BOUND STATES IN n DIMENSIONS (Especially $n = 1$ and $n = 2$)

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ABSTRACT

We stress that in contradiction with what happens in space dimensions $n \geq 3$, there is no strict bound on the number of bound states with the same structure as the semi-classical estimate for large coupling constant and give, in two dimensions, examples of weak potentials with one or infinitely many bound states. We derive bounds for one and two dimensions which have the "right" coupling constant behaviour for large coupling.

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1 Introduction

First, I would like to apologize for the fact that I am speaking on something which, in two respects, is outside the subject of this workshop:
(i) I speak of two-body bound states while we speak here of few-bodies which means at least three;
(ii) the most important results I present are in one- and two-space dimensions.

My excuse for (i) is that control of two-body bound states is useful for more body bound states. For instance the proof that the system proton-electron-muon is unstable uses two-body results. For (ii) it is well known that two-space dimensions are very often realized in condensed matter physics.

2 Large coupling constant behaviour

If we take a potential gV, the number of negative energy bound states in n dimensions is given by the semi-classical expression for large g

$$N_{total} \simeq c_n \; g^{n/2} \int d^n x |V^-(x)|^{n/2} \; , \tag{1}$$

where V^- designates the negative part of the potential, and c_n the semi-classical constant is given by

$$c_n = \frac{2^{-n} \pi^{-n/2}}{\Gamma(1 + \frac{n}{2})} \tag{2}$$

This holds for any n, provided V is sufficiently smooth and sufficiently rapidly decreasing at infinity. For $n = 1$, it was established by Chadan [1] and for general n by Martin [2] and some years later by Tamura [3].

3 Strict bounds for $n \geq 3$, absence of bounds for $n = 1, 2$

For $n \geq 3$, it was established by Lieb [4], Cwikel [5] and Rozenblum [6], and later by others [7] that besides the asymptotic estimate (1) there is a strict bound of the same form

$$N_{total} < b_n \; g^{n/2} \int d^n x |V^-(x)|^{n/2} \tag{3}$$

where b_n is strictly larger than c_n, even for very large n, as shown by Glaser, Grosse and Martin [8].

On the contrary, in one and two dimensions, the situation is drastically different.

For $n = 1$ and $n = 2$ an arbitrarily weak attractive potential has at least one bound state. For $n = 1$ this is easy to demonstrate using a gaussian wave function. For $n = 2$ it is also true but more delicate [9]. The most elegant proof is given by Yang and De Llano [10] using a trial function $\exp -(r + r_0)^\alpha$, α very close to zero.

It is in fact sufficient, both for $n = 1$ and $n = 2$ to have an arbitrarily weak, globally attractive potential, i.e., such that

$$\int V(x) d^n x < 0 \tag{4}$$

to have a bound state.

One can go further than that, i.e. construct examples in which an arbitrarily weak potential has infinitely many bound states. One such example, inspired by Jean-Marc Richard, is, in two dimensions

$$V = -\sum_{n=1}^{\infty} \quad g_n \delta(|x - x_n| - 1) \tag{5}$$

g_n decreasing, g_1 arbitrarily small, x_n adequately chosen. One can manage to have (5) satisfying

$$\int |V| d^2x < \infty \ , \tag{6}$$

and even

$$\int |V| \Big(\ln(2 + |x|) \Big)^{1-\epsilon} \quad d^2x < \infty \ , \tag{7}$$

$\epsilon > 0$ arbitrarily small.

One can construct other examples where the delta function is replaced by a finite circular ditch.

However, this or these bound states which occur for arbitrarily weak potentials are incredibly weakly bound. We have been able to show that the ground state energy, $\epsilon = -\kappa^2$, satisfies, for a potential gV

$$\kappa < \exp \left(- \frac{2\pi - g \int \ln^+ \left(\frac{1}{x}\right) \ V_R \ d^2x - \frac{1}{2} g \int |V^-| d^2x}{g \int |V^-| d^2x} \right) \tag{8}$$

where $\ln^+(t) = \ln(t)$ for $t > 1$, $=0$ for $t < 1$, and V_R is the circular decreasing rearrangement of $|V^-|$.

If some of you do not know what is a circular decreasing rearrangement, let me say that seeing Mont Blanc it is easy to understand. You take the “Carte Vallot”. The Mont Blanc has level lines. You replace these level lines by circles centered at the origin enclosing the same area as the original level lines. In this way you manufacture the rearranged Mont Blanc. You may have to group together disconnected level lines if necessary.

In a recent preprint, Nieto [11] has calculated that in units where $\hbar = c = 2m = 1$, a square well of radius 1, with a strength 0.1 has a bound state with an energy of 10^{-18} which, incidentally, is a particular case of (8) except for the non-dominant terms.

4 Bounds on the number of bound states in one dimension

Here and in the next sections we use extensively the fact that the number of negative energy bound states is equal to the number of nodes of the zero energy wave function. In one dimension let $x_1 x_2 \dots x_K \dots x_{K+1} \dots x_N$ be these nodes and assume that $x_K < 0 < x_{K+1}$. Since the Schrödinger equation

$$-\frac{d^2\psi}{dx^2} + V(x)\psi = 0$$

looks like a radial reduced three-dimensional equation with angular momentum zero, we can use known bounds and apply them to the intervals $-\infty \to x_K$, $x_{K+1} \to \infty$. So the Bargman bound [12]

$$N_\ell < \frac{1}{2\ell+1} \int_0^\infty r\ V^-(r) dr (q) \tag{9}$$

which, for $\ell = 0$ reduces to

$$N_0 < \int_0^\infty r\ V^-(r) dr\ ,$$

gives us

$$\begin{aligned} N - K - 1 \quad &< \textstyle\int_{x_K}^\infty (x - x_K) V^-(x) dx \\ &< \textstyle\int_0^\infty x V^-(x) dx \end{aligned}$$

and

$$K < \int_{-\infty}^0 |x| V^-(x) dx$$

so that

$$N(m=0) < 1 + \int_{-\infty}^{+\infty} |x| V^-(x) dx \tag{10}$$

Similarly, the bound obtained by one of us (A.M.) in three dimensions for $\ell = 0$ [13]:

$$N(3dim, \ell = 0) < \left[\int_0^\infty r^2 |V^-| dr \int_0^\infty |V^-| dr \right]^{1/4} \tag{11}$$

leads to

$$N(m=0) < 1 + \sqrt{2} \left[\int_{-\infty}^{+\infty} x |V^-| dx \int_{-\infty}^{+\infty} |V^-| dx \right]^{1/4} \tag{12}$$

This bound has the advantage that if we replace V by gV it is <u>linear</u> in g, like the semi-classical estimate for large g.

5 Bounds in two-space dimensions

We start with the <u>radial</u> case for which the reduced Schrödinger equation at zero energy is

$$\left[-\frac{d^2}{dr^2} + \frac{m^2 - 1/4}{r^2} + V(r) \right] u = 0 \tag{13}$$

It is in fact more convenient in the $m = 0$ case to work directly with the non-reduced form:

$$-\frac{d^2\psi}{dr^2} - \frac{1}{r}\ \frac{d\psi}{dr} + V(r)\psi = 0 \tag{14}$$

Details of our aproach will be given in a future publication. Let me just say that we integrate the Schrödinger from one node to the next and find a lower bound for the quantities

$$\int_{x_K}^{x_{K+1}} \ln\left(\frac{x}{x_K}\right) V^-(x)x\ \ dx$$

and

$$\int_{x_K}^{x_{K+1}} \ln\left(\frac{x_{K+1}}{x}\right) V^-(x)x\ \ dx\quad .$$

From this we get

$$N(m=0) < 1 + \int_0^\infty r \left|\ln\left(\frac{R}{r}\right)\right|\quad |V^-(r)|\ \ dr \tag{15}$$

where R is arbitrary. If we call

$$I(R) = \int_0^\infty r|\ln\left(\frac{R}{r}\right)|\ \ |V^-(r)|dr \tag{16}$$

we can minimize with respect to R, and get $I_{min} = I(R_0)$ such that

$$\int_0^{R_0} r|V^-(r)|dr = \int_{R_0}^\infty r|V^-(r)|dr \tag{17}$$

It is interesting to compare I_{min} with the bound on $N(m = 0)$ obtained years ago by Newton [14] and Setô [15].

$$N(m=0) < \frac{\frac{1}{2}\int r\ dr\ r'\ dr'\ \ |V^-(r)|(V^-(r')|\ \ |\ln\left(\frac{r}{r'}\right)|}{\int r\ dr\ |V^-(r)|} \tag{18}$$

It is easy to see, from the mean value theorem that the right-hand side of (18) is larger than $1/2I_{min}$. It is less trivial to prove that it is less or equal to I_{min}. So we get

$$\frac{1}{2}I_{min} < N(\text{Newton} - \text{Setô}) \leq I_{min} \tag{19}$$

which means that for $m = 0$ our result is not quite as good, but simpler. However, both bounds are optimal in the sense that one can approach arbitrarily close to saturation.

For $m \neq 0$, the easiest thing to do is notice that the equation (13) looks like a radial reduced Schrödinger equation in three dimensions with an angular momentum $\ell = m - 1/2$. Thus, for $m \neq 0$ we can use the Bargmann bound, which is valid not only for integer ℓ but any real $\ell > -1/2$. However, if we want to calculate the TOTAL number of bound states, we find, summing up the contributions from the various values of m that if we take a potential gV we find a bound which behaves like $g \ln g$ for large g, i.e., increases qualitatively faster than the semi-classical bound. To avoid this we use a trick which was invented many years ago by Glaser, Grosse and one of us (A.M.) [8]. We transform Eq. (13) by making the change of variables

$$z = \ln r, v = r^2 V \ , \tag{20}$$

into

$$\left(-\frac{d^2}{dz^2} + v(z)\right)\phi(z) = -\left(m^2 - \frac{1}{4}\right)\phi(z) \tag{21}$$

this is a one-dimensional Schrödinger equation whose eigenvalues are

$$e_i = -\left(m_i^2 - \frac{1}{4}\right) \tag{22}$$

Each m_i (seen from the point of view of Eq. (13)), is on a "Regge trajectory", which means that if we disregard the $m = 0$ bound states we have bound states at

$$1, 2, \ldots, [m_i] - 1, [m_i]$$

where $[m_i]$ is the integer part of m_i.

Each bound state has a multiplicity two corresponding to $\pm|m|$. Hence the total number of $m \neq 0$ bound states is less than

$$2\sum m_i < 2 \times \frac{2}{\sqrt{3}} \sum |e_i|^{1/2} \ , \tag{23}$$

since $e_i/m_i^2 \geq 3/4$. Finding a bound on the right-hand side is a standard moment problem in one dimension first considered by Lieb and Thirring [16]. Weidl [17] has been able to prove the inequality

$$\sum |e_i|^{1/2} < C \int |v(z)| dz \ ,$$

and Hundertmark, Lieb and Thomas [18] have found the best possible constant in this inequality

$$\sum |e_i|^{1/2} < \frac{1}{2} \int_{-\infty}^{+\infty} |v(z)| \ dz \tag{24}$$

Hence using (20), (23) and for instance our bound (15) for $m = 0$ we get for the total number of bound states in n potential gV

$$\begin{aligned} N_{TOT} &< 1 + g \int r \ dr |\ln\left(\frac{R}{r}\right)| \ \ |V^- r| \\ &+ g\frac{2}{\sqrt{3}} \int r \ dr |V^-(r)| \end{aligned} \tag{25}$$

i.e., a bound <u>linear</u> in g.

We can get now a bound on the total number of bound states for a potential without symmetry: choose an origin and define

$$\begin{aligned} B(r) &= -\inf_{\theta} V(r,\theta) \\ B(r) &= 0, \text{ if } \inf_{\theta} V(r,\theta) > 0 \end{aligned} \tag{26}$$

(with obvious notations in V).

From the monotonicity of the eigenvalues with respect to the potential we get

$$\begin{aligned} N_{TOT} &< 1 + g \int r\ dr |\ln\left(\frac{R}{r}\right)| B(r) \\ &+ g\frac{2}{\sqrt{3}} \int r\ dr\ B(r) \end{aligned} \tag{27}$$

It is clear that this bound will be good if V has only one singular point which can be chosen as origin. Otherwise it may be very bad or even meaningless. Our conjecture is that there must be a bound like

$$\begin{aligned} N_{TOT} &< 1 + C_1 \int d^2x\ \ V_R(x) \left(\ln|\frac{x_0}{x}|\right)^+ \\ &+ C_2 \int d^2x |V^-(x)| \left(\ln|\frac{x}{x_0}|\right)^+ \\ &+ C_3 \int d^2x |V^-(x)| \end{aligned}$$

where V_R is the rearrangement of V^-.

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