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New analytic results for electroweak baryon number violation

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Abstract: Real-time anomalous fermion number violation has been investigated for massless chiral fermions in spherically symmetric $SU(2)$ Yang-Mills gauge field backgrounds which can be weakly dissipative or even nondissipative. Restricting consideration to spherically symmetric fermion fields, a relation has been found between the spectral flow of the Dirac Hamiltonian and two characteristics of the background gauge field. This new result may be relevant to electroweak baryon number violation in the early universe.

1. Introduction

The electroweak Standard Model displays an anomalous violation of baryon (B) and lepton (L) number [1, 2]:

$$\Delta B = \Delta L = N_{\text{fam}}\, \Delta N_{\text{CS}}\ , \tag{1.1}$$

with N_{fam} the number of families and ΔN_{CS} the change of Chern-Simons number of the $SU(2)$ Yang–Mills gauge fields. Strictly speaking, this relation holds only for transitions from (near-)vacuum to (near-)vacuum. The total energy E of the process is then far below the top of the energy barrier associated with the Sphaleron [3], that is, $E << E_{\text{Sphal}} \approx 10\,\text{TeV}$ for the zero-temperature case.

In fact, 't Hooft calculated the *tunneling* process, using the Euclidean (imaginary-time) path integral formalism. For finite action gauge field configurations, the topological charge

$$Q \equiv \int_{\mathbb{R}^4} \mathrm{d}^4x\ \tilde{q}(x) \equiv -\frac{1}{32\,\pi^2}\int_{\mathbb{R}^4} \mathrm{d}^4x\ \epsilon^{KLMN}\,\text{tr}\,(F_{KL}F_{MN}) \tag{1.2}$$

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takes on integer values,

$$Q[\,A_{\text{finite action}}\,] = \Delta N_{\rm CS} \in \mathbb{Z}\;. \tag{1.3}$$

The reason is that the gauge fields at infinity are pure gauge and Q corresponds to the winding number of the map

$$S^3\big|_{x^2=\infty} \rightarrow SU(2) \sim S^3\;, \tag{1.4}$$

which equals the change of Chern–Simons number $\Delta N_{\rm CS}$. This integer Q is then identified with the number of fermions created or annihilated in the tunneling process.

But the topological charge Q is, in general, a noninteger for physical processes in Minkowski spacetime. The question, then, is what determines the real-time electroweak fermion number violation $\Delta(B+L)$, in particular for the processes of the early universe at temperatures above the electroweak phase transition ($T > T_c \approx 10^2\,$GeV)?

Here, we report on results obtained by a direct investigation of the Dirac equation for a restricted set of gauge fields and vanishing Yukawa coupling to the Higgs field.

2. Main result

In this talk, we consider pure $SU(2)$ Yang–Mills theory with a single isodoublet of left-handed fermions. The starting point is the eigenvalue equation of the time-dependent Dirac Hamiltonian:

$$H(t,\vec{x}\,)\,\Psi(t,\vec{x}\,) = E(t)\,\Psi(t,\vec{x}\,)\;. \tag{2.1}$$

Then fermion number violation is related to the SPECTRAL FLOW $\mathcal{F}_H$ of the Dirac Hamiltonian H. The general definition of spectral flow is as follows: $\mathcal{F}[\,t_f,t_i\,]$ is the number of eigenvalues crossing zero from below minus the number of eigenvalues crossing zero from above, for the time interval $[\,t_i,t_f\,]$ with $t_i < t_f$. Henceforth, we simply write $\mathcal{F}$ for $\mathcal{F}_H$. See Fig.1 for an example and Ref. [4] for further details and references.

For *strongly dissipative* gauge fields, the following result holds [4, 5, 6]:

$$\lim_{t_i\rightarrow-\infty}\;\lim_{t_f\rightarrow+\infty}\;\mathcal{F}[\,t_f,t_i\,] = \Delta N_{\rm CS}\;, \tag{2.2}$$

which corresponds to Eq. (1.1) above.

For *weakly- or non-dissipative spherically symmetric* gauge fields, a careful analysis of the zero-eigenvalue equation (2.1) gives [7]:

$$\mathcal{F}[\,t_f,t_i\,] = \Delta N_\chi[\,t_f,t_i\,] + \Delta N_\Theta[\,t_f,t_i\,]\;, \tag{2.3}$$

with $\Delta N_\chi = \Delta N_{\rm CS}$ for near-vacuum fields. The new contribution ΔN_Θ is called the "twist factor" of the spherically symmetric gauge field, whereas ΔN_χ is called the "winding factor." A spherically symmetric $SU(2)$ gauge field solution is called *strongly dissipative*, if both the (3+1)-dimensional and (1+1)-dimensional energy densities approach zero uniformly for large times ($t \rightarrow \pm\infty$), and *weakly dissipative*, if the (3+1)-dimensional energy density dissipates with time, but not the (1+1)-dimensional energy density.

The result (2.3) multiplied by $2\,N_{\rm fam}$ answers, in part, the question about $\Delta(B+L)$ raised in the penultimate paragraph of the previous section. In addition, we expect that

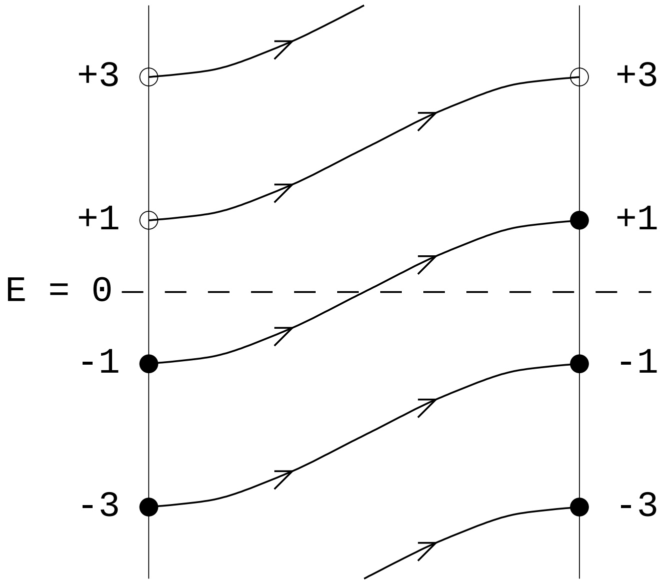

Figure 1: Sketch of the spectral flow of the eigenvalues $E(t)$ of the time-dependent Dirac Hamiltonian, with $\mathcal{F}[\,t_f, t_i\,] = +1$. Filling the (infinite) Dirac sea at the initial time t_i results in one extra fermion at the final time t_f.

$\Delta(B-L) = 0$, since all left-handed isodoublets of the Standard Model contribute equally, at least for the spherically symmetric fields considered. In the rest of the talk, we will explain the origin of the result (2.3), which holds for a single left-handed isodoublet, and discuss an explicit verification for (weakly dissipative) Lüscher-Schechter gauge fields [8].

3. Spherically symmetric *Ansatz*

The action of chiral $SU(2)$ Yang–Mills theory over Minkowski spacetime (indices M, N running over 0, ... , 3) is given by

$$S = \int_{\mathbb{R}^4} \mathrm{d}^4x \, \left(-\frac{1}{2\,g^2}\,\mathrm{tr}\left(F^{MN}F_{MN}\right) + \sum_{f=1}^{N_F} \bar{\Psi}_f\,\Gamma^M D_M \Psi_f \right) \,, \tag{3.1}$$

with the Dirac matrices Γ^M and the further definitions

$$F_{MN} \equiv \partial_M A_N - \partial_N A_M + A_M A_N - A_N A_M \;, \quad A_M \equiv A_M^a\,\sigma^a/(2i) \;,$$

$$D_M \equiv \partial_M + A_M\,\mathrm{P}_L \;, \quad \mathrm{P}_L \equiv (1-\Gamma_5)/2 \;, \quad \Gamma_5 \equiv -i\,\Gamma^0\Gamma^1\Gamma^2\Gamma^3 \;.$$

The spherical *Ansatz* implements invariance under spatial rotations, modulo $SU(2)$ gauge transformations. For $N_F = 1$, this gives an effective (1+1)-dimensional $U(1)$ gauge field theory with gauge fields $a_0(t,r)$ and $a_1(t,r)$, a complex Higgs-like scalar $\chi(t,r)$ and a single two-component Dirac spinor $\psi(t,r)$.

Explicitly, the effective (1+1)-dimensional $U(1)$ gauge field theory (indices μ, ν running over 0, 1) is given by

$$S = \left(4\pi/g^2\right) \int_{-\infty}^{+\infty} \mathrm{d}t \int_0^{\infty} \mathrm{d}r \, \Big(\tfrac{1}{4}\, r^2 f^{\mu\nu} f_{\mu\nu} + |D_\mu \chi|^2 + \tfrac{1}{2} \left(|\chi|^2 - 1\right)^2 / r^2$$

$$+ \, g^2 \, \bar{\psi} \left[\gamma^\mu D_\mu + (\mathrm{Re}\, \chi + i \gamma_5 \, \mathrm{Im}\, \chi)/r \right] \psi \Big) \,, \tag{3.2}$$

with the definitions

$$f_{\mu\nu} \equiv \partial_\mu a_\nu - \partial_\nu a_\mu \,, \quad D_\mu \chi \equiv (\partial_\mu - i a_\mu)\, \chi \,, \quad D_\mu \psi \equiv (\partial_\mu + i a_\mu \gamma_5 / 2)\, \psi \,,$$

$$\gamma^0 \equiv i \sigma^1, \quad \gamma^1 \equiv -\sigma^3, \quad \gamma_5 \equiv -\gamma^0 \gamma^1 \equiv \sigma^2,$$

and σ^a, $a = 1, 2, 3$, the 2×2 Pauli spin matrices. The fermionic part of the action (3.2) then gives the precise form of the Hamiltonian considered in the eigenvalue problem (2.1).

4. Winding and twist factors

For a given spherically symmetric $SU(2)$ gauge field at time t, the gauge condition

$$\chi(t, 0) = \chi(t, \infty) = 1 \tag{4.1}$$

results in a closed loop in the χ-plane. Now write $\chi(t, r)$ in polar notation:

$$\chi(t, r) = \rho(t, r) \;\; \exp\left[\, i \varphi(t, r) \,\right] \,, \quad \rho(t, r) \geq 0 \,. \tag{4.2}$$

Then define the <u>WINDING NUMBER</u> at a fixed time t by

$$N_\chi(t) \equiv \left[\, \varphi(t, \infty) - \varphi(t, 0) \,\right] / (2\pi) \tag{4.3}$$

and the <u>WINDING FACTOR</u> between an initial time t_i and final time t_f by

$$\Delta N_\chi [t_f, t_i] \equiv N_\chi(t_f) - N_\chi(t_i) \,. \tag{4.4}$$

For near-vacuum fields, it can be verified that $\Delta N_\chi = \Delta N_{\mathrm{CS}}$.

After a unitary transformation and the elimination of an irrelevant phase factor, the resulting real Dirac spinor $\tilde{\psi}(t, r)$ can be written as

$$\tilde{\psi}(t, r) \equiv |\tilde{\psi}(t, r)| \begin{pmatrix} \sin \Theta(t, r) \\ \cos \Theta(t, r) \end{pmatrix} . \tag{4.5}$$

The zero-eigenvalue equation (2.1) at fixed t now gives two coupled differential equations:

$$\partial_r \Theta(t, r) = -\lambda(t, r) \, \sin 2\Theta(t, r) + \mathcal{R}(t, r) \,, \quad \partial_r |\tilde{\psi}(t, r)| = |\tilde{\psi}(t, r)| \, \lambda(t, r) \, \cos 2\Theta(t, r) \,, \tag{4.6}$$

with boundary conditions

$$\Theta(t, 0) = 0 \,, \quad |\tilde{\psi}(t, 0)| = 0 \,, \tag{4.7}$$

and definitions

$$\lambda(t, r) \equiv \rho(t, r) / r \,, \quad \mathcal{R}(t, r) \equiv \left[\, a_1(t, r) - \partial_r \varphi(t, r) \,\right] / 2 \,. \tag{4.8}$$

It has been shown in Ref. [7] that the first differential equation in (4.6) can be transformed into a generalized Riccati equation. For the (unique) solution $\Theta(t,r)$ of this equation, define the SPINOR TWIST NUMBER at a fixed time t by

$$N_\Theta(t) \equiv [\,\Theta(t,\infty) - \Theta(t,0)\,]/\pi \ , \tag{4.9}$$

and the TWIST FACTOR between an initial time t_i and final time t_f by

$$\Delta N_\Theta[t_f,t_i] \equiv N_\Theta(t_f) - N_\Theta(t_i) \ . \tag{4.10}$$

Remark that the twist factor ΔN_Θ measures an *intrinsic property* of the spherically symmetric gauge field configuration:

$$\Delta N_\Theta[t_f,t_i] = \frac{1}{\pi}\int_0^\infty \mathrm{d}r \int_{t_i}^{t_f} \mathrm{d}t\; \partial_t\,\partial_r \Theta(t,r) \ , \tag{4.11}$$

with $\Theta(t,r)$ an explicitly known functional of the background gauge field, $\Theta = \Theta[\,\chi, a_1\,]$. Whether or not there exists a more direct way to obtain ΔN_Θ remains an open question.

5. Relation between spectral flow and gauge field background

Now consider an arbitrary fermion zero mode at $t = t^*$. A straightforward perturbative analysis of the time dependence of the zero-eigenvalue equation (2.1) for the Dirac Hamiltonian gives *locally*

$$\mathrm{sgn}\left[\left.\frac{\mathrm{d}E}{\mathrm{d}t}\right|_{t=t^*} \right] = \delta N_\chi|_{t=t^*} + \delta N_\Theta|_{t=t^*} \ , \tag{5.1}$$

with the general definition

$$\delta N|_{t=t^*} \equiv \lim_{\epsilon\downarrow 0}\,[\,N(t^*+\epsilon) - N(t^*-\epsilon)\,] \ . \tag{5.2}$$

For a finite time interval $[\,t_i, t_f\,]$ with $t_i < t_f$, this results in the *over-all* spectral flow:

$$\mathcal{F}[\,t_f,t_i\,] = \Delta N_\chi[\,t_f,t_i\,] + \Delta N_\Theta[\,t_f,t_i\,] \ , \tag{5.3}$$

in terms of the winding and twist factors defined by Eqs. (4.4) and (4.10), respectively. The results (5.1) and (5.3) hold for generic gauge field backgrounds; see Ref. [7].

The relation (5.3) has the form of an index theorem, with a property of the fermions on the left-hand side and a characteristic of the gauge fields on the right-hand side; cf. Ref. [4]. Next, we turn to an explicit verification of this relation.

6. Spectral flow for Lüscher-Schechter gauge field backgrounds

Lüscher and Schechter have independently obtained analytic solutions for $SU(2)$ gauge field theory, which describe collapsing and re-expanding shells of energy [8]. In this section, we discuss the exact fermion zero modes and the corresponding spectral flow for some of these gauge field backgrounds. Throughout, the same (arbitrary) mass scale is used to make the spacetime coordinates and energy dimensionless.

The starting point for the Lüscher-Schechter (LS) solutions is the following spherically symmetric *Ansatz*:

$$a_\mu = -q(\tau)\,\partial_\mu w\;, \quad \mathrm{Re}\,\chi = 1 + q(\tau)\,\cos^2 w\;, \quad \mathrm{Im}\,\chi = \tfrac{1}{2}\,q(\tau)\,\sin 2w\;, \tag{6.1}$$

in terms of the new coordinates

$$\tau \equiv \mathrm{sgn}(t)\;\arccos\left(\frac{1+r^2-t^2}{\sqrt{(1+t^2-r^2)^2+4\,r^2}}\right), \quad w \equiv \arctan\left(\frac{1-r^2+t^2}{2\,r}\right). \tag{6.2}$$

In order to obtain a solution of the full $SU(2)$ Yang–Mills field equations, the *Ansatz* function $q(\tau)$ must solve the following second-order ordinary differential equation:

$$\ddot{q} + 2\,q\,(q+1)\,(q+2) = 0\;, \tag{6.3}$$

where a dot indicates a derivative with respect to τ. Introducing the energy parameter

$$\epsilon \equiv \tfrac{1}{2}\,\dot{q}^2 + \tfrac{1}{2}\,q^2(q+2)^2\;, \tag{6.4}$$

the explicit solution for $\epsilon > 1/2$ is [8]:

$$q(\tau) = -1 + \sqrt{1+\sqrt{2\epsilon}}\;\mathrm{cn}\Big[\sqrt[4]{8\epsilon}\,(\tau-\tau_0)|\,m\Big]\;, \tag{6.5}$$

in terms of the Jacobi elliptic function $\mathrm{cn}[u|m]$ with modulus $m \equiv \big(1+\sqrt{2\epsilon}\big)\,/\,\big(2\sqrt{2\epsilon}\big) < 1$. This solution has $\Delta N_\chi \neq 0$. (Note that ΔN_χ is always 0 for $\epsilon < 1/2$; cf. Ref. [6].)

For moderately large energies, we have verified [7] that the spectral flow is given by

$$\mathcal{F}[\infty,-\infty\,] = \Delta N_\chi[\infty,-\infty\,]\;, \quad \text{for} \quad \epsilon < \epsilon^* \approx 5.37071\;. \tag{6.6}$$

But for $\epsilon > \epsilon^*$ the situation changes drastically.

Henceforth, we consider the particular LS solution with parameters

$$\epsilon = 20\,, \quad \tau_0 = -K(m)\,/\,\sqrt[4]{8\epsilon} \approx -0.54197\;, \tag{6.7}$$

and topological charge

$$Q \approx -0.13\;. \tag{6.8}$$

It turns out that the Higgs-like field $\chi(t,r)$ vanishes at three spacetime points:

$$(t_{-1},r_{-1}) \approx (-1.89, 2.14)\;, \quad (t_0,r_0) = (0,1)\;, \quad (t_{+1},r_{+1}) \approx (+1.89, 2.14)\;. \tag{6.9}$$

For the corresponding time slices $t = t_{-1}$, t_0, t_{+1}, there are exact fermion zero modes; see Figs. 3 and 6 of Ref. [7]. Surprisingly, there are also fermion zero modes at the time slices

$$t = \pm\,t_a \approx \pm\,2.92\;, \tag{6.10}$$

see Fig. 7 of Ref. [7].

Altogether, there are five fermion zero modes. A direct calculation gives for the slopes:

$$\begin{aligned} &\left.\frac{\mathrm{d}E}{\mathrm{d}t}\right|_{t=-t_a} = \left.\frac{\mathrm{d}E}{\mathrm{d}t}\right|_{t=+t_a} \approx -0.03 < 0\;, \\ &\left.\frac{\mathrm{d}E}{\mathrm{d}t}\right|_{t=t_{-1}} = \left.\frac{\mathrm{d}E}{\mathrm{d}t}\right|_{t=t_{+1}} \approx +0.08 > 0\;, \quad \left.\frac{\mathrm{d}E}{\mathrm{d}t}\right|_{t=t_0} \approx -5.00 < 0\;. \end{aligned} \tag{6.11}$$

From these level crossings, we have for the spectral flow (starting from $t = -t_a$ and ending at $t = +t_a$):

$$\mathcal{F}[\infty, -\infty\,] = -1 + 1 - 1 + 1 - 1 = -1\,. \tag{6.12}$$

On the other hand, the gauge field background has

$$\Delta N_\chi[\infty, -\infty\,] = +1\;, \quad \Delta N_\Theta[\infty, -\infty\,] = -2\;. \tag{6.13}$$

Equations (6.12) and (6.13) together verify our index theorem (5.3):

$$\mathcal{F}[\infty, -\infty\,] = \Delta N_\chi[\infty, -\infty\,] + \Delta N_\Theta[\infty, -\infty\,]\;. \tag{6.14}$$

This result is a direct generalization of Eq. (6.6), since $\Delta N_\Theta = 0$ for $\epsilon < \epsilon^*$. As mentioned in the Introduction, the anomalous change of fermion number, $\Delta(B + L)$, is necessarily an integer and this is indeed the case for the result (6.14), whereas the corresponding topological charge Q from Eq. (6.8) is definitely a noninteger.

7. Conclusions

Restricting to spherically symmetric fields, we have established a new relation, Eq. (5.3), between the spectral flow of the Dirac Hamiltonian and two characteristics of the background gauge fields, the winding and twist factors.

This result holds in particular for weakly dissipative or nondissipative gauge field backgrounds, as existed in the early universe. Fundamentally, these new effects appear because of the long-range behavior of the gauge fields. Recall that the standard result $\partial_\mu J^\mu_{B+L} \propto \tilde{q}$, which implies Eq. (1.1) of the Introduction, has been derived [1, 2] from Feynman perturbation theory, with the interactions at infinity "turned off."

The main outstanding problem now is to understand the role of the twist factor in the full (3+1)-dimensional $SU(2)$ Yang–Mills theory, not just the subspace of spherically symmetric configurations. Any good idea would be most welcome!

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