h'^{\nu}_{a}\right)h'_{\mu b},\\ \Gamma^{(*)\mu}_{\nu\lambda} &= \Gamma^{(*)}_{\nu\kappa\lambda}g^{\kappa\mu} \end{aligned}$$

also for the tetrads h'^{μ}_{a} and h'^{μ}_{b} valid the relation $h'^{\mu}_{a} = L^{b}_{a}h'^{\mu}_{b}$ (4.1) of [121]), then taking into account (10.8) we take the transformation formula of spin connection coefficients $\omega^{[*]}_{ab\lambda}$,

$$\begin{aligned} a)\quad \omega^{[*]\prime}_{ab\lambda} &= L^c_a L^d_b \omega^{[*]}_{cd\lambda} + \frac{\partial^{[*]} L^c_a}{\partial x^\lambda} h_{cd} L^d_b, \\ b)\quad \theta^{[*]\prime}_{ab\lambda} &= L^c_a L^d_b \theta^{[*]}_{cd\lambda} + \frac{\partial^{[*]} L^c_a}{\partial y^\lambda} n_{cd} L^d_b, \\ c)\quad \theta^{[*]\prime\beta}_{ab} &= \Lambda^{-1\beta}_{\gamma} \left[\theta^{[*]\gamma}_{cd} L^c_a L^d_b + \frac{\partial^{[*]} L^c_a}{\partial \xi_\gamma} L^d_b n_{dc} \right], \\ d)\quad \theta^{[*]\prime}_{ab\beta} &= \Lambda^{\gamma}{}_{\beta} \left[\theta^{[*]}_{cd\gamma} L^c_a L^d_b + \frac{\partial^{[*]} L^c_a}{\partial \overline{\xi}^\gamma} L^d_b n_{dc} \right], \end{aligned} \tag{10.9}$$

where the connection coefficients $\Gamma^{(*)\mu}_{\nu\lambda}, \Gamma^{(*)'\mu}_{\nu\lambda}$ are Lorentz–scalar and similar procedures are considered for the transformed connection coefficients of $\theta^{[*]}_{ab\lambda}, \theta^{[*]\beta}_{ab}, \theta^{[*]}_{ab\beta}$, using the relations (3.23) b),c), of [123].

Next, we shall derive the transformation character of the spin connection coefficients $\left(\Gamma^{(*)\kappa}_{\tau\lambda}, C^{(*)\kappa}_{\tau\lambda}, \overline{C}^{\prime(*)\kappa\alpha}_{\tau}, C^{\prime(*)\kappa\alpha}_{\tau}\right)$ under Lorentz transformations. If we take the relation (3.6) a) of [123],

$$N_{\tau\lambda} = \Gamma^{(*)\kappa}_{\tau\lambda} \xi_\kappa \tag{10.10}$$

and

$$N'_{\tau\lambda} = \Gamma^{(*)'\kappa}_{\tau\lambda} \xi'_\kappa, \tag{10.11}$$

and we substitute (10.5) a) in (10.11), then we get the required transformation formula,

$$\begin{aligned} a)\quad \Gamma^{(*)'\delta}_{\alpha\lambda} &= \Lambda^{-1\delta}_{\varepsilon} \Lambda^{\beta}_{\alpha} \Gamma^{(*)\varepsilon}_{\beta\lambda} + \frac{\partial^{[*]} \Lambda^{\varepsilon}_{\alpha}}{\partial x^\lambda} \Lambda^{-1\delta}_{\varepsilon}, \\ b)\quad C^{(*)'\delta}_{\alpha\lambda} &= \Lambda^{-1\delta}_{\varepsilon} \Lambda^{\beta}_{\alpha} C^{(*)\varepsilon}_{\beta\lambda} + \frac{\partial^{[*]} \Lambda^{\varepsilon}_{\alpha}}{\partial y^\lambda} \Lambda^{-1\delta}_{\varepsilon}, \\ c)\quad \widetilde{C}^{(*)\delta\rho}_{\varepsilon} &= \left[\Lambda^{\gamma}_{\varepsilon} \widetilde{C}^{(*)\beta\alpha}_{\gamma} \Lambda^{-1\delta}_{\beta} + \frac{\partial^{[*]} \Lambda^{\varepsilon}_{\alpha}}{\partial \xi_\alpha} \Lambda^{-1\delta}_{\gamma} \right] \Lambda^{-1\rho}_{\alpha}, \\ d)\quad C^{(*)'\delta}_{\varepsilon\rho} &= \Lambda^{\alpha}_{\rho} \left[\Lambda^{\gamma}_{\varepsilon} C^{(*)\beta}_{\gamma\alpha} \Lambda^{-1\delta}_{\beta} + \frac{\partial^{[*]} \Lambda^{\gamma}_{\varepsilon}}{\partial \overline{\xi}^\alpha} \Lambda^{-1\delta}_{\gamma} \right]. \end{aligned} \tag{10.12}$$

Finally, from (3.20) of [123] and (10.5), (10.12), arbitrary terms a_λ, b_λ,

$\overline{\beta}^{\alpha}, \beta_{\alpha}$ are transformed as follows

$$\begin{aligned}
a_{\lambda} &= a^{'}_{\lambda} + \overline{\beta}'^{\alpha}\left(\frac{\partial\Lambda^{\alpha}_{\ \beta}}{\partial x^{\lambda}}\right)\xi_{\beta} + \overline{\xi}^{\beta}\left(\frac{\partial\Lambda^{-1\alpha}_{\ \ \beta}}{\partial x^{\lambda}}\right)\beta'_{\alpha},\\
b_{\lambda} &= b^{'}_{\lambda} + \overline{\beta}'^{\beta}\left(\frac{\partial\Lambda^{\gamma}_{\ \beta}}{\partial y^{\lambda}}\right)\xi_{\gamma} + \overline{\xi}^{\gamma}\left(\frac{\partial^{[*]}\Lambda^{-1\beta}_{\ \ \gamma}}{\partial y^{\lambda}}\right)\beta'_{\beta},\\
\overline{\beta}^{\alpha} &= \overline{\beta}'^{\gamma}\Lambda^{\alpha}_{\ \gamma} + \overline{\beta}'^{\gamma}\left(\frac{\partial\Lambda^{\gamma}_{\ \beta}}{\partial\xi_{\alpha}}\right)\xi_{\varepsilon} + \overline{\xi}^{\varepsilon}\left(\frac{\partial^{[*]}\Lambda^{-1\gamma}_{\ \ \varepsilon}}{\partial y^{\lambda}}\right)\beta'_{\gamma},\\
\beta_{\alpha} &= \Lambda^{-1\gamma}_{\ \ \alpha}\beta'_{\gamma} + \overline{\beta}'^{\gamma}\left(\frac{\partial\Lambda^{\gamma}_{\ \gamma}}{\partial\overline{\xi}^{\alpha}}\right)\xi_{\varepsilon} + \overline{\xi}^{\varepsilon}\left(\frac{\partial^{[*]}\Lambda^{-1\gamma}_{\ \ \varepsilon}}{\partial\overline{\xi}^{\alpha}}\right)\beta'_{\gamma}.
\end{aligned}$$

10.2 Curvature

In this section we shall present the form of the curvature of the above–mentioned spaces. There must exist ten kinds of curvature tensors corresponding to four kind of covariant derivatives with respect to $x^i, y^{\lambda}, \xi_{\alpha}, \overline{\xi}^{\alpha}$, (coordinates, vector variables, spinor variables).

If we denote with M, n the number of curvatures and the kind of covariant derivatives, then we have generally, $N = n(n+1)/2$. In our case $N = 10, n = 4$. Like in [121] (paragraph 5), here, they appear three different expressions of the above–mentioned ten curvature tensors which are closely related to each other. The relation between ten curvature tensors $T^{\mu}_{\nu XY}$ and ten spin–curvature tensors T_{abXY} will be the following:

$$T_{abXY} = T^{\mu}_{\nu XY}h^{\nu}_{a}h_{\mu b} \tag{10.13}$$

which arises from integrability conditions of the partial differential equations (cf. (3.22) of [121]).

The curvature tensors which are calculated below come from the Ricci identities [136, 96], as well as the commutation formula of the $[*]$–differential operators $\partial^{[*]}/\partial x^{\lambda}$ and $\partial^{[*]\prime}/\partial y^{\lambda}$.

The curvature tensors $T^{\mu}_{\nu XY}$ are defined as follows

$$\begin{aligned}
R^{\mu}_{\nu\lambda\kappa} &= \frac{\partial^{[*]}\Gamma^{(*)\mu}_{\nu\lambda}}{\partial x^{\kappa}} - \frac{\partial^{[*]}\Gamma^{(*)\mu}_{\nu\kappa}}{\partial x^{\lambda}} + \Gamma^{(*)\tau}_{\nu\lambda}\Gamma^{(*)\mu}_{\tau\kappa} - \Gamma^{(*)\tau}_{\nu\kappa}\Gamma^{(*)\mu}_{\tau\lambda}\\
&\quad - \left(A^{[*]}_{\gamma\lambda\kappa}\overline{C}^{[*]\mu\gamma}_{\nu} + \widehat{A}^{[*]\gamma}_{\lambda\kappa}\overline{C}^{[*]\mu}_{\nu\gamma} + \check{A}^{[*]\tau}_{\lambda\kappa}C^{[*]\mu}_{\nu\tau}\right),
\end{aligned}$$

where $A^{[*]}_{\gamma\lambda\kappa}, \widehat{A}^{[*]\gamma}_{\lambda\kappa}, \breve{A}^{[*]\tau}_{\lambda\kappa}$ are given by

$$
\begin{aligned}
A^{[*]}_{\gamma\lambda\kappa} &= A_{\gamma\lambda\kappa} - C^{0\xi}_{\gamma} A_{\xi\lambda\kappa} - \hat{A}^{\xi}_{\lambda\kappa} C^{0}_{\gamma\xi} - \breve{A}^{\xi}_{\lambda\kappa} C^{0}_{\gamma\xi}, \\
\widehat{A}^{[*]\gamma}_{\lambda\kappa} &= A^{[*]\gamma}_{\lambda\kappa} + \tilde{C}^{\gamma\xi}_{0} A_{\xi\lambda\kappa} - \hat{A}^{\xi}_{\lambda\kappa} C^{\gamma}_{0\xi} + \breve{A}^{\xi}_{\lambda\kappa} C^{0}_{\gamma\xi}, \\
\breve{A}^{[*]\rho}_{\lambda\kappa} &= \breve{A}^{\rho}_{\lambda\kappa} + \left(\bar{C}^{\rho\xi}_{\tau} y^{\tau}\right) A_{\xi\lambda\kappa} + \hat{A}^{\xi}_{\lambda\kappa} \left(C^{\rho}_{\tau\xi} y^{\tau}\right), \\
A_{\gamma\lambda\kappa} &= \frac{\partial^{[*]} N_{\gamma\lambda}}{\partial x^{\kappa}} - \frac{\partial^{[*]} N_{\gamma\kappa}}{\partial x^{\lambda}}, \\
\hat{A}^{\gamma}_{\lambda\kappa} &= \frac{\partial^{[*]} \bar{N}^{\gamma}_{\lambda}}{\partial x^{\kappa}} - \frac{\partial^{[*]} \bar{N}^{\gamma}_{\kappa}}{\partial x^{\lambda}}, \\
\breve{A}^{[*]\rho}_{\lambda\kappa} &= \frac{\partial^{[*]}}{\partial x^{\kappa}} \left[- \left(\Gamma^{\rho}_{\tau\lambda} + \bar{C}^{\rho\alpha}_{\tau} N_{\alpha\lambda} + \bar{N}^{\alpha}_{\lambda} C^{\rho}_{\tau\alpha} \right) y^{\tau} \right] \\
&\quad - \frac{\partial^{[*]}}{\partial x^{\lambda}} \left[- \left(\Gamma^{\rho}_{\tau\kappa} + \bar{C}^{\rho\alpha}_{\tau} N_{\alpha\kappa} + \bar{N}^{\alpha}_{\kappa} C^{\rho}_{\tau\alpha} \right) y^{\tau} \right].
\end{aligned}
$$

Similarly, the curvatures $P^{\mu}_{\nu\lambda\alpha}$ and $W^{\mu}_{\nu\lambda\alpha}$ can be defined as follows

$$
\begin{aligned}
P^{\mu}_{\nu\lambda\alpha} &= \frac{\partial^{[*]} \Gamma^{(*)\mu}_{\nu\lambda}}{\partial \xi^{\alpha}} - \frac{\partial^{[*]} C^{(*)\mu}_{\nu\alpha}}{\partial x^{\lambda}} + \Gamma^{(*)\tau}_{\nu\lambda} C^{(*)\mu}_{\tau\alpha} - \Gamma^{(*)\mu}_{\tau\lambda} C^{(*)\tau}_{\nu\alpha} \\
&\quad - \left(E^{[*]}_{\gamma\lambda\alpha} \overline{C}^{[*]\mu\gamma}_{\nu} + \widehat{E}^{[*]\gamma}_{\lambda\alpha} \overline{C}^{[*]\mu}_{\nu\gamma} + \breve{E}^{[*]\tau}_{\lambda\kappa} C^{[*]\mu}_{\nu\tau} \right), \\
R^{\mu}_{\nu\lambda\kappa} &= \frac{\partial^{[*]} C^{(*)\mu}_{\nu\lambda}}{\partial x^{\kappa}} - \frac{\partial^{[*]} \Gamma^{(*)\mu}_{\nu\kappa}}{\partial y^{\lambda}} + \Gamma^{(*)i}_{\nu\lambda} C^{(*)\mu}_{i\kappa} - \Gamma^{(*)\mu}_{i\kappa} C^{(*)i}_{\tau\lambda} \\
&\quad - \left(D^{[*]}_{\gamma\lambda\kappa} C^{[*]\mu\gamma}_{\nu} + \widehat{D}^{[*]\gamma}_{\lambda\kappa} C^{[*]\mu}_{\nu\gamma} + \breve{D}^{[*]\tau}_{\lambda\kappa} C^{[*]\mu}_{\nu\tau} \right).
\end{aligned}
$$

The quantities $E^{[*]}_{\gamma\lambda\alpha}, \widehat{E}^{[*]\gamma}_{\lambda\alpha}, \breve{E}^{[*]\tau}_{\lambda\kappa}$ and $D^{[*]}_{\gamma\lambda\kappa}, \widehat{D}^{[*]\gamma}_{\lambda\kappa}, \breve{D}^{[*]\tau}_{\lambda\kappa}$ are defined respectively to $A^{[*]}_{\gamma\lambda\kappa}, \widehat{A}^{[*]\gamma}_{\lambda\kappa}, \breve{A}^{[*]\tau}_{\lambda\kappa}$. A a matter of fact the expressions are too big to be presented for all ten curvature tensors, we prefer to give an algorithm for the general case, presenting the following the Table 10.1 of symbols for nonlinear connection.

In general for each of the ten curvature tensors, we have

$$
\begin{aligned}
\mathbf{T}^{\mu}_{\nu XY} &= \frac{\partial^{[*]} Con\mathbf{X}^{\mu}_{\nu X}}{\partial Y} - \frac{\partial^{[*]} Con\mathbf{Y}^{\mu}_{\nu X}}{\partial X} \\
&\quad + Con\mathbf{X}^{\mu}_{\nu X} Con\mathbf{Y}^{\mu}_{\tau Y} - Con\mathbf{Y}^{\mu}_{\nu Y} Con\mathbf{X}^{\mu}_{\tau X} \\
&\quad - \left(\mathbf{A}^{[*]}_{\gamma XY} \overline{C}^{[*]\mu\gamma}_{\nu} + \widehat{\mathbf{A}}^{[*]\gamma}_{XY} C^{[*]\mu}_{\nu\gamma} + \breve{\mathbf{A}}^{[*]\tau}_{XY} C^{[*]\mu}_{\nu\tau} \right),
\end{aligned}
\tag{10.14}
$$

coordinate vector spinors	connection coefficients	$\mathbf{N}_{\gamma Y}$	$\hat{\mathbf{N}}_X^Y$	$-\bar{\mathbf{N}}_X^k$
x^i	$\Gamma_{\nu\lambda}^{(*)\mu}$	$N_{\alpha\lambda}$	$\bar{N}_\lambda^\alpha$	$\left(\Gamma_{\tau\lambda}^\rho + \bar{C}_\tau^{\rho\alpha} N_{\alpha\lambda} + \bar{N}_\lambda^\alpha C_{\tau\alpha}^\rho\right) y^\tau$
y^λ	$C_{\nu\alpha}^{(*)\mu}$	$n_{\alpha\lambda}$	$\bar{n}_\lambda^\alpha$	$\left(C_{\tau\lambda}^\rho + \bar{C}_\tau^{\rho\alpha} n_{\alpha\lambda} + \bar{n}_\lambda^\alpha C_{\tau\alpha}^\rho\right) y^\tau$
ξ_α	$\overline{C}_\nu^{[*]\mu\gamma}$	$n_\beta^{0\alpha}$	$\bar{n}_0^{\beta\alpha}$	$\left(\bar{C}_\tau^{\rho\alpha} + \bar{C}_\tau^{\rho\alpha} n_\beta^{0\alpha} + \bar{n}_0^{\beta\alpha} C_{\tau\beta}^\rho\right) y^\tau$
$\overline{\xi}^\alpha$	$C_{\nu\lambda}^{(*)\mu}$	$n_{\beta\alpha}^0$	$n_{0\alpha}^\beta$	$\left(C_{\tau\alpha}^\rho + \bar{C}_\tau^{\rho\beta} n_{\beta\alpha}^0 + n_{0\alpha}^\beta C_{\tau\beta}^\rho\right) y^\tau$

Table 10.1: Nonlinear connections

where the coefficients are given by

$$\begin{aligned}
\mathbf{A}_{\gamma XY}^{[*]} &= \mathbf{A}_{\gamma\lambda\kappa} - C_\gamma^{0\xi}\mathbf{A}_{\xi XY} - \hat{\mathbf{A}}_{XY}^\xi C_{\gamma\xi}^0 - \breve{\mathbf{A}}_{XY}^\xi C_{\gamma\xi}^0,\\
\widehat{\mathbf{A}}_{XY}^{[*]\gamma} &= \mathbf{A}_{XY}^{[*]\gamma} + \tilde{C}_0^{\gamma\xi}\mathbf{A}_{\xi XY} - \hat{\mathbf{A}}_{XY}^\xi C_{0\xi}^\gamma + \breve{\mathbf{A}}_{XY}^\xi C_{\gamma\xi}^0,\\
\breve{\mathbf{A}}_{XY}^{[*]\rho} &= \breve{\mathbf{A}}_{XY}^\rho + \left(\bar{C}_\tau^{\rho\xi} y^\tau\right)\mathbf{A}_{\xi XY} + \hat{\mathbf{A}}_{XY}^\xi\left(C_{\tau\xi}^\rho y^\tau\right),\\
\mathbf{A}_{\gamma XY} &= \frac{\partial^{[*]}\mathbf{N}_{\gamma X}}{\partial Y} - \frac{\partial^{[*]}\mathbf{N}_{\gamma Y}}{\partial X},\\
\hat{\mathbf{A}}_{XY}^\gamma &= \frac{\partial^{[*]}\bar{\mathbf{N}}_X^\gamma}{\partial Y} - \frac{\partial^{[*]}\bar{\mathbf{N}}_Y^\gamma}{\partial X},\\
\breve{\mathbf{A}}_{\lambda\kappa}^{[*]\rho} &= \frac{\partial^{[*]}\check{\mathbf{N}}_X^\rho}{\partial Y} - \frac{\partial^{[*]}\check{\mathbf{N}}_Y^\rho}{\partial X},
\end{aligned}$$

$Con\mathbf{X}_{\nu X}^\mu$ represent the connection coefficients $\left(\Gamma_{\nu\kappa}^{(*)\mu}, C_{\nu\gamma}^{[*]\mu}, \overline{C}_\nu^{[*]\mu\gamma}, C_{\nu\alpha}^{[*]\mu}\right)$. We can write down all ten curvatures using the algorithm presented the above and adopt the following symbolism:

We can write down all the spin–curvature tensors using the symbolism of Table 10.2 with appropriate indices. The spin curvature tensros T_{abXY} are defined in (10.15). According the Tables 10.2 and 10.3 our general formula becomes

$$\begin{aligned}
T_{abXY} &= \frac{\partial^{[*]} sp.ConX_{abX}}{\partial Y} - \frac{\partial^{[*]} sp.ConX_{abY}}{\partial X} \qquad (10.15)\\
&+ sp.ConX_{acX}\ sp.ConY_{bY}^c - sp.ConY_{acY}\ sp.ConX_{bX}^c\\
&- \left(\mathbf{A}_{\gamma XY}^{[*]}\ \theta_{ab}^{[*]\gamma} + \widehat{\mathbf{A}}_{XY}^{[*]\gamma}\theta_{ab\gamma}^{[*]} + \breve{\mathbf{A}}_{XY}^{[*]\tau}\theta_{ab\tau}^{[*]}\right),
\end{aligned}$$

where $sp.ConX_{abX}$ represent the spin connection coefficients $\omega_{ab\lambda}^{[*]}$, $\theta_{ab\lambda}^{[*]}$, $\overline{\theta}_{ab}^{[*]\alpha}$, $\theta_{ab\lambda}^{[*]}$ with before defined $\mathbf{A}_{\gamma XY}^{[*]}, \widehat{\mathbf{A}}_{XY}^{[*]\gamma}, \breve{\mathbf{A}}_{XY}^{[*]\tau}$.

These spinor–curvature tensors will also appear in Ricci' formulae for a Lorentz vector field. To examine the transformation character of the curvature tensors it is convenient to divide them into the parts $T^{(0)\mu}_{\nu XY}$ and $T^{(1)\mu}_{\nu XY}$,

$$T^{\mu}_{\nu XY} = T^{(0)\mu}_{\nu XY} - T^{(1)\mu}_{\nu XY},$$

where

$$\begin{aligned} T^{(0)\mu}_{\nu XY} &= \frac{\partial^{[*]} ConX^{\mu}_{\nu X}}{\partial Y} - \frac{\partial^{[*]} ConY^{\mu}_{\nu Y}}{\partial X} \\ &\quad + ConX^{\tau}_{\nu X} ConY^{\mu}_{\tau Y} - ConY^{\mu}_{\nu Y} ConX^{\mu}_{\tau X}, \\ T^{(1)\mu}_{\nu XY} &= \mathbf{A}^{[*]}_{\gamma XY} \overline{C}^{[*]\mu\gamma}_{\nu} + \widehat{\mathbf{A}}^{[*]\gamma}_{XY} C^{[*]\mu}_{\nu\gamma} + \breve{\mathbf{A}}^{[*]\tau}_{XY} C^{[*]\mu}_{\nu\tau}. \end{aligned}$$

The curvature tensors $T^{(1)\mu}_{\nu XY}$ are expected to have the same transformation character as $T^{(0)\mu}_{\nu XY}$ and $T^{\mu}_{\nu XY}$ and are confirmed to transform as tensors or spinors under general coordinate transformations and local Lorentz transformations by formulae (10.3), (10.5) and (10.12). The arbitrary terms of spin connection coefficients are contained only in the parts $T^{(1)\mu}_{\nu XY}$, the arbitrariness disappear completely by virtue of the homogeneity of $\Gamma^{(*)\mu}_{\nu\kappa}, C^{[*]\mu}_{\nu\gamma}, \overline{C}^{[*]\mu\gamma}_{\nu}, C^{[*]\mu}_{\nu\alpha}$. Therefore, $T^{\mu}_{\nu XY}$ as well as $T^{(1)\mu}_{\nu XY}$ are defined unambiguously. The following conditions are imposed on $T^{(0)\mu}_{\nu XY}$ and $T^{(1)\mu}_{\nu XY}$ and, therefore, on $T^{\mu}_{\nu XY}$.

$X-Y$	$T^{\mu}_{\nu XY}$	$T^{\delta}_{\varepsilon XY}$	T_{XY}	$\mathbf{A}^{[*]}$	$\widehat{\mathbf{A}}^{[*]}$	$\breve{\mathbf{A}}^{[*]}$
$x-x$	R	X	φ	A	$\hat{A}$	$\check{A}$
$x-\xi$	P	$\overline{\Xi}$	Ψ	E	$\hat{E}$	$\check{E}$
$x-\bar{\xi}$	$\overline{P}$	Ξ	$\overline{\Psi}$	F	$\hat{F}$	$\check{F}$
$x-y$	W	Ψ	x	D	$\hat{D}$	$\check{D}$
$\xi-\xi$	$\overline{Q}$	O	ρ	B	$\hat{B}$	$\check{B}$
$\xi-\bar{\xi}$	S	K	μ	V	$\hat{V}$	$\check{V}$
$\xi-y$	Ω	$\widetilde{U}$	ν	G	$\hat{G}$	$\check{G}$
$\bar{\xi}-\bar{\xi}$	Q	O	$\overline{\rho}$	J	$\hat{J}$	$\check{J}$
$\bar{\xi}-y$	$\overline{\Omega}$	U	$\overline{\nu}$	Φ	$\hat{\Phi}$	$\check{\Phi}$
$y-y$	Z	Y	υ	H	$\hat{H}$	$\check{H}$

Table 10.2: Curvatures

Contractions of $\overline{\xi}, \xi, y^{\lambda}$ with the curvature tensors give the following:

$$\begin{aligned} \overline{\xi}^{\alpha} T^{(\,)}_{(\overline{\xi}^{\alpha}, x^{\lambda})} &= 0, \ \overline{\xi}^{\alpha} T^{(\,)}_{(\overline{\xi}^{\alpha}, \xi_{\alpha})} = 0, \\ \overline{\xi}^{\alpha} T^{(\,)}_{(\overline{\xi}^{\alpha}, y^{\lambda})} &= 0, \ \overline{\xi}^{\alpha} T^{(\,)}_{(\overline{\xi}^{\alpha}, \overline{\xi}^{\alpha})} = 0, \end{aligned} \tag{10.16}$$

The above mentioned structures and properties of curvature tensors $T^{\mu}_{\nu XY}$ are transformed to those of spin–curvature tensors T_{abXY} through the relations (10.13). Also, the integrability conditions of the partial differential equations of Ricci formulae for a spinor field, led to another spin–curvature tensors $T^{\delta}_{\varepsilon XY}$ which are related to T_{abXY} by the relation of

$$T^{\delta}_{\varepsilon XY} = \frac{1}{2} T_{abXY} \ \left(S^{ab}\right)^{\delta}_{\varepsilon} + iT_{XY} \ I^{\delta}_{\varepsilon},$$

where I^{δ}_{ε} is the unit matrix, $T^{\delta}_{\varepsilon XY}, T_{XY}$ are defined by (10.17) and (10.18) respectively, T_{abXY} are given by (10.15) and S^{ab} and (3.18) of [123].

coordinate vector spinors	Spin connection coefficients 1	Spin connection coefficients 2	coef. Xx
x^{λ}	$\omega^{[*]}_{ab\lambda}$	$\Gamma^{(*)\mu}_{\nu k}$	$a^{[*]}_{\lambda}$
y^{λ}	$\theta^{[*]}_{ab\lambda}$	$C^{(*)\mu}_{\nu\gamma}$	$b^{[*]}_{\lambda}$
ξ_{α}	$\overline{\theta}^{[*]\alpha}_{ab}$	$\overline{C}^{(*)\mu\gamma}_{\nu}$	$\overline{\beta}^{[*]\alpha}$
$\overline{\xi}^{\alpha}$	$\theta^{[*]}_{ab\lambda}$	$C^{(*)\mu}_{\nu\alpha}$	$\beta^{[*]}_{\alpha}$

Table 10.3: Spin Connections

Again, in order to present the spin–curvature tensors $T^{\delta}_{\varepsilon XY}$ we are going to use an algorithm along with appropriate columns in the Tables 10.2 and 10.3. The general formula is

$$\begin{aligned} T^{\delta}_{\varepsilon XY} &= \frac{\partial^{[*]} sp.ConX^{\delta}_{\varepsilon X}}{\partial Y} - \frac{\partial^{[*]} sp.ConY^{\delta}_{\varepsilon Y}}{\partial X} \\ &\quad + sp.ConX^{j}_{\varepsilon X} \ sp.ConY^{\delta}_{jY} - sp.ConY^{j}_{\varepsilon Y} \ sp.ConX^{\delta}_{jX} \\ &\quad - \left(\mathbf{A}^{[*]}_{jXY} \ \widetilde{C}^{[*]\delta j}_{\varepsilon} + \widehat{\mathbf{A}}^{[*]j}_{XY} C^{[*]\delta j}_{\varepsilon} + \breve{\mathbf{A}}^{[*]\tau}_{XY} C^{[*]\delta}_{\varepsilon\tau}\right), \end{aligned} \tag{10.17}$$

where $sp.ConX^{j}_{\varepsilon X}$ represent the spin connection coefficients and $\mathbf{A}^{[*]}$ are defined as before.

The spin–curvature tensors T_{XY} consisting of the arbitrary terms of $\Gamma^{(*)\mu}_{\nu\kappa}$, $C^{[*]\mu}_{\nu\gamma}$, $\overline{C}^{[*]\mu\gamma}_{\nu}$, $C^{[*]\mu}_{\nu\alpha}$ are defined as follows

$$\begin{aligned} T_{XY} &= \frac{\partial^{[*]} coefX_X}{\partial Y} - \frac{\partial^{[*]} coefY_Y}{\partial X} \\ &\quad + i\left(coefX_X \ coefY_Y - coefY_Y \ coefX_X\right) \\ &\quad - \left(\mathbf{A}^{[*]}_{\gamma XY} \ \overline{\beta}^{[*]\gamma}_{XY} + \widehat{\mathbf{A}}^{[*]\gamma}_{XY} \beta^{[*]\gamma}_{XY} + \breve{\mathbf{A}}^{[*]\tau}_{XY} b^{[*]}_{\tau}\right), \end{aligned} \tag{10.18}$$

where the $coefX_X$ are defined in Table 10.3. If we want to write down all ten spin–curvature tensors T_{XY} we must use the corresponding column in Table 10.2. These objects are defined uniquely on account of the conditions (3.27) or (3.28) of [123] and the homogeneity properties of $\Gamma^{(*)\mu}_{\nu\kappa}, C^{[*]\mu}_{\nu\gamma}, \overline{C}^{[*]\mu\gamma}_{\nu}, C^{[*]\mu}_{\nu\alpha}$. There are imposed on T_{XY} conditions similar to (10.16): that is contractions of $\overline{\xi}, \xi, y^{\lambda}$ with the spin–curvature T_{XY} results

$$\overline{\xi}^{\alpha}\Psi_{\lambda\alpha} = 0. \tag{10.19}$$

Now, from (10.13), (10.17) together with (10.16), (10.19), it is easily shown that the similar conditions to (10.14) on $T^{\mu}_{\nu XY}$ must be imposed on $T^{\delta}_{\varepsilon XY}$.

Chapter 11

Field Equations in Spinor Variables

11.1 Introduction

The introduction of a metric $g_{\mu\nu}(x, \omega)$ that depends on the position variables x as well as on the spinor variables ω assigns a non-Riemannian structure to the space and provides it with torsion. This procedure enables the construction of a non-localized (bi–local) gravitational field, identical to the one proposed by Yukawa [211] that allows a more general description of elementary particles. Further arguments have been developed by some other authors [78, 121, 152]. In our context each point of the space-time is characterized by the influence of two fields: an external one which is the conventional field in Einstein's sense, and an internal one due to the introduction of the spinor variables. These fields are expected to play the role of a geometrical unification of the fields. If ω is represented by a vector y, then we work in the Finslerian framework [14, 77, 109]. A more general case of the gauge approach in the framework of Finsler and Lagrange geometry has been studied e.g. in [15, 17, 28, 109, 112, 115].

In the following, we consider a space-time and we denote its metric tensor by

$$g_{\mu\nu}(Z^M),$$

(here $Z^M = (x^\mu, \xi_\alpha, \bar{\xi}^\alpha), x^\mu, \xi_\alpha, \bar{\xi}^\alpha$ represent the position and the 4-spinor variables $\bar{\xi}$ denotes the Dirac conjugate of ξ) [152]. With the Greek letters λ, μ, ν and α, β, γ we denote the space-time indices and the spinor indices, also Latin letters α, b, c are used for the Lorenz (flat) indices. The (*)-

differential operators $\partial^{(*)}_M$ are defined as

$$\partial^{(*)}_M = \frac{\partial^{(*)}}{\partial Z^M} = \left(\frac{\partial^{(*)}}{\partial x^\mu}, \frac{\partial^{(*)}}{\partial \xi_\alpha}, \frac{\partial^{(*)}}{\partial \bar{\xi}^\alpha}\right), \tag{11.1}$$

with

$$\begin{aligned}
\frac{\partial^{(*)}}{\partial x^\lambda} &= \frac{\partial}{\partial x^\lambda} + N_{\alpha\lambda}\frac{\partial}{\partial \xi^\alpha} + \overline{N}^\alpha_\lambda \frac{\partial}{\partial \bar{\xi}^\alpha}, \\
\frac{\partial^{(*)}}{\partial \xi_\alpha} &= \frac{\partial}{\partial \xi_\alpha} + \tilde{\eta}^{0\alpha}_\beta \frac{\partial}{\partial \xi_\alpha} + \tilde{\eta}^{\beta\alpha}_0 \frac{\partial}{\partial \bar{\xi}^\beta}, \\
\frac{\partial^{(*)}}{\partial \bar{\xi}^\alpha} &= \frac{\partial}{\partial \bar{\xi}^\alpha} + \eta^0_{\beta\alpha}\frac{\partial}{\partial \xi_\beta} + \eta^\beta_{0\alpha}\frac{\partial}{\partial \bar{\xi}^\beta},
\end{aligned}$$

here $N_{\alpha\lambda}, \overline{N}^\alpha_\lambda, \tilde{\eta}^{0\alpha}_\beta, \tilde{\beta\alpha}_0, \eta^0_{\beta\alpha}, \eta^\beta_{0\alpha}$ are the nonlinear connections [121].

In our study, field equations are obtained from a Lagrangian density of the form

$$L(\Psi^{(A)}, \partial^{(*)}_M \Psi^{(A)}), \tag{11.2}$$

here $\Psi^{(A)}$ is the set

$$\Psi^{(A)} = \{h^a_\mu(x,\xi,\bar{\xi}), \omega^{(*)ab}_\mu(x,\xi,\bar{\xi}), \theta^{(*)ab}_\alpha(x,\xi,\bar{\xi}), \bar{\theta}^{(*)ab\alpha}(x,\xi,\bar{\xi})\}.$$

Thus L is a function of the tetrad field, of the spin connection coefficients and of their (*)-derivatives. the variables $h, \omega^{(*)}, \theta^{(*)}, \bar{\theta}^{(*)}$ are considered as independent.

It is known that gravity can be described by the tetrad field and the Lorenz connection coefficients [134]. The variation of the Palatini action with respect to h and ω yields a set of two equations:

$$\begin{aligned}
R^a_\mu - \frac{1}{2}Rh^a_\mu &= 0 \qquad (a) \\
D_\mu[h(h^\nu_a h^\mu_b - h^\nu_b h^\mu_a)] &= 0 \qquad (b)
\end{aligned} \tag{11.3}$$

R^a_μ is the determinant of the tetrad h^a_μ and D_μ is the gauge covariant derivative

$$D_\mu = \partial_\mu + \sum \omega_\mu,$$

where the sum is taken over all Lorentz indices.

In spaces whose metric tensor depends on spinor variables, an analogous method can be applied, but instead of one connection we have three connections:

$$\omega_\mu^{(*)}(x,\xi,\bar{\xi}), \qquad \theta_a^{(*)}(x,\xi,\bar{\xi}), \qquad \bar{\theta}^{(*)\alpha}(x,\xi,\bar{\xi}).$$

So we choose a Lagrangian density of the form (11.2) from which four equations are obtained. The analogous gauge covariant derivatives of D_μ appear naturally as

$$\begin{aligned} a)\ D_\mu^{(*)} &= \partial_\mu^{(*)} + \sum \omega_\mu^{(*)}, \\ b)\ D_\alpha^{(*)} &= \partial_\alpha^{(*)} + \sum \theta_\alpha^{(*)}, \\ c)\ D^{(*)\alpha} &= \partial^{(*)\alpha} + \sum \bar{\theta}^{(*)\alpha}. \end{aligned} \qquad (11.4)$$

Transformation laws of the connection coefficients $\omega_{ab\lambda}^{(*)}(x,\xi,\bar{\xi}), \theta_{ab\alpha}^{(*)}(x,\xi,\bar{\xi})$ and $\bar{\theta}_{ab}^{(*)\alpha}(x,\xi,\bar{\xi})$ under local Lorenz transformations are the expected transformation laws for the gauge potentials [134]

$$\begin{aligned} a)\ \omega_{ab\lambda}^{(*)} &= L_a^c L_b^d \omega_{cd\lambda}^{(*)} + \frac{\partial^{(*)} L_\alpha^c}{\partial x^\lambda} L_{bc}, \\ b)\ \bar{\theta}_{ab}^{(*)'a} &= \left[L_a^c L_b^d \bar{\theta}_{cd}^{(*)\beta} + \frac{\partial^{(*)} L_\alpha^c}{\partial \xi_\beta} L_{bc} \right] (\Lambda^{-1})_\beta^\alpha, \\ c)\ \theta_{ab\alpha}^{(*)'} &= \Lambda_\alpha^\beta \left[L_a^c L_b^d \theta_{cd\beta}^{(*)} + \frac{\partial^{(*)} L_\alpha^c}{\partial \bar{\xi}_\beta} L_{bc} \right]. \end{aligned} \qquad (11.5)$$

The matrices L and Λ belong to the vector and spinor representations of the Lorentz group, respectively.

11.2 Derivation of the field equations

The field equations will be the Euler-Lagrange equations for a given Lagrangian. We postulate the explicit form of the Lagrangian density

$$L(\Psi^{(A)}, \partial_M^{(*)} \psi^{(A)}). \qquad (11.6)$$

But first we observe that the metric tensor $g_{\mu\nu}$ and the tetrad h_μ^a are related by (cf. [134])

$$a)\ g_{\mu\nu}(x,\xi,\bar{\xi}) = h_\mu^a h_\nu^b \eta_{ab},$$

$$b)\ g^{\mu\nu}(x,\xi,\bar{\xi}) = h^{\mu}_{a}h^{\nu}_{b}\eta^{ab}, \tag{11.7}$$

where η_{ab} is the Minkowski metric tensor and it is of the form $diag(+1,-1,-1,-1)$. From the relations [203]:

$$a)\quad g = -h^2, \qquad b)\quad dg = gg^{\mu\nu}dg_{\mu\nu}, \tag{11.8}$$

and using (11.7), we get

$$\frac{\partial h}{\partial h^{\mu}_{a}} = -\frac{1}{2}hh^{a}_{\mu}, \tag{11.9}$$

where g =det$(g_{\mu\nu})$.

Now we postulate the Lagrangian density in the form

$$L = h(R + P + Q + S), \tag{11.10}$$

where R, P, Q, S are the scalar curvatures obtained by contraction of the spin curvature tensors:

$$\begin{aligned} R &= h^{\mu}_{a}h^{\nu}_{b}R^{ab}_{\mu\nu},\ P = h^{\mu}_{a}h^{\nu}_{b}P^{a}_{c\mu\alpha}\overline{P}^{bc\alpha}_{\nu}, \\ Q &= Q_{ab\beta\alpha}\widetilde{Q}^{ab\beta\alpha},\ S = S^{\alpha}_{ab\beta}S^{ab\beta}_{\alpha}. \end{aligned} \tag{11.11}$$

The spin curvature tensors are given by the components

$$\begin{aligned} P^{ab}_{\lambda\alpha} &= \frac{\partial^{(*)}\omega^{(*)ab}_{\lambda}}{\partial\bar{\xi}^{\alpha}} - \frac{\partial^{(*)}\theta^{(*)ab}_{\alpha}}{\partial x^{\lambda}} \\ &+\omega^{(*)a}_{c\lambda}\theta^{(*)cb}_{\alpha} - \theta^{(*)a}_{c\alpha}\omega^{(*)cb}_{\lambda} - (\bar{\theta}^{ab\beta}E_{\beta\lambda\alpha} + F^{\beta}_{\lambda\alpha}\theta^{ab}_{\beta}), \end{aligned} \tag{11.12}$$

$$
\begin{aligned}
\overline{P}^{ab\alpha}_{\lambda} &= \frac{\partial^{(*)}\omega^{(*)ab}_{\lambda}}{\partial\xi_{\alpha}} - \frac{\partial^{(*)}\bar{\theta}^{(*)ab\alpha}}{\partial x^{\lambda}} \\
&\quad + \omega^{(*)a}_{c\lambda}\bar{\theta}^{(*)a\alpha b} - \bar{\theta}^{(*)a\alpha}_{c}\omega^{(*)cb}_{\lambda} - (\bar{\theta}^{ab\beta}\widetilde{F}^{\alpha}_{\beta\lambda} + \widetilde{E}^{\beta\alpha}_{\lambda}\theta^{ab}_{\beta}), \\
S^{ab\alpha}_{\beta} &= \frac{\partial^{(*)}\omega^{(*)ab}_{\beta}}{\partial\xi_{\alpha}} - \frac{\partial^{(*)}\bar{\theta}^{(*)ab\alpha}}{\partial\bar{\xi}^{\beta}} \\
&\quad + \theta^{(*)a}_{c\beta}\bar{\theta}^{(*)c\alpha b} - \bar{\theta}^{(*)a\alpha}_{c}\theta^{(*)cb}_{\beta} - (\theta^{ab}_{\gamma}G^{\alpha}_{\gamma\beta} + H^{\gamma\alpha}_{\beta}\theta^{ab}_{\gamma}), \\
R^{ab}_{\mu\nu} &= \frac{\partial^{(*)}\omega^{(*)ab}_{\mu}}{\partial x^{\nu}} - \frac{\partial^{(*)}\theta^{(*)ab}_{\nu}}{\partial x^{\mu}} \\
&\quad + \omega^{(*)ac}_{\mu}\omega^{(*)b}_{c\nu} - \omega^{(*)ac}_{\nu}\omega^{(*)b}_{c\mu} - (\bar{\theta}^{ab\beta}A_{\beta\mu\nu} + \overline{A}^{\beta}_{\mu\nu}\theta^{ab}_{\beta}), \\
\widetilde{Q}^{\beta\alpha}_{ab} &= \frac{\partial^{(*)}\bar{\theta}^{(*)\beta}_{ab}}{\partial\xi_{\alpha}} - \frac{\partial^{(*)}\bar{\theta}^{(*)\alpha}_{ab}}{\partial\xi_{\beta}} \\
&\quad + \bar{\theta}^{(*)\beta}_{ac}\theta^{(*)ca}_{b} - \bar{\theta}^{(*)\alpha}_{ac}\bar{\theta}^{(*)c\beta}_{b} - (\bar{\theta}^{\gamma}_{ab}\widetilde{K}^{\beta\alpha}_{\gamma} + \widetilde{J}^{\gamma\beta\alpha}\theta_{ab\gamma}), \\
Q^{ab}_{\beta\alpha} &= \frac{\partial^{(*)}\theta^{(*)ab}_{\beta}}{\partial\bar{\xi}^{\alpha}} - \frac{\partial^{(*)}\theta^{(*)ab}_{\alpha}}{\partial x^{\lambda}} \\
&\quad + \theta^{(*)a}_{c\beta}\theta^{(*)c}_{b\alpha} - \theta^{(*)a}_{c\alpha}\theta^{(*)cb}_{\beta} - (\theta^{ab\gamma}J_{\gamma\beta\alpha} + K^{\gamma}_{\beta\alpha}\theta^{ab}_{\gamma}),
\end{aligned}
$$

where the coefficients are defined

$$
\begin{aligned}
A_{\beta\mu\nu} &= \frac{\partial^{(*)}N_{\beta\mu}}{\partial x^{\nu}} - \frac{\partial^{(*)}N_{\beta\nu}}{\partial x^{\mu}}, \overline{A}^{\beta}_{\mu\nu} = \frac{\partial^{(*)}\overline{N}^{\beta}_{\mu}}{\partial x^{\nu}} - \frac{\partial^{(*)}\overline{N}^{\beta}_{\nu}}{\partial x^{\mu}}, \\
E_{\beta\lambda\alpha} &= \frac{\partial^{(*)}N_{\beta\lambda}}{\partial\bar{\xi}^{\alpha}} - \frac{\partial^{(*)}\eta^{0}_{\beta\alpha}}{\partial x^{\lambda}}, F^{\beta}_{\lambda\alpha} = \frac{\partial^{(*)}\overline{N}^{\beta}_{\lambda}}{\partial\bar{\xi}^{\alpha}} - \frac{\partial^{(*)}\eta^{\beta}_{0\alpha}}{\partial x^{\lambda}}, \\
\widetilde{F}^{\alpha}_{\beta\lambda} &= \frac{\partial^{(*)}N_{\beta\lambda}}{\partial\xi_{\alpha}} - \frac{\partial^{(*)}\tilde{\eta}^{0\alpha}_{\beta}}{\partial x^{\lambda}}, \widetilde{E}^{\beta\alpha}_{\lambda} = \frac{\partial^{(*)}\overline{N}^{\beta}_{\lambda}}{\partial\xi_{\alpha}} - \frac{\partial^{(*)}\tilde{\eta}^{\beta\alpha}_{0}}{\partial x^{\lambda}}, \\
G^{\alpha}_{\gamma\beta} &= \frac{\partial^{(*)}\eta^{0}_{\gamma\beta}}{\partial\xi_{\alpha}} - \frac{\partial^{(*)}\tilde{\eta}^{0\alpha}_{\gamma}}{\partial\bar{\xi}^{\beta}}, H^{\gamma\alpha}_{\beta} = \frac{\partial^{(*)}\eta^{\gamma}_{0\beta}}{\partial\xi_{\alpha}} - \frac{\partial^{(*)}\tilde{\eta}^{\gamma\alpha}_{0}}{\partial\bar{\xi}^{\beta}}, \\
J_{\gamma\beta\alpha} &= \frac{\partial^{(*)}\eta^{0}_{\gamma\beta}}{\partial\bar{\xi}^{\alpha}} - \frac{\partial^{(*)}\eta^{0}_{\gamma\alpha}}{\partial\bar{\xi}^{\beta}}, K^{\gamma}_{\beta\alpha} = \frac{\partial^{(*)}\eta^{\gamma}_{0\beta}}{\partial\bar{\xi}^{\alpha}} - \frac{\partial^{(*)}\eta^{\gamma}_{0\alpha}}{\partial\bar{\xi}^{\beta}}, \\
\widetilde{K}^{\beta\alpha}_{\gamma} &= \frac{\partial^{(*)}\eta^{0\beta}_{\gamma}}{\partial\xi_{\alpha}} - \frac{\partial^{(*)}\tilde{\eta}^{0\alpha}_{\gamma}}{\partial\xi_{\beta}}, \widetilde{J}^{\gamma\beta\alpha} = \frac{\partial^{(*)}\tilde{\eta}^{\gamma\beta}_{0}}{\partial\xi_{\alpha}} - \frac{\partial^{(*)}\tilde{\eta}^{\gamma\alpha}_{0}}{\partial\xi_{\beta}}.
\end{aligned}
$$

The Lagrangian (11.10) is the only possible scalar that can be made from the curvature tensors (11.12) and it must be the sum of the first-order quantity R and the second–order quantities P, Q and S. The mixing of the quantities of different order is not impossible. It is known that the Einstein-

Maxwell Lagrangian is the sum of the first-order quantity R and the second-order quantity $F_{\mu\nu}F^{\mu\nu}$. So, our Lagrangian (11.10) is physically acceptable.

The Euler–Lagrange equations for the objects

$$\Psi^{(A)} = \{h^\mu, \omega_\mu^{(*)}, \theta_\alpha^{(*)}, \bar{\theta}^{(*)\alpha}\}$$

are of the form

$$\partial_M^{(*)}\left(\frac{\partial L}{\partial(\partial_M^{(*)}\Psi^{(A)})}\right) - \frac{\partial L}{\partial \Psi^{(A)}} = 0, \tag{11.13}$$

where $\partial_M^{(*)}$ was defined in (11.1). From the variation of L with respect to the tetrad we have

$$\frac{\partial L}{\partial h_b^\nu} = 0. \tag{11.14}$$

Taking into account (11.8), and (11.9) we get the equation

$$H_\nu^b - \frac{1}{2}h_\nu^b = 0, \tag{11.15}$$

where

$$H_\nu^b = R_\nu^b + P_\nu^b = h_a^\mu R_{\mu\nu}^{ab} + h_a^\mu P_{c\mu\alpha}^a \overline{P}_\nu^{bc\alpha}, \tag{11.16}$$

and

$$H = R + P. \tag{11.17}$$

From the variation of L with respect to $\omega_\mu^{(*)}$ we get

$$\partial_\mu^{(*)} \frac{\partial L}{\partial(\partial_\mu^{(*)}\omega_\nu^{(*)ab})} + \partial^{(*)\alpha} \frac{\partial L}{\partial(\partial^{(*)\alpha}\omega_\nu^{(*)ab})} + \partial_\alpha^{(*)} \frac{\partial L}{\partial(\partial_\alpha^{(*)}\omega_\nu^{(*)ab})} - \frac{\partial L}{\partial \omega_\nu^{(*)ab}} = 0. \tag{11.18}$$

The spin-connection coefficients $\omega_\nu^{(*)ab}$ are contained in R and P:

$$h(R+P) = hh_a^\mu h_b^\nu (R_{\mu\nu}^{ab} + P_{c\mu\alpha}^a \overline{P}_\nu^{bc\alpha}).$$

From relation (11.18) we get the following variation of the term hR with respect to $\omega_\mu^{(*)}$:

$$\partial_\mu^{(*)} \frac{\partial(hR)}{\partial(\partial_\mu^{(*)}\omega_\nu^{(*)ab})} + \partial^{(*)\alpha} \frac{\partial(hR)}{\partial(\partial^{(*)\alpha}\omega_\nu^{(*)ab})} + \partial_\alpha^{(*)} \frac{\partial(hR)}{\partial(\partial_\alpha^{(*)}\omega_\nu^{(*)ab})} - \frac{\partial(hR)}{\partial \omega_\nu^{(*)ab}}. \tag{11.19}$$

By a direct calculation, the first term of (11.19) can be written as

$$\partial^{(*)}_\mu [h(h^\nu_a h^\mu_b - h^\mu_a h^\nu_b)].$$

The second and the third terms of (11.19) are equal to zero. The fourth term equals

$$h(h^\nu_c h^\mu_b - h^\nu_b h^\mu_c)\omega^{(*)c}_{a\mu} + h(h^\nu_c h^\mu_a - h^\nu_a h^\mu_c)\omega^{(*)c}_{b\mu}. \tag{11.20}$$

Consequently, the first and the fourth terms can be rewritten as

$$D^{(*)}_\mu [h(h^\nu_a h^\mu_b - h^\nu_b h^\mu_a)], \tag{11.21}$$

where we have used the gauge covariant derivative $D^{(*)}_\mu$ from (11.4). Contribution from the P-part is equal to

$$\partial^{(*)}_\mu \frac{\partial(hP)}{\partial(\partial^{(*)}_\mu \omega^{(*)ab}_\nu)} + \partial^{(*)\alpha} \frac{\partial(hP)}{\partial(\partial^{(*)\alpha}\omega^{(*)ab}_\nu)} + \partial^{(*)}_\alpha \frac{\partial(hP)}{\partial(\partial^{(*)}_\alpha \omega^{(*)ab}_\nu)} - \frac{\partial(hP)}{\partial\omega^{(*)ab}_\nu}. \tag{11.22}$$

The first term of (11.22) is equal to zero.The second and the third terms can be written as

$$\partial^{(*)\alpha}(hh^\mu_c h^\nu_a P^c_{b\mu\alpha}), \tag{11.23}$$

$$\partial^{(*)}_\alpha (hh^\mu_c h^\nu_a \overline{P}^{c\alpha}_{b\mu}), \tag{11.24}$$

respectively. The fourth term may be written as

$$hh^\nu_a h^\mu_l \theta^{(*)}_{bc\alpha}\overline{P}^{cl\alpha}_\mu - hh^\nu_l h^\mu_k \theta^{(*)l}_{a\alpha}\overline{P}^{k\alpha}_{b\mu} - hh^\mu_l h^\nu_a P^l_{c\mu\alpha}\bar{\theta}^{(*)c\alpha}_b - hh^\nu_k h^\mu_l P^l_{b\mu\alpha}\bar{\theta}^{(*)k\alpha}_a. \tag{11.25}$$

The sum of (11.23), (11.24) and (11.25) is equal to

$$D^{(*)}_\alpha (hh^\nu_a h^\mu_l \overline{P}^{l\alpha}_{b\mu}) + D^{(*)\alpha}(hh^\nu_a h^\mu_l P^{l\alpha}_{b\mu}). \tag{11.26}$$

So, (11.22) is written in the form

$$D^{(*)}_\mu [h(h^\nu_a h^\mu_b - h^\nu_b h^\mu_a)] + D^{(*)}_\alpha (hh^\nu_a h^\mu_l \overline{P}^{l\alpha}_{b\mu}) + D^{(*)\alpha}(hh^\nu_a h^\mu_l \overline{P}^{l\alpha}_{b\mu}) = 0. \tag{11.27}$$

Taking the variation of L with respect to $\theta^{(*)}_\alpha$ we have contributions from $(P + Q + S)$. The field equation is

$$\partial^{(*)}_\mu \frac{\partial(hL)}{\partial(\partial^{(*)}_\mu \theta^{(*)ab}_\nu)} + \partial^{(*)\alpha} \frac{\partial(hL)}{\partial(\partial^{(*)\alpha}\theta^{(*)ab}_\nu)} + \partial^{(*)}_\alpha \frac{\partial(hL)}{\partial(\partial^{(*)}_\alpha \theta^{(*)ab}_\nu)} - \frac{\partial(hL)}{\partial\theta^{(*)ab}_\nu} = 0. \tag{11.28}$$

We proceed in the same way as before. The contribution from the hP term is

$$-D^{(*)}_{\mu}(hh^{\mu}_{a}h^{\nu}_{c}\overline{P}^{ca}_{b\nu}). \tag{11.29}$$

The contribution from the hQ term gives

$$D^{(*)}_{\beta}(2h\widetilde{Q}^{[\alpha\beta]}_{ab}). \tag{11.30}$$

Similarly, the hS term yields

$$2D^{(*)\beta}(hS^{\alpha}_{ab\beta}). \tag{11.31}$$

So, the third equation is written in the form

$$D^{(*)}_{\mu}(hh^{\mu}_{a}h^{\nu}_{c}\overline{P}^{ca}_{b\nu}) - D^{(*)}_{\beta}(2h\widetilde{Q}^{[\alpha\beta]}_{ab}) - 2D^{(*)\beta}(hS^{\alpha}_{ab\beta}) = 0. \tag{11.32}$$

Finally, the variation with respect to $\theta^{(*)\alpha}$ yields the equation "conjugate" to (11.32)

$$D^{(*)}_{\mu}(hh^{\mu}_{c}h^{\nu}_{a}P^{c}_{b\nu\alpha}) - D^{(*)\beta}(2hQ_{ab[\alpha\beta]}) - 2D^{(*)}_{\beta}(hS^{\beta}_{ab\alpha}) = 0. \tag{11.33}$$

11.3 Generalized Conformally Flat Spaces

In this Section we study the form of the spin-connection coefficients, spin-curvature tensors, and the field equations for generalized conformally flat spaces (GCFS) $(M, g_{\mu\nu}(x,\xi,\bar{\xi}) = e^{2\sigma(x,\xi,\bar{\xi})}\eta_{\mu\nu})$, where $\eta_{\mu\nu}$ represents the Lorenz metric tensor $\eta_{\mu\nu}$ =diag$(+,-,-,-)$, and $\xi,\bar{\xi}$ are internal variables. The case of conformally related metrics of the Riemannian and the generalized Lagrange spaces has been extensively studied in [112, 115]. It is remarkable that in the above mentioned GCFS spaces, some spin–connection and spin-curvature tensors vanish.

As pointed out in [121], the absolute differential DV^{μ} of a vector field $V^{\mu}(x,\xi.\bar{\xi})$ is expressed in terms of the coefficients

$$\{\Gamma^{\mu}_{\nu\lambda}, \overline{C}^{\mu\alpha}_{\nu}, C^{\mu}_{\nu\alpha}\}. \tag{11.34}$$

Considering the absolute differentials of the spinor variables $\xi_{\alpha},\bar{\xi}^{\alpha}$:

$$D\xi_{\alpha} = d\xi_{\alpha} - N_{\alpha\lambda}dx^{\lambda} - \tilde{\eta}^{0\beta}_{\alpha}D\xi_{\beta} - D\bar{\xi}^{\beta}\eta^{0}_{\alpha\beta},$$
$$D\bar{\xi}^{\alpha} = d\bar{\xi}^{\alpha} - \overline{N}^{\alpha}_{\lambda}dx^{\lambda} - \tilde{\eta}^{\alpha\beta}_{0}D\xi_{\beta} - D\bar{\xi}^{\beta}\eta^{\alpha}_{0\beta},$$

which depend on the nonlinear connections:

$$\{N_{\alpha\lambda}, \overline{N}^{\alpha}_{\lambda}, \tilde{\eta}^{0\beta}_{\alpha}, \tilde{\eta}^{\alpha\beta}_{0}\eta^{0}_{\alpha\beta}, \eta^{\alpha}_{0\beta}\}, \tag{11.35}$$

and expressing DV^{μ} in terms of dx^{λ},$D\xi_{\alpha}$,$D\bar{\xi}^{\alpha}$, we obtain the connection coefficients

$$\{\Gamma^{(*)\mu}_{\nu\alpha}, \overline{C}^{(*)\mu\alpha}_{\nu}, C^{(*)\mu}_{\nu\alpha}\} \tag{11.36}$$

related to the coefficients (11.34) via the non-linear connections (11.35) [121].

By imposing the postulates of the length preservation for the parallel vector fields and symmetry of the derived coefficients

$$\{\Gamma^{(*)}_{\nu\mu\lambda}, \overline{C}^{\alpha}_{\nu\mu}, C_{\nu\mu\alpha}\} \tag{11.37}$$

in the first two tensor indices, we have the relations:

$$\begin{gathered}
\Gamma^{(*)}_{\nu\mu\lambda} = \frac{1}{2}\left(\frac{\partial^{(*)} g_{\mu\nu}}{\partial x^{\lambda}} - \frac{\partial^{(*)} g_{\nu\lambda}}{\partial x^{\mu}}\right), \\
\overline{C}^{\alpha}_{\nu\mu} = \frac{1}{2}\frac{\partial g_{\nu\mu}}{\partial \xi_{\alpha}}, \qquad C_{\nu\mu\alpha} = \frac{1}{2}\frac{\partial g_{\nu\mu}}{\partial \bar{\xi}^{\alpha}},
\end{gathered} \tag{11.38}$$

where $\tau_{\{\mu\nu\}} = \tau_{\mu\nu} + \tau_{\nu\mu}$.

Theorem 11.1. *For the GCFS spaces we infer the following:*
(a) The coefficients (11.37) have the explicit form

$$\Gamma^{*}_{\nu\mu\lambda} = e^{2\sigma}(\eta_{\mu}\{_{\nu}\sigma^{*}_{\lambda}\} - \eta_{\nu\lambda}\sigma^{*}_{\mu}), \quad \overline{C}^{\mu\alpha}_{\nu} = \delta^{\mu}_{\nu}\sigma^{\alpha}, \quad C^{\mu}_{\nu\alpha} = \bar{\sigma}_{\alpha}\delta^{\mu}_{\nu}, \tag{11.39}$$

where $\sigma^{\alpha} = \partial\sigma/\partial\xi_{\alpha}, \bar{\sigma}_{\alpha} = \partial\sigma/\partial\bar{\xi}^{\alpha}, \sigma^{*}_{\lambda} = \partial^{*}\sigma/\partial x^{\lambda}$ *are the derivation operators of scalar fields involving the coefficients (11.35).*
(b) The following relations hold:

$$\begin{aligned}
\Gamma^{\mu}_{\nu\lambda} &= \Gamma^{*\mu}_{\nu\lambda} - \delta^{\mu}_{\nu}\sigma^{\alpha}N_{\alpha\lambda} - \delta^{\mu}_{\nu}\bar{\sigma}_{\alpha}\overline{N}^{\alpha}_{\lambda}, \\
\overline{C}^{*\mu\alpha}_{\nu} &= \overline{C}^{\mu\alpha}_{\nu} + \delta^{\mu}_{\nu}\sigma^{\beta}\tilde{\eta}^{0\alpha}_{\beta} + \delta^{\mu}_{\nu}\bar{\sigma}_{\beta}\tilde{\eta}^{\beta\alpha}_{0}, \\
C^{*\mu}_{\nu\alpha} &= C^{\mu}_{\nu\alpha} + \delta^{\mu}_{\nu}\sigma^{\beta}\eta^{0}_{\beta\alpha} + \delta^{\mu}_{\nu}\eta^{\beta}_{0\alpha}.
\end{aligned} \tag{11.40}$$

Proof. Computational, using the consequences (11.38) of the above postulates and identifying the absolute differentials expressed in terms of (11.34) and (11.36) . □

Considering the absolute differentials of a Dirac spinor field $\psi(x, \xi, \bar{\xi})$ and of its adjoint $\bar{\psi}(x, \xi, \bar{\xi})$ we have the coefficients

$$\{\Gamma^{\beta}_{\gamma\lambda}, \widetilde{C}^{\beta\alpha}_{\gamma}, C^{\beta}_{\gamma\alpha}\}. \tag{11.41}$$

Expressing $D\psi$ and $D\bar{\psi}$ in terms of $dx^\lambda, D\xi_\alpha, D\bar{\xi}^\alpha$, we are led to the spin-connection coefficients I:

$$\{\Gamma^{*\beta}_{\gamma\lambda}, \widetilde{C}^{*\beta\alpha}_{\gamma}, C^{*\beta}_{\gamma\alpha}\} \tag{11.42}$$

connected to (11.41) [121]. In a similar manner, the absolute differential of a Lorenz vector $V^a(x, \xi, \bar{\xi})$ produces the coefficients

$$\{\omega_{ba\lambda}, \bar{\theta}^\alpha_{ba}, \theta_{ba\alpha}\}, \tag{11.43}$$

where the raising and lowering of the indices $a, b, \ldots = 1, \ldots, 4$ are performed by means of η_{ab}, and also the spin-connection coefficients II:

$$\{\omega^*_{ba\lambda}, \bar{\theta}^{*\alpha}_{ba}, \theta^*_{ba\alpha}\} \tag{11.44}$$

related to the coefficients (11.43) and (11.36) [121]. Similarly to the previous work of Takano and Ono [121], we shall postulate the invariance of length of the parallel Lorentz vector fields, and the vanishing of the absolute differentials and covariant derivatives of the tetrads h^μ_α,which involve the connection coefficients (11.36) and (11.44).

In the GCFS, the tetrads are given by $h^a_\mu(x, \xi, \bar{\xi}) = e^{\sigma(x,\xi,\bar{\xi})}\delta^a_\mu$ and lead to the dual entities $h^\mu_a(x, \xi, \bar{\xi}) = e^{-\sigma(x,\xi,\bar{\xi})}\delta^\mu_a$. In general, the above postulates produce the relations:

$$\begin{aligned} \omega_{ab\lambda} &= \left(\frac{\partial h^\mu_a}{\partial x^\lambda} + \Gamma^\mu_{\nu\lambda} h^\nu_a\right) h_{\mu b}, \\ \bar{\theta}^\alpha_{ab} &= \left(\frac{\partial h^\mu_a}{\partial \xi_\alpha} + \overline{C}^{\mu\alpha}_{\nu} h^\nu_a\right) h_{\mu b}, \\ \theta_{ab\alpha} &= \left(\frac{\partial h^\mu_a}{\partial \bar{\xi}^\alpha} + C^\mu_{\nu\alpha} h^\nu_a\right) h_{\mu b}, \end{aligned} \tag{11.45}$$

$$\omega^*_{ab\lambda} = \left(\frac{\partial^* h^\mu_a}{\partial x^\lambda} + \Gamma^{*\mu}_{\nu\lambda} h^\nu_a\right) h_{\mu b}. \tag{11.46}$$

For the GCFS case we are led to

Theorem 11.2. *The spin–connection coefficients (II) and the coefficients (11.43) are subject to*

$$\omega_{ba\lambda} = h_{\mu a}\Gamma^\mu_{b\lambda} - \sigma_\lambda \eta_{ba}, \quad \bar{\theta}^\alpha_{ab} = 0, \quad \theta_{ab\alpha} = 0, \tag{11.47}$$

$$\omega^*_{ba\lambda} = \eta_{\lambda(a}\sigma^*_{b)}, \quad \bar{\theta}^{*\alpha}_{ab} = 0, \quad \theta^*_{ab\alpha} = 0, \tag{11.48}$$

$$\omega^*_{ba\lambda} = \omega_{ba\lambda}, \tag{11.49}$$

where $h_{\mu a} = e^\sigma \eta_{\mu a}$ and $T_{(ab)} = T_{ab} - T_{ba}$.

Proof. Relations (11.45) imply (11.47) ; (11.46) and

$$\omega^*_{ba\lambda} = \omega_{ba\lambda} + \theta^{\beta}_{ba} N_{\beta\lambda} + \overline{N}^{\beta}_{\lambda}\theta_{ba\beta}$$

produce (11.49) and

$$\bar{\theta}^{*\alpha}_{ba} = \bar{\theta}^{\alpha}_{ba} + \bar{\theta}^{\beta}_{ba}\tilde{\eta}^{0\alpha}_{\beta} + \tilde{\eta}^{\beta\alpha}_{0}\bar{\theta}_{ba\beta}, \quad \theta^*_{ba\alpha} = \theta_{ba\alpha} + \bar{\theta}^{\beta}_{ba}\eta^{0}_{\beta\alpha} + \eta^{\beta}_{0\alpha}\theta_{ba\beta}.$$

So, we infer (11.47) and (11.48) . □

The connections (11.36) and (11.42) give rise to 8 curvature tensors as described in (5.2) of [121]. But also the spin-connections (II) connected to (11.36) lead to six spin-curvature tensors (11.12)

$$\{R_{ab\lambda\mu}, P_{ab\lambda\alpha}, \overline{P}^{\alpha}_{ab\lambda}, S^{\alpha}_{ab\beta}, Q_{ab\beta\alpha}, \widetilde{Q}^{\beta\alpha}_{ab}\}. \tag{11.50}$$

Taking into account Theorems 11.1 and 11.2 we can express these tensors as follows.

Theorem 11.3. *In the GCFS spaces the spin-curvature tensors are given by*

$$R_{ab\lambda\mu} = \eta_{\lambda(b}\sigma^*_{\mu a)} + \eta_{\mu(a}\sigma^*_{\lambda b)} + \eta_{\mu(b}\sigma^*_{\lambda}\sigma^*_{a)} \tag{11.51}$$
$$+\eta_{\lambda(a}\sigma^*_{\mu}\sigma^*_{b)} + \eta_{(\mu a}\eta_{\lambda)b}\eta^{cd}\sigma^*_{c}\sigma^*_{d},$$
$$P_{ab\lambda\alpha} = \eta_{\lambda(b}\sigma^*_{\alpha a)}, \qquad \overline{P}^{\alpha}_{ab\lambda} = \eta_{\lambda(b}\sigma^{*\alpha}_{a)},$$
$$S^{\alpha}_{ab\beta} = 0, \quad Q_{ab\beta\alpha} = 0, \quad \widetilde{Q}^{\beta\alpha}_{ab} = 0, \tag{11.52}$$

where $\sigma^*_{xy} = \partial^{*2}\sigma/\partial x^x \partial x^y$; $x, y = \{\lambda, \alpha, a\}$ *and* $\eta_{\lambda(b}\sigma^*_{\mu a)} = \eta_{\lambda b}\sigma^{*\alpha}_{\mu a} - \eta_{\lambda a}\sigma^*_{\mu b}$.

Proof. Relations (11.52) are directly implied by (11.48) and (11.49). (11.39) leads to (11.51) after a straightforward calculation. Also, using Theorem 11.2, we infer that

$$P_{ab\lambda\alpha} = \omega^*_{ab\lambda,\alpha}, \qquad \overline{P}_{ab\lambda\alpha} = \omega^*_{ab\lambda,\alpha}, \tag{11.53}$$

where

$$\omega^*_{ab\lambda,\alpha} = \frac{\partial^*\omega_{ab\lambda}}{\partial\bar{\xi}^{\alpha}}, \qquad \omega^*_{ab\lambda,\alpha} = \frac{\partial^*\omega_{ab\lambda}}{\partial\xi_{\alpha}}.$$

Then (11.54) leads to (11.51) and (11.52). □

Relations (11.52) are directly implied by (11.47)–(11.49). The relations (11.39) leads to (11.51) after a straightforward calculations. Also, using Theorem 11.3, we infer that

$$P_{ab\lambda\alpha} = \frac{\partial^* \omega_{ab\lambda}}{\partial \overline{\xi}^{\alpha}}, \quad \overline{P}_{ab\lambda\alpha} = \frac{\partial^* \omega_{ab\lambda}}{\partial \xi_{\alpha}}. \tag{11.54}$$

Then (11.48) leads to (11.51) and (11.52).

As a consequence of this theorem we state the following

Corollary 11.1. *In the GCFS space $(M, g_{\mu\nu})$ the Ricci tensor fields have the form*

$$\begin{aligned} R^d_\mu &= e^{-\sigma}(2\eta^{bd}\sigma^*_\mu\sigma^*_b - 2\eta^{bd}\sigma^*_{\mu b} - \delta^d_\mu\eta^{a\lambda}\sigma^*_{\lambda a} - 2\delta^d_\mu\eta^{ef}\sigma^*_c\sigma^*_f), \\ P^b_\nu &= -3e^{-\sigma}(\eta^{bc}\sigma^*_{\alpha c}\sigma^{*\alpha}_\nu - \sigma^*_{\alpha\nu}\sigma^{*\alpha}_e\eta^{eb}). \end{aligned} \tag{11.55}$$

Proof. Using Theorem 11.3 $R^d_\mu = h^\lambda_c R^{cd}_{\lambda\mu}, P^b_\nu = h^\mu_a P^a_{c\mu\alpha}\overline{P}^{bc\alpha}_\nu$ we obtain relations (11.55) . □

Remark (1) It follows that the scalar curvature takes the form

$$R = R^d_\mu h^\mu_d = -6e^{-2\sigma}(\eta^{bd}\sigma^*_{db} + \eta^{ef}\sigma^*_e\sigma^*_f). \tag{11.56}$$

Furthermore, it can be easily seen that

$$P = P^b_\nu h^\nu_b \equiv 0. \tag{11.57}$$

As we have previously remarked, the scalar curvature fields

$$Q = Q_{ab\beta\alpha}\widetilde{Q}^{ab\beta\alpha} \text{ and } S = S_{ab\alpha\beta}S^{ab\alpha\beta}$$

vanish identically. Then the employed Lagrangian density (11.10)

$$L = h(R + P + Q + S), \qquad det(g_{\mu\nu}) = -h^2,$$

reduces to $L = e^\sigma(R + P)$ and depends on the fields
$\varphi \in \{h^b_\nu, \omega^*_{ab\lambda}, \theta^*_{ab\alpha}, \bar{\theta}^{*\alpha}_{ab}\}$. The Euler-Lagrange equations

$$\partial^*_M\left(\frac{\partial L}{\partial(\partial^*_M\varphi)}\right) - \frac{\partial L}{\partial\varphi} = 0 \tag{11.58}$$

for these fields produce the field equations (11.15), (11.27), (11.32) and (11.33).

We shall obtain their form for the GCFS as follows.

Theorem 11.4. *The field equations for the GCFS are*

$$\begin{aligned} \delta^d_\mu \eta^{ef}(2\sigma^*_{ef} - \sigma^*_e\sigma^*_f) + 2\eta^{bd}(\sigma^*_{\mu b} - \sigma^*_\mu\sigma^*_b) & \\ +3\eta^{ed}\sigma^*_{\alpha\mu}\sigma^{*\alpha}_e - 3\eta^{dc}\sigma^*_{\alpha c}\sigma^{*\alpha}_\mu &= 0, \qquad (F1) \\ \sigma^*_{(b}\delta^\nu_{a)} - 3\sigma^*_\alpha\sigma^{*\alpha}_{(b}\delta^\nu_{a)} - 3\delta^\nu_a\sigma^{*\alpha}_{\alpha b} &= 0, \qquad (F2) \\ 2\sigma^*_a\sigma^*_{\alpha\alpha\beta} &= 0, \qquad (F3) \\ 2\sigma^*_\mu\eta^{\mu d}\eta_{\alpha b}\sigma^*_{\alpha d} - 2\sigma^*_a\sigma^*_{\alpha b} + \eta^{\mu d}\eta_{ab}\sigma^*_{\mu\alpha d} - \sigma^*_{a\alpha b} &= 0, \qquad (F4) \end{aligned}$$

where we have put $\sigma^{*\beta}_{\alpha b} = \partial^{*3}\sigma/\partial\xi_\beta\partial\bar{\xi}^\alpha\partial x^b$.

Proof. By virtue of relations (11.15) and (11.16), and using Corollary 11.1 and Remark (1), we get (F1).

Considering Theorem 11.2 we infer that $D^*_\alpha = \partial^*_\alpha$ and $D^{*\alpha} = \partial^{*\alpha}$. Also from $\omega^{(*)}_{\alpha b\lambda} = \omega_{\alpha b\lambda} = -\omega_{ba\lambda}$, we derive $D^*_\mu = \partial^*_\mu$. Taking into account (11.27) , we obtain relation (F2) by a straightforward computation. Also, by means of Theorem 11.3 and noticing that $\overline{P}^{\mu\alpha}_{b\mu} = -3\sigma^{*\alpha}_b$, after substituting to (11.32), we infer (F3). Finally, from (11.33) we derive (F4). □

11.4 Geodesics and geodesic deviation

We shall now give the form of geodesics in spaces with the $g_{\mu\nu}(x, \xi, \bar{\xi})$ metric.

A curve c in a space $(M, g_{\mu\nu}(x, \xi, \bar{\xi}))$ is defined as a smooth mapping $c : I \to U \subset M : t \to (x(t), \xi(t), \bar{\xi}(t))$, where U is an open set of M and t is an arbitrary parameter.

Definition 11.1. *A curve c is a geodesic if it satisfies the set of equations:*

$$\begin{aligned} \frac{D\dot{x}^\mu}{ds} &\equiv \frac{d^2x^\mu}{ds^2} + \dot{x}^\nu(\Gamma^\mu_{\nu\lambda x}\dot{x}^\lambda + \overline{C}^{\mu\alpha}_\nu\xi_\alpha + C^\mu_{\nu\alpha}\bar{\xi}^\alpha) = 0, \qquad (a) \\ \frac{D^2\xi_\alpha}{ds^2} &\equiv \frac{D}{ds}[\dot{\xi}_\alpha - \xi_\gamma(\Gamma^\gamma_{\alpha\lambda}\dot{x}^\lambda + \widetilde{C}^{\gamma\beta}_\alpha\dot{\xi}_\beta + C^\gamma_{\alpha\beta}\bar{\xi}^\beta)] = 0, \qquad (b) \\ \frac{D^2\bar{\xi}^\alpha}{ds^2} &\equiv \frac{D}{ds}[\bar{\xi}^\alpha + \bar{\xi}^\gamma(\Gamma^\alpha_{\gamma\lambda}\dot{x}^\lambda + \widetilde{C}^{\alpha\beta}_\gamma\dot{\xi}_\beta + C^\alpha_{\gamma\beta}\bar{\xi}^\beta)] = 0, \qquad (c) \end{aligned} \tag{11.59}$$

where $\dot{x}^\mu = dx^\mu/ds, \dot{\xi}_\alpha = d\xi_\alpha/ds, \bar{\xi}^\alpha = d\bar{\xi}^\alpha/ds$, and the coefficients $\Gamma^\mu_{\nu\lambda}$, $\Gamma^\gamma_{\alpha\lambda}$, $\overline{C}^{\mu\alpha}_\nu$, $\widetilde{C}^{\alpha\beta}_\gamma, C^\mu_{\nu\alpha}, C^\gamma_{\alpha\beta}$ satisfy the postulates imposed by Y. Takano and T. Ono [121].

Proposition 11.1. *(a) If $\overline{C}^{\mu\alpha}_\nu = 0$ and $C^\mu_{\nu\alpha} = 0$, then $\Gamma^\mu_{\nu\lambda} = \Gamma^\mu_{\lambda\nu}$ and relation (11.59) becomes*

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\lambda}(x, \xi(x), \bar{\xi}(x))\frac{dx^\nu}{ds}\frac{dx^\lambda}{ds} = 0. \tag{11.60}$$

(b) For the GCFS, equation (11.59) has the form

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda + \dot{x}^\mu(\sigma^\alpha \dot{xi}_\alpha + \bar{\sigma}_\alpha\bar{\xi}^\alpha) = 0. \tag{11.61}$$

In this case $\overline{C}^{\mu\alpha}_\nu = 0, C^\mu_{\nu\alpha} = 0$ *hold true iff* $\sigma^\alpha = \bar{\sigma}_\alpha = 0$, *i.e., for* σ *depending only on* x.

Proof. Equations (11.60) and (11.61) are consequences of Definition (11.59) (a) and relations (11.39) . □

Remark (2): The spinor parts of equations (11.60) and (11.61) also write as

$$\begin{aligned} &\ddot{\xi}_\alpha - \dot{\xi}_\gamma T^\gamma_\alpha - \xi_\gamma \dot{T}^\gamma_\alpha - (\dot{\xi}_\gamma - \xi_\delta T^\delta_\gamma)T^\gamma_\alpha = 0, \\ &\bar{\xi}^\alpha + \bar{\xi}^\alpha \overline{T}^\gamma_\alpha + \bar{\xi}^\gamma \overline{T}^\alpha_\gamma + (\bar{\xi}^\gamma + \bar{\xi}^\delta \overline{T}^\gamma_\delta)T^\alpha_\gamma = 0, \end{aligned} \tag{11.62}$$

where

$$T^\gamma_\alpha \equiv \Gamma^\gamma_{\alpha\lambda}\dot{x}^\lambda + \widetilde{C}^{\gamma\beta}_\alpha \dot{\xi}_\beta + C^\gamma_{\alpha\beta}\bar{\xi}^\beta = \overline{T}^\gamma_\alpha.$$

Having the equations of geodesics, it remains to derive the equations of geodesic deviation of our spaces. This geodesic deviation can be given a physical meaning if we consider two very close geodesic curves and the curvature tensor is Riemannian.

In the general case of GCFS, the spinor variables are independent of the position, so it is difficult to convey a physical meaning to the equations of geodesic deviation. For this reason it is convenient to study the deviation of the geodesics in the case where the spinor field $\xi_\alpha = \xi_\alpha(x^\mu)$(and $\bar{\xi}^\alpha = \bar{\xi}^\alpha(x^\mu)$) is defined on the manifold. This spinor field associates a spinor -and its conjugate-to every point of the space-time.

In this case, from Proposition (1) and relation (11.60) the Christoffel symbols $\Gamma^\mu_{\nu\lambda}$ are symmetric in the lower indices and the equation of geodesics is similar to the Riemannian one, except that the connection coefficients have the additional dependence on the spinors $\xi_\alpha(x^\mu), \bar{\xi}^\alpha(x^\mu)$. Thus our approach is more general. The equation of geodesic deviation in our case is given by

$$\frac{D^2\zeta^\lambda}{ds^2} + R^\lambda_{\mu\nu\varrho}\frac{dx^\mu}{ds}\frac{d\zeta^\nu}{ds}\frac{dx^\varrho}{ds} = 0. \tag{11.63}$$

The above curvature tensor $R^\lambda_{\mu\nu\varrho}(x, \xi(x), \bar{\xi}(x))$ has a modified Riemannian form. This equation has additional contributions from the spinor parts which enter the curvature tensor R and the covariant derivative. In (11.63) ζ^μ denotes the deviation vector, and s the arc length.

For the GCFS, the deviation equation has the above form, where the curvature tensor depends on the function $\sigma(x, \xi(x), \bar{\xi}(x))$ and its derivatives, as we have proved in Theorem 11.3, relation (11.51). After a direct calculation from (11.63) and

$$R^{\lambda}_{\mu\nu\varrho} = R_{ab\mu\varrho} h^{b\lambda} h^{a}_{\nu}, \tag{11.64}$$

where $h^{b\lambda} = e^{-\sigma}\eta^{b\lambda}, h^{a}_{\nu} = e^{\sigma}\delta^{a}_{\nu}$, we get the equation of geodesic deviation for the GCFS, with $\overline{C}^{\mu\alpha}_{\nu} = 0, C^{\mu}_{\nu\alpha} = 0$, in the form

$$\frac{D^2\zeta^2}{ds^2} + (\delta^{\lambda}_{(\mu}\sigma^{*}_{\varrho)\nu} + \eta_{\nu(\varrho}\sigma^{*}_{\mu)b}\eta^{b\lambda} + \delta^{\lambda}_{(\varrho}\sigma^{*}_{\mu)}\sigma^{*}_{\nu} + \\ + \eta_{\nu(\mu}\sigma^{*}_{\varrho)}\sigma^{*}_{b}h^{b\lambda} + \eta_{\nu(\varrho}\delta^{\lambda}_{\mu)}\eta^{cd}\sigma^{*}_{c}\sigma^{*}_{d})\frac{dx^{\mu}}{ds}\frac{d\zeta^{\nu}}{ds}\frac{dx^{\varrho}}{ds} = 0.$$

11.5 Conclusions

(a) We derived the gravitational field equations in spaces whose metric tensor depends on spinor variables. Equations (11.15) and (11.27) are generalizations of the conventional equations (11.3) a) and (11.3) b). They are reduced to equations (11.3) a) and (11.3) b) when the coefficients

$$(\omega^{(*)}_{\mu}, \theta^{(*)}_{\alpha}, \bar{\theta}^{(*)\alpha}) \rightarrow (\omega_{\mu}).$$

Relations (11.32) and (11.33) give rise to new results.

(b) Equations (F1)-(F4) represent the field equations on the GCFS $(M, g_{\mu\nu}(x, \xi, \bar{\xi}))$. The solutions of these equations are the subject of further concern. They represent an application of the gauge approach, for spaces with the metric $g(x, \xi, \bar{\xi})$, studied by two of the authors in [146, 147].

(c) The vanishing of the curvatures $S^{\alpha}_{ab\beta}, Q_{ab\beta\alpha}, \widetilde{Q}^{\beta\alpha}_{ab}$ (Theorem 11.3), reduces the 6 spin curvatures of the theory of Y. Takano and T. Ono to the three ones $R_{ab\lambda\mu}, P_{ab\lambda\alpha}, \overline{P}^{\alpha}_{ab\lambda}$. This simplifies considerably the study of the generalized conformally flat spaces.

Chapter 12

Gauge Gravity Over Sinor Bundles

12.1 Introduction

The concept of the nonlocalized field theory has already been developed in recent years by Japanese authors (see, for instance, [79]) in order to provide a unified description of elementary particles. In this approach, the internal variable is replaced by a spinor $\omega = (\xi, \bar{\xi})$ (ξ and its conjugate $\bar{\xi}$ are considered as independent variables).

The description of gravity through the introduction of variables $\omega_\mu^{ab}(x)$ as a gravitational potential (Lorentz connection coefficients) was proposed originally by Utiyama [155, 38]. He considered the Lorentz group as a local transformation group. The gravitational field is described by the tetrad $h_\mu^a(x)$ viewed as independent variables. With the help of these variables we may pass from a general system of coordinates to a local Lorentz ones.

The Einstein equation were derived in the context of Utiyama's approach, but this was not satisfactory because of the arbitrariness of the elements introduced. Later T. Kibble [64, 79, 87] introduced a gauge approach which enables the introduction of all gravitational variables. To achieve this goal it is important to use the Poincaré group (i.e. a group consisting of rotations, boosts and translations).

This group first assigns an exact meaning to the terms: "momentum", "energy", "mass" and "spin" used to determine characteristics of elementary particles. On the other hand, it is a gauge acting locally in the space-time. Thus, we may perform Poincaré transformations for a physical approach. Hence by treating the Poincaré group as a local group, we introduce the fundamental 1-form field ρ_μ taking in the Lie algebra of the Poincaré group.

In our present study the basic idea is to consider a spinor bundle with a base manifold M of a metric tensor $g_{\mu\nu}(x, \xi, \bar{\xi})$ that depends on the position coordinates x^k and the spinor (Dirac) variables $(\xi_\alpha, \bar{\xi}^\alpha) \in \mathbb{C}^4 \times \mathbb{C}^4$, where $\bar{\xi}^\alpha$ is the adjoint of ξ_α, an independent variable, similar to the one proposed by Y. Takano [152], and Y. Takano and T. Ono [121, 122, 123, 124]. The spinor bundle $S^{(1)}(M)$ is constructed from one of the principal fiber bundles with a fiber: $F = \mathbb{C}^4$.

Each fiber is diffeomorphic with one proper Lorentz group (which is produced by Lorentz transformations) and it entail a principal bundle $SL(4, \mathbb{C})$ over M, ($SL(4, \mathbb{C})$ consists of the group of ratations and boosts of unit determinant acting on a four-dimensional complex space, which is reducible to $(SL(2, \mathbb{C}))$.

The consideration of the d-connections that preserve the (hv)-distribution by the parallel translation (cf.[109, 116], in relation to the second order bundles $S^2(M) = M \times \mathbb{C}^{2\cdot 4}$ enables us to use a more general group $G^{(2)}$ called a structured group of all rotations and translations that is isomorphic to the Poincaré Lie algebra. Therefore, a *spinor* in $x \in M$ is an element of the spinor bundle $S^{(2)}(M)$.

$$(x^\mu, \xi_\alpha, \bar{\xi}^\alpha) \in S^{(2)}(M).$$

A *spinor field* is a section of $S^{(2)}(M)$.

Moreover, the fundamental gauge 1-form field mentioned above in connection with the spaces that possess metric tensor $g_{\mu\nu}(x, \xi, \bar{\xi})$ will take a similar but more general form than that proposed by other authors [97]. We shall define a nonlinear connection on $S^{(2)}(M)$ such as,

$$T(S^{(2)}M) = H(S^{(2)}M) \oplus \mathcal{F}^{(1)}(S^{(2)}M) \oplus \mathcal{F}^{(2)}(S^{(2)}M),$$

where H, $\mathcal{F}^{(1)}$, $\mathcal{F}^{(2)}$ represent the horizontal, vertical, and normal distribution. In a local base, for the horizontal distribution $H(S^{(2)}M)$ we have:

$$\rho_\mu(x, \xi, \bar{\xi}) = \frac{1}{2}\omega_\mu^{*ab} J_{ab} + h_\mu^a(x, \xi, \bar{\xi}) P_a,$$

where J_{ab}, P_a are the generators of the four-dimensional Poincaré group satisfying relations of the form:

$$[J_{ab}, J_{cd}] = n_{bc}J_{ad} - n_{bd}J_{ac} + n_{ad}J_{bc} - n_{ac}J_{bd},$$
$$[J_{ab}, P_c] = n_{bc}P_a - n_{ac}P_b, \quad [P_a, P_b] = 0, \quad J_{ab} + J_{ba} = 0.$$

The quantities $\omega_\mu^{(*)ab}$ represent the (Lorentz) spin connection coefficients and are considered as given, n_{ab} is the metric for the local Lorentz spaces

with signature $(+---)$.
These are connected with $g_{\mu\nu}$ by

$$g_{\mu\nu}h_a^\mu h_b^\nu = n_{ab}, \quad g^{\mu\nu} = n^{ab}h_a^\mu h_b^\nu,$$

where h_a^ν represents the tetrads. Similarly, for the vertical and normal distributions $\mathcal{F}^{(1)}(S^{(2)}M)$, $\mathcal{F}^{(2)}(S^{(2)}M)$ the fundamental 1-forms ζ_α, $\bar{\zeta}^\alpha$ are given by

$$\begin{aligned}
\zeta_\alpha &= \frac{1}{2}\Theta_\alpha^{(*)ab}J_{ab} + \Psi_\alpha^a P_a, \\
\bar{\zeta}^\alpha &= \frac{1}{2}\bar{\Theta}^{(*)\alpha ab}J_{ab} + \bar{\Psi}^{\alpha a}P_a,
\end{aligned}$$

where $\bar{\psi}^{\alpha a}$, ψ_α^a are the spin tetrad coefficients, and $\Theta_\alpha^{(*)ab}$, $\bar{\Theta}^{(*)\alpha ab}$ are the given spin connection coefficients which are determined in such a way that the absolute differential and the covariant derivatives of the metric tensor $g_{\mu\nu}(x, \xi, \bar{\xi})$ vanish identically.

We use the Greek letters $\lambda, \mu, \nu \ldots$ for space-time indices, λ, β, γ for spinors, and the Latin letters $a, b, c, \ldots$ for the Lorentz indices.

The general transformations of coordinates on $S^{(2)}(M)$ are:

$$x'^\mu = x'^\mu(x^\nu), \quad \xi'_\alpha = \xi'_\alpha(\xi_\beta, \bar{\xi}^\gamma), \quad \bar{\xi}'^\alpha = \bar{\xi}'^\alpha(\bar{\xi}^\beta, \xi_\gamma). \tag{12.1}$$

12.2 Connections

We define the following gauge covariant derivatives

$$\begin{aligned}
D_\mu^{(*)} &= \frac{\delta}{\delta x^\mu} + \frac{1}{2}\omega_\mu^{(*)ab}J_{ab}, \\
D^{(*)\alpha} &= \frac{\delta}{\delta \xi_\alpha} + \frac{1}{2}\bar{\Theta}^{(*)\alpha ab}J_{ab}, \\
D^{(*)\alpha} &= \frac{\delta}{\delta \bar{\xi}^\alpha} + \frac{1}{2}\Theta_\alpha^{(*)ab}J_{ab},
\end{aligned}$$

where

$$\begin{aligned}
\frac{\delta}{\delta x^\mu} &= \frac{\partial}{\partial x^\mu} + N_{\alpha\mu}\frac{\partial}{\partial \xi_\alpha} - \bar{N}_\mu^\alpha \frac{\partial}{\partial \bar{\xi}_\alpha}, \\
\frac{\delta}{\delta \xi_\alpha} &= \frac{\partial}{\partial \xi_\alpha} - \tilde{N}_0^{\alpha\beta}\frac{\partial}{\partial \bar{\xi}^\beta}.
\end{aligned}$$

$N_{\alpha\lambda}$, $\bar{N}_\lambda^\alpha$, $\tilde{N}_0^{\alpha\beta}$ are the nonlinear connections which we shall define below.

The covariant derivatives of the metric tensor $g_{\mu\nu}$ are all zero:

$$D^{(*)}_{\mu} g_{\kappa\lambda} = 0, \quad D^{(*)\alpha} g_{\kappa\lambda} = 0, \quad D^{(*)}_{\alpha} g_{\kappa\lambda} = 0.$$

The space-time frame $\delta/\delta x^{\mu}$ and the local Lorentz frame $\delta/\delta x^{a}$ are connected with

$$\frac{\delta}{\delta x^{\mu}} = h^{a}_{\mu} \frac{\delta}{\delta x^{a}}.$$

Similarly, the spin-tetrad coefficients ψ^{a}_{α} and adjoint $\bar{\psi}^{\alpha a}$ connect the spin frames, $\partial/\partial\xi_{\alpha}$, $\partial/\partial\bar{\xi}^{\alpha}$ with $\partial/\partial x^{a}$:

$$\frac{\partial}{\partial \xi_{\alpha}} = \bar{\psi}^{\alpha a} \frac{\partial}{\partial x^{a}},$$

$$\frac{\partial}{\partial \bar{\xi}^{\alpha}} = \psi^{a}_{\alpha} \frac{\partial}{\partial x^{a}}.$$

The absolute differential of an arbitrary contravariant vector X^{ν} is given by

$$DX^{\nu} = (D^{(*)}_{\mu} dx^{\mu} + D^{(*)\alpha} X^{\nu}) d\xi_{\alpha} + (D^{(*)}_{\alpha} X^{\nu}) d\bar{\xi}^{\alpha}.$$

12.2.1 Nonlinear connections

We give the nonlinear connections $N = \{N_{\beta\mu}, \tilde{N}^{0\alpha}_{\beta}, N^{0}_{\alpha\beta}, \bar{N}^{\beta}_{\mu}, \tilde{N}^{\beta\alpha}_{0}, N^{\beta}_{0\alpha}\}$ in the framework of our consideration in the following form:

$$\begin{aligned} N_{\beta\mu} &= \frac{1}{2}\omega^{(*)ab}_{\mu} J_{ab}\xi_{\beta}, \quad \tilde{N}^{0\alpha}_{\beta} = \frac{1}{2}\bar{\Theta}^{(*)\alpha ab} J_{ab}\xi_{\beta}, \\ N^{0}_{\alpha\beta} &= \frac{1}{2}\Theta^{(*)ab}_{\alpha} J_{ab}\xi_{\beta}, \quad \bar{N}^{\beta}_{\mu} = -\frac{1}{2}\omega^{(*)ab}_{\mu} J_{ab}\bar{\xi}^{\beta}, \\ \tilde{N}^{\alpha\beta}_{0} &= -\frac{1}{2}\bar{\Theta}^{(*)\alpha ab} J_{ab}\bar{\xi}^{\beta}, \quad N^{\beta}_{0\alpha} = -\frac{1}{2}\Theta^{(*)ab}_{\alpha} J_{ab}\bar{\xi}^{\beta}. \end{aligned} \tag{12.2}$$

The differentials of $D\xi_{\alpha}$, $D\bar{\xi}^{\alpha}$ can be written, after the relations (12.2), in the form:

$$\begin{aligned} D\xi_{\beta} &= d\xi_{\beta} + N^{0}_{\alpha\beta} d\bar{\xi}_{\alpha} + \tilde{N}^{0\alpha}_{\beta} d\xi_{\alpha} + N_{\beta\mu} dx^{\mu}, \\ D\bar{\xi}^{\beta} &= d\bar{\xi}^{\beta} + N^{\beta}_{0\alpha} d\bar{\xi}^{\alpha} - \tilde{N}^{\beta\alpha}_{0} d\xi_{\alpha} - \tilde{N}^{\beta}_{\mu} dx^{\mu}, \end{aligned} \tag{12.3}$$

The metric in the second order tangent bundle is given by the relation

$$G = g_{\kappa\lambda} dx^{\kappa} dx^{\lambda} + g_{ij}\delta y^{i}\delta y^{j} + g_{\alpha\beta}\delta u^{\alpha}\delta u^{\beta},$$

and the adapted frame

$$\frac{\partial}{\partial Z^A} = \left(\frac{\delta}{\delta x^\lambda} = \frac{\partial}{\partial x^\lambda} - N^i_\lambda \frac{\partial}{\partial y^i} - M^\alpha_\lambda \frac{\partial}{\partial u^\alpha}, \frac{\delta}{\delta y^i}, \frac{\partial}{\partial u^\alpha}\right)$$

where $\delta/\delta y^i = \partial/\partial y^i - L^\alpha_i \partial/\partial u^\alpha$.

Furthermore, the dual frame is

$$\delta Z^A = (dx^\kappa, \delta y^i + N^i_\lambda dx^\lambda, \delta u^\alpha = du^\alpha + L^\alpha_i dy^i + M^\alpha_\lambda dx^\lambda).$$

The metrical structure in the bundle will be defined as follows:

$$G = g_{\mu\nu}(x,\xi,\bar{\xi}) dx^\mu dx^\nu + g_{\alpha\beta}(x,\xi,\bar{\xi}) D\bar{\xi}^\alpha D\bar{\xi}^{*\beta} + g^{\alpha\beta} D\xi_\alpha D\xi^*_\beta.$$

an analogy with the previous adapted frame, a local adapted frame on a spinor bundle $S^{(2)}(M)$ will be defined as

$$\begin{aligned}
\left(\frac{\partial}{\partial \zeta^A}\right) &= \left\{\frac{\delta}{\delta x^\lambda}, \frac{\delta}{\delta \xi_\alpha}, \frac{\delta}{\delta \bar{\xi}^\alpha}\right\}, \\
\frac{\delta}{\delta x^\lambda} &= \frac{\partial}{\partial x^\lambda} + N_{\alpha\lambda}\frac{\partial}{\partial \xi_\alpha} - \bar{N}^\alpha_\lambda \frac{\partial}{\partial \bar{\xi}^\alpha}, \\
\frac{\delta}{\delta \xi_\alpha} &= \frac{\partial}{\partial \xi_\alpha} - \tilde{N}^{\beta\alpha}_0 \frac{\partial}{\partial \bar{\xi}^\beta},
\end{aligned}$$

and

$$\delta\zeta^A = \{dx^\kappa, D\xi_\beta, D\bar{\xi}^\beta\},$$

where the expressions $D\xi_\beta, D\bar{\xi}^\beta$ are given by (12.3). If we consider the connection coefficients Γ^A_{BC} given in the general case, then in the total space $S^{(2)}(M)$ we have

$$\Gamma^A_{BC} = \{\Gamma^{(*)\mu}_{\nu\rho}, C^\mu_{\nu\alpha}, \bar{C}^{\mu\alpha}_\nu, \bar{\Gamma}^{(*)\gamma}_{\beta\lambda}, C^{\gamma\alpha}_\beta, \tilde{C}^{\gamma\alpha}_\beta, \tilde{C}^\gamma_{\alpha\beta}, \Gamma^{(*)\beta}_{\alpha\lambda}, C^\gamma_{\alpha\beta}\}.$$

Considering that the connections are d–connections [109, 116] in an adapted base, we get the following relations

$$D_{\partial/\partial x^C}\frac{\partial}{\partial x^B} = \Gamma^A_{BC}\frac{\partial}{\partial x^A},$$

or, in explicit form,

$$
\begin{aligned}
D_{\delta/\delta x^{\rho}}\frac{\delta}{\delta x^{\nu}} &= \Gamma^{(*)\mu}_{\nu\rho}\frac{\delta}{\delta x^{\mu}}, \quad D_{\partial/\partial\bar{\xi}^{\alpha}}\frac{\delta}{\delta x^{\nu}} = C^{\mu}_{\nu\alpha}\frac{\delta}{\delta x^{\mu}},\\
D_{\delta/\delta\xi_{\alpha}}\frac{\delta}{\delta x^{\nu}} &= \bar{C}^{\mu\alpha}_{\nu}\frac{\delta}{\delta x^{\mu}}, \quad D_{\partial/\partial\bar{\xi}^{\alpha}}\frac{\delta}{\delta x^{\nu}} = \Gamma^{(*)\gamma}_{\nu\alpha}\frac{\delta}{\delta\xi^{\gamma}},\\
D_{\delta/\delta x^{\alpha}}\frac{\delta}{\delta x_{\beta}} &= \bar{\Gamma}^{(*)\beta}_{\lambda\gamma}\frac{\delta}{\delta\xi_{\gamma}}, \quad D_{\delta/\delta\xi_{\alpha}}\frac{\partial}{\partial\xi^{\beta}} = C^{\gamma\alpha}_{\beta}\frac{\partial}{\partial\xi^{\gamma}},\\
D_{\delta/\delta\xi_{\alpha}}\frac{\partial}{\partial\xi^{\beta}} &= C^{\gamma}_{\beta\alpha}\frac{\partial}{\partial\xi^{\gamma}}, \quad D_{\partial/\partial\bar{\xi}^{\alpha}}\frac{\delta}{\delta\xi_{\beta}} = \tilde{C}^{\beta}_{\alpha\gamma}\frac{\delta}{\delta\xi_{\gamma}},\\
D_{\delta/\delta\xi_{\alpha}}\frac{\delta}{\delta\xi_{\beta}} &= \tilde{C}^{\beta\alpha}_{\gamma}\frac{\delta}{\delta\xi_{\gamma}}.
\end{aligned}
$$

The covariant differentiation of tensor and spin-tensors of arbitrary rank may be classified into three types:

$$
\begin{aligned}
\bigtriangledown_{\lambda}T^{\mu\ldots}_{\nu\ldots} &= \frac{\delta T^{\mu\ldots}_{\nu\ldots}}{\delta x^{\lambda}} + \Gamma^{(*)\mu}_{\kappa\lambda}T^{\kappa\ldots}_{\nu\ldots} + \cdots - \Gamma^{(*)\kappa}_{\nu\lambda}T^{\mu\ldots}_{\kappa\ldots},\\
\bigtriangledown^{\alpha}T^{\mu\ldots}_{\nu\ldots} &= \frac{\delta T^{\mu\ldots}_{\nu\ldots}}{\delta\xi_{\alpha}} + \bar{C}^{(*)\mu\alpha}_{\kappa}T^{\kappa\ldots}_{\nu\ldots} + \cdots - \bar{C}^{(*)\kappa\alpha}_{\nu}T^{\mu\ldots}_{\kappa\ldots},\\
\bigtriangledown_{\alpha}T^{\mu\ldots}_{\nu\ldots} &= \frac{\partial T^{\mu\ldots}_{\nu\ldots}}{\partial\xi^{\alpha}} + C^{(*)\mu}_{\kappa\alpha}T^{\kappa\ldots}_{\nu\ldots} + \cdots - C^{(*)\kappa}_{\nu\alpha}T^{\mu\ldots}_{\kappa\ldots},\\
\bigtriangledown_{\lambda}\Phi^{\alpha\ldots}_{\beta\ldots} &= \frac{\delta\Phi^{\alpha\ldots}_{\beta\ldots}}{\delta x^{\lambda}} - \Gamma^{(*)\gamma}_{\beta\lambda}\Phi^{\alpha\ldots}_{\gamma\ldots} - \cdots + \Phi^{\gamma\ldots}_{\beta\ldots}\Gamma^{(*)\alpha}_{\gamma\lambda} + \ldots,\\
\bigtriangledown^{\delta}\Phi^{\alpha\ldots}_{\beta\ldots} &= \frac{\delta\Phi^{\alpha\ldots}_{\beta\ldots}}{\delta\xi_{\delta}} - \tilde{C}^{(*)\gamma\delta}_{\beta}\Phi^{\alpha\ldots}_{\gamma\ldots} - \cdots + \Phi^{\gamma\ldots}_{\beta\ldots}\tilde{C}^{(*)\alpha\delta}_{\gamma} + \ldots,\\
\bigtriangledown_{\delta}\Phi^{\alpha\ldots}_{\beta\ldots} &= \frac{\partial\Phi^{\alpha\ldots}_{\beta\ldots}}{\partial\xi^{\delta}} - C^{(*)\gamma}_{\beta\delta}\Phi^{\alpha\ldots}_{\gamma\ldots} - \cdots + \Phi^{\gamma\ldots}_{\beta\ldots}C^{(*)\alpha}_{\gamma\delta} + \ldots,\\
\bigtriangledown^{(*)}_{\mu}V^{\alpha\ldots}_{c\ldots} &= \frac{\delta V^{\alpha\ldots}_{c\ldots}}{\delta x^{\mu}} + \omega^{(*)\alpha}_{\mu b}V^{b\ldots}_{c\ldots} + \cdots - \omega^{(*)b}_{\mu c}V^{\alpha\ldots}_{b\ldots},\\
\bigtriangledown^{(*)\alpha}V^{\alpha\ldots}_{c\ldots} &= \frac{\delta V^{\alpha\ldots}_{c\ldots}}{\delta\xi_{\alpha}} + \bar{\Theta}^{(*)a\alpha}_{b}V^{b\ldots}_{c\ldots} + \cdots - \bar{\Theta}^{(*)\alpha b}_{c}V^{\alpha\ldots}_{b\ldots},\\
\bigtriangledown^{(*)}_{\alpha}V^{\alpha\ldots}_{c\ldots} &= \frac{\partial V^{\alpha\ldots}_{c\ldots}}{\delta\xi^{\alpha}} + \Theta^{(*)a}_{\alpha b}V^{b\ldots}_{c\ldots} + \cdots - \Theta^{(*)b}_{\alpha c}V^{\alpha\ldots}_{b\ldots}.
\end{aligned}
$$

12.2.2 Lorentz transformation

We can get the Lorentz transformations of linear connections $\omega_\nu^{(*)ab}$, $\bar{\Theta}^{(*)\beta ab}$, $\Theta_\beta^{(*)ab}$ in the following form:

$$
\begin{aligned}
{\omega'}_\mu^{(*)ab} &= L_c^a L_d^b \omega_\mu^{(*)cd} + \frac{\delta L_c^a}{\delta x^\mu} L_d^b n^{cd}, \\
\bar{\Theta}^{(*)'\alpha ab} &= \left[L_c^a L_d^b \bar{\Theta}^{(*)\beta cd} + \frac{\delta L_c^a}{\delta \xi_\beta} L_d^b n^{cd} \right] \Lambda_\beta^{-1\alpha}, \\
\Theta_\alpha^{(*)'ab} &= \Lambda_\alpha^\beta \left[L_c^a L_d^b \bar{\Theta}_\beta^{(*)cd} + \frac{\partial L_c^a}{\partial \xi^\beta} L_d^b n^{cd} \right],
\end{aligned}
$$

Similarly, the Lorentz transformation law of nonlinear connection is given by:

$$
\begin{aligned}
\tilde{N}_{\beta\mu} &= \frac{1}{2} \omega_\mu^{(*)ab} J_{ab} \xi_\alpha L_\beta^\alpha + \frac{1}{2} n^{cd} \frac{\delta L_c^a}{\delta x^\mu} L_d^b J_{ab} \Lambda_\beta^\alpha \xi_\alpha \\
&= N_{\alpha\mu} \Lambda_\beta^\alpha + \frac{1}{2} n^{cd} \frac{\delta L_c^a}{\delta x^\mu} J'_{ab} \Lambda_\beta^\alpha \xi_\alpha,
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{N}_\beta^{0\alpha} &= \left[\tilde{N}_\gamma^{0\delta} \Lambda_\beta^\gamma + \frac{1}{2} n^{cd} \frac{\delta L_c^a}{\delta \xi_\delta} L_d^b J'_{ab} \Lambda_\beta^\gamma \xi_\gamma \right] \Lambda_\delta^{-1\alpha}, \\
\tilde{N}_{\alpha\beta}^0 &= \Lambda_\alpha^\delta \left[N_{\gamma\delta}^0 \Lambda_\beta^\gamma + \frac{1}{2} n^{cd} \frac{\partial L_c^a}{\partial \xi^\delta} L_d^b J'_{ab} \Lambda_\beta^\gamma \xi_\gamma \right], \\
\bar{N}_\mu^\beta &= N_\mu^\alpha \Lambda_\alpha^{1-\beta} - \frac{1}{2} n^{cd} \frac{\delta L_c^a}{\delta x_\mu} L_d^b J'_{ab} \bar{\xi}^\gamma \Lambda_\gamma^{-1\beta}, \\
\tilde{N}_0^{\alpha\beta} &= \left[\tilde{N}_0^{\gamma\delta} \Lambda_\gamma^{-1\beta} - \frac{1}{2} n^{cd} \frac{\delta L_c^a}{\delta \xi_\delta} L_d^b J'_{ab} \Lambda_\gamma^{-1\beta} \right] \Lambda_\delta^{-1\alpha}, \\
\tilde{N}_{0\alpha}^\beta &= \Lambda_\alpha^\delta \left[N_{0\delta}^\gamma \Lambda_\gamma^{-\beta} - \frac{1}{2} n^{cd} \frac{\partial L_c^a}{\partial \xi^\delta} L_d^b J'_{ab} \Lambda_\beta^\gamma \bar{\xi}^\gamma \Lambda_\gamma^{-1\beta} \right],
\end{aligned}
$$

where $J'_{ab} = L_{\hat{a}}^c L_b^d J_{cd}$.

12.3 Curvatures and torsions

From the covariant derivatives $D^{(*)}_{\mu}$, $D^{(*)\alpha}$, $D^{(*)}_{\alpha}$ we get six curvatures and torsions:

$$
\begin{aligned}
a) \quad \left[D^{(*)}_{\mu}, D^{(*)}_{\nu}\right] &= D^{(*)}_{\mu} D^{(*)}_{\nu} - D^{(*)}_{\nu} D^{(*)}_{\mu} = R^{a}_{\mu\nu} P_a + \frac{1}{2} R^{ab}_{\mu\nu} J_{ab}, \\
R^{a}_{\mu\nu} &= \frac{\delta h^{a}_{\mu}}{\delta x^{\nu}} - \frac{\delta h^{a}_{\nu}}{\delta x^{\mu}} + \omega^{(*)a}_{\mu b} h^{b}_{\nu} - \omega^{(*)a}_{\nu b} h^{b}_{\mu}, \\
R^{ab}_{\mu\nu} &= \frac{\delta \omega^{(*)ab}_{\mu}}{\delta x^{\nu}} - \frac{\delta \omega^{(*)ab}_{\nu}}{\delta x^{\mu}} + \omega^{(*)a\rho}_{\mu} \omega^{(*)b}_{\nu\rho} - \omega^{(}*)\rho a_{\nu} \omega^{(*)b}_{\mu\rho}, \\
b) \quad \left[D^{(*)}_{\mu}, D^{(*)}_{\alpha}\right] &= P^{a}_{\mu\alpha} P_a + \frac{1}{2} P^{ab}_{\mu\alpha} J_{ab}, \\
P^{ab}_{\mu\alpha} &= \frac{\delta \theta^{(*)ab}_{\alpha}}{\delta x^{\nu}} - \frac{\partial \omega^{(*)ab}_{\mu}}{\delta \bar{\xi}^{\alpha}} + \Theta^{(*)b}_{\alpha c} \omega^{(*)a}_{\mu} - \Theta^{(*)a}_{\alpha c} \omega^{(*)cb}_{\mu}, \\
P^{a}_{\mu\alpha} &= \frac{\delta \psi^{a}_{\alpha}}{\delta x^{\mu}} - \frac{\partial h^{a}_{\mu}}{\delta \bar{\xi}^{\alpha}} + \omega^{(*)a}_{\mu c} \psi^{c}_{\alpha} - \Theta^{(}*)a_{\alpha c} h^{c}_{\mu}, \\
c) \quad \left[D^{(*)}_{\mu}, D^{(*)}_{\alpha}\right] &= \bar{P}^{a\alpha}_{\mu} P_a + \frac{1}{2} \bar{P}^{ab\alpha}_{\mu} J_{ab}, \\
\bar{P}^{ab\alpha}_{\mu} &= \frac{\delta \Theta^{(*)\alpha ab}_{\alpha}}{\delta x^{\mu}} - \frac{\delta \omega^{(*)ab}_{\mu}}{\delta \xi_{\alpha}} + \bar{\Theta}^{(*)ab}_{c} \omega^{(*)ac}_{\mu} - \bar{\Theta}^{(*)\alpha a}_{c} \omega^{(*)cb}_{\mu}, \\
\bar{P}^{a\alpha}_{\mu\alpha} &= \frac{\delta \bar{\psi}^{\alpha a}_{\alpha}}{\delta x^{\mu}} - \frac{\delta h^{a}_{\mu}}{\delta \xi^{\alpha}} + \omega^{(*)a}_{\mu c} \bar{\psi}^{c\alpha} - \bar{\Theta}^{(*)\alpha a}_{c} h^{c}_{\mu}, \\
d) \quad \left[D^{(*)}_{\alpha}, D^{(*)\beta}\right] &= S^{\beta a}_{\alpha} P_a + \frac{1}{2} S^{ab\beta}_{\alpha} J_{ab}, \\
S^{\beta a}_{\alpha} &= \frac{\delta \bar{\psi}^{\beta a}}{\delta \bar{\xi}_{\alpha}} - \frac{\delta \psi^{a}_{\alpha}}{\delta \xi_{\beta}} + \bar{\Theta}^{(*)\beta ba} \psi_{ab} - \Theta^{(*)ab}_{\alpha} \bar{\psi}^{\beta}_{b}, \\
S^{ab\beta}_{\alpha} &= \frac{\partial \bar{\Theta}^{\beta ab}}{\partial \xi^{\alpha}} - \frac{\partial \Theta^{(*)ab}_{\alpha}}{\delta \xi_{\beta}} + \Theta^{(*)a}_{\alpha c} \bar{\Theta}^{(*)\beta cb} - \Theta^{(}*)b_{ac} \bar{\Theta}^{\beta ca}, \\
e) \quad \left[D^{(*)}_{\alpha}, D^{(*)}_{\beta}\right] &= Q^{a}_{\alpha} P_a + \frac{1}{2} Q^{ab}_{\alpha\beta} J_{ab}, \\
Q^{a}_{\alpha\beta} &= \frac{\partial \psi^{a}_{\beta}}{\partial \bar{\xi}^{\alpha}} - \frac{\partial \psi^{a}_{\alpha}}{\partial \bar{\xi}^{\beta}} + \Theta^{(*)ba}_{\beta} \psi_{ab} - \Theta^{(*)ab}_{\alpha} \psi_{\beta b}, \\
Q^{ab}_{\alpha\beta} &= \frac{\partial \theta^{(*)ab}_{\alpha}}{\partial \bar{\xi}^{\alpha}} - \frac{\partial \theta^{(*)ab}_{\alpha}}{\partial \bar{\xi}^{\beta}} + \Theta^{(*)a}_{\alpha c} \Theta^{(*)cb}_{\beta} - \Theta^{(}*)b_{\alpha c} \Theta^{(*)ca}_{\beta},
\end{aligned}
$$

$$
\begin{aligned}
f) \quad \left[D^{(*)\alpha}, D^{(*)\beta}\right] &= \tilde{Q}^{\alpha\beta a} P_a + \frac{1}{2}\tilde{Q}^{ab\alpha\beta} J_{ab}, \\
\tilde{Q}^{\alpha\beta a} &= \frac{\delta\psi^a_\beta}{\delta\xi_\alpha} - \frac{\delta\psi^a_\alpha}{\partial\xi_\beta} + \bar{\Theta}^{(*)\beta ba}\bar{\psi}^\alpha_b - \bar{\Theta}^{(*)\alpha ba}\bar{\psi}^\beta_b, \\
\tilde{Q}^{ab\alpha\beta} &= \frac{\delta\bar{\theta}^{\beta ab}}{\delta\xi_\alpha} - \frac{\delta\bar{\theta}^{\alpha ab}}{\partial\xi_\beta} + \bar{\Theta}^{(*)\beta ab}\bar{\Theta}^{(*)\alpha a}_c - \bar{\Theta}^{(*)\alpha b}_c\bar{\Theta}^{\beta ca},
\end{aligned}
$$

12.4 Field equations

We derive the field equations using the spin–tetrad frames in the Lagrangian form: $\mathcal{L}(h, \omega^{(*)}, \psi, \Theta^{(*)}, \bar{\psi}, \bar{\Theta}^{(*)})$. The method of derivation of equations is similar to Palatini's one.

We get the Lagrangian

$$\mathcal{L}(h, \omega^{(*)}, \psi, \Theta^{(*)}, \bar{\psi}, \bar{\Theta}^{(*)})$$

or

$$\mathcal{L}\psi^A, \delta_M\psi^A = h(R + P + \bar{P} + S + Q + \tilde{Q}), \tag{12.4}$$

where

$$\psi^A = \left(h^a_\mu(x,\xi,\bar{\xi}), \omega^{(*)ab}_\mu(x,\xi,\bar{\xi}), \psi^a_\alpha(x,\xi,\bar{\xi}), \bar{\psi}^{\alpha a}_{(\dots)}, \Theta^{(*)ab}_{\alpha(\dots)}, \bar{\Theta}^{(*)\alpha ab}_{(\dots)}\right),$$

$$\delta_M = \frac{\delta}{\delta z^M} = \left(\frac{\delta}{\delta x^m}, \frac{\delta}{\delta\xi_\alpha}, \frac{\delta}{\delta\bar{\xi}^\alpha}\right), \qquad z^M = (x^\mu, \xi_\alpha, \bar{\xi}^\alpha),$$

$$
\begin{aligned}
R &= h^\mu_a h^\nu_b R^{ab}_{\mu\nu}, \\
P &= h^\mu_a \bar{\psi}^\alpha_b P^{ab}_{\mu\alpha}, \qquad \bar{P} = h^\mu_a \bar{\psi}_{\alpha b}\bar{P}^{ab\alpha}_\mu, \\
Q &= Q^{ab}_{\alpha\beta}\bar{\psi}^\alpha_a\bar{\psi}^\beta_b, \qquad \tilde{Q} = \tilde{Q}^{ab\alpha\beta}\psi_{\alpha a}\psi_{\beta b}, \\
S &= \bar{\psi}^\alpha_a\psi_{\beta b}S^{ab\beta}_\alpha.
\end{aligned}
$$

The Euler–Lagrange equations are written in the form:

$$\frac{\delta\mathcal{L}}{\delta z^M} = \frac{\partial\mathcal{L}}{\partial(\delta_M\psi^{(A)})} - \frac{\partial\mathcal{L}}{\partial\psi^{(A)}} = 0.$$

From the relation (12.4), the variation of $\mathcal{L}$ with respect to $h^{\nu}b$ yields the equations

$$\begin{aligned}(R^a_\mu + P^a_\mu + \bar{P}^a_\mu) - \frac{1}{2}(R + P + \bar{P})h^a_\mu &= 0,\\ H^a_\mu - \frac{1}{2}Hh^a_\mu &= 0,\end{aligned}$$

where

$$P^a_\mu = \bar{\psi}^\alpha_b P^{ab}_{\mu\alpha}, \quad \bar{P}^a_\mu = \psi_{ab}\bar{P}^{ab\alpha}_\mu, \quad R^a_\mu = h^\nu_b R^{ab}_{\mu\nu},$$

and

$$H^a_\mu = R^a_\mu + P^a_\mu + \bar{P}^a_\mu, \quad H = R + P + \bar{P}.$$

From the variation of $\mathcal{L}$ with respect to $\omega^{(*)ab}_\mu$

$$\begin{aligned}\frac{\delta}{\delta x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\delta}{\delta x^\mu}\omega^{(*)ab}_\nu\right)}\right) + \frac{\delta}{\delta\xi_\alpha}\frac{\partial\mathcal{L}}{\partial\left(\frac{\delta\omega^{(*)ab}_\nu}{\delta\xi_\alpha}\right)}&\\ +\frac{\delta}{\delta\bar{\xi}^\alpha}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\delta}{\delta\bar{\xi}^\alpha}\omega^{(*)ab}_\nu\right)}\right) - \frac{\partial\mathcal{L}}{\partial\omega^{(*)ab}_\nu} &= 0,\end{aligned}$$

we get

$$\begin{aligned}D^{(*)}_\mu[h(h^\nu_a h^\mu_b - h^\nu_b h^\mu_a)] + D^{(*)}_\alpha[h(h^\nu_a\bar{\psi}^\alpha_b - h^\nu_a\bar{\psi}^\alpha_a)]&\\ +D^{(*)\alpha}[h(h^\nu_a\psi_{\alpha b} - h^\nu_b\psi^a_\alpha)] &= 0.\end{aligned}$$

The variations with respect to $\Theta^{(*)ab}_\alpha$, $\bar{\Theta}^{(*)\alpha ab}$ yield the relation

$$\frac{\delta}{\delta x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\delta\Omega^{(*)}}{\delta x^\mu}\right)}\right) + \frac{\delta}{\delta\xi_\alpha}\frac{\partial\mathcal{L}}{\partial\left(\frac{\delta\Omega^{(*)}}{\delta\xi_\alpha}\right)} + \frac{\delta}{\delta\bar{\xi}^\alpha}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\delta\Omega^{(*)}}{\delta\bar{\xi}^\alpha}\right)}\right) - \frac{\partial\mathcal{L}}{\partial\Omega^{(*)}} = 0,$$

with

$$\Omega^{(*)} = \left\{\Theta^{(*)ab}_\alpha, \bar{\Theta}^{(*)\alpha ab}\right\}$$

which gives us the equations:

$$\begin{aligned}D^{(*)}_\mu(hh^\mu_a\bar{\psi}^\alpha_a) - D^{(*)}_\beta(2h\bar{\psi}^\alpha_a\bar{\psi}^\beta_b) - 2D^{(*)\beta}(h\bar{\psi}^\alpha_a\psi_{\beta b}) &= 0,\\ D^{(*)}_\mu(hh^\mu_a\psi_{b\alpha}) - 2D^{(*)}_\beta(h\psi_{a\alpha}\bar{\psi}^\beta_b) - D^{(*)\beta}(2h\psi_{a\alpha}\psi_{b\beta}) &= 0.\end{aligned}$$

Finally, the variation of $\mathcal{L}$ with respect to the spin-tetrad coefficients $\bar{\psi}^{\alpha}_{a}$, $\psi^{\alpha a}$ derives the equations:

$$\begin{aligned} Q^{ab}_{\alpha\beta}\bar{\psi}^{\beta}_{b} + \frac{1}{2}S^{ab\beta}_{\alpha}\psi_{\beta b} + \frac{1}{2}P^{ba}_{\mu\alpha}h^{\mu}_{b} &= 0, \\ \tilde{Q}^{a\alpha} - \frac{1}{2}(S^{a\alpha} + \bar{P}^{a\alpha}) &= 0. \end{aligned}$$

12.5 Bianchi identities

From Jacobi identities,

$$\mathcal{Q}_{(XYZ)}\left[D^{(*)}_{X}, [D^{(*)}_{Y}, D^{(*)}_{Z}]\right] = 0,$$

we get $18(3\times 6)$ relations of different types. For each relation we derive two identities, namely 36 ones in total. Taking into account that

$$D^{(*)}_{\mu} = \frac{\delta}{\delta x^{\mu}} + \frac{1}{2}\omega^{(*)ab}_{\mu}J_{ab},$$

where

$$\begin{aligned} \frac{\delta}{\delta x^{\mu}} &= \frac{\partial}{\partial x^{\mu}} - N_{\mu\alpha}\frac{\partial}{\partial \xi_{\alpha}} - \bar{N}^{\alpha}_{\mu}\frac{\partial}{\partial \bar{\xi}^{\alpha}} \\ &= h^{a}_{\mu}P_{a} - N_{\mu\alpha}\bar{\Psi}^{\alpha a}P_{a} - \bar{N}^{\alpha}_{\mu}\Psi^{a}_{\alpha}P_{a} = A^{a}_{\mu}P_{a}, \\ A^{a}_{\mu} &= h^{a}_{\mu} - N_{\mu\alpha}\bar{\psi}^{\alpha a} - \bar{N}^{\alpha}_{\mu}, \quad P_{a} = \frac{\partial}{\partial x^{a}}, \end{aligned}$$

we can get

$$\begin{aligned} \left[D^{(*)}_{\mu}, [D^{(*)}_{\kappa}, D^{(*)}_{\lambda}]\right] &= \left[A^{c}_{\mu}P_{c}, \frac{1}{2}R^{ab}_{\kappa\lambda}J_{ab}\right] + [A^{c}_{\mu}P_{c}, R^{a}_{\kappa\lambda}P_{a}] \\ &\quad + \frac{1}{2}\omega^{(*)ab}_{\mu}R^{cd}_{\kappa\lambda}[J_{ab}, J_{cd}] + \frac{1}{2}\omega^{(*)ab}_{\mu}R^{c}_{\kappa\lambda}[J_{ab}, P_{c}]. \end{aligned} \tag{12.5}$$

The first term of the right hand side of (12.5) by straightforward calculations is written in the form

$$\left[A^{c}_{\mu}P_{c}, \frac{1}{2}R^{ab}_{\kappa\lambda}J_{ab}\right] = \frac{1}{2}\frac{\delta R^{ab}_{\kappa\lambda}}{\delta x^{\mu}}J_{ab} + R^{a}_{b\kappa\lambda}A^{b}_{\mu}P_{a}.$$

Similarly, the second, third, and fourth terms of (12.5) yield the relations

$$[A^{c}_{\mu}P_{c}, R^{a}_{\kappa\lambda}P_{a}] = \frac{\delta R^{a}_{\kappa\lambda}}{\delta x^{\mu}}P_{a} + A^{c}_{\mu}R^{a}_{\kappa\lambda}[P_{c}, P_{a}] = \frac{\delta R^{a}_{\kappa\lambda}}{\delta x^{\mu}}P_{a},$$

where we used the fact that $[P_c, P_a] = 0$ Also

$$\begin{aligned}\frac{1}{4}\omega_\mu^{(*)ab} R^{cd}_{\kappa\lambda}[J_{ab}, J_{cd}] &= \omega_\mu^{(*)ac} R^b_{c\kappa\lambda} J_{ab}, \\ \frac{1}{2}\omega_\mu^{(*)ab} R^{c}_{\kappa\lambda}[J_{ab}, P_{c}] &= \omega_\mu^{(*)ac} R^b_{\kappa\lambda} P_{a},\end{aligned}$$

so the relation (12.5) is written as

$$\begin{aligned}\left[D_\mu^{(*)}, [D_\kappa^{(*)}, D_\lambda^{(*)}]\right] &= \left(\frac{1}{2}\frac{\delta R^{ab}_{\kappa\lambda}}{\delta x^\mu} + \omega_\mu^{(*)ac} R^b_{c\kappa\lambda}\right) \\ &\quad + J_{ab} + \left(\frac{\delta R^{a}_{\kappa\lambda}}{\delta x^\mu} + R^a_{b\kappa\lambda} A^b_\mu + R^c_{\kappa\lambda}\omega_{\mu b}^{(*)a}\right) P_a.\end{aligned}$$

Defining

$$D_\mu R^{ab}_{\kappa\lambda} = \frac{1}{2}\frac{\delta R^{ab}_{\kappa\lambda}}{\delta x^\mu} + \omega_\mu^{(*)ac} R^b_{c\kappa\lambda}, \tag{12.6}$$

$$D_\mu R^{a}_{\kappa\lambda} = \frac{1}{2}\frac{\delta R^{a}_{\kappa\lambda}}{\delta x^\mu} + A^b_\mu + R^c_{\kappa\lambda}\omega_{\mu b}^{(*)a}, \tag{12.7}$$

we have the relations:

$$\begin{aligned}D_\mu R^{ab}_{\kappa\lambda} + D_\kappa R^{ab}_{\lambda\mu} + D_\lambda R^{ab}_{\mu\kappa} = 0, \\ D_\mu R^{a}_{\kappa\lambda} + D_\kappa R^{a}_{\lambda\mu} + D_\lambda R^{a}_{\mu\kappa} = 0.\end{aligned}$$

In the similar way, from

$$\mathcal{Q}_{(\alpha\beta\gamma)}\left[D_\alpha^{(*)}, [D_\beta^{(*)}, D_\gamma^{(*)}]\right] = 0$$

we get for the Q-curvature and torsion the identities below:

$$D_\alpha Q^{ab}_{\beta\gamma} + D_\beta Q^{ab}_{\gamma\alpha} + D_\gamma Q^{ab}_{\alpha\beta} = 0$$

and

$$D_\alpha Q^{a}_{\beta\gamma} + D_\beta Q^{a}_{\gamma\alpha} + D_\gamma Q^{a}_{\alpha\beta} = 0,$$

where we put

$$\begin{aligned}D_\alpha Q^{ab}_{\beta\gamma} &= \frac{1}{2}\frac{\partial Q^{ab}_{\beta\gamma}}{\partial\bar{\xi}^\alpha} + \Theta_\alpha^{(*)ac} Q^b_{c\beta\gamma}, \\ D_\alpha Q^{a}_{\beta\gamma} &= \frac{\partial Q^{a}_{\beta\gamma}}{\partial\bar{\xi}^\alpha} + Q^a_{b\beta\gamma}\Psi^b_\alpha + Q^b_{\beta\gamma}\Theta_{\alpha b}^{(*)a}.\end{aligned}$$

12.6 Yang-Mills fields

In this section, we study Yang-Mills fields and we derive the generalized Yang-Mills equations in the framework of our approach. In such a case we consider a vector field A

$$F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu + i[A_\mu, A_\nu] \tag{12.8}$$

represents the Yang-Mills field, A_μ is given by

$$A_\mu = A^i_\mu \tau_i, \qquad [\tau_i, \tau_j] = C^k_{ij}\tau_k, \tag{12.9}$$

the elements τ_i are the generators which satisfy the commutation relations of the Lie algebra, and D_μ represent the gauge covariant derivatives.

Using (12.8), (12.9) of the matrices A_μ we find that

$$F_{\mu\nu} = F^i_{\mu\nu}\tau_i,$$

where the field strengths are given by

$$F^k_{\mu\nu} = D_\mu A^k_\nu - D_\nu A^k_\mu + iA^i_\mu A^j_\nu C^k_{ij}.$$

Moreover, the generalized gauge field is defined by the quantities $F_{XY}, X, Y = \{\mu, \nu, \alpha, \beta\}$, that is

$$\begin{aligned}
[\tilde{D}_\mu, \tilde{D}_\alpha] &= [D_\mu, D_\alpha] + iF_{\mu\alpha},\\
[\tilde{D}_\mu, \tilde{D}^\alpha] &= [D_\mu, D^\alpha] + i\bar{F}^\alpha_\mu,\\
[\tilde{D}_\alpha, \tilde{D}_\beta] &= [D_\alpha, D_\beta] + iF^\beta_\alpha,\\
[\tilde{D}_\alpha, \tilde{D}^\beta] &= [D_\alpha, D^\beta] + iF_{\alpha\beta},\\
[\tilde{D}^\alpha, \tilde{D}^\beta] &= [D^\alpha, D^\beta] + iF^{\alpha\beta},
\end{aligned}$$

with

$$\begin{aligned}
F_{\mu\alpha} &= D_\mu A_\alpha - D_\alpha A_\mu + i[A_\mu, A_\alpha],\\
\bar{F}^\alpha_\mu &= D_\mu \bar{A}^\alpha - \bar{D}^\alpha A_\mu + i[A_\mu, \bar{A}^\alpha],\\
F^\beta_\alpha &= D_\alpha \bar{A}^\beta - \bar{D}^\beta A_\alpha + i[A_\alpha, \bar{A}^\beta],\\
F_{\alpha\beta} &= D_\alpha A_\beta - D_\beta A_\alpha + i[A_\alpha, A_\beta],\\
\bar{F}^{\alpha\beta} &= \bar{D}^\alpha \bar{A}^\beta - \bar{D}^\beta \bar{A}^\alpha + i[\bar{A}^\alpha, \bar{A}^\beta].
\end{aligned}$$

In our space $S^{(*)}(M)$ the Yang-Mills generalized action can be written in the form

$$S_{GF} = \int d^4x d^4\xi d^4\bar{\xi} h(trF_{\mu\nu}F^{\mu\nu} + trF_{\mu\alpha}\bar{F}^{\mu\alpha} + trF_{\alpha\beta}\bar{F}^{\alpha\beta} + trF^\beta_\alpha F^\alpha_\beta), \tag{12.10}$$

where $F_{\mu\nu}$ represent the internal quantities in the base manifold, F^{μ}_{α} the field in the tensor bundle and $F_{\alpha\beta}$ the internal quantities in the internal space.

In order to derive the generalized Yang-Mills equations we get the Lagrangian

$$\mathcal{L}_{YM}(A_X, D_X A_Y),$$

where $A_X = \{A_\mu, A_\alpha, A^\beta\}$ and $D_X A_Y$ represent

$$D_X A_Y = \{D_\mu A_\nu, D_\alpha A_\nu, \bar{D}^\alpha, D_\alpha A_\beta, \bar{D}^\alpha A_\beta, D_\mu A_\alpha, D_\mu A^\alpha\}.$$

Varying the action (12.10) and taking into account the Euler-Lagrange equations

$$D_X\left(\frac{\partial \mathcal{L}_{YM}}{\partial(D_X A_Y)}\right) - \frac{\partial \mathcal{L}_{YM}}{\partial A_Y} = 0, \tag{12.11}$$

obtain the generalized Yang-Mills equations in the following form:

$$\begin{aligned} \tilde{D}^\mu F_{\mu\nu} + \tilde{D}^\alpha F_{\alpha\nu} + \tilde{D}_\alpha \bar{F}^\alpha_\nu &= 0, \\ \tilde{D}_\mu F^{\mu\beta} + \tilde{D}_\alpha F^{\alpha\beta} + \tilde{D}^\alpha F^\beta_\alpha &= 0, \\ \tilde{D}_\mu F^\mu_\beta + \tilde{D}_\alpha F^\alpha_\beta + \tilde{D}^\alpha F_{\alpha\beta} &= 0, \end{aligned}$$

we used the trace properties of the operators τ_α with the normalization condition

$$tr(\tau^\alpha \tau^\beta) = \frac{1}{2}\delta^{\alpha\beta}.$$

12.7 Yang-Mills-Higgs field

In this last Section we shall give the form of Yang-Mills-Higgs field in a sufficiently generalized form. The usual case has been studied with the appropriate Lagrangian $\mathcal{L}$ ... the corresponding Euler–Lagrange equations.

Here, we get a scalar field ϕ of mass m which is valuated in the Lie algebra $\mathcal{G}$ of consideration and is defined by

$$\phi : M^{(4)} \times \mathbb{C}^4 \times \mathbb{C}^4 \to \mathcal{G}$$

$$\phi(x^\mu, \xi_\alpha, \bar{\xi}^\alpha \in \mathcal{G}.$$

... is in adjoint representations, its covariant derivatives are given by

$$\tilde{D}_\mu \phi = D_\mu \phi + [A_\mu, \phi], \tilde{D}_\alpha \phi = D_\alpha \phi + [A_\alpha, \phi], \tilde{D}^\alpha \phi = D^\alpha \phi + [A^\alpha, \phi].$$

The first of these relations, after taking into account (12.9), becomes

$$\tilde{D}_\mu \phi = D_\mu \phi + A^\alpha_\mu \phi^b C^c_{\alpha c} \tau_b; \tag{12.12}$$

for $\tilde{D}_\alpha \phi$, $\tilde{D}^\alpha \phi$ similar relations are produced.

The generalized Lagrangian is given by the following form:

$$\mathcal{L} = \mathcal{L}_{YM} - \frac{1}{2} tr(\tilde{D}_\mu \phi) - \frac{1}{2} tr(\tilde{D}_\alpha \phi)(\tilde{D}^\alpha \phi) + \frac{1}{2} m^2 tr \phi^2.$$

Using (12.12) and getting (12.11) for this Lagrangian $\mathcal{L}$, the generalized Yang-Mills-Higgs equations are as follows:

$$\begin{aligned} \tilde{D}^\mu F_{\mu\nu} + \tilde{D}^\alpha F_{\alpha\nu} + \tilde{D}_\alpha F^\alpha_\nu + [\phi, \tilde{D}_\nu \phi] &= 0, \\ \tilde{D}_\mu F^{\mu\beta} + \tilde{D}_\alpha F^{\alpha\beta} + \tilde{D}^\alpha F^\beta_\alpha + [\phi, \tilde{D}^\beta \phi] &= 0, \\ \tilde{D}_\mu F^\mu_\beta + \tilde{D}_\alpha F^\alpha_\beta + \tilde{D}^\alpha F_{\alpha\beta} + [\phi, \tilde{D}_\beta \phi] &= 0. \end{aligned}$$

These equations defines a Poincare like gravity theory on spaces where the metric tensor $g_{\mu\nu}(x, \omega)$ depends on internal independent variables $\omega = \left(\xi, \bar{\xi}\right)$.

Chapter 13

Spinor Bundle on Internal Deformed Systems

13.1 Introduction

It was formulated [140, 148] the concept of a spinor bundle $S^{(2)}M$ and its relation to the Poincarè group. This group, consisting of the set of rotations, boosts and translations, gives an exact meaning to the terms: "momentum", "energy", "mass", and "spin" and is used to determine characteristics of the elementary particles. Also, it is a gauge, acting locally in the space-time. Hence we may perform Poincarè transformations for a physical approach. In that study we considered a base manifold $(M, g_{\mu\nu}(x, \xi, \bar{\xi}))$ where the metric tensor depends on the position coordinates and the spinor (Dirac) variables $(\xi_\alpha, \bar{\xi^\alpha}) \in C^4 \times C^4$. A spinor bundle $S^{(1)}(M)$ can be constructed from one of the principal fiber bundles with fiber $F = C^4$. Each fiber is diffeomorphic with one proper Lorentz group.

In our study we apply an analogous method as in the theory of deformed bundles developed in [144], for the case of a **spinor bundle** $S^{(2)}M = M \times C^{4\cdot 2}$ in connection with a deformed **internal fiber** R. Namely our space has the form $S^{(2)}M \times R$. The consideration of Miron d - connections [109], which preserve the $h-$ and $v-$distributions is of vital importance in our approach, as in the previous work. This standpoint enables us to use a more general group $G^{(2)}$, called the structural group of all rotations and translations, that is isomorphic to the Poincarè Lie algebra. A **spinor** is an element of the spinor bundle $S^{(2)}(M) \times R$ where R represents the **internal fiber of deformation.**
The local variables are in this case

$$(x^\mu, \xi_\alpha, \bar{\xi}^\alpha, \lambda) \in S^{(2)}(M) \times R = \tilde{S}^{(2)}(M), \lambda \in R.$$

The non-linear connection on $\tilde{S}^{(2)}(M)$ is defined analogously, as for the vector bundles of order two [111, 120]

$$T(\tilde{S}^{(2)}M) = H(\tilde{S}^2 M) \oplus \mathcal{F}^{(1)}(\tilde{S}^{(2)}M) \oplus \mathcal{F}^{(2)}(\tilde{S}^{(2)}M) \oplus R,$$

where $\mathcal{H}$,$\mathcal{F}^{(1)}$,$\mathcal{F}^{(2)}$,R represent the horizontal, vertical, normal and deformation distributions respectively.

The fundamental gauge 1-form fields which take values from the Lie algebra of the Poincarè group will have the following form in the local bases of their corresponding distributions

$$\rho_\mu(x,\xi,\bar{\xi},\lambda) = \frac{1}{2}\omega_\mu^{*ab}\mathcal{J}_{ab} + h_\mu^a(x,\xi,\bar{\xi},\lambda)P_a \quad (13.1)$$

$$\zeta_\alpha = \frac{1}{2}\theta_\alpha^{(*)ab}J_{ab} + \psi_\alpha^a P_a \quad (13.2)$$

$$\bar{\zeta}^\alpha = \frac{1}{2}\bar{\theta}^{(*)\alpha ab}J_{ab} + \bar{\psi}^{\alpha a}P_a \quad (13.3)$$

$$\rho_o = \frac{1}{2}\omega_o^{ab}J_{ab} + L_o^a P_a \quad (13.4)$$

where, J_{ab}, P_a are the generators of the four–dimensional Poincarè group, namely the angular momentum and linear momentum, $\omega_\mu^{(*)ab}$ represent the Lorentz - spin connection coefficients, $\bar{\Psi}^{\alpha a}$, Ψ_α^a, $\theta_\alpha^{(*)ab}$, $\bar{\theta}^{(*)\alpha ab}$ are the given spin-tetrad and spin - connection coefficients, and L_o^a deformed tetrad coefficients. We use Greek letters $\lambda, \mu, \nu, \ldots$ for space-time indices, $\alpha, \beta, \gamma, \ldots$ for the spinor, $a, b, c, \ldots$ for Lorentz ones, and the index (o) represents the deformed variable; $\lambda, \alpha, a = 1, \ldots, 4$. The general transformations of coordinates on $\tilde{S}^{(2)}M$ are

$$x'^\mu = x'^\mu(x^\nu), \xi'_\alpha = \xi'_\alpha(\xi_\beta, \bar{\xi}^\beta), \bar{\xi}'^\alpha = \bar{\xi}'^\alpha(\bar{\xi}^\beta, \xi_\beta), \lambda' = \lambda \quad (13.5)$$

13.2 Connections

We define the following gauge covariant derivatives

$$(a)\ \mathcal{D}_\mu^{(*)} = \frac{\delta}{\delta x^\mu} + \frac{1}{2}\omega_\mu^{(*)ab}\mathcal{J}_{ab} \qquad (b)\ \mathcal{D}^{(*)\alpha} = \frac{\delta}{\delta \xi_\alpha} + \frac{1}{2}\bar{\Theta}^{(*)\alpha ab}\mathcal{J}_{ab}$$

$$(c)\ \mathcal{D}_\alpha^{(*)} = \frac{\partial}{\partial \bar{\xi}^\alpha} + \frac{1}{2}\Theta_\alpha^{(*)ab}\mathcal{J}_{ab}$$

$$(d)\ \mathcal{D}_o^{(*)} = \frac{\partial}{\partial \lambda} + \omega_o^{ab}\mathcal{J}_{ab} \quad (13.6)$$

where,

$$
\begin{aligned}
\frac{\delta}{\delta x^{\mu}} &= \frac{\partial}{\partial x^{\mu}} + \mathcal{N}_{\alpha\mu}\frac{\partial}{\partial \xi_{\alpha}} - \bar{\mathcal{N}}^{\alpha}_{\mu}\frac{\partial}{\partial \bar{\xi}^{\alpha}} - \mathcal{N}^{o}_{\mu}\frac{\partial}{\partial \lambda} \\
\frac{\delta}{\delta \xi_{\alpha}} &= \frac{\partial}{\partial \xi_{\alpha}} - \bar{\mathcal{N}}^{\alpha\beta}_{o}\frac{\partial}{\partial \bar{\xi}^{\beta}}, \frac{\partial}{\partial \lambda} = L^{\mu}_{o}\frac{\partial}{\partial x^{\mu}}.
\end{aligned} \tag{13.7}
$$

The nonlinear connection coefficients are defined further. The space-time, Lorentz, spin frames and the deformed frame are connected by the relations

$$
\begin{aligned}
(a)\ \frac{\delta}{\delta x^{\mu}} &= h^{a}_{\mu}\frac{\delta}{\delta x^{a}} \\
(b)\ \frac{\partial}{\partial \xi_{\ \alpha}} &= \bar{\Psi}^{\alpha a}\frac{\partial}{\partial x^{a}} (b')\ \frac{\partial}{\partial \bar{\xi}^{\alpha}} = \Psi^{a}_{\alpha}\frac{\partial}{\partial x^{a}} \\
(c)\ \frac{\partial}{\partial \lambda} &= L^{\mu}_{o}\frac{\partial}{\partial x^{\mu}}.
\end{aligned} \tag{13.8}
$$

The relation (13.8a) is a generalization of the well - known principle of equivalence in the total space of the spinor bundle $S^{(2)}M$. In addition, the relations (13.8a, b, b', c) represent a generalized form of the equivalence principle, since the considered deformed spinor bundle contains spinors as internal variables.

The absolute differential of an arbitrary contravariant vector X^{ν} in $\tilde{S}^{(2)}M$, has the form

$$
\mathcal{D}X^{\nu} = (\mathcal{D}^{(*)}_{\mu}X^{\nu})dx^{\mu} + (\mathcal{D}^{(*)\alpha}X^{\nu})d\xi_{\alpha} + (\mathcal{D}^{(*)}_{\alpha}X^{\nu})d\bar{\xi}^{\alpha} + (\mathcal{D}^{(*)}_{o}X^{\nu})d\lambda \tag{13.9}
$$

The differentials $\mathcal{D}\xi_{\alpha}$, $\mathcal{D}\bar{\xi}^{\alpha}$, $\mathcal{D}\lambda$ can be written

$$
\begin{aligned}
\mathcal{D}\xi_{\beta} &= d\xi_{\beta} + \mathcal{N}^{o}_{\alpha\beta}d\bar{\xi}^{\alpha} + \tilde{\mathcal{N}}^{o\alpha}_{\beta}d\xi_{\alpha} + \mathcal{N}_{\beta\mu}dx^{\mu} \\
\mathcal{D}\bar{\xi}^{\beta} &= d\bar{\xi}^{\beta} + \mathcal{N}^{\beta}_{o\alpha}d\bar{\xi}^{\alpha} - \tilde{\mathcal{N}}^{\beta\alpha}_{o}d\xi_{\alpha} - \bar{\mathcal{N}}^{\beta}_{\mu}dx^{\mu} \\
\mathcal{D}_{o}\lambda &= d\lambda + \mathcal{N}^{o}_{\kappa}dx^{\kappa} - \tilde{\mathcal{N}}^{\alpha}_{o}d\xi_{\alpha} - \mathcal{N}^{o}_{\alpha}d\bar{\xi}^{\alpha},
\end{aligned} \tag{13.10}
$$

where

$$
\mathcal{N} = \{\mathcal{N}^{o}_{\alpha\beta}, \mathcal{N}_{\beta\mu}, \tilde{\mathcal{N}}^{o\alpha}_{\beta}, \bar{\mathcal{N}}^{\beta}_{\mu}, \tilde{\mathcal{N}}^{\beta\alpha}_{o}, \mathcal{N}^{\beta}_{o\alpha}, \mathcal{N}^{o}_{\kappa}, \tilde{\mathcal{N}}^{\beta}_{o}, \mathcal{N}^{o}_{\alpha}\}
$$

represent the coefficients of the nonlinear connection which are given by

$$
\begin{aligned}
\mathcal{N}_{\beta\mu} &= \frac{1}{2}\omega^{(*)ab}_{\mu}\mathcal{J}_{ab}\xi_{\beta}, \qquad \bar{\mathcal{N}}^{\beta}_{\mu} = -\frac{1}{2}\omega^{(*)ab}_{\mu}\mathcal{J}_{ab}\bar{\xi}^{\beta}, \mathcal{N}^{o}_{\mu} = \frac{1}{2}\omega^{\alpha b}_{o\mu}\mathcal{J}_{ab} \\
\tilde{\mathcal{N}}^{o\alpha}_{\beta} &= \frac{1}{2}\bar{\theta}^{(*)\alpha ab}\mathcal{J}_{ab}\xi_{\beta},\ \tilde{\mathcal{N}}^{\alpha\beta}_{o} = -\frac{1}{2}\bar{\theta}^{(*)\alpha ab}\mathcal{J}_{ab}\bar{\xi}^{\beta},\ \tilde{\mathcal{N}}^{\alpha}_{o} = \frac{1}{2}\omega^{ab}_{o}\mathcal{J}_{ab}\bar{\xi}^{\alpha}, \\
\mathcal{N}^{o}_{\alpha\beta} &= \frac{1}{2}\theta^{(*)ab}_{\alpha}\mathcal{J}_{ab}\xi_{\beta},\ \mathcal{N}^{\beta}_{o\alpha} = -\frac{1}{2}\theta^{(*)ab}_{\alpha}\mathcal{J}_{ab}\bar{\xi}^{\beta},\ \mathcal{N}^{o}_{\alpha} = \frac{1}{2}\omega^{ab}_{o}\mathcal{J}_{ab}\xi_{\alpha}.
\end{aligned} \tag{13.11}
$$

The metrical structure in the deformed spinor bundle $\tilde{S}^{(2)}M$ has the form

$$\begin{aligned} G &= g_{\mu\nu}(x,\xi,\bar{\xi},\lambda)dx^{\mu}\otimes dx^{\nu} + g_{\alpha\beta}(x,\xi,\bar{\xi},\lambda)\mathcal{D}\bar{\xi}^{\alpha}\otimes\mathcal{D}\bar{\xi}^{*\beta} + \\ &\quad + g^{\alpha\beta}(x,\xi,\bar{\xi},\lambda)\mathcal{D}\xi_{\alpha}\otimes\mathcal{D}\xi^{*}_{\beta} + g_{oo}(x,\xi,\bar{\xi},\lambda)\mathcal{D}\lambda\otimes\mathcal{D}\lambda \qquad (13.12)\end{aligned}$$

where '*' denotes Hermitean conjugation. The local adapted frame is given by

$$\frac{\delta}{\delta\zeta^A} = \{\frac{\delta}{\delta x^{\lambda}}, \frac{\delta}{\delta\xi_{\alpha}}, \frac{\partial}{\partial\bar{\xi}^{\alpha}}, \frac{\partial}{\partial\lambda}\}$$

and the associated dual frame

$$\delta\zeta^A = \{\mathcal{D}x^{\kappa}\equiv dx^{\kappa}, \mathcal{D}\xi_{\beta}, \mathcal{D}\bar{\xi}^{\beta}, \mathcal{D}_o\lambda\}, \qquad (13.13)$$

where the terms $\frac{\delta}{\delta x^{\lambda}}, \frac{\delta}{\delta\xi_{\alpha}}, \mathcal{D}_o\lambda$, $\mathcal{D}x^{\kappa}, \mathcal{D}\xi_{\beta}, \mathcal{D}\bar{\xi}^{\beta}$, are provided by the relations (13.7), (13.9), (13.10).

The considered connection in $\tilde{S}^{(2)}(M)$ is a d-connection, having with respect to the adapted basis the coefficients

$$\begin{aligned} \Gamma^{A}_{BC} &= \{\Gamma^{(*)\mu}_{\nu\rho}, C^{\mu}_{\nu\alpha}, \bar{C}^{\mu\alpha}_{\nu}, \Gamma^{(*)\mu}_{\nu o}, \bar{\Gamma}^{(*)\alpha}_{\beta\lambda}, \tilde{C}^{\beta}_{\alpha\gamma}, \tilde{C}^{\beta\alpha}_{\gamma}, \bar{\Gamma}^{(*)\beta}_{o\gamma}, \\ &\quad \Gamma^{(*)\beta}_{\alpha\nu}, C^{\gamma}_{\beta\alpha}, C^{\gamma\alpha}_{\beta}, C^{\alpha}_{\beta o}, \Gamma^{(*)o}_{o\mu}, \bar{C}^{o\alpha}_{o}, C^{o}_{o\alpha}, L^{o}_{oo}\} \qquad (13.14)\end{aligned}$$

defined by the generic relations

$$\mathcal{D}_{\frac{\delta}{\delta z^C}}\frac{\delta}{\delta z^B} = \Gamma^{A}_{BC}\frac{\delta}{\delta z^A}, \qquad \frac{\delta}{\delta z^A}\in\left\{\frac{\delta}{\delta x^{\mu}}, \frac{\delta}{\delta\xi_{\alpha}}, \frac{\partial}{\partial\bar{\xi}^{\alpha}}, \frac{\partial}{\partial\lambda}\right\}. \qquad (13.15)$$

It preserves the distributions $\mathcal{H}$,$\mathcal{F}^{(1)}$, $\mathcal{F}^{(2)}$,R, and its coefficients are defined by

$$\begin{aligned}
\mathcal{D}_{\frac{\delta}{\delta x^{\rho}}}\frac{\delta}{\delta x^{\nu}} &= \Gamma^{(*)\mu}_{\nu\rho}\frac{\delta}{\delta x^{\mu}}, \qquad \mathcal{D}_{\frac{\partial}{\partial\bar{\xi}^{\alpha}}}\frac{\delta}{\delta x^{\nu}} = C^{\mu}_{\nu\alpha}\frac{\delta}{\delta x^{\mu}}, \\
\mathcal{D}_{\frac{\delta}{\delta\xi_{\alpha}}}\frac{\delta}{\delta x^{\nu}} &= \bar{C}^{\mu\alpha}_{\nu}\frac{\delta}{\delta x^{\mu}}, \qquad \mathcal{D}_{\frac{\partial}{\partial\lambda}}\frac{\delta}{\delta x^{\nu}} = \Gamma^{(*)\mu}_{\nu o}\frac{\delta}{\delta x^{\mu}}, \\
\mathcal{D}_{\frac{\delta}{\delta x^{\lambda}}}\frac{\delta}{\delta\xi_{\alpha}} &= \bar{\Gamma}^{(*)\alpha}_{\beta\lambda}\frac{\delta}{\delta\xi_{\beta}}, \qquad \mathcal{D}_{\frac{\partial}{\partial\bar{\xi}^{\alpha}}}\frac{\delta}{\delta\xi_{\beta}} = \tilde{C}^{\beta}_{\alpha\gamma}\frac{\delta}{\delta\xi_{\gamma}}, \\
\mathcal{D}_{\frac{\delta}{\delta\xi_{\alpha}}}\frac{\delta}{\delta\xi_{\beta}} &= \tilde{C}^{\beta\alpha}_{\gamma}\frac{\delta}{\delta\xi_{\gamma}}, \qquad \mathcal{D}_{\frac{\partial}{\partial\lambda}}\frac{\delta}{\delta\xi_{\beta}} = \bar{\Gamma}^{(*)\beta}_{o\gamma}\frac{\delta}{\delta\xi_{\gamma}}, \\
\mathcal{D}_{\frac{\delta}{\delta x^{\nu}}}\frac{\partial}{\partial\bar{\xi}^{\alpha}} &= \Gamma^{(*)\beta}_{\alpha\nu}\frac{\partial}{\partial\bar{\xi}^{\beta}}, \qquad \mathcal{D}_{\frac{\partial}{\partial\bar{\xi}^{\alpha}}}\frac{\partial}{\partial\bar{\xi}^{\beta}} = C^{\gamma}_{\beta\alpha}\frac{\partial}{\partial\bar{\xi}^{\gamma}}, \\
\mathcal{D}_{\frac{\delta}{\delta\xi_{\alpha}}}\frac{\partial}{\partial\bar{\xi}^{\beta}} &= C^{\gamma\alpha}_{\beta}\frac{\partial}{\partial\bar{\xi}^{\gamma}}, \qquad \mathcal{D}_{\frac{\partial}{\partial\lambda}}\frac{\partial}{\partial\bar{\xi}^{\beta}} = C^{\alpha}_{\beta o}\frac{\partial}{\partial\bar{\xi}^{\alpha}}, \\
\mathcal{D}_{\frac{\delta}{\delta x^{\mu}}}\frac{\partial}{\partial\lambda} &= \Gamma^{(*)o}_{o\mu}\frac{\partial}{\partial\lambda}, \qquad \mathcal{D}_{\frac{\delta}{\delta\xi_{\alpha}}}\frac{\partial}{\partial\lambda} = \bar{C}^{o\alpha}_{o}\frac{\partial}{\partial\lambda}\mathcal{D}_{\frac{\partial}{\partial\bar{\xi}^{\alpha}}}\frac{\partial}{\partial\lambda} = C^{o}_{o\alpha}\frac{\partial}{\partial\lambda}, \\
\mathcal{D}_{\frac{\partial}{\partial\lambda}}\frac{\partial}{\partial\lambda} &= L^{o}_{oo}\frac{\partial}{\partial\lambda}.
\end{aligned}$$

The covariant differentiation of tensors, spin-tensors and Lorentz - type tensors of arbitrary rank is defined as follows:

$$
\begin{aligned}
\bigtriangledown_\kappa T^{\mu\dots}_{\nu\dots} &= \frac{\delta T^{\mu\dots}_{\nu\dots}}{\delta x^\kappa} + \Gamma^{(*)\mu}_{\rho\kappa} T^{\rho\dots}_{\nu\dots} + \dots - \Gamma^{(*)\rho}_{\nu\kappa} T^{\mu\dots}_{\rho\dots} \\
\bigtriangledown^\alpha T^{\mu\dots}_{\nu\dots} &= \frac{\delta T^{\mu\dots}_{\nu\dots}}{\delta \xi_\alpha} + \bar{C}^{\mu\alpha}_{\kappa} T^{\kappa\dots}_{\nu\dots} + \dots - \bar{C}^{\kappa\alpha}_{\nu} T^{\mu\dots}_{\kappa\dots} \\
\bigtriangledown_\alpha T^{\mu\dots}_{\nu\dots} &= \frac{\partial T^{\mu\dots}_{\nu\dots}}{\partial \bar{\xi}^\alpha} + C^{\mu}_{\kappa\alpha} T^{\kappa\dots}_{\nu\dots} + \dots - \bar{C}^{\kappa}_{\nu\alpha} T^{\mu\dots}_{\kappa\dots} \\
\bigtriangledown_o T^{\mu\dots}_{\nu\dots} &= \frac{\partial T^{\mu\dots}_{\nu\dots}}{\partial \lambda} + \Gamma^{(*)\mu}_{\kappa o} T^{\kappa\dots}_{\nu\dots} - \dots + \Gamma^{(*)\kappa}_{\nu o} T^{\mu\dots}_{\kappa\dots} \\
\bigtriangledown_\kappa \Phi^{\alpha\dots}_{\beta\dots} &= \frac{\delta \Phi^{\alpha\dots}_{\beta\dots}}{\delta x^\kappa} - \bar{\Gamma}^{(*)\gamma}_{\beta\kappa} \Phi^{\alpha\dots}_{\gamma\dots} - \dots + \Phi^{\gamma\dots}_{\beta\dots} \bar{\Gamma}^{(*)\alpha\dots}_{\gamma\kappa} \\
\bigtriangledown^\delta \Phi^{\alpha\dots}_{\beta\dots} &= \frac{\delta \Phi^{\alpha\dots}_{\beta\dots}}{\delta \xi_\delta} - \tilde{C}^{\gamma\delta}_{\beta} \Phi^{\alpha\dots}_{\gamma\dots} - \dots + \Phi^{\gamma\dots}_{\beta\dots} \tilde{C}^{\alpha\delta}_{\gamma}
\end{aligned}
\tag{13.16}
$$

$$
\begin{aligned}
\bigtriangledown_\delta \Phi^{\alpha\dots}_{\beta\dots} &= \frac{\partial \Phi^{\alpha\dots}_{\beta\dots}}{\partial \bar{\xi}^\delta} - C^{\gamma}_{\beta\delta} \Phi^{\alpha\dots}_{\gamma\dots} - \dots + \Phi^{\gamma\dots}_{\beta\dots} C^{\alpha}_{\gamma\delta} \\
\bigtriangledown_o \Phi^{\alpha\dots}_{\beta\dots} &= \frac{\partial \Phi^{\alpha\dots}_{\beta\dots}}{\partial \lambda} - \bar{\Gamma}^{(*)\gamma}_{o\beta} \Phi^{\alpha\dots}_{\gamma\dots} - \dots + \Phi^{\gamma\dots}_{\beta\dots} \Gamma^{(*)\alpha}_{o\gamma} \\
\bigtriangledown^{(*)}_\mu V^{a\dots}_{c\dots} &= \frac{\delta V^{a\dots}_{c\dots}}{\delta x^\mu} + \omega^{(*)a}_{\mu b} V^{b\dots}_{c\dots} + \dots - \omega^{(*)b}_{\mu c} V^{a\dots}_{b\dots} \\
\bigtriangledown^{(*)\alpha} V^{a\dots}_{c\dots} &= \frac{\delta V^{a\dots}_{c\dots}}{\delta \xi_\alpha} + \dots + \bar{\theta}^{(*)a}_{\alpha b} V^{b\dots}_{c\dots} + \dots - \theta^{(*)b}_{\alpha c} V^{a\dots}_{b\dots} \\
\bigtriangledown^{(*)}_\alpha V^{a\dots}_{c\dots} &= \frac{\partial V^{a\dots}_{c\dots}}{\partial \bar{\xi}_\alpha} + \theta^{(*)a}_{\alpha b} V^{b\dots}_{c\dots} + \dots - \theta^{(*)b}_{\alpha c} V^{a\dots}_{b\dots} \\
\bigtriangledown^{(*)}_o V^{a\dots}_{c\dots} &= \frac{\partial V^{a\dots}_{c\dots}}{\partial \lambda} + \omega^{(*)a}_{ob} V^{b\dots}_{c\dots} + \dots - \omega^{(*)b}_{oc} V^{a\dots}_{b\dots}.
\end{aligned}
$$

The covariant derivatives of the metric tensor $g_{\mu\nu}$ are postulated to be zero:

$$
\mathcal{D}^{(*)}_\mu g_{\kappa\lambda} = 0, \mathcal{D}^{(*)\alpha} g_{\kappa\lambda} = 0, \mathcal{D}^{(*)}_\alpha g_{\kappa\lambda} = 0, \mathcal{D}^{(*)}_{(o)} g_{\kappa\lambda} = 0. \tag{13.17}
$$

13.3 Curvatures and Torsions

From the relations (13.6) we obtain the curvatures and torsions of the space $\tilde{S}^{(2)}M$

$$
[\mathcal{D}^{(*)}_\mu, \mathcal{D}^{(*)}_\nu] = \mathcal{D}^{(*)}_\mu \mathcal{D}^{(*)}_\nu - \mathcal{D}^{(*)}_\nu \mathcal{D}^{(*)}_\mu = R^a_{\mu\nu} P_a + \frac{1}{2} R^a_{\mu\nu} J_{ab} \tag{13.18}
$$

with their explicit expressions given by

$$
\begin{aligned}
R^{a}_{\mu\nu} &= \frac{\delta h^{a}_{\mu}}{\delta x^{\nu}} - \frac{\delta h^{a}_{\nu}}{\delta x^{\mu}} + \omega^{(*)a}_{\mu b} h^{b}_{\nu} - \omega^{(*)a}_{\nu b} h^{b}_{\mu}, \\
R^{ab}_{\mu\nu} &= \frac{\delta \omega^{(*)ab}_{\mu}}{\delta x^{\nu}} - \frac{\delta \omega^{(*)ab}_{\nu}}{\delta x^{\mu}} + \omega^{(*)a\rho}_{\mu} \omega^{(*)b}_{\nu\rho} - \omega^{(*)\rho a}_{\nu} \omega^{(*)b}_{\mu\rho}, \\
R^{ab}_{\mu\nu} &= \frac{\delta \omega^{(*)ab}_{\mu}}{\delta x^{\nu}} - \frac{\delta \omega^{(*)ab}_{\nu}}{\delta x^{\mu}} + \omega^{(*)a\rho}_{\mu} \omega^{(*)b}_{\nu\rho} - \omega^{(*)\rho a}_{\nu} \omega^{(*)b}_{\mu\rho},
\end{aligned}
$$

$$
[D^{(*)}_{\mu}, D^{(*)}_{\alpha}] = P^{a}_{\mu\alpha} P_a + \frac{1}{2} P^{ab}_{\mu\alpha} J_{ab} \tag{13.19}
$$

$$
\begin{aligned}
P^{ab}_{\mu\alpha} &= \frac{\delta \theta^{(*)ab}_{\alpha}}{\delta x^{\mu}} - \frac{\partial \omega^{(*)ab}_{\mu}}{\partial \bar{\xi}^{\alpha}} + \theta^{(*)b}_{\alpha c} \omega^{(*)ac}_{\mu} - \theta^{(*)a}_{\alpha c} \omega^{(*)cb}_{\mu}, \\
P^{a}_{\mu\alpha} &= \frac{\delta \psi^{a}_{\alpha}}{\delta x^{\mu}} - \frac{\partial h^{a}_{\mu}}{\partial \bar{\xi}^{\alpha}} + \omega^{(*)a}_{\mu c} \psi^{c}_{\alpha} - \theta^{(*)a}_{\alpha c} h^{c}_{\mu}, \\
P^{a}_{\mu\alpha} &= \frac{\delta \psi^{a}_{\alpha}}{\delta x^{\mu}} - \frac{\partial h^{a}_{\mu}}{\partial \bar{\xi}^{\alpha}} + \omega^{(*)a}_{\mu c} \psi^{c}_{\alpha} - \theta^{(*)a}_{\alpha c} h^{c}_{\mu}.
\end{aligned}
$$

Similarly to [148], the other four curvatures and torsions result from the commutation relations

$$
\begin{aligned}
[D^{(*)}_{\mu}, D^{\alpha}] &= \bar{P}^{a\alpha}_{\mu} P_a + \frac{1}{2} \bar{P}^{ab\alpha}_{\mu} J_{ab} && (13.20) \\
[D^{(*)}_{\alpha}, D^{(*)\beta}] &= S^{\beta a}_{\alpha} P_a + \frac{1}{2} S^{ab\beta}_{\alpha} J_{ab} && (13.21) \\
[D^{(*)}_{\alpha}, D^{(*)}_{\beta}] &= Q^{a}_{\alpha\beta} P_a + \frac{1}{2} Q^{ab}_{\alpha\beta} J_{ab} && (13.22) \\
[D^{(*)\alpha}, D^{(*)\beta}] &= \tilde{Q}^{\alpha\beta a} P_a + \frac{1}{2} \tilde{Q}^{ab\alpha\beta} J_{ab}. && (13.23)
\end{aligned}
$$

The contribution of the λ - covariant derivative $\mathcal{D}^{(*)}_{o}$ provides us the following curvatures and torsions

$$
[\mathcal{D}^{(*)}_{o}, \mathcal{D}^{(*)}_{\mu}] = R^{a}_{o\mu} P_a + \frac{1}{2} R^{ab}_{o\mu} J_{ab} \tag{13.24}
$$

$$
\begin{aligned}
R^{a}_{o\mu} &= \frac{\delta L^{a}_{\mu}}{\delta \lambda} - \frac{\delta h^{a}_{o}}{\delta x^{\mu}} + \omega^{(*)a}_{\mu b} L^{b}_{o} - \omega^{(*)a}_{bo} h^{b}_{\mu}, \\
R^{ab}_{o\mu} &= \frac{\delta \omega^{(*)ab}_{\mu}}{\delta \lambda} - \frac{\delta \omega^{(*)ab}_{o}}{\delta x^{\mu}} + \omega^{(*)a\rho}_{\mu} \omega^{(*)b}_{o\rho} - \omega^{(*)\rho a}_{o} \omega^{(*)b}_{\mu\rho}, \\
R^{ab}_{o\mu} &= \frac{\delta \omega^{(*)ab}_{\mu}}{\delta \lambda} - \frac{\delta \omega^{(*)ab}_{o}}{\delta x^{\mu}} + \omega^{(*)a\rho}_{\mu} \omega^{(*)b}_{o\rho} - \omega^{(*)\rho a}_{o} \omega^{(*)b}_{\mu\rho},
\end{aligned}
$$

$$[\mathcal{D}_o^{(*)}, \mathcal{D}_o^{(*)}] = 0, \qquad R^{ab}_{oo} = 0, \qquad R^a_{oo} = 0. \tag{13.25}$$

$$[\mathcal{D}_o^{(*)}, \mathcal{D}^{(*)\alpha}] = \bar{P}_o^{a\alpha} P_a + \frac{1}{2}\bar{P}_o^{ab\alpha} J_{ab} \tag{13.26}$$

$$\begin{aligned}
\bar{P}_o^{a\alpha} &= \frac{\partial \bar{\psi}^{a\alpha}}{\partial \lambda} - \frac{\delta L^a_o}{\delta \xi_\alpha} + \omega^{(*)a}_{oc}\bar{\psi}^{\alpha a} - \bar{\theta}^{(*)a\alpha}_c L^c_o, \\
\bar{P}_o^{ab\alpha} &= \frac{\partial \bar{\theta}^{(*)\alpha ab}}{\partial \lambda} - \frac{\delta \omega^{(*)ab}_o}{\delta \xi_\alpha} + \bar{\theta}^{(*)ab}_c \omega^{(*)ac}_\mu - \bar{\theta}^{(*)a\alpha}_c \omega^{(*)ab}_o, \\
\bar{P}_o^{ab\alpha} &= \frac{\partial \bar{\theta}^{(*)\alpha ab}}{\partial \lambda} - \frac{\delta \omega^{(*)ab}_o}{\delta \xi_\alpha} + \bar{\theta}^{(*)ab}_c \omega^{(*)ac}_\mu - \bar{\theta}^{(*)a\alpha}_c \omega^{(*)ab}_o,
\end{aligned}$$

$$[\mathcal{D}_o^{(*)}, \mathcal{D}_\alpha^{(*)}] = P^a_{o\alpha} P_a + \frac{1}{2} P^{ab}_{o\alpha} J_{ab}, \tag{13.27}$$

$$\begin{aligned}
P^a_{o\alpha} &= \frac{\partial \psi^a_\alpha}{\partial \lambda} - \frac{\partial L^a_o}{\partial \bar{\xi}^\alpha} + \omega^{(*)a}_{oc}\psi^c_\alpha - \theta^{(*)a}_{\alpha c} L^c_o, \\
P^{ab}_{o\alpha} &= \frac{\partial \theta^{(*)ab}_\alpha}{\partial \lambda} - \frac{\delta \omega^{(*)ab}_o}{\delta \bar{\xi}^\alpha} + \theta^{(*)b}_{ac}\omega^{(*)ac}_o - \theta^{(*)a}_{\alpha c}\omega^{(*)cb}_o, \\
P^{ab}_{o\alpha} &= \frac{\partial \theta^{(*)ab}_\alpha}{\partial \lambda} - \frac{\delta \omega^{(*)ab}_o}{\delta \bar{\xi}^\alpha} + \theta^{(*)b}_{ac}\omega^{(*)ac}_o - \theta^{(*)a}_{\alpha c}\omega^{(*)cb}_o.
\end{aligned}$$

13.4 Field Equations

In the following, we derive by means of the Palatini method the field equations, using a Lagrangian of the form

$$\mathcal{L} = h(R + P + \bar{P} + Q + \tilde{Q} + R_o + \bar{P}_o + Po) \tag{13.28}$$

which depends on the tetrads and on the connection coefficients,

$$\mathcal{L}(\kappa^A, \delta_M \kappa^A) = \mathcal{L}(h, \omega^{(*)}, \psi, \bar{\psi}, \theta^{(*)}, \bar{\theta}^{(*)}, \omega^{(*)}_o)$$

where,

$$\begin{aligned}
&\kappa^A \in \{(h^a_\mu(z), \omega^{(*)ab}_\mu(z), \psi^a_\alpha(z), \bar{\psi}^{\alpha a}(z), \theta^{(*)ab}_\alpha(z), \bar{\theta}^{(*)\alpha ab}(z), \omega^{(*)ab}_o(z))\}, \\
&\delta_M = \frac{\delta}{\delta z^M} \in \left\{ \frac{\delta}{\delta x^\mu}, \frac{\delta}{\delta \xi_\alpha}, \frac{\partial}{\partial \bar{\xi}^\alpha}, \frac{\partial}{\partial \lambda} \right\}, \; z = (z^M) = (x^\mu, \xi_\alpha, \bar{\xi}^\alpha, \lambda)
\end{aligned} \tag{13.29}$$

and

$$\begin{cases} R = h^{\mu}_{a} h^{\kappa}_{c} R^{c}_{\mu\kappa}, & P = h^{\mu}\psi^{\alpha}_{c} P^{c}_{\mu\alpha}, & \bar{P} = h^{\mu}\bar{\psi}_{\alpha c}\bar{P}^{c\alpha}_{\mu}, \\ Q = Q^{ab}_{\alpha\beta}\bar{\psi}^{\alpha}_{a}\bar{\psi}^{\beta}_{b}, & \tilde{Q} = \tilde{Q}^{ab\alpha\beta} = \psi_{\alpha a}\psi_{\beta b}, & S = \bar{\psi}^{\alpha}_{a}\psi_{\beta b} S^{ab\beta}_{\alpha}, \\ R_o = L^{o}_{\kappa} h^{\mu}_{c} h^{\kappa}_{a} R^{ac}_{o\mu}, & \bar{P}_o = L^{o}_{\kappa} h^{\kappa}_{a}\bar{\psi}_{\alpha c}\bar{P}^{ac\alpha}_{o}, & P_o = L^{o}_{\kappa} h^{\kappa}_{a}\bar{\psi}^{\alpha}_{c} P^{ac}_{o\alpha}. \end{cases}$$

The Euler-Lagrange equations are generally given by

$$\frac{\delta \mathcal{L}}{\delta \kappa^{(A)}} = \partial_M \left(\frac{\partial \mathcal{L}}{\partial(\partial_M \kappa^{(A)})} \right) - \frac{\partial \mathcal{L}}{\partial \kappa^{(A)}} = 0, \tag{13.30}$$

with $\partial_M = \frac{\partial}{\partial z^M} \in \left\{ \frac{\partial}{\partial x^\mu}, \frac{\partial}{\delta \xi_\alpha}, \frac{\partial}{\partial \bar{\xi}^\alpha}, \frac{\partial}{\partial \lambda} \right\}$.

From the relation (13.30), the variation of $\mathcal{L}$ with respect to the tetrads h^{ν}_{b} gives us the first field equation

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu h^{\nu}_{b})} + \partial^\alpha \frac{\partial \mathcal{L}}{\partial(\partial^\alpha h^{\nu}_{b})} + \partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha h^{\nu}_{b})} + \partial_o \frac{\partial \mathcal{L}}{\partial(\partial_o h^{\nu}_{b})} - \frac{\partial \mathcal{L}}{\partial h^{\nu}_{b}} = 0, \tag{13.31}$$

where we denoted $\partial_o = \frac{\partial}{\partial \lambda}$. Finally, after some calculations we get,

$$\tilde{H}^{b}_{\nu} - \frac{1}{2} h^{b}_{\nu} \tilde{H} = 0, \tag{13.32}$$

where we put,

$$\begin{aligned} \tilde{H} &= R + P + \bar{P} + Q + \tilde{Q} + S + R_o + \bar{P}_o + P_o, \\ \tilde{H}^{b}_{\nu} &= 2R^{b}_{\nu} + P^{b}_{\nu} + \bar{P}^{b}_{\nu} + R^{b}_{o\nu} + \bar{P}^{b}_{o\nu} + P^{b}_{(o)\nu}, \end{aligned}$$

$$\begin{cases} R^{b}_{\nu} = h^{\kappa}_{c} R^{bc}_{\nu\kappa}, & P^{b}_{\nu} = \bar{\psi}^{\alpha}_{c} P^{bc}_{\nu\alpha}, & R^{b}_{o\nu} = L^{o}_{\kappa} h^{\kappa} R^{b}_{o\nu} + L^{o}_{\nu} h^{\mu}_{c} R^{bc}_{o\mu}, \\ \bar{P}^{b}_{\nu} = \psi_{\alpha c}\bar{P}^{bc\alpha}_{\nu}, & \bar{P}^{b}_{o\nu} = L^{o}_{\nu}\bar{\psi}_{\alpha c}\bar{P}^{bc\alpha}_{o}, & P^{b}_{o\nu} = L^{o}_{\nu}\bar{\psi}^{\alpha}_{c} P^{bc}_{o\alpha} \end{cases}$$

The equation (13.32) is the **Einstein equation for empty space**, in the framework of our consideration. Also, the variation of $\mathcal{L}$ with respect to $\omega^{(*)ab}_{\mu}$ gives,

$$\partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu \omega^{(*)ab}_{\mu})} + \partial^\alpha \frac{\partial \mathcal{L}}{\partial(\partial^\alpha \omega^{(*)ab}_{\mu})} + \partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \omega^{(*)ab}_{\mu})} - \frac{\partial \mathcal{L}}{\partial \omega^{(*)ab}_{\mu}} = 0. \tag{13.33}$$

From this relation we get the second field equation in the following form,

$$\partial_\nu[h(h^{\mu}_{a} h^{\nu}_{b} - h^{\mu}_{b} h^{\nu}_{a})] - \partial^\alpha[h h^{\mu}_{a}\psi_{\alpha b}] - \partial_\alpha[h h^{\mu}_{a}\bar{\psi}^{\alpha}_{b}] - \partial_o[h L^{o}_{a} h^{\mu}_{b}] -$$

$$-h[\omega^{(*)d}_{\kappa(a}h^{(\kappa}_{b)}h^{\mu)}_{d}+h^{\mu}_{a}(\bar{\psi}^{\alpha}_{c}\theta^{(*)c}_{\alpha b}+\psi_{\alpha c}\bar{\theta}^{(*)\alpha c}_{b})+$$

$$+h^{\mu}_{d}(\bar{\psi}^{\alpha}_{b}\theta^{(*)d}_{\alpha a}+\psi_{\alpha b}\theta^{(*)\alpha d}_{a})-L^{o}_{\kappa}\omega^{(*)c}_{ob}h^{\mu}_{[c}h^{\kappa}_{a]}=0, \tag{13.34}$$

where the parantheses () and [] are used to denote symmetrization and anti-symmetrization respectively.

The variation of $\mathcal{L}$ with respect to ψ^{a}_{α} provides the field equation

$$\partial_{\nu}\frac{\partial\mathcal{L}}{\partial(\partial_{\nu}\psi^{a}_{\alpha})}+\partial^{\beta}\frac{\partial\mathcal{L}}{\partial(\partial^{\beta}\psi^{a}_{\alpha})}+\partial_{\beta}\frac{\partial\mathcal{L}}{\partial(\partial_{\beta}\psi^{a}_{\alpha})}+\partial_{o}\frac{\partial\mathcal{L}}{\partial(\partial_{o}\psi^{a}_{\alpha})}-\frac{\partial\mathcal{L}}{\partial\psi^{a}_{\alpha}}=0, \tag{13.35}$$

having the explicit form

$$\frac{1}{2}\bar{\psi}^{\beta}_{c}S^{c\alpha}_{\beta a}+\frac{1}{2}P^{ba}_{\mu\alpha}h^{\mu}_{b}+\tilde{Q}^{d\gamma\alpha}_{a}\psi_{\gamma d}=0. \tag{13.36}$$

From the variation with respect to $\bar{\psi}^{\alpha a}$

$$\partial_{\nu}\frac{\partial\mathcal{L}}{\partial(\partial_{\nu}\bar{\psi}^{\alpha a})}+\partial^{\beta}\frac{\partial\mathcal{L}}{\partial(\partial^{\beta}\bar{\psi}^{\alpha a})}+\partial_{\beta}\frac{\partial\mathcal{L}}{\partial(\partial_{\beta}\bar{\psi}^{\alpha a})}+\partial_{o}\frac{\partial\mathcal{L}}{\partial(\partial_{o}\bar{\psi}^{\alpha a})}-\frac{\partial\mathcal{L}}{\partial\bar{\psi}^{\alpha a}}=0, \tag{13.37}$$

we get the fourth field equation

$$\frac{1}{2}\bar{P}^{ba\alpha}_{\mu}h^{\mu}_{b}+\frac{1}{2}\psi_{\beta b}S^{b\beta}_{a\alpha}+\bar{\psi}^{\gamma}_{d}Q^{d}_{a\gamma\alpha}=0. \tag{13.38}$$

Finally, we write down the other three field equations which are derived from the variation of $\mathcal{L}$ with respect to the connection coefficients $\theta^{(*)ab}_{\alpha}$, $\bar{\theta}^{(*)\alpha ab}$ and $\omega^{(*)ab}_{(o)}$

$$\partial_{\mu}(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\Omega^{(\star)})})+\partial^{\beta}(\frac{\partial\mathcal{L}}{\partial(\partial^{\beta}\Omega^{(\star)})})+\partial_{\beta}(\frac{\partial\mathcal{L}}{\partial(\partial_{\beta}\Omega^{(\star)})})+\partial_{o}(\frac{\partial\mathcal{L}}{\partial(\partial_{o}\Omega^{(\star)})})=0 \tag{13.39}$$

with $\Omega^{(*)}\in\{\theta^{(*)ab}_{\alpha},\bar{\theta}^{(*)\alpha ab},\omega^{(*)ab}_{(o)}\}$, which have the corresponding detailed forms

$$\begin{aligned}
\partial_{\nu}(hh^{\nu}_{a}\psi^{\alpha}_{b})+\partial_{\beta}(h\psi^{(\beta}_{a}\bar{\psi}^{\alpha)}_{b})-\partial^{\beta}(h\bar{\psi}^{\alpha}_{a}\psi_{\beta b})+\partial_{o}(hL^{o}_{\kappa}h^{\kappa}_{a}\bar{\psi}^{\alpha}_{b})- && \quad (13.40)\\
-h[\omega^{(*)d}_{\mu b}h^{\mu}_{(d}\psi^{\alpha}_{a)}+\bar{\psi}^{[\gamma}_{d}\bar{\psi}^{\alpha]}_{(a}\theta^{(*)d}_{b)\gamma}+\theta^{(*)\beta d}_{b}\bar{\psi}^{\alpha}_{(a}\psi_{d)\beta}+L^{o}_{\kappa}\omega^{(*)}_{ob}h^{\kappa}_{(d}\bar{\psi}^{\alpha}_{a)}] &= 0,\\
-h[\omega^{(*)d}_{\mu b}h^{\mu}_{(d}\psi^{\alpha}_{a)}+\bar{\psi}^{[\gamma}_{d}\bar{\psi}^{\alpha]}_{(a}\theta^{(*)d}_{b)\gamma}+\theta^{(*)\beta d}_{b}\bar{\psi}^{\alpha}_{(a}\psi_{d)\beta}+L^{o}_{\kappa}\omega^{(*)}_{ob}h^{\kappa}_{(d}\bar{\psi}^{\alpha}_{a)}] &= 0
\end{aligned}$$

$$\begin{cases} \partial_\nu(hh^\nu_a\bar{\psi}_{\alpha b})+ & \partial^\beta(h\psi_{a[\beta}\psi_{\alpha]b})+ & \partial_\beta(h\bar{\psi}^\beta_a\psi_{\alpha b})+ \\ \partial_o(hL^o_\kappa h^\kappa_a\psi_{\alpha b})- & h[\omega^{(*)d}_{b\mu}h^\mu_{[d}\bar{\psi}_{a]\alpha}+ & \theta^{(*)d}_{\beta a}\bar{\psi}^\beta_{[d}\psi_{b]\alpha}+ \\ 2\theta^{(*)\gamma d}_a\psi_{\gamma[d}\psi_{b]\alpha}+ & L^o_\kappa(h^\kappa_d\bar{\psi}_{\alpha a}\omega^{(*)d}_{ob}- & \bar{\psi}_{\alpha d}h^\kappa_b\omega^{(*)d}_{oa})] = 0 \end{cases} \tag{13.41}$$

$$\begin{cases} hL^o_\kappa[h^\mu_a h^\kappa_d\omega^{(*)d}_{\mu b}- & h^\mu_d h^\kappa_b\omega^{(*)d}_{\mu a}+ & h^\kappa_a\bar{\psi}^\alpha_c\theta^{(*)c}_{\alpha b}- \\ \theta^{(*)d}_{\alpha a}\bar{\psi}^\alpha_b h^\kappa_d+ & h^\kappa_a\bar{\psi}_{\alpha c}\bar{\theta}^{(*)\alpha c}_b- & \bar{\theta}^{(*)\alpha d}_a\bar{\psi}_{\alpha b}h^\kappa_d]+ \\ \partial_\nu[hL^o_\kappa h^\nu_b h^\kappa_\alpha]+ & \partial_\beta[hL^o_\kappa h^\kappa_a\bar{\psi}^\beta_b]+ & \partial^\beta[hL^o_\kappa h^\kappa_a\bar{\psi}_{\beta b}] = 0 \end{cases} \tag{13.42}$$

Chapter 14

Bianchi Identities, Gauge–Higgs Fields and Deformed Bundles

14.1 Introduction

In the works [148],[143] the concepts of spinor bundle $S^{(2)}M$ as well as of deformed spinor bundle $\tilde{S}^{(2)}M$ of order two, were intorduced in the framework of a geometrical generalization of the proper spinor bundles as they have been studied from different authors e.g. [39],[201],[200].

The study of fundamental geometrical subjects as well as the gauge covariant derivatives, connections field equations e.t.c. in a deformed spinor bundle $\tilde{S}^{(2)}M$, has been developed in a sufficiently generalized approach. [143] In these spaces the internal variables or the gauge variables of fibration have been substituted by the internal (Dirac) variables $\omega = (\xi, \bar{\xi})$. In addition, another central point of our consideration is that of the internal fibres C^4.

The initial spinor bundle $(S^{(2)}M, \pi, F)$, $\pi : S^{(2)}M \to M$ was constructed from the one of the principal fibre bundles with fibre $F = C^4$ (C^4 denotes the complex space) and M the base manifold of space-time events of signature $(+, -, -, -)$. A spinor in $x \in M$ is an element of the spinor bundle $S^{(2)}M$ [148],

$$(x^\mu, \xi_\alpha, \bar{\xi}^\alpha) \in S^{(2)}M.$$

A spinor field is section of $S^{(2)}M$. A generalization of the spinor bundle $S^{(2)}M$ in an internal deformed system, has been given in the work [143] .The form of this bundle determined as

$$\tilde{S}^{(2)}M = \tilde{S}^{(2)}M \times R$$

where R represents the internal on dimension fibre of deformation. The metrical structure in the deformed spinor bundle $\tilde{S}^{(2)}M$ has th form:

$$\begin{aligned} G &= g_{\mu\nu}(x,\xi,\bar{\xi})dx^{\mu} \otimes dx^{\nu} + g_{\alpha\beta}(x,\xi,\bar{\xi},\lambda)D\bar{\xi}^{\alpha} \otimes D\bar{\xi}^{*\beta} + \\ &\quad + g^{\alpha\beta}(x,\xi,\bar{\xi},\lambda)D\xi_{\alpha} \otimes D\xi^{*}_{\beta} + g_{0,0}(x,\xi,\bar{\xi},\lambda)D\lambda \otimes D\lambda \end{aligned} \tag{14.1}$$

where "$*$" denotes Hermitean conjugation. The local adapted frame is given by:

$$\frac{\delta}{\delta\zeta^{A}} = \{\frac{\delta}{\delta x^{\lambda}}, \frac{\delta}{\delta\xi_{\alpha}}, \frac{\delta}{\delta\bar{\xi}^{\alpha}}, \frac{\partial}{\partial\lambda}\} \tag{14.2}$$

and the associated dual frame:

$$\delta\zeta^{A} = \{Dx^{K} \equiv dx^{K}, D\xi_{\beta}D\bar{\xi}_{\beta}, D_{0\lambda}\} \tag{14.3}$$

where the terms $\frac{\delta}{\delta\chi^{\lambda}}, \frac{\delta}{\delta\xi_{\alpha}}D_{0\lambda}, Dx^{K}, D\xi_{\beta}, D\bar{\xi}^{\beta}$ are provided by the relations (6)-(7) of [143].

The considered connection in $\tilde{S}^{(2)}M$ is a d-connection [109] having with respect to the adapted basis the coefficients(cf. [143]).

$$\begin{aligned} \Gamma^{A}_{BC} &= \{\Gamma^{(*)\mu}_{\nu\rho}, C^{\mu}_{\nu\alpha}, \bar{C}^{\mu\alpha}_{\nu}, \Gamma^{(*)\mu}_{\nu 0}, \bar{\Gamma}^{(*)\alpha}_{\beta\lambda}, \tilde{C}^{\beta}_{\alpha\gamma}, \tilde{C}^{\beta\alpha}_{\gamma}, \\ &\quad \tilde{\Gamma}^{(*)\beta}_{\alpha\gamma}, \Gamma^{(*)\beta}_{\alpha\nu}, C^{\gamma}_{\beta\alpha}, C^{\gamma\alpha}_{\beta}, C^{\alpha}_{\beta 0}, \Gamma^{(*)0}_{0\mu}, \bar{C}^{0\alpha}_{)}, C^{0}_{0\alpha}, L^{0}_{00}\}. \end{aligned} \tag{14.4}$$

The metric G of relation (14.1) could be considered as a definite physical application like the one given by R. Miron and G. Atanasiu for Lagrange spaces [111] for the case of spinor bundles of order two. According to our approach on $\tilde{S}^{(2)}M$ the internal variables ξ, $\bar{\xi}$ play a crucial role similar to the variables $y^{(1)}, y^{(2)}$ of the vector bundles of order two.

The non-linear connection on $\tilde{S}^{(2)}M$ is defined analogously to the vector bundles at order two (cf. [111]) but in a gauge covariant form:

$$T(\tilde{S}^{(2)}M) = H(\tilde{S}^{(2)}M) \oplus F^{(1)}(\tilde{S}^{(2)}M) \oplus F^{(2)}(\tilde{S}^{(2)}M) \oplus R \tag{14.5}$$

where $H, F^{(1)}, F^{(2)}, R$ represent the horizontal vertical normal and deformation distributions respectively.

In the following we study the Bianchi identities and Yang-Mills-Higgs fields on $\tilde{S}^{(2)}M$ bundle..

14.2 Bianchi Identities

In order to study the Bianchi Identities (kinematic constraints) it is necessary to use the Jacobi identities:

$$S_{(XYZ)}[\tilde{D}_X^{(*)},[\tilde{D}_Y^{(*)},\tilde{D}_Z^{(*)}]]=0 \tag{14.6}$$

There are forty-eight Bianchi relations derived from twenty-four different types of Jacobi identities. Two of these relations are identically zero. Therefore remain forty-six Bianchi relations. We will give now some characteristic cases of the Bianchi identities.

Similarly to our previous work [143], the gauge covariant derivative will take the form

$$\tilde{D}_\mu^{(*)}=\frac{\tilde{\delta}}{\delta x^\mu}+\frac{1}{2}\omega_\mu^{(*)ab}J_{ab} \tag{14.7}$$

here

$$\frac{\tilde{\delta}}{\delta x^\mu}=\frac{\partial}{\partial x^\mu}-N_{\alpha\mu}\frac{\partial}{\partial\xi_\alpha}-\bar{N}_\mu^\alpha\frac{\partial}{\partial\bar{\xi}^\alpha}-\bar{N}_\mu^0\frac{\partial}{\partial\lambda}$$

or

$$\frac{\tilde{\delta}}{\delta x^\mu}=\tilde{A}_\mu^a P_a$$

with

$$\begin{aligned}\tilde{A}_\mu^a &= A_\mu^a-N_\mu^0L_0^kL_k^a,\;\; P_a=\frac{\partial}{\partial x^a},\\ A_\mu^a &= h_\mu^a-N_{\alpha\mu}\bar{\psi}^{\alpha a}-\bar{N}_\mu^\alpha\psi_\alpha^a.\end{aligned}$$

After some calculations we get:

$$\begin{aligned}[\tilde{D}_\mu^{(*)},[\tilde{D}_\kappa^{(*)},\tilde{D}_\lambda^{(*)}]] &= (\frac{\delta\tilde{R}_{\kappa\lambda}^a}{\partial x^\mu}+\tilde{R}_{b\kappa\lambda}^aA_\mu^b+\omega_{\mu c}^{(*)ab}\tilde{R}_{\kappa\lambda}^c)P_a\\ &\quad+(\frac{1}{2}\frac{\delta\tilde{R}_{\kappa\lambda}^{ce}}{\partial x^\mu}+\omega_\mu^{(*)cd}\tilde{R}_{d\kappa\lambda}^e)J_{ce}\end{aligned} \tag{14.8}$$

and $\omega_\mu^{(*)ab}$ represent the Lorentz-spin connection coefficients. We define also:

$$\tilde{D}_\mu\tilde{R}_{\kappa\lambda}^{ce} = \frac{1}{2}\frac{\tilde{\delta}\tilde{R}_{\kappa\lambda}^{ce}}{\partial x^\mu}+\omega_\mu^{(*)cd}\tilde{R}_{d\kappa\lambda}^e \tag{14.9}$$

$$\tilde{D}_\mu^e\tilde{R}_{\kappa\lambda}^a = \frac{\tilde{\delta}\tilde{R}_{\kappa\lambda}^\alpha}{\delta x^\mu}+\tilde{R}_{b\kappa\lambda}^\alpha\tilde{A}_\mu^b+\omega_{\mu c}^{(*)ab}\tilde{R}_{\kappa\lambda}^c \tag{14.10}$$

By cyclic permutation of the independent generators J_{ce}, P_a we get the following Bianchi identities,

$$\tilde{D}_\mu \tilde{R}^a_{\kappa\lambda} + \tilde{D}_\kappa \tilde{R}^a_{\lambda\mu} + \tilde{D}_\lambda \tilde{R}^a_{\mu\kappa} = 0 \tag{14.11}$$

$$\tilde{D}_\mu \tilde{R}^{ce}_{\kappa\lambda} + \tilde{D}_\kappa \tilde{R}^{ce}_{\lambda\mu} + \tilde{D}_\lambda \tilde{R}^{ce}_{\mu\kappa} = 0 \tag{14.12}$$

Using the Jacobi identities $Q_(\alpha, \beta, \gamma)[\tilde{D}^{(*)}_\alpha, [\tilde{D}^{(*)}_\beta, \tilde{D}^{(*)}_\gamma]] = 0$ the Bianchi identities with respect to spinor quantities produce the relations,

$$\tilde{D}_\alpha Q^{ab}_{\beta\gamma} + \tilde{D}_\beta Q^{ab}_{\gamma\alpha} + \tilde{D}_\gamma Q^{ab}_{\alpha\beta} = 0, \tag{14.13}$$
$$\tilde{D}_\alpha Q^{a}_{\beta\gamma} + \tilde{D}_\beta Q^{a}_{\gamma\alpha} + \tilde{D}_\gamma Q^{a}_{\alpha\beta} = 0. \tag{14.14}$$

The new Jacobi identity, due to λ, has the form

$$[D^{(*)}_0, [D^{(*)}_0, D^{(*)}_0]] = 0 \tag{14.15}$$

which yields us no Bianchi identity.

Bianchi identities of mixed type give us the kinematic constraint which encompass space-time, spinors and deformed gauge covariant derivatives. In that case from the Jacobi identities

$$Q_{\mu\alpha 0}[\tilde{D}^{(*)}_\mu, [\tilde{D}^{(*)}_\alpha, \tilde{D}^{(*)}_0]] = 0$$

we get the relations

$$[\tilde{D}^{(*)}_\mu, [\tilde{D}^{(*)}_\alpha, \tilde{D}^{(*)}_0]] = (\frac{\tilde{\delta} \tilde{P}^d_{0\alpha}}{\delta x^\mu} + \tilde{P}^d_{c0\alpha} A^e_\mu + \omega^{(*)d}_{\mu a} \tilde{P}^a_{0\alpha}) P_d$$
$$+(\frac{1}{2}\frac{\tilde{\delta} \tilde{P}^{cd}_{0\alpha}}{\delta x^\mu} + \omega^{(*)c}_{\mu\alpha} \tilde{P}^{ad}_{0\alpha}) J_{cd} \tag{14.16}$$
$$[\tilde{D}^{(*)}_\alpha, [\tilde{D}^{(*)}_0, \tilde{D}^{(*)}_\mu]] = (\frac{\partial \tilde{P}^d_{\mu 0}}{\partial \bar{\xi}^\alpha} + \tilde{P}^d_{c\mu 0} A^c_\alpha + *\omega^{(*)d}_{\alpha a} \tilde{P}^a_{\mu 0}) P_d$$
$$+(\frac{1}{2}\frac{\partial \tilde{P}^{cd}_{\mu 0}}{\partial \bar{\xi}^\alpha} + \omega^{(*)c}_{\alpha a} \tilde{P}^{ad}_{\mu 0}) J_{cd} \tag{14.17}$$
$$[\tilde{D}^{(*)}_0, [\tilde{D}^{(*)}_\mu, \tilde{D}^{(*)}_\alpha]] = (\frac{\partial \tilde{P}^d_{\alpha\mu}}{\partial \lambda} + \tilde{P}^d_{c\alpha\mu} A^c_0 + \omega^{(*)d}_{0a} \tilde{P}^a_{\alpha\mu}) P_d$$
$$+(\frac{1}{2}\frac{\partial \tilde{P}^{cd}_{\alpha\mu}}{\partial \lambda} + \omega^{(*)c}_{0a} \tilde{P}^{ad}_{\alpha\mu}) J_{cd} \tag{14.18}$$

where,

$$\begin{aligned}\tilde{D}^{(*)}_{\mu} &= \frac{\tilde{\delta}}{\delta x^{\mu}} + \frac{1}{2}\omega^{(*)ab}_{\mu} J_{ab},\ \tilde{D}^{(*)}_{\alpha} = \frac{\partial}{\partial\bar{\xi}^{\alpha}} + \frac{1}{2}\Theta^{(*)ab}_{\alpha} J_{ab}, \\ \tilde{D}^{(*)}_{0} &= \frac{\partial}{\partial\lambda} + \omega^{ab}_{0} J_{ab}, \frac{\partial}{\partial\lambda} = L^{\mu}_{0} h^{a}_{\mu} P_{a}, \frac{\partial}{\partial\bar{\xi}^{\alpha}} = \psi^{a}_{\alpha} P_{a}\end{aligned}$$

Now we put,

$$\tilde{D}_{\mu}\tilde{P}^{d}_{0\alpha} = \frac{\tilde{\delta}\tilde{P}^{d}_{0\alpha}}{\delta x^{\mu}} + \tilde{P}^{d}_{c\mu 0} A^{c}_{\mu} + \omega^{(*)d}_{\alpha a}\tilde{P}^{a}_{\mu 0}, \tag{14.19}$$

$$\tilde{D}_{\alpha}\tilde{P}^{d}_{\mu 0} = \frac{\partial\tilde{P}^{d}_{\mu 0}}{\partial\bar{\xi}^{\alpha}} + \tilde{P}^{d}_{c\mu 0} A^{c}_{\mu} + \omega^{(*)d}_{\alpha a}\tilde{P}^{a}_{\mu 0}, \tag{14.20}$$

$$\tilde{D}_{0}\tilde{P}^{d}_{\alpha\mu} = \frac{\partial\tilde{P}^{d}_{\alpha\mu}}{\partial\lambda} + \tilde{P}^{d}_{c\alpha\mu} A^{c}_{0} + \omega^{(*)d}_{0a}\tilde{P}^{a}_{\alpha\mu}. \tag{14.21}$$

in virtue of (14.14), (14.15) and (14.16) we get the identity

$$\tilde{D}_{\mu}\tilde{P}^{d}_{0\alpha} + \tilde{D}_{\alpha}\tilde{P}^{d}_{\mu 0} + \tilde{D}_{0}\tilde{P}^{d}_{\alpha\mu} = 0 \tag{14.22}$$

Similarly we define

$$\tilde{D}_{\mu}\tilde{P}^{cd}_{0\alpha} = \frac{1}{2}\frac{\partial\tilde{P}^{cd}_{0\alpha}}{\partial x^{\mu}} + \omega^{(*)c}_{\mu a}\tilde{P}^{ad}_{0\alpha}, \tag{14.23}$$

$$\tilde{D}_{\alpha}\tilde{P}^{cd}_{\mu 0} = \frac{1}{2}\frac{\partial\tilde{P}^{cd}_{\mu 0}}{\partial\bar{\xi}^{\alpha}} + \omega^{(*)c}_{\alpha a}\tilde{P}^{ad}_{\mu 0}, \tag{14.24}$$

$$\tilde{D}_{0}\tilde{P}^{cd}_{\alpha\mu} = \frac{1}{2}\frac{\partial\tilde{P}^{cd}_{\alpha\mu}}{\partial\lambda} + \omega^{(*)c}_{0\alpha}\tilde{P}^{ad}_{\alpha\mu}. \tag{14.25}$$

From (14.18)–(14.20) we get

$$\tilde{D}_{\mu}\tilde{P}^{cd}_{0\alpha} + \tilde{D}_{\alpha}\tilde{P}^{cd}_{\mu 0} + \tilde{D}_{0}\tilde{P}^{cd}_{\alpha\mu} = 0 \tag{14.26}$$

14.3 Yang-Mills-Higgs equations.

The study of Yang-Mills-Higgs equations within the framework of the geometrical structure of $\tilde{S}^{(2)}(M)$-bundle that contains the one-dimensional fibre as an internal deformed system can characterize the Higgs field which is studied in the elementary particle physics. In our description we are allowed to choose a scalar from the internal deformed fibre of the spinor bundle $\tilde{S}^{(2)}(M)$. Its contribution to the Lagrangian density provides us with the generated Yang-Mills-Higgs equations.

In the following we define a gauge potential $\tilde{A} = (A_\mu, A_\alpha, \bar{A}^\alpha, \varphi)$ with space-time and spinor components, $\varphi : R \longrightarrow g$ which takes its values in a Lie Algebra g.

$$\begin{aligned}
&\tilde{A} : \tilde{S}(M) \longrightarrow g \\
&\tilde{A}_X = A^i_X \tau_i, [\tau_i, \tau_j] = C^k_{ij}\tau_k \\
&\tilde{A}_X = \{A_\mu, A_\alpha, \bar{A}^\alpha, \varphi\}
\end{aligned}$$

where the elements τ_i are the components which satisfy the commutation relations of the Lie algebra. Then $\tilde{A}$ is called a g-valued spinor gauge potential.

We can define the gauge covariant derivatives:

$$\begin{aligned}
\hat{D}_\mu &= \tilde{D}_\mu + iA_\mu \\
\hat{D}_\alpha &= \tilde{D}_\alpha + iA_\alpha \\
\hat{D}^\alpha = \tilde{\bar{D}}^\alpha &+ iA_\mu
\end{aligned} \tag{14.27}$$

In virtue of the preceding relations we get the following theorem:

Theorem 14.1. *The commutators of gauge covariant derivatives on a $\tilde{S}^{(2)}M$ deformed bundle are given by the relations:*

$$\begin{aligned}
&a) \quad [\hat{D}_\mu, \hat{D}_\nu] = [\tilde{D}_\mu, \tilde{D}_\nu] + i\tilde{F}_{\mu\nu} \\
&b) \quad [\hat{D}_\mu, \hat{\bar{D}}^\alpha] = [\tilde{D}_\mu, \tilde{\bar{D}}^\alpha] + i\tilde{F}^\alpha_\mu \\
&c) \quad [\hat{D}_\alpha, \hat{\bar{D}}^\beta] = [\tilde{D}_\alpha, \tilde{\bar{D}}^\beta] + i\tilde{F}^\beta_\alpha \\
&d) \quad [\hat{D}_\alpha, \hat{\bar{D}}_\beta] = [\tilde{D}_\alpha, \tilde{\bar{D}}_\beta] + i\tilde{F}_{\alpha\beta} \\
&e) \quad [\hat{D}_\mu, \hat{\bar{D}}_\alpha] = [\tilde{D}_\mu, \tilde{\bar{D}}_\alpha] + i\tilde{F}_{\mu\alpha} \\
&f) \quad [\hat{\bar{D}}^\alpha, \hat{\bar{D}}^\beta] = [\tilde{\bar{D}}^\alpha, \tilde{\bar{D}}^\beta] + i\tilde{\bar{F}}^{\alpha\beta}
\end{aligned} \tag{14.28}$$

The curvature two-forms $\tilde{F}_{XY}, \tilde{F}^{XY}, F^X_Y, X, Y = \{\alpha, \beta, \mu, \nu\}$ are the g-valued field strengths on $\tilde{S}^{(2)}M$ and they have the following form:

$$\begin{aligned}
\tilde{F}_{\mu\nu} &= \tilde{D}_\mu A_\nu - \tilde{D}_\nu A_\mu + i[A_\mu, A_\nu] \\
\tilde{F}_{\mu\alpha} &= \tilde{D}_\mu A_\alpha - \tilde{D}_\alpha A_\mu + i[A_\alpha, A_\mu] \\
\tilde{\bar{F}}^\beta_\alpha &= \tilde{D}_\alpha \bar{A}^\beta - \tilde{\bar{D}}^\beta \bar{A}_\alpha + i[A_\alpha, \bar{A}^\beta] \\
\tilde{\bar{F}}^\alpha_\mu &= \tilde{D}_\mu \bar{A}^\alpha - \tilde{\bar{D}}_\alpha A_\mu + i[A_\mu, \bar{A}^\alpha] \\
\tilde{F}_{\alpha\beta} &= \tilde{D}_\alpha A_\beta - \tilde{D}_\beta A_\alpha + i[A_\alpha, A_\beta] \\
\tilde{\bar{F}}^{\alpha\beta} &= \tilde{\bar{D}}^\alpha \bar{A}^\beta - \tilde{\bar{D}}^\beta \bar{A}_\alpha + i[\bar{A}^\alpha, \bar{A}^\beta]
\end{aligned} \tag{14.29}$$

The appropriate Lagrangian density of Yang-Mills(Higgs) can be written in the form

$$\begin{aligned}\tilde{L} &= tr(\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} + tr(\tilde{F}_{\mu\alpha}\tilde{\bar{F}}^{\mu\alpha} + tr(\tilde{F}_{\alpha\beta}\tilde{\bar{F}}^{\alpha\beta} + tr(\tilde{\bar{F}}^{\beta}_{\alpha}\tilde{\bar{F}}^{\alpha}_{\beta} \\ &\quad +\frac{1}{2}m^2\varphi^2 - \frac{1}{2}tr[(\hat{D}_\mu\varphi)(\hat{D}^\mu\varphi)] - \frac{1}{2}tr[(\hat{D}_\alpha\varphi)(\hat{D}^\mu\varphi)] \qquad (14.30)\end{aligned}$$

In our case the Yang-Mills(Higgs) generalized action can be written in the form

$$\tilde{\mathcal{I}}_{YM(H)} = \int \tilde{\mathcal{L}} d^4x d^4\xi d^4\bar{\xi} d\lambda \qquad (14.31)$$

From the Euler-Lagrange equations

$$\frac{\delta\tilde{L}}{\delta A_Y} = \tilde{D}_X(\frac{\partial\tilde{L}}{\partial(\tilde{D}_X A_Y)}) - \frac{\partial\tilde{L}}{\partial A_Y} = 0 \qquad (14.32)$$

the variation of $\tilde{L}$ with respect to A_λ is

$$\tilde{D}_k(\frac{\partial\tilde{L}}{\partial(\tilde{D}_k A_\lambda)}) + \tilde{D}_\beta(\frac{\partial\tilde{L}}{\partial(\tilde{D}_\beta A_\lambda)}) + \bar{D}^\beta(\frac{\partial\tilde{L}}{\partial(\bar{D}_\beta A_\lambda)}) - \frac{\partial\tilde{L}}{\partial A_\lambda} = 0 \qquad (14.33)$$

and it will give us after some straightforward calculations the equation:

$$\hat{D}_k\tilde{F}^{k\lambda} + \hat{D}_\beta\bar{F}^{\lambda\beta} + \bar{D}^\beta\tilde{F}^\lambda_\beta + [\varphi, \hat{D}^\lambda\varphi] = 0 \qquad (14.34)$$

Similarly from the variation of $\tilde{L}$ with respect to A_α and $\tilde{A}^\beta$ we associate the equations:

$$\begin{aligned}\hat{D}_k\bar{F}^{k\gamma} + \hat{D}_\delta\bar{F}^\gamma_\delta + \bar{D}^\delta\bar{F}^\gamma_\delta + [\varphi, \hat{D}^\gamma\varphi] &= 0, \qquad (14.35)\\ \hat{D}_k\tilde{F}^k_\gamma + \hat{D}_\delta\tilde{F}^\delta_\gamma + \bar{D}^\delta\tilde{F}_{\delta\gamma} + [\varphi, \bar{D}_\gamma\varphi] &= 0. \qquad (14.36)\end{aligned}$$

So we can state the following theorem:

Theorem 14.2. *The Yang-Mills-Higgs equations of $\tilde{S}^{(2)}M$-bundle are given by the relations (14.30)–(14.31).*

14.4 Field Equations of an Internal Deformed System

Considering on the deformed spinor bundle $S^{(2)}M \times R$ a nonlinear connection and a gauge d-connection, the authors obtain the equivalence principle and the explicit expressions of the field equations corresponding to a Utiyama gauge invariant Lagrangian density produced by the corresponding scalars of curvature; these results extend the corresponding ones for $S^{(2)}M$.

The concept of a spinor bundle $S^{(2)}M$ and its relation to the Poincarè group were introduced in [140, 148]. This group, consisting of the set of rotations, boosts and translations, gives an exact meaning to the terms: "momentum", "energy", "mass", and "spin" and is used to determine characteristics of the elementary particles; also, it is a gauge, acting locally in the space-time. Hence we may perform Poincarè transformations for a physical approach. In [140], the metric tensor of the base manifold $(M, g_{\mu\nu}(x,\xi,\bar{\xi}))$, depends on the position coordinates and on the spinor (Dirac) variables $(\xi_\alpha, \bar{\xi^\alpha}) \in C^4 \times C^4$. A spinor bundle $S^{(1)}(M)$ can be constructed from one of the principal fiber bundles with fiber $F = C^4$. Each fiber is diffeomorphic with one proper Lorentz group. In this study we apply for the space $S^{(2)}M \times R$ an analogous method as in the theory of deformed bundles developed in [6], for the case of a **spinor bundle** $S^{(2)}M = M \times C^{4 \cdot 2}$ in connection with a deformed **internal fibre** R. The consideration of Miron d - connections [109], which preserve the $h-$ and $v-$distributions is essential in our approach, as in the previous work: this standpoint enables using a more general group $G^{(2)}$, called the structural group of all rotations and translations, that is isomorphic to the Poincarè Lie algebra. A **spinor** is an element of the spinor bundle $S^{(2)}(M) \times R$ where R represents the **internal fibre of deformation.** The local variables are in this case

$$(x^\mu, \xi_\alpha, \bar{\xi}^\alpha, \lambda) \in S^{(2)}(M) \times R = \tilde{S}^{(2)}(M), \lambda \in R.$$

The non-linear connection on $\tilde{S}^{(2)}(M)$ is defined analogously, as for the vector bundles of order two [111, 120]

$$T(\tilde{S}^{(2)}M) = H(\tilde{S}^2 M) \oplus \mathcal{F}^{(1)}(\tilde{S}^{(2)}M) \oplus \mathcal{F}^{(2)}(\tilde{S}^{(2)}M) \oplus R,$$

where $\mathcal{H}$,$\mathcal{F}^{(1)}$,$\mathcal{F}^{(2)}$,R represent the horizontal, vertical, normal and deformation distributions respectively.

We introduce the fundamental gauge 1-form fields which take values from the Lie algebra of the Poincarè group and denote by J_{ab}, P_a the generators of the four-dimensional Poincarè group (namely the angular momentum and linear momentum), by $\omega_\mu^{(*)ab}$ - the Lorentz - spin connection coefficients, $\bar{\Psi}^{\alpha a}$, Ψ_α^a, $\theta_\alpha^{(*)ab}$, $\bar{\theta}^{(*)\alpha ab}$ - the spin-tetrad and spin - connection coefficients, and L_o^a - the deformed tetrad coefficients. We use Greek letters $\lambda, \mu, \nu, \ldots$ for space-time indices, $\alpha, \beta, \gamma, \ldots$ for the spinor, $a, b, c, \ldots$ for Lorentz ones, and the index (o) represents the deformed variable; $\lambda, \alpha,\ a = 1, \ldots, 4$. The general transformations of coordinates on $\tilde{S}^{(2)}M$ are

$$x'^{\mu} = x'^{\mu}(x^{\nu}), \xi'_{\alpha} = \xi'_{\alpha}(\xi_{\beta}, \bar{\xi}^{\beta}), \bar{\xi}'^{\alpha} = \bar{\xi}'^{\alpha}(\bar{\xi}^{\beta}, \xi_{\beta}), \lambda' = \lambda$$

Like in [140, 148] we define the following gauge covariant derivatives, including the new derivative corresponding to the deformation-parameter.

The space-time, Lorentz, spin frames and the deformed frame are shown to be connected by a set of the relations which generalize the well - known principle of equivalence in the total space of the spinor bundle $S^{(2)}M$.

The deformed spinor bundle $\tilde{S}^{(2)}M$ is endowed with a metrical structure.The considered connection in $\tilde{S}^{(2)}(M)$ is a d-connection; it preserves the distributions $\mathcal{H}$,$\mathcal{F}^{(1)}$,$\mathcal{F}^{(2)}$,R, and is assumed to be metrical.

The covariant differentiation of tensors, spin-tensors and Lorentz - type tensors of arbitary rank is defined as in [140, 148]; also, are present the supplementary derivation laws relative to the deformation component.

From the anticommutation relations of the adapted basis, we obtain the curvatures and torsions of the space $\tilde{S}^{(2)}M$

$$R^{a}_{\mu\nu}, R^{ab}_{\mu\nu}, P^{ab}_{\mu\alpha}, P^{a}_{\mu\alpha}$$

and, similarly to [148], other four curvatures and torsions. The contribution of the λ - covariant derivative $\mathcal{D}^{(*)}_{o}$ provides us the following curvatures and torsions

$$R^{a}_{o\mu}, R^{ab}_{o\mu}, R^{ab}_{oo} = 0, R^{a}_{oo} = 0, \bar{P}^{a\alpha}_{o}, \bar{P}^{ab\alpha}_{o}, P^{a}_{o\alpha}, P^{ab}_{o\alpha}.$$

In the following, are derived the field equations, by means of the Palatini method, using a Lagrangian of the form

$$\mathcal{L} = h(R + P + \bar{P} + Q + \tilde{Q} + R_o + \bar{P}_o + Po)$$

which depends on the tetrads and on the connection coefficients,

$$\mathcal{L}(\kappa^A, \delta_M \kappa^A) = \mathcal{L}(h, \omega^{(*)}, \psi, \bar{\psi}, \theta^{(*)}, \bar{\theta}^{(*)}, \omega^{(*)}_{o})$$

where,

$$\kappa^A \in \{(h^a_{\mu}(z), \omega^{(*)ab}_{\mu}(z), \psi^a_{\alpha}(z), \bar{\psi}^{\alpha a}(z), \theta^{(*)ab}_{\alpha}(z), \bar{\theta}^{(*)\alpha ab}(z), \omega^{(*)ab}_{o}(z))\},$$

$$\delta_M = \frac{\delta}{\delta z^M} \in \left\{\frac{\delta}{\delta x^{\mu}}, \frac{\delta}{\delta \xi_{\alpha}}, \frac{\partial}{\partial \bar{\xi}^{\alpha}}, \frac{\partial}{\partial \lambda}\right\}, \qquad z = (z^M) = (x^{\mu}, \xi_{\alpha}, \bar{\xi}^{\alpha}, \lambda)$$

and

$$\begin{cases} R = h^\mu_a h^\kappa_c R^c_{\mu\kappa}, & P = h^\mu \psi^\alpha_c P^c_{\mu\alpha}, & \bar{P} = h^\mu \bar{\psi}_{\alpha c} \bar{P}^{c\alpha}_\mu, \\ Q = Q^{ab}_{\alpha\beta} \bar{\psi}^\alpha_a \bar{\psi}^\beta_b, & \tilde{Q} = \tilde{Q}^{ab\alpha\beta} = \psi_{\alpha a}\psi_{\beta b}, & S = \bar{\psi}^\alpha_a \psi_{\beta b} S^{ab\beta}_\alpha, \\ R_o = L^o_\kappa h^\mu_c h^\kappa_a R^{ac}_{o\mu}, & \bar{P}_o = L^o_\kappa h^\kappa_a \bar{\psi}_{\alpha c} \bar{P}^{ac\alpha}_o, & P_o = L^o_\kappa h^\kappa_a \bar{\psi}^\alpha_c P^{ac}_{o\alpha}. \end{cases}$$

The Euler-Lagrange equations are generally given by

$$\frac{\delta \mathcal{L}}{\delta \kappa^{(A)}} = \partial_M \left(\frac{\partial \mathcal{L}}{\partial(\partial_M \kappa^{(A)})} \right) - \frac{\partial \mathcal{L}}{\partial \kappa^{(A)}} = 0,$$

with $\partial_M = \frac{\partial}{\partial z^M} \in \left\{ \frac{\partial}{\partial x^\mu}, \frac{\partial}{\delta \xi_\alpha}, \frac{\partial}{\partial \bar{\xi}^\alpha}, \frac{\partial}{\partial \lambda} \right\}$.

The variation of $\mathcal{L}$ with respect to the tetrads h^ν_b gives us the first field equation

$$\tilde{H}^b_\nu - \frac{1}{2} h^b_\nu \tilde{H} = 0,$$

where we put,

$$\begin{aligned} \tilde{H} &= R + P + \bar{P} + Q + \tilde{Q} + S + R_o + \bar{P}_o + P_o, \\ \tilde{H}^b_\nu &= 2R^b_\nu + P^b_\nu + \bar{P}^b_\nu + R^b_{o\nu} + \bar{P}^b_{o\nu} + P^b_{(o)\nu}, \\ R^b_\nu &= h^\kappa_c R^{bc}_{\nu\kappa}, \qquad P^b_\nu = \bar{\psi}^\alpha_c P^{bc}_{\nu\alpha}, \qquad R^b_{o\nu} = L^o_\kappa h^\kappa R^b_{o\nu} + L^o_\nu h^\mu_c R^{bc}_{o\mu}, \\ \bar{P}^b_\nu &= \psi_{\alpha c} \bar{P}^{bc\alpha}_\nu, \qquad \bar{P}^b_{o\nu} = L^o_\nu \bar{\psi}_{\alpha c} \bar{P}^{bc\alpha}_o, \qquad P^b_{o\nu} = L^o_\nu \bar{\psi}^\alpha_c P^{bc}_{o\alpha}. \end{aligned}$$

The equation (14.4) is the **Einstein equation for empty space**, in the framework of our consideration. Also, the variation of $\mathcal{L}$ with respect to $\omega^{(*)ab}_\mu$ gives the second field equation

$$\begin{aligned} \partial_\nu [h(h^\mu_a h^\nu_b - h^\mu_b h^\nu_a)] - \partial^\alpha [h h^\mu_a \psi_{\alpha b}] - \partial_\alpha [h h^\mu_a \bar{\psi}^\alpha_b] - \partial_o [h L^o_a h^\mu_b] - \\ - h[\omega^{(*)d}_{\kappa(a} h^{(\kappa}_{b)} h^{\mu)}_d + h^\mu_a (\bar{\psi}^\alpha_c \theta^{(*)c}_{\alpha b} + \psi_{\alpha c} \bar{\theta}^{(*)\alpha c}_b) + \\ + h^\mu_d (\bar{\psi}^\alpha_b \theta^{(*)d}_{\alpha a} + \psi_{\alpha b} \theta^{(*)\alpha d}_a) - L^o_\kappa \omega^{(*)c}_{ob} h^\mu_{[c} h^\kappa_{a]} &= 0, \end{aligned}$$

where the parantheses (...) and [...] are used to denote symmetrization and antisymmetrization respectively.

The variation of $\mathcal{L}$ with respect to ψ^a_α provides the field equation

$$\frac{1}{2} \bar{\psi}^\beta_c S^{c\alpha}_{\beta a} + \frac{1}{2} P^{ba}_{\mu\alpha} h^\mu_b + \tilde{Q}^{d\gamma\alpha}_a \psi_{\gamma d} = 0.$$

From the variation with respect to $\bar{\psi}^{\alpha a}$ we get the fourth field equation

$$\frac{1}{2}\bar{P}^{ba\alpha}_{\mu}h^{\mu}_{b}+\frac{1}{2}\psi_{\beta b}S^{b\beta}_{a\alpha}+\bar{\psi}^{\gamma}_{d}Q^{d}_{a\gamma\alpha}=0.$$

Finally, are obtained the explicit expressions of the other three field equations, by means of the variation of $\mathcal{L}$ with respect to the connection coefficients $\theta^{(*)ab}_{\alpha}$, $\bar{\theta}^{(*)\alpha ab}$ and $\omega^{(*)ab}_{(o)}$.

Chapter 15

Tensor and Spinor Equivalence on Generalized Metric Tangent Bundles

15.1 Introduction

The theory of spinors on pseudo-Riemannian spaces has been recognized by many authors, e.g. [128, 39, 200] for the important role it has played from the mathematical and physical point of view.

The spinors that we are dealing with here, are associated with the group $SL(2, C)$. In particular $SL(2, C)$ acts on C^2. Each elements of C^2 represents a two-component spinor. This group is the covering group of the Lorentz group in which the tensors are described [39]. The correspondence between spinors and tensors is achieved by means of mixed quantities initially introduced by Infeld and Van der Waerden.

The correspondence of tensors and spinors establishes a homomoerhism between the Lorentz group and the covering group $SL(2, C)$.

In the following, we give some important relations between spinors and tensors on a general manifold of metric $g_{\mu\nu}$.

Let $\sigma : S \otimes \bar{S} \rightarrow V^4$ be a homomorphism between spinor spaces $S, \bar{S}$ and four-vectors belonging to the V^4 space, then the components of σ, which are called the *Pauli-spinor matrices*, are given by

$$
\begin{aligned}
\sigma^0_{AB'} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma^1_{AB'} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\sigma^2_{AB'} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad \sigma^3_{AB'} = \frac{1}{\sqrt{2}} \begin{pmatrix} & 0 \\ 0 & -1 \end{pmatrix}
\end{aligned} \tag{15.1}
$$

The hermitian spinorial equivalent notation of $\sigma^{\mu}_{AB'}$ is given by $\sigma^{\mu}_{AB'} = \bar{\sigma}^{\mu}_{BA'} = \bar{\sigma}^{\mu}_{B'A}$. Greek letters $\mu, \nu, \cdots$ represent the usual space-time indices taking the values $0, 1, 2, 3$ and the Roman capital indices A, B, A', B' are the spinor indices taking the values $0, 1$. The tensor indices are raised and lowered by means of the metric tensor, whereas the raising and lowering of spinor indices is given by the *spinor metric tensors* ε_{AC}, $\varepsilon_{B'C'}$ which are of skew-symmetric form. Thus, for two spinors ξ^A, $n^{A'}$ we have the relations, moreover we have,

$$\xi^A n_A = \xi^A n^B \varepsilon_{BA} = -\xi^A \varepsilon_{AB} n^B = -\xi_B n^B.$$

For a real vector V_μ its spinor equivalent it

$$V_{AB'} = V_\mu \sigma^{\mu}_{AB'}, \tag{15.2}$$

where $\sigma^{\mu}_{AB'}$ are given by the relation (15.1). Also, the following formulas are satisfied,

$$\sigma^{\mu}_{AB'} \sigma^{\nu AB'} = g^{\mu\nu}, \quad \sigma^{\mu}_{AB'} \sigma^{AB'}_{\nu} = \delta^{\mu}_{\nu}.$$

The spinor equivalent of a tensor $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = \sigma^{AB'}_{\mu} \sigma^{CD'}_{\nu} T_{AB'CD'}$$

and the tensor corresponding to the spinor $T_{AB'CD'}$ is,

$$T_{AB'CD'} = \sigma^{\mu}_{AB'} \sigma^{\nu}_{CD'} T_{\mu\nu}.$$

The relationship between the matrices σ^ν and the geometric tensor $g_{\mu\nu}$, as well as its spinor equivalent are

$$\begin{aligned} g_{\mu\nu} \sigma^{\mu}_{AB'} \sigma^{\nu}_{CD'} &= \varepsilon_{AC} \varepsilon_{B'D'}, \\ g_{AB'CD'} &= \sigma^{\mu}_{AB'} \sigma^{\nu}_{CD'} g_{\mu\nu} = \varepsilon_{AC} \varepsilon_{B'D'}, \\ g^{AB'CD'} &= \sigma^{AB'}_{\mu} \sigma^{CD'}_{\nu} g^{\mu\nu} = \varepsilon^{AC} \varepsilon^{B'D'}. \end{aligned} \tag{15.3}$$

The complex conjugation of the spinor $S_{AB'}$ is

$$\overline{S_{AB'}} = \bar{S}_{A'B}.$$

Furthermore, for a real vector V_μ the spinor hermitian equivalence yields $\bar{V}_{B'A} = V_{AB'}$. If a vector y^k is a null-vector,

$$y^k y_k = g_{k\lambda} y^k y^\lambda = 0, \tag{15.4}$$

then its spinor equivalent will take the form

$$y^k = \sigma^k_{AB'}\theta^A\bar{\theta}^{B'}, \tag{15.5}$$

where,$\theta^A, \bar{\theta}^{B'}$ represents the two-component spinors of $SL(2,C)$ group.

In the Riemannian space, the covariant derivative of x-dependent spinors will take the form

$$\begin{aligned} D_\mu\xi^A &= \frac{\partial\xi^A}{\partial x^\mu} + L^A_{B\mu}\xi^B,\ D_\mu\bar{\xi}^{A'} = \frac{\partial\bar{\xi}^{A'}}{\partial x^\mu} + \bar{L}^{A'}_{B'\mu}\bar{\xi}^{B'}, \\ D_\mu\xi_A &= \frac{\partial\xi_A}{\partial x^\mu} - L^B_{A\mu}\xi_B,\ D_\mu\bar{\xi}_{A'} = \frac{\partial\bar{\xi}_{A'}}{\partial x^\mu} + \bar{L}^{B'}_{A'\mu}\bar{\xi}_{B'}, \end{aligned}$$

where $\xi^A, \bar{\xi}^{A'}, \bar{\xi}_{A'}$ represents two–components spinors and $L^A_{B\mu}, \bar{L}^{A'}_{B'\mu}$ are the spinor affine connections. In the case that we have spinors with two indices, the covariant derivative will be in the form

$$D_\mu\xi^{AB'} = \frac{\partial\xi^{AB'}}{\partial\xi^\mu} + L^A_{C_\mu}\xi^{CB'} + \bar{L}^{B'}_{C'_\mu}\xi^{AC'}.$$

Applying this formula to the spinor metric tensors $\varepsilon_{AC}, \varepsilon_{B'C'}$ we get

$$D_\mu\varepsilon_{AB} = \frac{\partial\varepsilon_{AB}}{\partial x^\mu} - L^C_{A_\mu}\varepsilon_{CB} - L^C_{B_\mu}\varepsilon_{AC}. \tag{15.6}$$

If

$$D_\mu\varepsilon_{AB} = 0,$$

we shall say that the spinor connection coefficients $L^A_{B_\mu}$ are *metrical* together with the relations

$$D_\mu\sigma^\nu_{AB'} = 0, \quad D_\mu\varepsilon^{AB} = 0, \quad D_\mu\varepsilon_{A'B'} = 0, \quad D_\mu\varepsilon^{A'B'} = 0. \tag{15.7}$$

From the relation (15.6) we immediately obtain

$$L_{BA_\mu} = L_{AB_\mu}$$

where we used the relation

$$L_{AB_\mu} = L^C_{B_\mu}\varepsilon_{CA}.$$

Also from the relation (15.7) we have

$$D_\mu\sigma^\nu_{AB'} = \partial_\mu\sigma^\nu_{AB'} + L^\nu_{\mu\rho}\sigma^\rho_{AB'} - L^C_{A_\mu}\sigma^\nu_{CB'} - \bar{L}^{D'}_{B'_\mu}\sigma^\nu_{AD'} = 0.$$

15.2 Generalization of the Equivalent of Two Component–Spinors with Tensors

The above mentioned well-known procedure for $SL(2,C)$ group between spinors and tensors in a pseudo-Riemannian space-time can be applied to more generalized metric spaces or bundles. For example G. Asanov [14] applied this method for Finsler spaces (FS), where the two-component spinors $n(x,y)$ depend on the position and direction variables or $n(x^i, z^\alpha$, with $z\alpha$ a scalar for a gauge approach. Concerning this approach some results were given relatively to the gauge covariant derivative of spinors and the Finslerian tetrad. In our present study we give the relation between spinors of $SL(2,C)$ group and tensors in the framework of Lagrabge spaces (LS).

The expansion for the covariant derivatives, connections non-linear connections, torsions and curvatures are the main purpose of our approach.

In the following, we shall study the case that the vectors of LS are null-vectors and consequently fulfill the relation (15.5). In Finsler type space-time the metric tensor $g_{ij}(x,y)$ depends on the position and directional variables, where the vector y may be identified with the frame velocity ([14] ch. t). So, a vector v^i will be called *null* if

$$g_{ij}(x,v)v^i v^j = 0. \tag{15.8}$$

In this case there is no unique solution for the light-cone [80, 26]. The problem of causality is solved considering the velocity as a parameter and the motion of a particle in Finsler space is described by a pair (x,y). The metric form in such a case will be given by

$$ds^2 = g_{ij}(x,v)dx^i dx^j.$$

When a particle is moving in the tangent bundle of a Finsler (Lagrange) space-time its line-element will be given by

$$\begin{aligned} d\sigma^2 &= G_{ab}dx^a dx^b \qquad (15.9)\\ &= g^{(0)}_{ij}(x,y)dx^i dx^j + g^{(1)}_{\alpha\beta}(x,y)\delta y^\alpha \delta y^\beta, \quad \left(y^\alpha = \frac{dx^\alpha}{dt}\right), \end{aligned}$$

where the indices i,j and α,β taking the values $1,2,3,4$ and

$$\delta y^\alpha = dy^\alpha + \mathcal{N}^\alpha_j dx^j.$$

Thus we have

Theorem 15.1. *The null-geodesic condition (15.8) is satisfied for a particle moving in the tangent bundle of Finsler space-time of metric* $d\sigma^2$ *(15.9) with the assumption, the velocity* v *is taken as a parameter of the absolute parallelism*

$$\delta y^\alpha = 0.$$

The previous treatment of null-vectors in Finsler spaces can also be considered for Lagrange spaces involving Lagrangians which are not homogeneous [109, 26]. The introduction of spinors $\theta, \bar{\theta}$ *of the covering group* $SL(2, C)$ *in the metric tensor* $g(x, \theta, \bar{\theta})$ *under the correspondence between spinors and tensors in LS,*

$$(x, y) \to (x, V_{AB'} \to (x, \theta^A, \bar{\theta}^{A'})$$

preserves the anisotropy of space with torsions. in this case all objects depend on the position and spinors, e.g. the Pauli matrices $\tilde{\sigma}^i_{AA'}(x, \theta, \bar{\theta})$*. Such an approach can be developed for a second-order spinor bundle applying the method analogous to [144]. In virtue of relation (15.4), a null vector in spinor form can be characterized by*

$$g_{AA'BB'}\theta^A\bar{\theta}^{A'}\theta^B\bar{\theta}^{B'} = \tilde{\sigma}^i_{AA'}\tilde{\sigma}^j_{BB'}g_{ij}\theta^A\bar{\theta}^{A'}\theta^B\bar{\theta}^{B'} = 0. \tag{15.10}$$

Proposition 15.1. *In a tangent bundle of metric (Finsler, Lagrange)*

$$G = g_{ij}(x, y)dx^i dx^j + h_{ab}(x, y)\delta y^a \delta y^b,$$

if the vector y *is a null, then the corresponding spinor metric of the bundle will be given in the form*

$$G = g_{AA'BB'}d\theta^A d\bar{\theta}^{A'} d\theta^B d\bar{\theta}^{B'} + h_{AA'BB'}\delta(\theta^B\bar{\theta}^{B'})\delta(\theta^A\bar{\theta}^{A'}) \tag{15.11}$$

or equivalently

$$G = g_{AA'BB'}d\theta^A d\bar{\theta}^{A'} d\theta^B d\bar{\theta}^{B'} + h_{AA'BB'}\delta y^{AA'}\delta y^{BB'},$$

where $y^{AA'} = \theta^A\bar{\theta}^{A'}$*, when* y *is null vector (cf. [39]).*

Proof. The relation (15.11) is obvious by virtue of (15.3) and (15.5).

Remark. A generalized spinor can be considered as the equare root of a Finsler (Lagrange) null vector.

15.3 Adapted Frames and Linear Connections

In the general case of a LS, the spinor equivalent to the metric tensor

$$g_{ij} = \frac{\partial^2 L}{\partial y^i \partial y^j}, \quad L = \frac{1}{2}F^2$$

is given by

$$g_{ij} = \tilde{\sigma}_i^{AA'} \tilde{\sigma}_j^{BB'} g_{AA'BB'}.$$

The corresponding Lagrangian will be $\bar{L} : M \times C^2 \times C^2 \to R$, with the property $\bar{L}(x, \theta, \bar{\theta}) = L(x, y)$, where L represents the Lagrangian in a Lagrange space. We can adopt the spinor equivalent form of the adapted frames and their duals in a LS,

$$\left(\frac{\delta}{\delta x^\mu}, \frac{\partial}{\partial y^i}\right) \to \left(\frac{\delta}{\delta x^\mu}, \frac{\partial}{\partial \theta^A}, \frac{\partial}{\partial \bar{\theta}^{A'}}\right), \qquad (dx^\mu, \delta y^i) \to (dx^\mu, \delta\theta^A, \delta\bar{\theta}^{A'})$$

as well as the spinor counterpart of the non-linear connection $\mathcal{N}_\mu^i$ of a LS,

$$\mathcal{N}_\mu^i \to (N_\mu^A, \bar{N}_\mu^{A'}).$$

The geometrical objects $\delta\theta^A, \delta\bar{\theta}^{A'}$ are given by

$$\delta\theta^A = d\theta^A + N_\mu^A dx^\mu, \qquad \delta\bar{\theta}^{A'} = d\bar{\theta}^{A'} + \bar{N}_\mu^{A'} dx^\mu. \tag{15.12}$$

In virtue of (15.2), the bases $\partial_\mu, \partial_{AA'}$ are related as follows

$$\partial_\mu = \tilde{\sigma}_\mu^{AA'} \partial_{AA'}, \tag{15.13}$$

where $\partial_\mu = \frac{\partial}{\partial x^\mu}$ and $\partial_{AA'} = \frac{\partial}{\partial \theta^A} \frac{\partial}{\partial \bar{\theta}^{A'}}$.

Theorem 15.2. *In a Lagrange space the spinor equivalent of the adapted basis $(\delta/\delta x^\mu, \partial/\partial y^\alpha)$ and its dual $(dx^\mu, \delta y^\alpha)$ are given by*

$$a) \quad \frac{\delta}{\delta x^\mu} = \tilde{\sigma}_\mu^{AA'} \partial_A \partial_{A'} - N_\mu^A \partial_A - \bar{N}_\mu^{A'} \partial_{A'} \tag{15.14}$$

$$b) \quad \partial_P \tilde{\sigma}_P^{AA'} = \partial_{AA'}, \qquad P = \{i, \alpha\}$$

$$c) \quad dx^\mu = \tilde{\sigma}_{AA'}^\mu d\theta^A d\bar{\theta}^{A'}$$

$$d) \quad \delta y^\alpha = (\bar{\theta}^{A'} d\theta^A + \theta^A d\bar{\theta}^{A'}) \tilde{\sigma}_{AA'}^\alpha + (\bar{\theta}^{A'} N_\gamma^A + \theta^A \bar{N}_\gamma^{A'}) \tilde{\sigma}_{AA'}^\gamma d\theta^B d\theta^{\bar{B}'}$$

Proof. The relations (15.14) are derived from (15.12) and (15.13).

Proposition 15.2. *The null-geodesic equation of spinor equivalence in a LS or FS is given by*

$$dy^{\alpha} = \tilde{\sigma}^{\alpha}_{AA'}(\bar{\theta}^{A'} d\theta^{A} + \theta^{A} d\bar{\theta}^{A'}), \quad N^{\alpha}_{j} = \tilde{\sigma}^{\alpha}_{AA'}(\bar{\theta}^{A'} N^{A}_{j} + \theta^{A} \bar{N}^{A'}_{j}). \tag{15.15}$$

Proof. The relation (15.15) is obvious because of (15.14) d).

Proposition 15.3. *The null-geodesic equation of spinor equivalence in a LS or FS is given by*

$$\bar{\theta}^{A'} d\theta^{A}(\tilde{\sigma}^{\mu}_{AA'} N^{A}_{\mu} d\bar{\theta}^{A'} + 1) + \theta^{A} d\bar{\theta}^{A'}(\tilde{\sigma}^{A'}_{AA'} \bar{N}^{A'}_{\mu} d\theta^{A} + 1) = 0. \tag{15.16}$$

Proof. In virtue of relations (15.10) and (15.14) c,d) we obtain the relation (15.16).

Affine connections and affine spinor connections are defined in the frames of LS by the following formulas

$$\begin{aligned}
&D_{\delta/\delta x^{\mu}}\left(\frac{\delta}{\delta x^{\nu}}\right) = L^{k}_{\nu\mu}\frac{\delta}{\delta x^{k}}, \quad D_{\delta/\delta x^{\mu}}\left(\frac{\partial}{\partial\theta^{A}}\right) = L^{B}_{A\mu}\frac{\partial}{\partial\theta^{B}},\\
&D_{\delta/\delta x^{\mu}}\left(\frac{\partial}{\partial\bar{\theta}^{A'}}\right) = \bar{L}^{B'}_{A'\mu}\frac{\partial}{\partial\bar{\theta}^{B'}}, \quad D_{\partial/\partial\theta^{A}}\left(\frac{\delta}{\delta x^{\mu}}\right) = C^{\nu}_{\mu A}\frac{\delta}{\delta x^{\nu}},\\
&D_{\partial/\partial\theta^{A}}\left(\frac{\partial}{\partial\bar{\theta}^{B'}}\right) = C^{C'}_{B'A}\frac{\partial}{\partial\theta^{C'}}, \quad D_{\partial/\partial\bar{\theta}^{A'}}\left(\frac{\partial}{\partial\theta^{B}}\right) = \bar{C}^{C}_{BA'}\frac{\partial}{\partial\theta^{C}},\\
&D_{\partial/\partial\bar{\theta}^{A'}}\left(\frac{\partial}{\partial\bar{\theta}^{B'}}\right) = \bar{C}^{C'}_{B'A'}\frac{\partial}{\partial\bar{\theta}^{C}}, \quad D_{\partial/\partial\theta^{A}}\left(\frac{\partial}{\partial\theta^{B}}\right) = C^{C}_{BA}\frac{\partial}{\partial\theta^{C}},\\
&D_{\partial/\partial\bar{\theta}^{A'}}\left(\frac{\delta}{\delta x^{\mu}}\right) = \bar{C}^{\nu}_{\mu A'}\frac{\delta}{\delta x^{\nu}}.
\end{aligned} \tag{15.17}$$

We can give the covariant derivatives of the higher order generalized spinors $\zeta^{AB'}_{BA'}(x,\theta,\bar{\theta})$,

$$\begin{aligned}
\triangle_{\mu}\zeta^{AB'}_{BA'} &= \frac{\delta\zeta^{AB'\ldots}_{BA'\ldots}}{\delta x^{\mu}} + L^{A}_{C\mu}\zeta^{CB'\ldots}_{BA'\ldots} + \bar{L}^{B'}_{C'\mu}\zeta^{AC'\ldots}_{BA'\ldots} - L^{C}_{B\mu}\zeta^{AB'\ldots}_{CA'\ldots} - \bar{L}^{C'}_{\mu A'}\zeta^{AB'\ldots}_{BC'\ldots}\\
\triangle_{E}\zeta^{AB'}_{BA'} &= \frac{\partial\zeta^{AB'\ldots}_{BA'\ldots}}{\partial\theta^{E}} + C^{A}_{CE}\zeta^{CB'\ldots}_{BA'\ldots} + \bar{C}^{B'}_{C'E}\zeta^{AC'\ldots}_{BA'\ldots} - C^{C}_{BE}\zeta^{AB'\ldots}_{CA'\ldots} - \bar{C}^{C'}_{EA'}\zeta^{AB'\ldots}_{BC'\ldots}\\
\triangle_{Z'}\zeta^{AB'}_{BA'} &= \frac{\partial\zeta^{AB'\ldots}_{BA'\ldots}}{\partial\bar{\theta}^{Z'}} + \bar{C}^{A}_{CZ'}\zeta^{CB'\ldots}_{BA'\ldots} + \bar{C}^{B'}_{C'Z'}\zeta^{AC'\ldots}_{BA'\ldots} - \bar{C}^{C'}_{Z'A'}\zeta^{AB'\ldots}_{BC'\ldots}.
\end{aligned} \tag{15.18}$$

Proposition 15.4. *If the connections defined by the relations (15.17) are of the Cartan-type, then the spinor equivalent relations are given by*

$$\begin{aligned} \bar{\theta}^{A'}\frac{\delta\theta^{A}}{\delta x^{k}} + L^{A}_{Ck}\theta^{C}\bar{\theta}^{A'} + \theta^{A}\frac{\delta\bar{\theta}^{A'}}{\delta x^{k}} + \bar{L}^{A'}_{C'k}\bar{\theta}^{C'}\theta^{A} &= 0, \\ (\tilde{\sigma}^{AA'}_{\beta})^{-1}(\bar{\theta}^{A'} \triangle_{E}\theta^{A} + \theta^{A}\triangle_{E}\bar{\theta}^{A'}) &= 1, \\ (\tilde{\sigma}^{AA'}_{\gamma})^{-1}(\bar{\theta}^{A'} \triangle_{Z}\theta^{A} + \theta^{A}\triangle_{Z'}\bar{\theta}^{A'}) &= 1. \end{aligned} \qquad (15.19)$$

Proof. Applying the relations (15.18) to a null vector y with the Cartan-type properties $y^{\alpha}_{|k} = 0$ and $y^{\alpha} \mid_{\beta} = \delta^{\alpha}_{\beta}$ [116, 14], and taking into account the (15.2) a), (15.5) we obtain the relations (15.19). (As we have mentioned previously the y-covariant derivative has corresponded to the spinor covariant derivatives).

15.4 Torsions and Curvatures

The spinor torsions corresponding to the torsions of LS are given by an analogous method to that one we derived in [144] for a deformed bundle. The torsion tensor field T of a D-connection is given by

$$T(X,Y) = D_X Y - D_Y X - [X,Y]$$

.

Relatively to an adapted frame we have the relations

$$\begin{aligned}
&\text{a)}\quad T\left(\frac{\delta}{\delta x^{k}},\frac{\delta}{\delta x^{\lambda}}\right) = T^{\mu}_{\lambda k}\frac{\delta}{\delta x^{\mu}} + T^{A}_{\lambda k}\frac{\partial}{\partial\theta^{A}} + \bar{T}^{A'}_{\lambda k}\frac{\partial}{\partial\bar{\theta}^{A'}}\\
&\text{b)}\quad T\left(\frac{\partial}{\partial\theta^{A}},\frac{\delta}{\delta x^{\mu}}\right) = T^{\nu}_{\mu A}\frac{\delta}{\delta x^{\nu}} + T^{B}_{\mu A}\frac{\delta}{\delta\theta^{B}} + \bar{T}^{B'}_{\mu A}\frac{\partial}{\partial\bar{\theta}^{B'}}\\
&\text{c)}\quad T\left(\frac{\partial}{\partial\bar{\theta}^{A'}},\frac{\delta}{\delta x^{\mu}}\right) = T^{\nu}_{\mu A'}\frac{\delta}{\delta x^{\nu}} + T^{B}_{\mu A'}\frac{\partial}{\partial\theta^{B}} + \bar{T}^{B'}_{\mu A'}\frac{\partial}{\partial\bar{\theta}^{B'}}\\
&\text{d)}\quad T\left(\frac{\partial}{\partial\theta^{A}},\frac{\partial}{\partial\bar{\theta}^{B}}\right) = T^{\mu}_{BA}\frac{\delta}{\delta x^{\mu}} + T^{C}_{BA}\frac{\partial}{\partial\theta^{C}} + \bar{T}^{C'}_{BA}\frac{\partial}{\partial\theta^{C'}}\\
&\text{e)}\quad T\left(\frac{\partial}{\partial\theta^{A}},\frac{\partial}{\partial\bar{\theta}^{B'}}\right) = T^{\mu}_{B'A}\frac{\delta}{\delta x^{\mu}} + T^{C}_{B'A}\frac{\partial}{\partial\theta^{C}} + \bar{T}^{C'}_{B'A}\frac{\partial}{\partial\bar{\theta}^{C'}}\\
&\text{f)}\quad T\left(\frac{\partial}{\partial\bar{\theta}^{A'}},\frac{\partial}{\partial\bar{\theta}^{B}}\right) = T^{\mu}_{BA'}\frac{\delta}{\delta x^{\mu}} + T^{C}_{BA'}\frac{\partial}{\partial\theta^{C}} + \bar{T}^{C'}_{B'A}\frac{\partial}{\partial\bar{\theta}^{C'}}\\
&\text{g)}\quad T\left(\frac{\partial}{\partial\bar{\theta}^{A'}},\frac{\partial}{\partial\bar{\theta}^{B'}}\right) = T^{\mu}_{B'A'}\frac{\delta}{\delta x^{\mu}} + T^{C}_{B'A'}\frac{\partial}{\partial\theta^{C}} + \bar{T}^{C'}_{B'A'}\frac{\partial}{\partial\bar{\theta}^{C'}}.
\end{aligned} \qquad (15.20)$$

The torsion (15.20) a) can be written in the form

$$D_{\delta/\delta x^\mu}\frac{\delta}{\delta x^\nu} - D_{\delta/\delta x^\nu}\frac{\delta}{\delta x^\mu} - \left[\frac{\delta}{\delta x^\mu},\frac{\delta}{\delta x^\nu}\right] = \tag{15.30}$$

$$L^\lambda_{\nu\mu}\frac{\delta}{\delta x^\lambda} - L^\lambda_{\mu\nu}\frac{\delta}{\delta x^\lambda} - R^A_{\mu\nu}\frac{\partial}{\partial\theta^A} - V^{A'}_{\mu\nu}\frac{\partial}{\partial\bar{\theta}^{A'}},$$

where the brackets have the form

$$[\delta/\delta x^\mu, \delta/\delta x^\nu] = R^A_{\mu\nu}\frac{\partial}{\partial\theta^A} + V^{A'}_{\mu\nu}\frac{\partial}{\partial\bar{\theta}^{A'}}, \tag{15.31}$$

and $\delta/\delta x^\mu$, $R^A_{\mu\nu}$, $V^{A'}_{\mu\nu}$ are given by

$$\frac{\delta}{\delta x^k} = \frac{\partial}{\partial x^k} - \mathcal{N}^A_k\frac{\partial}{\partial\theta^A} - \bar{N}^{A'}_k\frac{\partial}{\partial\bar{\theta}^{A'}}, \tag{15.32}$$

$$R^A_{\mu\nu} = \frac{\delta N^A_\mu}{\delta x^\nu} - \frac{\delta\mathcal{N}^A_\nu}{\delta x^\mu},\ V^{A'}_{\mu\nu} = \frac{\delta N^{A'}}{\delta x^\nu} - \frac{\delta N^{A'}}{\delta x^\mu}.$$

The terms $R^A_{\mu\nu}$, $V^{A'}_{\mu\nu}$ represents *the spinor-curvatures of non-linear connections* N^A_ν, $N^{A'}_\mu$. In virtue of the relations (15.20), (15.30), (15.31) we obtain

$$T^\lambda_{\mu\nu} = L^\lambda_{\mu\nu} - L^\lambda_{\nu\mu},\quad T^A_{\nu\mu} = -R^A_{\mu\nu},\quad \bar{T}^{A'}_{\nu\mu} = -V^{A'}_{\mu\nu}.$$

Similarly from the relations (15.20) b)- g), comparing with the torsion in the following form,

$$T\left(\frac{\delta}{\delta Y^P},\frac{\delta}{\delta Y^Q}\right) = D_{\delta/\delta Y^P}\frac{\delta}{\delta Y^Q} - D_{\delta/\delta Y^Q}\frac{\delta}{\delta Y^P} - \left[\frac{\delta}{\delta Y^P},\frac{\delta}{\delta Y^Q}\right]$$

we can obtain the relations

$$\begin{aligned}
T^\lambda_{A\mu} &= C^\lambda_{A\mu},\quad T^A_{B\mu} = \frac{\partial N^A_\mu}{\partial\theta^B} - L^A_{B\mu} \\
T^{A'}_{A\mu} &= -\tilde{Y}^{A'}_{A\mu},\quad T^\lambda_{AB} = T^\lambda_{AA'} \\
T^l_{AB} &= C^l_{AB} - C^l_{BA},\quad \bar{T}^{A'}_{AB} = -R^{A'}_{AB},\quad T^\lambda_{\mu A'} = -\bar{C}^\lambda_{A'\mu} \\
T^A_{\mu A'} &= -\frac{\partial N^A_\mu}{\partial\bar{\theta}^{A'}},\quad T^{A'}_{\mu B'} = C^{A'}_{\mu B'} = C^{A'}_{B'\mu} - P^{A'}_{\mu B} \\
T^B_{AA'} &= -C^B_{AA'},\quad T^{A'}_{AB'} = C^{A'}_{AB'} - \frac{\partial C^{A'}_A}{\partial\bar{\theta}^{B'}},
\end{aligned} \tag{15.33}$$

where we have put

$$\begin{aligned}
\frac{\delta}{\delta Y^P} &= \left\{\frac{\partial}{\partial\theta^A},\frac{\partial}{\partial\bar{\theta}^{A'}}\right\},\frac{\delta}{\delta Y^Q} = \left\{\frac{\delta}{\delta x^\mu},\frac{\partial}{\partial\bar{\theta}^\Delta}\right\}, \\
\Delta &= B, B' \text{ and } C^{A'}_A = C^{A'}_A\theta^B.
\end{aligned}$$

So, we obtain the following:

Proposition 15.5. *In the adapted basis of a generalized metric tangent bundle the spinor equivalent of coefficients of the torsion T of a D-connection, are given by the relations (15.32)-(15.33).*

Proposition 15.6. *D-connection has no torsion if and only if all terms of the relation (15.33) are equal to zero.*

The curvature tensor field R of a D-connection has the form

$$R(X,Y)Z = [D_X, D_Y]Z - D_{[X,Y]}Z \ \forall X,Y,Z \in \mathcal{X}(TM).$$

The coefficients of the curvature tensor and the corresponding spinor curvature tensors in spinor bundle are given by

$$\begin{aligned}
R^k_{\lambda\nu\mu} &= \frac{\delta L^k_{\lambda\mu}}{\delta x^\mu} - \frac{\delta L^k_{\lambda\mu}}{\delta x^\nu} + L^\rho_{\lambda\nu}L^k_{\rho\mu} - L^\rho_{\lambda\mu}L^k_{\rho\nu} - R^A_{\mu\nu}C^k_{A\lambda} - V^{A'}_{\mu\nu}\bar{C}^k_{A'\lambda} \qquad (15.34)\\
R^B_{A\nu\mu} &= \frac{\delta L^B_{\nu A}}{\delta x^\mu} - \frac{\delta L^B_{\mu A}}{\delta x^\nu} + L^\rho_{A\nu}L^B_{\rho\mu} - L^\rho_{A\mu}L^B_{\rho\nu} - R^\rho_{\mu\nu}L^B_{A\rho} - V^{A'}_{\mu\nu}\bar{C}^B_{A'A}\\
R^{B'}_{A'\nu\mu} &= \frac{\delta \bar{L}^{B'}_{\nu A'}}{\delta x^\mu} - \frac{\delta \bar{L}^{B'}_{\mu A'}}{\delta x^\nu} + \bar{L}^{B'}_{\nu A'}L_\mu - L^{B'}_{A'\nu}L_\mu - R^A_{\mu\nu}C^{B'}_{A'A} - V^{D'}_{\mu\nu}\bar{C}^{B'}_{D'A'}\\
P^k_{\nu\mu A} &= \frac{\delta L^k_{\nu\mu}}{\delta \theta^A} - \frac{\delta C^k_{A\nu}}{\partial x^\mu} + L^\lambda_{\nu\mu}C^k_{A\lambda} - C^\lambda_{A\nu}L^k_{\lambda\mu} + \frac{\partial N^E_\mu}{\partial \theta^A}C^k_{E\nu} + \tilde{Y}'^k_{\mu A A'\nu}\\
P^l_{AB\mu} &= \frac{\partial L^l_{A\mu}}{\partial \theta^B} - \frac{\delta C^l_{AB}}{\delta x^\mu} + L^k_{A\mu}C^l_{kB} - C^k_{AB}L^l_{k\mu} + \frac{\partial N^n_\mu}{\partial \theta^A}C^l_{Bn} + \tilde{Y}^{A'}_{\mu A}C^l_{A'B}\\
P^{B'}_{A'A\mu} &= \frac{\partial L^{B'}_{A'\mu}}{\partial \theta^A} - \frac{\delta C^{B'}_{A'A}}{\delta x^\mu} + L^B_{A'\mu}C^{B'}_{BA} - C^B_{AA'}L^{B'}_{B\mu}\\
&\quad + \frac{\delta N^E_\mu}{\partial \theta^A}C^{B'}_{A'E} + \tilde{Y}^{E'}_{\mu A}\bar{C}^{B'}_{E'A'}\\
S^k_{\mu AB} &= \frac{\partial C^k_{\mu A}}{\partial \theta^B} - \frac{\partial C^k_{\mu B}}{\partial \theta^A} + C^\lambda_{\mu A}C^k_{\lambda B} - C^\lambda_{\mu B}C^k_{\lambda A} - R^{A'}_{AB}C^k_{A'\lambda}\\
S^m_{lAB} &= \frac{\partial C^m_{lA}}{\partial \theta^B} - \frac{\partial C^m_{lB}}{\partial \theta^A} + C^n_{lA}C^m_{nB} - C^n_{lB}C^m_{nA} - R^{A'}_{AB}\bar{C}^m_{A'l}\\
S^{B'}_{A'AB} &= \frac{\partial C^{B'}_{A'A}}{\partial \theta^B} - \frac{\partial C^{B'}_{A'B}}{\partial \theta^A} + C^{D'}_{A'A}C^{B'}_{D'B} - C^{D'}_{A'B}C^{B'}_{D'B} - R^{D'}_{AB}\bar{C}^{B'}_{D'A'}\\
I^k_{\nu A'\mu} &= \frac{\delta \bar{C}^k_{A'\nu}}{\delta x^\mu} - \frac{\partial L^k_{\nu\mu}}{\partial \bar{\theta}^{A'}} + C^\rho_{A'\nu}L^k_{\rho\mu} - L^\rho_{\nu\mu}\bar{C}^k_{A'\rho} - \frac{\partial N^A_\mu}{\partial \bar{\theta}^{A'}}C^k_{A\nu} - \bar{L}^{A'}_{A'\mu}C^k_{A\nu}\\
I^B_{AA'\mu} &= \frac{\delta C^B_{A'A}}{\delta x^\mu} - \frac{\partial L^B_{A\mu}}{\partial \bar{\theta}^{A'}} + C^\rho_{A'A}L^B_{\rho\mu} - L^\rho_{A\mu}C^B_{A'\rho} - \frac{\partial N^\rho_\mu}{\partial \bar{\theta}^{A'}}C^B_{A\rho} - \bar{L}^{D'}_{A'\mu}\bar{C}^B_{D'A}\\
I^{B'}_{A'C'\mu} &= \frac{\delta \bar{C}^{B'}_{A'C'}}{\delta x^\mu} - \frac{\partial \bar{L}^{B'}_{A'\mu}}{\partial \bar{\theta}^{C'}} + \bar{C}^{B'}_{A'D'}\bar{L}^{D'}_{C'\mu} - \bar{L}^{B'}_{E'\mu}\bar{C}^{E'}_{A'B'}\\
&\quad - \frac{\partial N^\rho_\mu}{\partial \bar{\theta}^{A'}}\bar{L}^{B'}_{C'\rho} - \bar{L}^{D'}_{A'\mu}\bar{C}^{B'}_{D'C'}
\end{aligned}$$

$$\begin{aligned}
J^{k}_{\nu A'B} &= \frac{\partial C^{k}_{A'\nu}}{\partial\theta^{B}} - \frac{\partial C^{k}_{B\nu}}{\partial\bar{\theta}^{A'}} + C^{\rho}_{A'\nu}C^{k}_{B\rho} - C^{\rho}_{B\nu}C^{k}_{A'\rho} - \frac{\partial L^{D'}_{B}}{\partial\bar{\theta}^{A'}}C^{k}_{D'\nu}\\
J^{rho}_{AA'B} &= \frac{\partial C^{\rho}_{A'B}}{\partial\theta^{A}} - \frac{\partial C^{\rho}_{AB}}{\partial\bar{\theta}^{A'}} + C^{k}_{A'A}C^{\rho}_{kB} - C^{k}_{AB}C^{\rho}_{A'k} - \frac{\partial L^{D'}_{A}}{\partial\bar{\theta}^{A'}}C^{\rho}_{D'B}\\
J^{B}_{A'C'A} &= \frac{\partial \bar{C}^{B'}_{A'C'}}{\partial\theta^{A}} - \frac{\partial C^{B'}_{A'A}}{\partial\bar{\theta}^{A'}} + C^{B'}_{A'D}C^{D}_{C'A} - C^{B'}_{A'E}C^{E}_{C'A} - \frac{\partial L^{D'}_{A}}{\partial\bar{\theta}^{A'}}\bar{C}^{B'}_{C'D'}\\
K^{\nu}_{\mu A'B'} &= K^{B}_{AA'B'} = K^{B'}_{A'C'D'} = 0.
\end{aligned}$$

So we have

Theorem 15.3. *The coefficients of the curvatures of a D-connection are given by the relation (15.34).*

Theorem 15.4. *In a tangent bundle a D-connection has no curvature if and only if all the coefficients (15.34) of the curvatures are equal to zero.*

Finally, we note that the gravitational field can be described by virtue of the corresponding spinorial form of the metric tensor equivalent to the spinor bundle. This will be the object of our future study.

Bibliography

[1] M. Anastasiei, *Structures spinorielles sur le varietes hilbertiennes, C. R. Acad. Sci. Paris* **A284** (1977) A943–A946

[2] M. Anastasiei, *Vector Bundles. Einstei Equations, An. şt. Univ. Iaşi, s I a* **32** (1986) 17–24

[3] M. Anastasiei, *Conservation laws in* (V, H)*–bundle model of relativity, Tensor, N. S.* **46** (1987) 323–328

[4] M. Anastasiei, in *Coloquium on Differential Geometry, 25-30 July 1994* (Lajos Kossuth University, Debrecen, Hungary, 1994) p.1

[5] P. L. Antonelly (ed.) *Finslerian Geometries: A Meeting of Minds* FTPH no. 101 (Kluwer Academic Publishers, Dordrecht, Boston, London, 1999)

[6] P. L. Antonelli and D. Hrimiuc, *Diffusions and Laplacians on Lagrange manifolds*, in: *The theory of Finslerian Laplacians and Applications* N 459 (Kluwer Academic Publishers, Dordrecht, Boston, London, 1998), pp. 123–131

[7] P. L. Antonelli, R. S. Ingarden and M. Matsumoto, *The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology*, FTPH no. 58 (Kluwer Academic Publishers, Dordrecht, Boston, London, 1993)

[8] P. L. Antonelli and B. Lackey (eds.) *The theory of Finslerian Laplacians and Applications* N 459 (Kluwer Academic Publishers, Dordrecht, Boston, London, 1998)

[9] P. L. Antonelli and R. Miron (eds), *Lagrange and Finsler Geometry, Applications to Physics and Biology*, FTPH no. 76 (Kluwer Academic Publishers, Dordrecht, Boston, London, 1996)

[10] P. L. Antonelli and T. J. Zastawniak, *Diffusion on Finsler manifoplds*, *Proc. XXV Symposium on Mathematical Physics* (Torun, 1992); Rep. Math. Phys. **33** (1993) 303–315

[11] P. L. Antonelli and T. J. Zastawniak, *Introduction to diffusion on Finsler manifolds*, in: *Lagrange geometry, Finsler spaces and noise applied in biology and physics*, *Math. Comput. Modelling* **20** (1994) 109–116

[12] P. L. Antonelli and T. J. Zastawniak (eds.), *Fundamentals of Finslerian Diffusion with Applications* N 101 (Kluwer Academic Publishers, Dordrecht, Boston, London, 1999)

[13] A. K. Aringazin and G. S. Asanov, *Problems of Finslerian Theory of Gauge Fields and GravitationRep. Math. Phys.* **25** (1988) 35–93

[14] G. S. Asanov, *Finsler Geometry, Relativity and Gauge Theories* (Reidel, Boston, 1985)

[15] G. S. Asanov, *Finslerian fibered extensions of gauge field theory: Lie–invariance fibres and effective Lagrangian Rep. Math. Phys.* **26** (1988) 367–399

[16] G. S. Asanov, *Fibered Generalization of the Gauge Field Theory. Finslerian and Jet Gauge Fields* (Moscow University, 1989) [in Russian]

[17] G. S. Asanov, it Finslerian and higher–order structure of space–time Lie–invariance on fibers, *Rep. Math. Phys.* **28** (1989) 263–287

[18] G. S. Asanov, *Finsler extension of Lorentz transformations*, *Rep. Math. Phys.* **42** (1989) 273–296

[19] G. S. Asanov and S. F. Ponomarenko, *Finslerovo Rassloenie nad Prostranstvom–Vremenem, assotsiiruemye kalibrovochnye polya i sveaznosti* [in Russian], *Finsler Bundle on Space–Time. Associated Gauge Fields and Connections* (Ştiinţa, Chişinău, 1988)

[20] A. Ashtekar, *New Hamiltonian formulation of general relativity, Phys. Rev.* **D36** (1987) 1587–1602

[21] M. F. Atiyah, R. Bott and A. Shapiro, *Clifford modules, Topology* **3** (1964) 3–38

[22] M. F. Atiyah and N. Hitchin, *Low energy scattering of nonabelian monopoles*, *Phys. Lett.* **A107** (1985) 21.

[23] A. F. Bais and P. Batenburg,*Fermion dynamics in the Kaluza–Klein monopole geometry, Nucl. Phys.* **B245** (1984) 469.

[24] D. Bao, S. S. Chern and Z. Shen, *An Introduction to Riemannian–Finsler Geometry* (Graduate Texts in Mathematics; 200) (Springer–Verlag, 2000)

[25] W. Barthel, *Nichtlineare Zusammenhange und Deren Holonomie Gruppen J. Reine Angew. Math.* **212** (1963) 120–149

[26] R. Beil, *Comparison of unified field theories, Tensor N. S.* **56** (1995) 175–183.

[27] A. Bejancu, *Finsler Geometry and Applications* (Ellis Horwood, Chichester, England, 1990)

[28] A. Bejancu, *Preprint Series in Mathematics of "A. Miller" Mathematical Seminary*, Iasi, Romania. Univeristy "Al. I. Cuza", Iasi, 3 (1990), pp. 1–24.

[29] A. Bejancu and H. R. Farran, *Geometry of Pseudo–Finsler Submanifolds* (Kluwer Academic Publishers, Dordrecht, Boston, London, 2000)

[30] V. Belinski and E. Verdaguer, *Gravitational Solitons* (Cambridge University Press, 2001)

[31] V. A. Belinskii and V. E. Zakharov, *Integration of the Einstein equation by means of the inverse scattering problem technique and construction of exact soliton solutions*, *Soviet. Phys. JETP*, **48** (1978) 985–994 [translated from: *Zh. Eksper. Teoret. Fiz.* **75** (1978) 1955–1971, in Russian]

[32] L. Berwald, *Math. Z.* **25** (1926) 40; Correction, *Math. Z.* **26** (1927) 176

[33] I. M. Benn and R. W. Tucker, *An Introduction to Spinors and Geometry with Applications in Physics* (Adam Hilger: New York, 1989).

[34] N. D. Birrel and P. C. Davies, *Quantum Fields in Curved Space* (Cambridge Univ. Press, 1982).

[35] Bishop R D and Crittenden R J 1964 *Geometry of Manifolds* (New York, Academic Press).

[36] F. Bloore and P. A. Horv´athy, *Helicity–supersymmetry of dyons*, *J. Math. Phys.* **33** (1992) 1869–1877.

[37] G. Yu. Bogoslovsky, *Theory of Locally-Anisotropic Space-Time* (Izdatel'stvo Moscovskogo Universiteta, Moscow, 1992) [in Russian]

[38] M. Carmeli, *Group Theory and General Relativity* (Mc. Graw–Hill, NY, 1977).

[39] M. Carmeli, *Classical Fields: General Relativity and Gauge Theory* (John Wiley and Sons, New York, 1982).

[40] S. M. Carroll et al., *Phys. Rev. Lett* **87** (2001) 141601

[41] E. Cartan, *Les Espaces de Finsler* (Hermann, Paris, 1935)

[42] E. Cartan, *Exposés de Géométrie* in *Series Actualités Scientifiques et Industrielles***79** (1936); reprinted (Herman, Paris, 1971)

[43] E. Cartan, *Leçons sur la théorie des spineurs.* Tome I: *Les spineurs de l'espace á* $n > 3$ *dimensions. Les Spineurs en gèometrie reimannienne* (Hermann, Paris, 1938); E. Cartan, *The Theory of Spinors* (Dover Publications: New York, 1966)

[44] E. Cartan, *Les Systems Differentielles Exterieurs et Lewrs Application Geometricques,* (Herman and Cie Editeur, Paris, 1945)

[45] C. Chevalley, *The Construction and Study of Certain Important Algebras,* (Publications of Mathematical Society, Tokyo, 1955)

[46] W. K. Clifford, *A preliminary sketch of biquaternions, Mathematical Papers* (London, 1882), p. 171

[47] A. Comtet and P. A. Horv´athy, *The Dirac equation in Taub NUT space*, *Phys. Lett.* **B349** (1995) 49–56

[48] B. Cordani, L. Gy. Feher and P. A. Horvathy, $O(4,2)$ *dynamical symmetry of the Kaluza–Klein monopole*, *Phys. Lett.* **B201** (1988) 481

[49] I. I. Cotăescu and M. Visinescu, *Runge–Lenz operator for Dirac field in Taub–NUT background*, *Phys. Lett.* **B 502** (2001) 229–234

[50] A. Crumeyrolle, *Structures spinorielles Ann. Inst. H. Poincare* **11** (1969) 19–55

[51] A. Crumeyrolle, *Groupes de spinorialite*, *Ann. Inst. H. Poincare* **14** (1971) 309–323

[52] C. Csaki, J. Erlich and C. Grojean, *Nucl. Phys.* **B604** (2001) 312–340

[53] H. Dehnen and E. Hitzer, $SU(2) \times U(1)$ *gauge gravity*, *Int. J. Theor. Phys.* **34** (1995) 1981–2001

[54] P. A. M. Dirac, *Proc. Roy. Soc.* A, **117** (1928) 610.

[55] V. S. Dryuma, *Ob analiticheskom reshenii dvumernogo uravnenia Kortevega– de Vriza (KdV)*, *Pis'ma v JETP,* **19** (1974) 753–757 [in Russian]; it On analytical solution of two dimensional Korteveg– de Vrise equation (KdV), *Sov. Phys. JETP* **19** (1974) 387–390 [English translation]

[56] G. Ellis and D. Hawking, *The Large Scale Structure of Space–Time* (Cambridge University Press, 1973)

[57] Z. F. Ezawa and A. Iwazaki, *Monopole–fermion dynamics and the Rubakov effect in Kaluza–Klein theories*, *Phys. Lett. B* **138** (1984) 81–86

[58] L. Gy. Feher and P. A. Horvathy, *Dynamical symmetry of monopole scattering*, *Phys. Lett.* **B183** (1987) 182–1986; *Erratum "Dynamical symmetry of monopole scattering* **B188** (1987) 512

[59] P. Finsler, *Über Kurven und Flächen in Allgemeiner Rämen* (Dissertation, Göttingen, 1918); reprinted (Birkhäuser, Basel, 1951).

[60] V. Fock, *Zs. f. Phys.* **57** (1929) 261

[61] B. Harrison, *Backlund transformation for the Ernst equation of general relativity*, *Phys. Rev. Lett.* **41** (1978) 1197–1200

[62] S. W. Hawking, *Gravitational instantons*, *Phys. Lett. A* **60** (1977) 81–83

[63] F M Hehl, J D Mc Grea, E W Mielke, and Y Ne'eman, *Metric–affine gauge theory of gravity: field equations, Noether identities, world spinors, and breaking of dilaton invariance*, *Phys. Rep.* **258** (1995) 1–171

[64] R. Hermann, *Lie Groups for Physicists* (Benjamin, NY, 1966).

[65] J. Hladik, *Les spineurs en physique* (Masson, Paris, 1996); [English translation] *Spinors in Physics* (Springer–Verlag, New York, Berlin, Heidelberg, 1999).

[66] E. D'Hoker and L. Vinet, *Supersymmetry of the Pauli equation in the presence of a magnetic monopole*, *Phys. Lett. B* **137** (1984) 72–74

[67] J. W. van Holten, *Supersymmetry and Geometry of Taub NUT*, *Phys. Lett.* **B 342** (1995) 47–52

[68] D. Hrimiuc, *Diffusion on Lagrange manifolds*, Proc. of the 25th National Conference on Geometry and Topology (Iasi, 1995); Annal. St. Univ. "Al. I. Cuza" Iasi, Mat. (N. S.) **42** (1996), suppl. 242–255

[69] D. Hrimiuc, *Diffusion on the total space of a vector bundle*, *Balkan J. Geom. Appl.* **1** (1996) 53–62

[70] H. F. Goenner H F and G. Yu. Bogoslovsky, *A class of anisotropic (Finsler-) space–time geometries*, *Gen. Rel. Grav.* **31** (1999) 1383–1394

[71] G. W. Gibbons and N. S. Manton, *Classical and quantum dynamics of BPS monopoles*, *Nucl. Phys.* **B274** (1986) 183–224

[72] G. W. Gibbons and P. J. Ruback, *The hidden symmetries of Taub–NUT and monompole scattering*, *Phys. Lett.* **B188** (1987) 226–230

[73] G. W. Gibbons and P. J. Ruback, *The hidden symmetries of mulicentre metrics*, *Commun. Math. Phys.* **115** (1988) 267–300

[74] A. Godbillon, *Elements de Topologie Algebraique* (Herman, Paris, 1971)

[75] L. P. Grishchuk, A. N. Petrov and A. D. Popova, *Exact theory of the (Einstein) gravitational field in an arbitrary background space–time*, *Commun. Math. Phys.* **94** (1984) 379–396

[76] D. J. Gross and M. J. Perry, *Magnetic monopoles in Kaluza–Klein theories*, *Nucl. Phys. B* **226** (1983) 29–49

[77] S. Ikeda, *A geometrical consideration in a "nonlocal" field theory*, *Rep. Math. Phys.* **18** (1980) 103–110

[78] S. Ikeda, *Nuovo Cimento* **98B** (1987).

[79] S. Ikeda, *On the theory of gravitational field nonlocalized by the internal variables, IV*, *Nuovo Cimento* **108B** (1993) 397–402

[80] H. Ishikawa, *Note on Finslerian relativity*, *J. Math. Phys.* **22** (1981) 995–1004

[81] B. B. Kadomtsev and V. I. Petviashvili, *Dokl. Akad. Nauk SSSR,* **192,** 753 (1970)

[82] E. Kamke, *Differential Gleichungen, Losungsmethoden und Lonsungen: I. Gewohnliche Differentialgleichungen* (Leipzig, 1959)

[83] M. Karoubi, *K-Theory* (Springer, Berlin, 1978)

[84] A. Kawaguchi, *Beziehung zweischen einen metrischen linearen Uertragung und iener nicht–metrischen in einem allgemeinen metricen Raume,* Akad. Wetensch. Amsterdam, Proc. 40 (1937) 596–601; A. Kawaguchi, On the Theory of Non–linear Connections, I, II, Tensor, N. S., **2** (1952) 123–142; **6** (1956) 165–199

[85] M. Kawaguchi, *An introduction to the theory of higher order spaces I: The theory of Kawaguchi spaces*, *RAAG Memoirs* **Vol. 3**, 1962

[86] J. Kern, *Lagrange Geometry, Arch. Math.* **25** (1974) 438-443.

[87] T. Kibble, *J. Math. Phys.* **2** (1961) 212

[88] M. Kobayashi and A. Sugamoto, *Fermions in the background of the Kaluza–Klein monopole*, *Progr. Theor. Phys.* **72** (1984) 122–133

[89] G. A. Korn and T. M. Korn *Mathematical Handbook* (McGraw–Hill Book Company, 1968)

[90] G. Liebbrandt, Phys. Rev. Lett. **41** (1978) 435

[91] C. P. Luehr and M. Rosenbaum, *Spinor connections in general relativity*, *J. Math. Phys.* **15** (1974) 1120–1137

[92] C. P. Luehr and M. Rosenbaum, *Graviton as an internal gauge theory of the Poincare group*, *J. Math. Phys.* **21** (1980) 1432–1438

[93] A. Macias and H. Dehnen, *Dirac field in the five dimensional Kaluza–Klein theory*, *Class. Quantum Grav.* **8** (1991) 203–208

[94] N. S. Manton, *A remark on the scattering of the BPS monopole*, *Phys. Lett.* **B110** (1985) 54–56

[95] N. S. Manton, *Monopole interactions at long range*, *Phys. Lett. B* **154** (1985) 397–400; *Erratum,* **B157** (1985) 475

[96] M. Matsumoto, *Foundations of Finsler Geometry and Special Finsler Spaces* (Kaisisha, Shigaken, 1986).

[97] P. Menotti and A. Pelissetto, *Poincare, de Sitter, and conformal gravity on the lattice*, *Phys. Rev. D* **35** (1987) 1194–1204

[98] E. V. Mielke, *Geometrodynamics of Gauge Fields — On the Geometry of Yang–Mills and Gravitational Gauge Theories* (Akademie–Verlag, Berlin, 1987).

[99] R. Miron, *A Lagrangian Theory of Relativity,* (I, II) *Analele Şt. Univ. "Al. I. Cuza", Iaşi,* Romania, s.1 math. f.2 and f.3, **32** (1986), 37-62, 7-16.

[100] R. Miron, *Subspaces in Genearalized Lagrange Spaces, Analele Şt. Univ. "Al. I. Cuza", Iaşi,* Romania, s.1 math. **33** (1986), 137-149, 7-16.

[101] R. Miron, *Cartan spaces in a new point of view by considering them from as duals of Finsler spaces*, *Tensor N. S.* **46** (1987) 330–334.

[102] R. Miron, *The geometry of Cartan spaces, Progr. of Math. , India* **22** (1 & 2) (1989), 1–38.

[103] R. Miron, *Hamilton geometry*, Univ. Timişoara, Sem. Mecanica, **3** (1987), 54.

[104] R. Miron, *Sur la geometrie de espaces Hamilton*, *C. R. Acad. Sci. Paris,* ser.1 **306** (1988), 195-198.

[105] R. Miron, *Hamilton geometry*, *Analele Şt. Univ. "Al. I. Cuza", Iaşi,* Romania, s.1 math. **35** (1989), 33-67.

[106] R. Miron, *The Geometry of Higher–Order Lagrange Spaces, Application to Mechanics and Physics*, FTPH no. 82 (Kluwer Academic Publishers, Dordrecht, Boston, London, 1997)

[107] R. Miron, *The Geometry of Higher–Order Finsler Spaces* (Hadronic Press, Palm Harbor, USA, 1998)

[108] R. Miron and M. Anastasiei, *Vector Bundles. Lagrange Spaces. Application in Relativity* (Academiei, Romania, 1987) [in Romanian]; [English translation] no. 1 (Geometry Balkan Press, Bucharest, 1997).

[109] R. Miron and M. Anastasiei, *The Geometry of Lagrange Spaces: Theory and Applications*, FTPH no. 59 (Kluwer Academic Publishers, Dordrecht, Boston, London, 1994)

[110] R. Miron and Gh. Atanasiu, *Compendium sur les Espaces Lagrange D'ordre Supérieur, Seminarul de Mecanică. Universitatea din Timişoara. Facultatea de Matematică, 1994* **40** p. 1; *Revue Roumaine de Mathematiques Pures et Appliquee* **XLI**, N^{of} 3–4 (1996) 205; 237; 251.

[111] R. Miron and Gh. Atanasiu, *Lagrange geometry of second order*, in: Lagange geometry, Finsler spaces and noise applied in biology and physics, *Math. Comput. Modelling.* **20** (Pergamon Press, 1994), pp. 41–56

[112] R. Miron and V. Balan, in: *Proc. Nat. Sem. of Finsler Spaces*, (Braşov, Romania, 1992), preprint.

[113] R. Miron, D. Hrimiuc, H. Shimada and V. S. Sabau, *The Geometry of Hamilton and Lagrange Spaces* (Kluwer Academic Publishers, Dordrecht, Boston, London, 2000).

[114] R. Miron and M. Radivoiovici–Tatoiu, *Extended Lagrangian Theory of Electro magnetism*, *Rep. Math. Phys.* **27** (1989) 193–229

[115] R. Miron, R. K. Tavakol, V. Balan, and I. Roxburgh, *Geometry of space–time and generalized Lagrange gauge theory*, *Publ. Math. Debrecen* **42** (1993) 215–224

[116] R. Miron, S. Watanabe and S. Ikeda, *Some connections on Tangent bundle and their applications to the general relativity*, *Tensor N. S.* **46** (1987) 8–22

[117] C. W. Misner, K. S. Thorne and J. A. Wheeler *Gravitation* (Freeman, 1973)

[118] I. Mocioiu, M. Pospelov and R. Roiban, *Phys. Lett.* **B 489** (2000) 390–420

[119] M. Morand, *Géométrie Spinorielle* (Masson, Paris, 1973)

[120] G. Munteanu and Gh. Atanasiu, *On Miron–connection in Lagrange spaces of second order*, *Tensor N. S.* **50** (1994) pp. 241–247

[121] T. Ono and Y. Takano, *The differential geometry of spaces whose metric depends on spinor variables and the theory of spinor gauge fields*, *Tensor N. S.* **49** (1990) 65–80

[122] T. Ono and Y. Takano, *The differential geometry of spaces whose metric depends on spinor variables and the theory of spinor gauge fields, II* , *Tensor N. S.* **49** (1990) 253–258

[123] T. Ono and Y. Takano, *The differential geometry of spaces whose metric depends on spinor variables and the theory of spinor gauge fields, III* , *Tensor N. S.* **49** (1990) 269–279

[124] T. Ono and Y. Takano, *Remarks on the spinor gauge field theory*, *Tensor N. S.* **52** (1993) 56–60

[125] J. M. Overduin and P. S. Wesson, *Remarks on the spinor gauge field theory* , *Phys. Rep.* **283** (1997) 303–378

[126] W. Pauli, *Zur Quantenmecchanik des magnetischen Elektrons Z. Physik* **43** (1927) 601

[127] R. Penrose, *Structure of Space-time,* in: *Battelle Rencontres, 1967 Lectures in Mathematics and Physics* eds. C. M. DeWitt and J. A. Wheeler (Benjamin, New York)

[128] R. Penrose and W. Rindler, *Spinors and Space-Time, vol. 1, Two-Spinor Calculus and Relativistic Fields* (Cambridge University Press, Cambridge, 1984).

[129] R. Penrose and W. Rindler, *Spinors and Space-Time, vol. 2, Spinor and Twistor Methods in Space-Time Geometry* (Cambridge University Press, Cambridge, 1986).

[130] R. Percacci and S. Randjbar–Daemi, *Kaluza–Klein theories on bundles with homogeneous fibers*, *J. Math. Phys.* **24** (1983) 807–814

[131] Ponomarev V N, Barvinsky A O, and Obukhov Yu N 1985 *Geometrodynamical Methods and the Gauge Approach to the Theory of Gravitational Interactions* (Energoatomizdat, Moscow, 1985).

[132] D. A. Popov, *On the theory of Yang–Mills fields*, *Teoret. Mat. Fiz.* **24** (1975) 347–356 [in Russian]; *Theoret. abd Math. Phys.* **24** (1975) 879–885 [English translation]

[133] D. A. Popov and L. I. Daikhin, *Einstein spaces, and Yang–Mills fields*, *Dokl. Acad. Nauk SSSR* **225** 790–793 [in Russian]; *Soviet Physics. Dokl.* **20** (1976) 818-820 [English translation]

[134] P. Ramon, *Field Theory: A Modern Primer* (Benjamin, Reading, 1981).

[135] B. Riemann, *Über die Hypothesen, welche der Geometrie zugrunde liegen.* Habilitationsvortrag 1854. Ges. math. Werke, 272–287 (Leipzig, 1892); Reproduced: (Dover Publications, 1953).

[136] H. Rund, *The Differential Geometry of Finsler Spaces* (Springer–Verlag, Berlin, 1959)

[137] A. Salam and J. Strathdee, *On Kaluza–Klein theory*, Ann. Phys. (NY) **141** (1982) 316–352

[138] E. Schrödinger, *Diracsches Elektron im Schwerfeld*, *Sitzungsb. Akad. f. Physik* **57** (1929) 261

[139] R. D. Sorkin, *Kaluza–Klein monopole*, *Phys. Rev. Lett.* **51** (1983) 87–90

[140] P. Stavrinos, *On the Differential Geometry of Nonlocalized Field Theory, Poincare Gravity*, Proceedings of the 8th Nat. Sem. on Finsler, Lagrange and Hamilton Spaces (1994).

[141] P. Stavrinos, *Bianchi Identities, Yang–Mills and Higgs Field Produced on $\tilde{S}^{(2)}M$–Deformed Bundle*, Balkan Journal of Geometry and Its Applications, **1** (1996) 75–82.

[142] P. Stavrinos, V. Balan, P. Manouselis and N. Prezas, *Rep. Math. Phys.*, **37** (1996) 163–175.

[143] P. Stavrinos, V. Balan, P. Manouselis and N. Prezas, Analele Stiintifice ale *Universitatii "Al. I. Cuza"*, Iasi, **XLIII** s. I. a. Matematica, f.1 (1997) 51–62.

[144] P. Stavrinos, S. Ikeda, *A Geometrical Structure in the λ–parameter Family of Generalized Metric Spaces* Tensor *N. S.* **56** (1995) 158–165.

[145] P. Stavrinos and S. Koutroubis, *Curvature and Lorentz transformations of spaces whose metric tensor depends on vector and spinor variables*, Tensor, *N. S.* , **55** (1994) 11–19.

[146] P. Stavrinos and P. Manouselis, Bulletin Applied Mathematics *(BAM)*, Technical Univ. of Budap. 923, Vol. LXIX (1993), pp. 25–36.

[147] P. Stavrinos and P. Manouselis, in. *Proc. of the 7th Nat. Seminar of Finsler Spaces* Brasov, 1994, preprint.

[148] P. Stavrinos and P. Manouselis, *Nonlocalized Field Theory ofver Spinor Bundles: Poincare Gravity and Yang–Mills Fields, Rep. Math. Phys.* **36** (1995) 293–306

[149] P. Stavrinos and P. Manouselis, *Tensor and Spinor Equivalence on Generalized Metric Tangent Bundles*, Balkan Journal of Geometry and *Its Applications,* **1** (1996) 119–130.

[150] P. Stavrinos, N. Prezias, P. Manouselis and V. Balan, *Development of Field Euquations of an Internal Deformed System* The Proceedings of *the XXV-th* National Conference on Differential Geometry and Topol*ogy,* (Iassy, 1995).

[151] J. L. Synge, *Relativity: The General Theory* (North Holland, Amsterdam, 1960).

[152] Y. Takano, *The differential geometry of spaces whose metric tensor depends on spinor variables and the theory of spinor gauge fields*, Tensor, *N. S.* , **40** (1983) 249–260.

[153] A. A. Tseytlin, *Poincare and de Sitter gauge theories of gravity with propagating torsion*, *Phys. Rev.* D **26** 3327–3341

[154] A. Turtoi, *Applications of Algebra and Geometry in Spinors Theory* (Editura Tehnocă, Bucureşti, 1989) [in Romanian]

[155] R. Utiyama, *Invariant theoretical interpretation of interactions*, *Phys. Rev.* **101** (1956) 1597–16007

[156] S. Vacaru, *Twistor–gauge interpretation of the Einstein-Hilbert equations* Vestnik Moscovskogo Universiteta, Fizica i Astronomia **28** (1987) 5-12 [in Russian]

[157] S. Vacaru, *Nearly Geodesic Mappings, Twistors and Conservation Laws in Gravitational Theories,* in: Lobachevsky and Modern Geometry, *part II,* ed. V. Bajanov and all, (University Press, Kazani, Tatarstan, Russia, 1992), p.65-66

[158] S. Vacaru, *Nearly Geodesic Mappings of Curved Spaces: An Approach to Twistors and Quantum Gravity,* in: Abstracts of Contributed Papers to the *13th* International Conference on General Relativity and

Gravitation, Eds. Pedro W. Lamberty and Omar E. Ortiz (Cordoba, Argentina, 1992) p. 118

[159] S. I. Vacaru, *Stochastic Calculus on Generalized Lagrange Spaces,* in: *The Program of the Iasi Academic Days, October 6-9, 1994* (Academia Romana, Filiala Iasi, 1994), p.30

[160] S. Vacaru, *Stochastic Differential Equations on Locally Anisotropic Superspaces,* in: *Abstracts of the Romanian National Conference on Physics,* (Baia Mare, 1995)

[161] S. Vacaru, *Spinor and Gauge Fields in Multidimensional and/or Locally Anisotropic Gravity,* in: *Abstracts of the Romanian National Conference on Physics,* (Baia Mare, 1995)

[162] S. Vacaru, *Clifford structures and spinors on spaces with local anisotropy Buletinul Academiei de Ştiinţe a Republicii Moldova, Fizica şi Tehnica* [Izvestia Akademii Nauk Respubliki Moldova, seria fizica i tehnika] **3** (1995) 53–62

[163] S. Vacaru, *Spinor structures and nonlinear connections in vector bundles, generalized Lagrange and Finsler spaces, J. Math. Phys.* **37** (1996) 508–523

[164] S. Vacaru, *Nearly autoparallel maps, tensor integral and conservation laws on locally anisotropic spaces;* in: *Workshop on "Fundamental Open Problems in Mathematics, Physics and Engineering at the Turn of the Millenium" Beijing, China, August 28, 1997, Vol I–III,* (Hadronic Press, Palm Harbor, FL, 1999) 67–103; gr–qc/9604017.

[165] S. Vacaru, *Spinors in Higher Dimensional and Locally Anisotropic Spaces,* gr–qc/9604015

[166] S. Vacaru, *Spinors, Nonlinear Connections and Nearly Autoparallel Maps of Generalized Finsler Spaces*, dg-ga/9609004

[167] S. Vacaru, *Stochastic Differential Equations on Spaces with Local Anisotropy*, *Buletinul Academiei de Stiinte a Republicii Moldova, Fizica si Tehnica [Izvestia Academii Nauk Respubliky Moldova, fizica i tehnika]*, **3** (1996) 13-25

[168] S. Vacaru, *Stochastic processes and diffusion on spaces with local anisotropy*: gr-qc/9604014; S. Vacaru, *Locally ansiotropic stochastic processes in fiber bundles*, *Proceedings of Workshop on Global Analysis,*

Differential Geometry and Lie Algebra (Thessaloniki, 1995), pp. 123–144

[169] S. Vacaru, *Generalized Lagrange and Finsler Supergravity*, gr–qc/ 9604016; *Locally Anisotropic Interactions,* I, II, III, hep–th/ 9607194, 9607195, 9607196

[170] S. Vacaru, *Locally Anisotropic Gravity and Strings, Ann. Phys.* (N. Y.), **256** (1997) 39–61

[171] S. Vacaru, *Superstrings in Higher Order Extensions of Finsler Superspaces*, *Nucl. Phys.* B **434** (1997) 590–656

[172] S. Vacaru, *Interactions, Strings and Isotopies in Higher Order Anisotropic Superspaces* (Hadronic Press, Palm Harbour, USA) [summary in physics / 9706038]

[173] S. Vacaru, *Spinors and Field Interactions in Higher Order Anisotropic Spaces, J. Higher Energy Phys.* **9809** (1998) 011

[174] S. Vacaru, *Gauge Gravity and Conservation Laws in Higher Order Anisotropic Spaces,* hep-th/9810229.

[175] S. Vacaru, *Thermodynamic geometry and locally anisotropic black holes,* gr-qc/9905053

[176] S. Vacaru, *Locally Anisotropic Black Holes in Einstein Gravity,* gr–qc/0001020

[177] S. Vacaru, *Anholonomic Soliton–Dilaton and Black Hole Solutions in Generalized Relativity*, *J. Higher Energy Phys.* **0104** (2001) 009

[178] S. Vacaru, *Stochastic Processes and Thermodynamics on Curved Spaces Ann. Phys. (Leipzig)*, **9** (2000), Special Issue, 175–176, gr–qc/0001057

[179] S. Vacaru, *Locally Anisotropic Kinetic Processes and Thermodynamics in Curved Spaces, Ann. Phys. (NY)* **290** (2001) 83–123, gr–qc/0001060

[180] S. Vacaru, *Gauge and Einstein Gravity from Non–Abelian Gauge Models on Noncommutative Spaces, Phys. Lett.* **B 498** (2001) 74–86

[181] S. Vacaru, *Off-Diagonal 5D Metrics and Mass Hierarchies with Anisotropies and Running Constants,* hep–th/ 0106268

[182] S. Vacaru, *A New Method of Constructing Black Hole Solutions in Einstein and 5D Dimension Gravity*, hep-th/0110250.

[183] S. Vacaru, *Black Tori in Einstein and 5D Gravity,* hep-th/0110284.

[184]) S. I. Vacaru, I. A. Chiosa and Nadejda A. Vicol, Locally Anisotropic Supergravity and Gauge Gravity on Noncommutative Spaces, in: NATO Advanced Research Workshop Proceedings "Noncommutative Structures in Mathematics and Physics", eds. S. Duplij and J. Wess, September 23-27, Kyiv, Ukraine (Kuwer Academic Publishers, 2001), 229 - 244

[185] S. Vacaru and H. Dehnen, *Locally Anisotropic Structures and Nonlinear Connections in Einstein and Gauge Gravity*, gr–qc/0009039; H. Dehnen and S. Vacaru, *Nonlinear Connections and Nearly Autorparallel Maps in General Relativity*, gr–qc/0009038.

[186] S. Vacaru and Yu. Goncharenko, *Yang–Mills fields and gauge gravity on generalized Lagrange and Finsler spaces*, Int. J. Theor. Phys. **34** (1995) 1955–1978

[187] S. Vacaru and E. Gaburov, *Anisotropic Black Holes in Einstein and Brane Gravity*, hep-th/0108065

[188] S. Vacaru and D. Gontsa, *Off-Diagonal Metrics and Anisotropic Brane Inflation,* hep-th/0109114

[189] S. Vacaru and S. Ostaf, *Twistors and nearly autoparallel maps,* 25–30 July 1994 (Lajos Kossuth University, Debrecen, Hungary) p. 56

[190] S. Vacaru and S. Ostaf, *Locally Anisotropic Spinors and Twistors* sl Contributions to the 14 th International Conference on General Relativity and Gravitation, Florence, August 6-12, 1995 (GR14, Florence, 1995)

[191] S. Vacaru and S. Ostaf, *Nearly autoparallel maps of Lagrange and Finsler spaces,* in Lagrange and Finsler Geometry, eds. P. L. Antonelli and R. Miron, (Kluwer Academic Publishers, Dordrecht, Boston, London, 1996) 241–253

[192] S. Vacaru and S. Ostaf, *Twistors and Nearly Autoparallel Maps* Rep. Math. Phys. **37** (1996) 309-324

[193] S. Vacaru, S. Ostaf, Yu. Goncharenko and A. Doina, *Nearly autoparallel maps of Lagrange spaces,* Buletinul Academiei de Ştiinţe a Republicii Moldova, Fizica şi Tehnica **3** (1994) 42–53

[194] S. Vacaru and F. C. Popa, *Dirac Spinor Waves and Solitons in Anisotropic Taub–NUT Spaces*, *Class. Quant. Grav.* **18** (2001) 4921–4938

[195] S. Vacaru, D. Singleton, V. Botan and D. Dotenco, *Locally Anisotropic Wormholes and Flux Tubes in 5D Gravity, Phys. Lett. B* **519** (2001) 249-259

[196] S. Vacaru, P. Stavrinos and E. Gaburov, *Anholonomic Triads and New Classes of (2+1)-Dimensional Black Hole solutions*, gr-qc/0106068

[197] Vacaru, P. Stavrinos and Denis Gontsa, *Anholonomic Frames and Thermodynamic Geometry of 3D Black Holes*, gr-qc/0106069

[198] S. Vacaru, P. Stavrinos and N. Vicol, *Spinors in Hamilton and Cartan Spaces*, Contribution at International Conference on "Finsler, Lagrange and Hamilton Spaces", Al. I. Cuza University, Iaşi, August 26–31, 2001, Romania.

[199] S. Vacaru and O. Tintareanu–Mircea, *Anholonomic Frames, Generalized Killing Equations, and Anisotropic Taub NUT spinning spaces*, hep–th/0004075

[200] R. S. Ward and R. O. Wells, *Twistor Geometry and Field Theory* (Cambridge University Press, 1990).

[201] R. Wald, *General Relativity* (University of Chicago Press, Chicago, London, 1984).

[202] R. P. Walner, *General Relativity and Gravitation* **17** (1985) 1081

[203] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, New York, 1972)

[204] J. Wess and J. Bagger, *Supersymmetry and Supergravity* (Prinston University Press, 1983)

[205] P. West, *Introduction to Supersymmetry and Supergravity* (World Scientific, 1986)

[206] H. Weyl, *Elektron und Gravitation Z. Physik* **56** (1929) 330.

[207] G. B. Whitham, *Comments on some recent multi–soliton solutions*, *J. Phys. A. Math.* **12** (1979) L1–L3

[208] C. M. Will, *Theory and Experiment in Gravitational Physics*, (Cambridge University Press, 1993)

[209] E. Witten, *An interpretation of classical Yang–Mills theory*, *Phys. Lett. B* **77** (1978) 394–398

[210] K. Yano and S. I. Ishihara, *Tangent and Cotangent Bundles. Differential Geometry* (Marcel Dekker, New York, 1973)

[211] H. Yukawa, *Quantum theory of non–local fields. Part I. Free fields*, *Phys. Rev.* **77** (1950) 219–226

[212] V. E. Zakharov and A. B. Shabat, Funk. Analiz i Ego Prilojenia [Funct. Analysis and its Applications] **8,** 43 (1974)