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A REMARK ON NON-ABELIAN CLASSICAL KINETIC THEORY

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It is known that non-Abelian classical kinetic theory reproduces the Hard Thermal/Dense Loop (HTL/HDL) effective action of QCD, obtained after integrating out the hardest momentum scales from the system, as well as the first higher dimensional operator beyond the HTL/HDL level. We discuss here its applicability at still higher orders, by comparing the exact classical effective action obtained in the static limit, with the 1-loop quantum effective potential. We remark that while correct types of operators arise, the classical colour algebra reproduces correctly the prefactor of the 4-point function $\mathrm{Tr}\, A_0^4$ only for matter in asymptotically high dimensional colour representations.

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1. Introduction

Most observables of QCD at a finite temperature T and chemical potential μ are not computable in perturbation theory beyond a certain order, due to severe infrared problems [1]. What can be done systematically however is the construction of various effective theories, obtained by integrating out only the hardest scales from the system. For static observables this leads to the concept of a dimensionally reduced effective theory [2], whose construction and non-perturbative properties have been studied in great detail (for reviews, see [3]).

For non-static observables, on the other hand, the relevant effective description is the Hard Thermal and/or Dense Loop (HTL/HDL) theory [4, 5] or, equivalently, its local reformulation [6, 7] as classical kinetic theory [8, 9]. These constructions and their non-perturbative properties have however so far not been studied to a similar beyond-the-leading-order level as those of the dimensionally reduced theory (for a review of the current status, see [10]).

Very recently, there was a new positive indication on the effectiveness of the kinetic description: it was noted that it reproduces also the first higher order operator beyond the HTL/HDL level [11], representing an effective (charge conjugation violating) three-gluon interaction. In this brief note, we wish to report on some properties of the classical kinetic theory at still higher orders. We point out that in this case agreement with quantum field theory is only obtained for matter in high dimensional colour representations.

2. Formulation of classical kinetic theory

We start by reviewing briefly the formulation of classical kinetic theory, used to describe how "hard" particles (quarks and gluons with momenta $\sim \max(T,\mu)$), behave in the background of a "soft" ($\sim \max(gT, g\mu)$, where g is the gauge coupling) gauge field configuration A^a_μ.

The starting point is to consider the hard modes as classical point particles carrying a colour charge Q^a, with dynamics governed by the Wong equations [8]. When the effect of collisions is neglected, the 1-particle distribution function obeys the Boltzmann equation [9]

$$p^\mu \left(\frac{\partial}{\partial x^\mu} + g f^{abc} Q^a A^b_\mu \frac{\partial}{\partial Q^c} - g Q^a F^a_{\mu\nu} \frac{\partial}{\partial p_\nu} \right) f(x,p,Q) = 0. \tag{1}$$

Our sign conventions correspond to QCD with a covariant derivative in the fundamental representation $D_\mu = \partial_\mu - igT^a A^a_\mu$, and in the adjoint $\mathcal{D}^{ab}_\mu = \delta^{ab}\partial_\mu + g f^{acb} A^c_\mu$; the field strength is $F_{\mu\nu} = (i/g)[D_\mu, D_\nu]$. We take N_f flavours of massless quarks and antiquarks, each carrying two helicities. For antiquarks one should replace $Q \to -Q$ in Eq. (1), and for gluons one should take the adjoint representation, $Q \to Q_{\text{adjoint}} \equiv \tilde{Q}$.

For quarks/antiquarks the boundary condition for the solutions of Eq. (1) is that for a vanishing gauge field background (say, at infinity),

$$f(x,p,Q) \to \delta_+(p^2)\, n_f(\omega_p \mp \mu), \quad \delta_+(p^2) \equiv 2\theta(p_0)\,(2\pi)\,\delta(p^2)\,, \tag{2}$$

where n_f is the Fermi-Dirac distribution function, and $\omega_p = |\mathbf{p}|$. For gluons $f(x,p,Q) \to \delta_+(p^2)\, n_b(\omega_p)$, where n_b is the Bose-Einstein distribution function.

The solution of Eq. (1) defines a current induced by the coloured particles,

$$j^a_\nu(x) = g \sum_{\substack{\text{helicities}\\ \text{species}}} \int_{p,Q} p_\nu Q^a f(x,p,Q), \tag{3}$$

with

$$\int_p = \int \frac{dp_0}{(2\pi)} \int_{\mathbf{p}}, \quad \int_{\mathbf{p}} = \int \frac{d^3p}{(2\pi)^3}, \quad \int_Q = \int dQ. \tag{4}$$

Here dQ is the colour measure, which contains delta functions fixing the representation dependent Casimirs (see, e.g., [12]); we shall return to its properties presently. The current in Eq. (3), in turn, defines via

$$j^\nu_a = -\frac{\delta}{\delta A^a_\nu} \delta S_M, \tag{5}$$

an action $S_M = S_M^{\text{local}} + \delta S_M$, where $S_M^{\text{local}} = -\int_x (1/2) \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu}$.

There are actually two somewhat different formulations of the kinetic theory, corresponding to the orders in which the integrals $\int_p, \int_Q$ are to be carried out in the above. We shall mostly carry out $\int_p$ first. However, if one takes the moments $\int_Q (...), \int_Q Q^a (...)$ of Eq. (1), imposing by hand the QCD type relation (say, for the fundamental representation)

$$\int_Q Q^a Q^b f(x,p,Q) = \frac{1}{2N_c} \delta^{ab} \bar{f}(x,p) + \frac{1}{2} d^{abc} f^c(x,p), \tag{6}$$

with $\bar{f} = \int_Q f, \quad f^a = \int_Q Q^a f$, then one gets a closed set of equations for $\bar{f}, f^a$ [9]:

$$p \cdot \partial \bar{f} - g p^\mu F^a_{\mu\nu} \frac{\partial f^a}{\partial p_\nu} = 0, \tag{7}$$

$$(p \cdot \mathcal{D})^{ab} f^b + \frac{g}{2} d^{abc} p^\mu F^b_{\mu\nu} \frac{\partial f^c}{\partial p_\nu} - \frac{g}{2N_c} p^\mu F^a_{\mu\nu} \frac{\partial \bar{f}}{\partial p_\nu} = 0. \tag{8}$$

We refer to this as the second formulation. Since the colour hierarchy is truncated by Eq. (6), the first and second formulations only agree for the low order colour moments.

3. Exact solutions for special backgrounds

Let us now recall that, for some special background field configurations, one can find an ansatz leading to an exact solution of Eq. (1). Such solutions have previously been discussed in the Abelian case [13, 14] and, for static backgrounds, in the non-Abelian [9] (see also [15]). However no comparison has been made with QCD, as far as we know.

We will search for a solution of Eq. (1) in a form that only depends on the canonical momenta. Introducing the shorthands

$$A^a Q^a \equiv A, \quad A^a \tilde{Q}^a \equiv \tilde{A}, \tag{9}$$

we take for quarks (with a set of α's to be specified presently)

$$f(x,p,Q) = \delta_+(p^2)\, F(p_\alpha + gA_\alpha). \tag{10}$$

Plugging into Eq. (1), we immediately find that if α's exist such that the condition $\partial_\alpha A^a_\nu = 0$ is satisfied for all a, ν, then the ansatz in Eq. (10) solves Eq. (1). Similar solutions hold for antiquarks, by replacing $Q \to -Q$, and for gluons, by replacing $Q \to \tilde{Q}$.

Apart from Eq. (1), the ansatz should also satisfy the boundary condition in Eq. (2). Thus, in the static limit ($\partial_0 A^a_\nu = 0$), we obtain (for quarks)

$$f(x,p,Q) = \delta_+(p^2)\, n_f(p_0 + gA_0 - \mu). \tag{11}$$

An exact solution could also be obtained in the homogeneous limit, $\partial_i A^a_\nu = 0$, by replacing $p_0 + gA_0$ in Eq. (11) by $|\mathbf{p} + g\mathbf{A}|$.

One may of course ask whether these exact solutions are the most general ones, given the boundary conditions in Eq. (2). It is at least easy to verify iteratively (without any ansatz) that they do agree with the perturbative solution in gQ^a of Eq. (1) around $f^{(0)}$ defined by Eq. (2): writing $f = f^{(0)} + f^{(1)} + f^{(2)} + \ldots$, the $n^{\rm th}$ order solution is obtained by solving

$$p^\mu \hat{D}_\mu f^{(n)} = \hat{L} f^{(n-1)}\ , \tag{12}$$

where $\hat{D}_\mu = \partial_\mu + g f^{abc} Q^a A^b_\mu \partial^c_Q$, $\hat{L} = g p^\mu Q^a F^a_{\mu\nu} \partial^\nu_p$.

4. Comparison with quantum field theory

Let us now see what kind of an effective action, S_M, the solutions found in the previous section lead to, and compare with quantum field theory.

Plugging Eq. (11) into Eq. (3) and taking into account the N_f flavours and the two helicities, we obtain, say, for the quark and antiquark contribution to the gauge current,

$$j^a_0 = 2gN_f \int_{\mathbf{p},Q} Q^a \Big[n_f(\omega_p + gA_0 - \mu) - n_f(\omega_p - gA_0 + \mu) \Big], \tag{13}$$

$$j^a_i = 0. \tag{14}$$

Solving now Eq. (5) and writing $\delta S^f_M = \int_x \delta\mathcal{L}^f_M$, we obtain

$$\delta\mathcal{L}^f_M = 2N_f T \int_{\mathbf{p},Q} \Big[\ln\Big(1 + e^{(-\omega_p - gA_0 + \mu)/T}\Big) + \ln\Big(1 + e^{(-\omega_p + gA_0 - \mu)/T}\Big) \Big]. \tag{15}$$

The integral over $\mathbf{p}$ is easily carried out, and we finally arrive at

$$\begin{aligned} \delta\mathcal{L}^f_M = {} & N_f \int_Q \Big[\Big(\frac{7}{2}\frac{\pi^2}{90} T^4 + \frac{1}{6}\mu^2 T^2 + \frac{1}{12\pi^2}\mu^4 \Big) \\ & - g\frac{\mu}{3}\Big(T^2 + \frac{\mu^2}{\pi^2}\Big) A_0 + \frac{g^2}{2}\Big(\frac{T^2}{3} + \frac{\mu^2}{\pi^2}\Big) A_0^2 - \mu\frac{g^3}{3\pi^2} A_0^3 + \frac{g^4}{12\pi^2} A_0^4 \Big]. \end{aligned} \tag{16}$$

We have for completeness kept here even the field independent part, accounting for the leading 1-loop ("free") expression for the fermionic contribution to the pressure of QCD.

Going to Euclidean metric by writing $A_0^M = iA_0^E$, $\mathcal{L}_E = -\mathcal{L}_M(A_0^M \to iA_0^E)$, we observe immediately that Eq. (16) *would* agree with the fermionic contribution to the dimensionally reduced effective action [16] for A_0; or, equivalently, with the full 1-loop fermionic contribution to the effective potential for the phase of the Polyakov line [17]; *provided* that

$$\int_Q Q^{a_1} Q^{a_2} \ldots Q^{a_n} \equiv (-1)^n \Big[\mathrm{Tr}\, T^{a_1} T^{a_2} \ldots T^{a_n} \Big]_{\text{symmetric part}}, \quad n = 0, ..., 4. \tag{17}$$

(The trivial factor $(-1)^n$ could be removed by inverting the sign convention for μ here, or in [16].) Thus the question is whether Eq. (17) is satisfied. It is easy to show that it is for $n \leq 3$, while in general it is not for $n = 4$.

Indeed, since the integration measure is gauge invariant [7] and the Q^a's commute, the result of the left-hand-side of Eq. (17) must have a covariant structure. For SU(2) this is of the form

$$\int_Q Q^{a_1} Q^{a_2} Q^{a_3} Q^{a_4} = L(R)(\delta^{a_1 a_2}\delta^{a_3 a_4} + \delta^{a_1 a_3}\delta^{a_2 a_4} + \delta^{a_1 a_4}\delta^{a_2 a_3}) \; . \tag{18}$$

The integral here can be carried out even explicitly, for SU(2). Alternatively, to fix the constant $L(R)$, we can contract this equation with $\delta^{a_1 a_2}\delta^{a_3 a_4}$, and sum over indices. Then the integral can easily be performed, due to the constraint $\delta(Q^a Q^a - C_2(R))$ where $C_2(R)$ is the quadratic Casimir, and the normalisation $\int_Q = d_R$ where d_R is the dimension of the representation. One finds $L(R) = (d_R/15) C_2^2(R)$. Therefore, in the classical effective action the piece quartic in A_0 reads

$$\int_Q A_0^4 = \frac{1}{5} d_R\, C_2^2(R)\, (A_0^a A_0^a)^2 \,, \tag{19}$$

while the quantum result, for an arbitrary representation of SU(2), can be seen to be

$$\mathrm{Tr}\, A_0^4 = \frac{1}{5} d_R\, C_2(R) \left(C_2(R) - \frac{1}{3} \right) (A_0^a A_0^a)^2 \,. \tag{20}$$

Writing $d_R = 2j+1$, $C_2(R) = j\,(j+1)$, we see that only for high values of j is the classical result in Eq. (19) a good approximation of the quantum one. (Suggestions along these lines were made already in [8, 9].) While the expressions given here hold for $N_c = 2$, we expect similar conclusions for any SU(N_c), whereas for U(1) there is a perfect match.

The reason why operators below the quartic one are correctly reproduced, is that the integral $\int_Q(...)$ involves explicitly the quadratic and cubic Casimirs, which fix the symmetric parts of the traces of two or three SU(N_c) generators. The quartic term, on the other hand, depends on the commutation/anticommutation relations, and it is then at this order that one sees a difference between the quantum and classical colour algebras.

While we have here focused our attention on the "first" formulation of classical kinetic theory, a similar conclusion can be reached for the "second" formulation, based on Eqs. (7), (8), although the numerical discrepancy we find is different. In this case we do not know of an exact solution, but solve the equations iteratively, as was done to second order in [11]. At the third order, relevant for $\mathrm{Tr}\, A_0^4$, we no longer find a solution local in space, unless we take A_0 completely constant. Then, the result is too large by a factor three, for SU(2).

5. Conclusions

We have recalled in this brief note that for some special backgrounds, like a static one, the Boltzmann equation in Eq. (1) can be solved exactly, and the corresponding gauge field effective action can be computed. We have then compared the result with the 1-loop dimensionally reduced effective action for quantum field theory. We remark that while the quadratic and cubic terms are correctly reproduced, the classical non-Abelian colour algebra generically fails at the next level. The relative discrepancy is the smaller the higher the dimensionality of the colour representation; however, for the physical QCD case, some more elaborate formulation seems to be required in order to have an exact match also at this level (see, e.g., [18]).

Of course, all this does not imply that classical kinetic theory could not be a useful tool for a non-perturbative determination of many important observables of QCD, such as plasmon frequencies, damping rates, or physics related to the colour conductivity, for which effects from the quartic coupling are subdominant. Nevertheless, our observation should underline the need for a better beyond-the-leading-order understanding of the proper effective description of real-time dynamics in the QCD plasma, which would ideally at the same time also allow for a proper fixing of the renormalisation scale in the gauge coupling g as in the dimensionally reduced theory [19], as well as for a systematic approach to the continuum limit [20].

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