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We consider the perturbative description of saturation based on the nonlinear QCD evolution equation of Balitsky and Kovchegov (BK). Although the nonlinear corrections lead to saturation of the scattering amplitude locally in impact parameter space, we show that they do not unitarize the total cross section. The total cross section for the scattering of a strongly interacting probe on a hadronic target is found to grow exponentially with rapidity $t = ln(s/s_0)$, $\sigma \propto \exp\{\frac{\alpha_s N_c}{2\pi}\epsilon t\}$ where ϵ is a number of order unity. The origin of this violation of unitarity is the presence of long range Coulomb fields away from the saturation region. The growth of these fields with rapidity is not tempered by the nonlinearity of the BK equation.

Understanding the growth of total scattering cross sections with energy $\sqrt{s}$ is a longstanding problem. The unitarity, or Froissart bound states that the total inelastic cross section for the scattering of a hadronic projectile on a hadronic target can not grow faster than

$$\sigma < \pi d^2 \ln^2(s/s_0)\,, \tag{1}$$

where d is some typical hadronic scale and $t = ln(s/s_0)$ the rapidity. While QCD, the theory of hadronic interactions, is a unitary theory and therefore satisfies this unitarity bound, there is no guarantee that perturbative calculations preserve this property. In fact, the linear perturbative evolution equation due to Balitsky, Fadin, Kuraeev and Lipatov (BFKL) implies an exponential growth of σ with t, thus violating unitarity.

Following the pioneering works of [1,2], there has been recent progress in high energy hadronic scattering in a derivation of a nonlinear evolution equation [3–5] which tames the BFKL-type growth. These equations resum the nonlinear corrections to the QCD evolution with rapidity (energy) to all orders in partonic density and to first order in the QCD coupling. While previous studies of these equations assumed translational invariance in the impact parameter plane, here we explore for the first time their impact parameter dependence. For the equations first derived by Balitsky [3], we show that the total cross section does not unitarize but grows exponentially with t.

The first in the hierarchy of nonlinear BK evolution equations which govern the evolution of correlation functions of the gluon fields in the target with t, reads [3]

$$\frac{d}{dt}\mathrm{Tr}\langle 1 - U^\dagger(x)U(y)\rangle = \frac{\alpha_s}{2\pi^2}\int d^2z\,\frac{(x-y)^2}{(x-z)^2(y-z)^2}$$
$$\langle N_c \mathrm{Tr}[U^\dagger(x)U(y)] - \mathrm{Tr}[U^\dagger(x)U(z)]\mathrm{Tr}[U^\dagger(z)U(y)]\rangle\,, \tag{2}$$

where $U(x)$ is the eikonal scattering amplitude of a fundamental probe on the target characterized by some distribution of gluon fields A_μ. In the gauge used in [3]

$$U(x) = P\exp\left[i\int dx^- T^a A^+_a(x)\right]. \tag{3}$$

The averaging in (2) is taken over the ensemble characterizing the target. In the large N_c limit, (2) simplifies to a closed equation for the scattering probability $N(x,y)$ of a colour singlet dipole with charges at points x, y

$$N(x,y) = \frac{1}{N_c}\mathrm{Tr}\langle 1 - U^\dagger(x)U(y)\rangle\,. \tag{4}$$

This equation was independently derived by Kovchegov [4] in the colour dipole approach of [7]

$$\frac{d}{dt}N(x,y) = \frac{\alpha_s N_c}{2\pi^2}\int d^2z\,\frac{(x-y)^2}{(x-z)^2(y-z)^2}$$
$$[N(x,z) + N(y,z) - N(x,y) - N(x,z)N(z,y)]\,. \tag{5}$$

Weigert [5] suceeded to reformulate Balitsky's hierarchy in terms of a nonlinear stochastic process,

$$\frac{dU(x)}{dt} = gU(x)iT^a\!\int\!\frac{d^2z}{\sqrt{4\pi^3}}\frac{(x-z)_i}{(x-z)^2}\Big[1 - \tilde{U}^\dagger(x)\tilde{U}(z)\Big]^{ab}\xi^b_i(z)$$
$$-\frac{i\alpha_s}{2\pi^2}\int d^2z\,\frac{1}{(x-z)^2}\,\mathrm{Tr}[T^a\tilde{U}^\dagger(x)\tilde{U}(z)]\,. \tag{6}$$

Here $U(x)$ and $\tilde{U}(x)$ are the unitary matrices (3) in the fundamental and adjoint representations, respectively. The noise is characterised by Gaussian local correlations

$$\langle \xi^a_i(t',z')\xi^b_j(t'',z'')\rangle = \delta^{ab}\delta_{ij}\delta(t'-t'')\delta(z'-z'')\,. \tag{7}$$

This simple Langevin equation gives rise to an infinite number of equations for correlators of U which coincide with those derived in [3].

Nonlinear evolution equations were also derived in the Wilson renormalization group approach [6]. While results from this approach coincide with (2,5) in a certain limit, the question whether they are generally equivalent or incorporate different physics is still open. Here, we focus entirely on the BK equations (2,5,6).

From the first numerical [8–10] and analytical [11,12] studies of the BK eqs. (2,5) the following consistent picture emerges: Suppose one starts the evolution from the initial condition of small target fields (or $N(x,y) \ll 1$ for all x, y). Then initially the evolution follows the BFKL equation, since the nonlinear term in (5) is negligible. As the scattering probability approaches unity, the nonlinear term kicks in and eventually the growth stops as the RHS of (5) vanishes for $N(x,y) = 1$. The larger dipoles $[(x-y)^2 \gg 1/Q_s^2(t)]$ saturate earlier, the smaller dipoles follow at later "time" t. These features are contained in the simple parametrization [13]

$$N(x,y) = 1 - \exp\Big[-(x-y)^2 Q_s^2(t)\Big]\,. \tag{8}$$

The saturation momentum $Q_s(t)$ is a growing function of rapidity. Its exact dependence on rapidity is not known, but both, the numerical results [8] and simple theoretical estimates [14,12] are consistent with the exponential growth of the form

$$Q_s(t) = \Lambda \exp\Big[\alpha_s c t\Big] \tag{9}$$

with c of order unity. This physical picture has been anticipated several years ago in [14].

While the BFKL equation leads to an unphysical exponential growth of the scattering probability $N(x,y)$ with t, the nonlinearities of eqs. (2,5) tame this growth such that $N(x,y) < 1$, as required for a probability. This is commonly refered to as "unitarization". However, the "saturation" of the scattering probability at fixed impact parameter does not insure that the total scattering cross section is unitary. We hence refer to the above phenomenon more accurately as "saturation".

To calculate the total inelastic cross section one has to integrate the scattering probability over the impact parameter. Thus in the saturation regime

$$\sigma = \pi R^2(t)\,, \tag{10}$$

where $R(t)$ is the size of the region in the transverse plain for which the scattering probability for hadronic size "dipoles" is unity. To satisfy the Froissart bound the radius $R(t)$ should grow at most linearly with t. We now present two simple calculations which establish that within the BK evolution the growth of the radius with rapidity is exponential.

First consider the Langevin equation (6). Assume that initially, at rapidity t_0 the target is black within some radius R_0. This means that for $|z| < R_0$ the matrix $U(z)$ fluctuates very strongly so that it covers the whole group space. We concentrate on a point x which is initially outside of this black region. The matrix $U(x)$ then is close to unity. Thus there is no correlation between $U(x)$ and $U(z)$, and the second term on the right hand side of (6) can be set to zero. This is the random phase approximation introduced in [5] and used later in [12]. As the target field ensemble evolves in rapidity, the radius of the black region grows. As long as the point x stays outside the black region we can approximate the Langevin equation by (we drop colour indices which are inessential to our argument)

$$\frac{d}{dt}U(x) = -\sqrt{\frac{\alpha_s N_c}{\pi^2}}\int_{|z|<R} d^2z \frac{(x-z)_i}{(x-z)^2}\xi_i(z)\,. \tag{11}$$

This equation neglects contributions to the derivative of U that come from gluons originating from the sources outside the black region. Those contributions would enhance the growth of U, and so by omitting them we underestimate the rate of growth of the radius of the black region. The formal solution of eq. (11) is

$$1 - U(x,t) = \sqrt{\frac{\alpha_s N_c}{\pi^2}}\int_{t_0}^{t} d\tau \int_{|z|<R(\tau)} d^2z \frac{(x-z)_i}{(x-z)^2}\xi_i(z)\,. \tag{12}$$

Squaring it and averaging over the noise term gives

$$\langle (1-U(x,t))^2\rangle = \frac{\alpha_s N_c}{\pi^2}\int_{t_0}^{t} d\tau \int_{|z|<R(\tau)} \frac{d^2z}{(x-z)^2}\,. \tag{13}$$

As long as x is outside the black region we can approximate the integral on the right hand side by

$$\int_{|z|<R(\tau)} d^2z \frac{1}{(x-z)^2} = \pi \frac{R^2(\tau)}{x^2}\,, \tag{14}$$

and eq.(13) becomes

$$\langle (1-U(x,t))^2\rangle = \frac{\alpha_s N_c}{\pi}\frac{1}{x^2}\int_{t_0}^{t} d\tau R^2(\tau)\,. \tag{15}$$

Now as the black region grows, eventually it reaches the point x. At this rapidity the matrix $U(x)$ will start fluctuating with the amplitude of order one. Thus when $R(t) = |x|$, the left hand side of eq. (15) becomes a number of order one, which we call $1/\epsilon$. We thus have an approximate equation for $R(t)$

$$\frac{1}{\epsilon}R^2(t) = \frac{\alpha_s N_c}{\pi}\int_{t_0}^{t} d\tau R^2(\tau)\,. \tag{16}$$

At large rapidities therefore the radius of the black region is exponentially large

$$R(t) = R(t_0)\exp\Big[\frac{\alpha_s N_c}{2\pi}\epsilon(t-t_0)\Big]\,. \tag{17}$$

This is our main result.

We note that while the approximations leading to eq. (16) cease to be valid when the point x is on the boundary of the black region, this does not affect our main conclusion. First, eq. (14) is an underestimate of the integral, thus underestimating the growth of R. Second, when x is on the boundary of the black region and z in the black region, although the factors $(1-U(x)U^\dagger(z))$ and $U(x)$ in eq.(6) are not strictly unity, they are still of order one for allmost all points z. Thus, although we can not determine the exact numerical value of ϵ, the functional form of the solution as well as its parametric dependence is given correctly by eq.(17).

Note that eq. (6) refers to the evolution of matrices $U(x)$. This can be thought of as evolution of the scattering amplitude of a coloured probe. The preceding derivation thus refers to the growth with rapidity of the cross section for scattering of a coloured probe. In a confining theory this is not a physical quantity. However the physics of BK equation does not incorporate effects of confinement, and therefore the cross section for a colourless dipole within the BK framework must grow in the same way. To establish this point, and to make more explicit the relation between ϵ and the BFKL dynamics, we now present an alternative derivation of eq. (17).

To this end we consider the BK evolution as the evolution of the projectile [4]. Suppose at the initial energy the projectile is a colour dipole of size x_0. It scatters on a hadronic target of some size $R_{\rm target}$. As is explicit in [4], as the energy is increased the projectile wave function evolves according to the BFKL equation. Thus at rapidity t the density of dipoles of size x at transverse distance r from the original dipole is given by the BFKL expression (see for example [15]):

$$n(x_0,x,r,t)=\frac{32}{x^2}\frac{\ln\frac{16r^2}{x_0x}}{(\pi a^2t)^{3/2}}\exp\Big[\omega t-\ln\frac{16r^2}{x_0x}-\frac{\ln^2\frac{16r^2}{x_0x}}{a^2t}\Big] \tag{18}$$

with $\omega=4\ln 2N_c\alpha_s/\pi$ and $a^2=14\zeta(3)N_c\alpha_s/\pi$. When the density of dipoles at a given impact parameter is greater than one, multiple scatterings become important. Thus the scattering probability is not proportional to n, but is an infinite series containing all multiple scattering terms [4].

For our argument, it is only important that once the density of dipoles at some impact parameter r becomes larger than some fixed critical number, the scattering amplitude at this impact parameter saturates. The exact value of this number depends on the target, but importantly it does not depend on rapidity. Thus the total cross section is given by the square of the largest impact parameter at which the dipole density in the projectile wave function is of order unity. In order to estimate this directly from eq. (18), we must choose the dipole size x in (18) to be the smallest size which is saturated on the target at initial rapidity. Within the ansatz of eq. (8) this would be $x=Q_s^{-1}(t_0)$. We then find

$$R^2(t)=\frac{1}{16}\frac{x_0}{Q_s(t_0)}\exp[\frac{\alpha_sN_c}{\pi}\epsilon t]\,, \tag{19}$$

$$\epsilon=7\,\zeta(3)\Big[-1+\sqrt{1+8\ln 2/7\zeta(3)}\Big]\,. \tag{20}$$

Thus, also for a coulour singlet projectile, we arrive again at the exponential growth of the cross section.

The exact value of ϵ given in eq. (20) should not be taken too seriously. The explicit form of the dipole density eq. (18) was derived by a saddle point integration, and as such is valid only for $\ln\frac{16r^2}{x_0x}<\alpha_st$. This condition is not satisfied by eq. (19). However, even beyond the saddle point approximation the density has the form

$$n(x_0,x,r,t)=\frac{1}{x^2}\ln\frac{16r^2}{x_0x}\exp\Big[\alpha_stF(\frac{\ln\frac{16r^2}{x_0x}}{\alpha_st})\Big]\,. \tag{21}$$

The relevant condition is $F=0$. Thus, while our calculation does not specify the numerical value of ϵ, the correct solution parametrically is the same as (19).

In the target rest frame, this violation of unitarity by the BK evolution can be understood as follows: Start with a single dipole scattering on the hadronic target of transverse size $R_{\rm target}$. With increasing energy the projectile dipole emits additional dipoles strictly according to the BFKL evolution. The density as well as the transverse size of the projectile state thus grows. The increase in density leads to increasing importance of multiple scatterings which are properly accounted for in the BK derivation. This ensures that the scattering probability saturates locally. In the saturation regime, as long as the size of the projectile state $R(t)$ is smaller than the target size $R_{\rm target}$, the cross section grows essentially due to surface effects,

$$\sigma=\pi R^2_{\rm target}+2\pi R_{\rm target}x_0\exp\Big[\frac{\alpha_sN_c}{2\pi}\epsilon t\Big]\,. \tag{22}$$

Thus as long as $\alpha_s\epsilon t<\ln\frac{R_{\rm target}}{x_0}$, the cross section is practically geometrical. However once the energy is high enough so that the projectile size is larger than that of the target, the total cross section is determined by the former and grows exponentially with the logarithm of energy according to eq. (19).

This also illustrates that the applicability of the BK evolution crucially depends on the nature of the target. If the target is thick enough, so that the multiple scatterings become important before the growth of the projectile radius does, and if the target is wide enough, so that saturation occurs before the projectile radius swells beyond that of the target, then there is an intermediate regime in which the inelastic cross section remains practically constant and equal to $\pi R^2_{\rm target}$. Then BK applies. However, if the target is a nucleon, neither one of these conditions is satisfied. Thus the tainted infrared behaviour of the BFKL evolution of the projectile will show up right away and will invalidate the application of the BK equation.

In order to discuss the violation of unitary from the point of view of the evolution of target fields, we now go back to the stochastic process (6). The RHS of (6) describes the total Coulomb (Weizsäcker-Williams) field at point x due to the colour charge sources at points z. Since the noise is stochastic, the colour sources are completely uncorrelated both in the transverse plain and in rapidity. For this random source, the square of the total colour charge is proportional to the area, and this is precisely the factor R^2 in eq. (15). The incoming dipole thus scatters on the Coulomb field created by the large incoherent colour charge. Because the Coulomb field is long range, the whole bulk of the region populated by the sources contributes to the evolution and leads to rapid growth of R. If the field created by the sources was screened by some mass, the evolution would be unitary. To illustrate this point, we substitute the Coulomb field $(x-z)_i/(x-z)^2$ in (eq.6) by an exponentially decaying field $m\exp\{-m|x-z|\}$. It is straightforward to perform now the same analysis as before. Eq. (14) is replaced by

$$\int_{|z|<R(\tau)} d^2z\, m^2 \exp\{-m|x-z|\} = \exp\{-m|x-R|\}\,. \quad (23)$$

This leads to the substitution $R^2 \to \exp\{mR\}$ in all subsequent equations with the end result that

$$R(t) = \alpha_s \frac{\epsilon}{m} t\,, \quad (24)$$

which in fact saturates the Froissart bound. Thus the reason for the violation of unitarity is that the evolution is driven by the emission of the long range Coulomb field from a large number of *incoherent* colour sources in the target.

Cutting off the Coulomb field is not the only possibility to cure this problem. Another option is that the sources of the colour charge in the high density regime cease to be incoherent. If they have correlations ensuring that the total colour charge in a region of fixed size L is zero, then the incoming dipole would feel the Coulomb field only within the fixed distance L from the black region. Thus the new charges produced by the evolution would only "split off" the edges of the black region rather than from its bulk. This scenario is equivalent to exponential decay of the field, and will lead to a unitary evolution. In a confining theory like QCD, it is likely to be materialized. We note that the desirability of such colour charge correlations was stressed in a somewhat different context in [16].

Although such charge correlations do not arise in the BK evolution, it is not *a priori* clear that they are not present in a more complete semiperturbative framework which still does not take into account the physics of confinement at low energies. In fact, the BK framework is incomplete inasmuch as it takes the evolution of the projectile wave function to be pure BFKL. One expects that once the density of gluons in this wave function becomes large, interactions should lead to saturation effects on the wave function level, i.e., the density of the dipoles should grow slower than eq. (18). Such corrections should still be semiperturbative, in the sense that they are present at small α_s. However, for scattering on "small" targets, they will become important at the same energy as the multiple scattering terms resummed in eq. (2,5). These wave function saturation effects may lead to charge correlations of the type necessary to unitarize the total cross section.

We finally note that the unitarity bound for Deeply Inelastic Scattering is different. In this case, the projectile is a virtual photon without fixed hadronic size. For transverse polarization, its perturbatively known wave function is $\Phi^2(r) \propto \alpha_{em}\frac{1}{r^2}$ for $r^2 \ll Q^{-2}$. In such a projectile not all dipoles saturate at the same energy. The main contribution to the scattering probability comes from dipoles of size $r > Q_s^{-1}(t)$ which are saturated. At high energy (i.e. for $Q_s^{-1} \ll Q^{-1}$),

$$N(\gamma^*) = \int_{Q_s^{-2}<r^2<Q^{-2}} d^2r\,\Phi^2(r) \propto \alpha_{em} \ln Q_s/Q\,. \quad (25)$$

With the exponential dependence (9) of Q_s on rapidity this translates into $N(\gamma^*) \propto \alpha_{em}\alpha_s \ln s/s_0$, and therefore

$$\sigma_{\rm DIS} \propto \alpha_{em}\alpha_s \pi R^2(t) t\,. \quad (26)$$

The DIS cross section has an extra power of t relative to the cross section of a purely hadronic process. This extra power of t is consistent with the numerical results of [10].

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