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The Microscopic Dirac Operator Spectrum

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We review the exact results for microscopic Dirac operator spectra based on either Random Matrix Theory, or, equivalently, chiral Lagrangians. Implications for lattice calculations are discussed.

1. Introduction

Exact statements can be made about the distributions of the smallest Dirac operator eigenvalues in gauge theories with spontaneous chiral symmetry breaking. This extends previous exact results for fermion zero modes in topologically non-trivial gauge field backgrounds to an infinite series of non-zero modes. In this short review we focus exclusively on these exact results for the smallest Dirac operator eigenvalues in four dimensions, and discuss why they are so important for lattice gauge theory.

The one single assumption that has to be made is that chiral symmetry is spontaneously broken. Actually, this statement needs a small qualification because the group of chiral symmetries depends on the number of fermions, and on the representation carried by them. We can be completely general and consider N_f (Dirac) fermions in an arbitrary representation r of the gauge group $\mathcal{G}$ with vectorlike couplings. As one could suspect from Wigner's classification of representations, there are just three symmetry breaking classes to consider. It all depends on whether the representation is complex, real or pseudo-real. The case of complex representations is not only the physically most relevant (QCD belongs to this class), it is also the most simple. In the two other cases the initial symmetry is larger than one would naively have expected, namely $\mathrm{SU}(2N_f)$ rather than just $\mathrm{SU}_L(N_f)\times\mathrm{SU}_R(N_f)$. The expected patterns of chiral symmetry breaking are as follows, depending on the representation r:

- *Complex*: $\mathrm{SU}_L(N_f)\times\mathrm{SU}_R(N_f) \to \mathrm{SU}(N_f)$.
- *Pseudo-real*: $\mathrm{SU}(2N_f) \to \mathrm{Sp}(2N_f)$.
- *Real*: $\mathrm{SU}(2N_f) \to \mathrm{SO}(2N_f)$.

By Goldstone's theorem, the cosets of the above symmetry breaking patterns determine the low-energy properties of these theories. Note that precise details about the gauge group $\mathcal{G}$ do not enter at all. We are here getting a first bite of the universality that turns out to govern the lowest-lying Dirac operator eigenvalues.

As explained by Verbaarschot [1], there exists a remarkable relation between the three symmetry breaking classes above and the classification of chiral Random Matrix Theories ensembles. They are labelled by the so-called Dyson indices β as follows: The coset $\mathrm{SU}(N_f)$ of complex fermion representations corresponds to the *chiral Unitary Ensemble* chUE ($\beta = 2$), the coset $\mathrm{SU}(2N_f)/\mathrm{Sp}(2N_f)$ of pseudo-real representations corresponds to the *chiral Orthogonal Ensemble* chOE ($\beta = 1$), while the coset $\mathrm{SU}(2N_f)/\mathrm{SO}(2N_f)$ of real representations corresponds to the *chiral Symplectic Ensemble* chSE of $\beta = 4$. This exhausts the Dyson classification.[2]

These symmetry breaking patterns concern continuum fermions. Staggered fermions, away from the continuum, have unusual patterns of chiral symmetry breaking: all real and pseudo-real representations are precisely swapped. This has been known for a while for special cases (see, *e.g.* ref. [2]), and has very recently been shown to

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[2]The labelling in terms of the integers β is related to a certain number occuring in the three different Random Matrix Theories; there is thus no "missing" class of $\beta = 3$!

hold in general [3]. Another peculiarity of staggered fermions is their apparent insensitivity to gauge field topology, which we shall return to in detail below.

The relationship to Random Matrix Theory has caused some confusion, although it can be stated very clearly. To get there, the theory first has to be formulated for fixed topological charge ν. The order parameter for spontaneous chiral symmetry breaking is formally defined by $\Sigma \equiv \lim_{m\to 0}\lim_{V\to\infty}|\langle\bar{\psi}\psi\rangle|$. To zoom in on the smallest Dirac operator eigenvalues, and to see how the chiral condensate is formed, one takes the chiral limit in an unorthodox way [4]. First, the euclidean four-volume V is kept finite, and it is convenient to work at fixed ultraviolet cut-off Λ. We are thus temporarily working with bare quantities which will have to be renormalized eventually. Both the finite volume and UV cut-off are ideal set-ups from the point of view of lattice gauge theory.

Next, the quark masses are chosen so that the pseudo-Goldstone bosons ("pions") are too light to fit inside the box, *i.e.* $m_\pi \ll 1/L$, where L is the linear extent of the 4-volume V. Now V is sent to infinity while the combination $m_i\Sigma V$ (m_i being quark masses) is kept finite – it is always possible to satisfy these two constraints by taking m_i small enough. Because L is taken much larger than $1/\Lambda_{QCD}$, the euclidean partition function is dominated by the pions. Higher-mass states do contribute to the partition function, but as the volume is sent to infinity their contribution will eventually, for large enough volume, be exponentially suppressed, of order $\exp[-ML]$, with $ML \gg 1$. In contrast, the pions do not yield exponentially suppressed contributions: their masses can be tuned to zero, and we precisely require $m_\pi L \ll 1$ throughout. It is in this sense that the results which will be reviewed below are exact: given any required degree of accuracy, this accuracy can be achieved by simply tuning V and m_i. The exact results hold in the limit.

Leutwyler and Smilga proposed to combine the above ingredients of $V \gg 1/m_\pi^4$ and fixed topology [4]. Because the euclidean partition function is dominated by the pions, it follows from the coset of chiral symmetry breaking. It is a non-renormalizable chiral Lagrangian with, in principle, an infinite number of terms. However, because of the peculiar finite-volume regime chosen, only the zero-momentum modes need to be considered (again an approximation that can be made as accurate as we wish by tuning m_i and V) [5]. Let us focus mainly on the symmetry breaking class relevant for QCD. Then the effective partition function is dominated by one single term in the effective Lagrangian:

$$Z \;=\; \int_{SU(N_f)} dU \exp\left[V\Sigma \mathrm{ReTr}(e^{i\theta/N_f}\mathcal{M}U^\dagger)\right] \quad (1)$$

One crucial point here is the dependence on the vacuum angle θ. Because of the anomaly, the partition function does not depend on the quark mass matrix $\mathcal{M}$ and θ separately. A chiral rotation (independent phase rotations of the left and right handed fields) gives a linear shift in θ, and since the partition function is left unchanged by such a change of integration variables, it can only depend on the invariant combination $\exp[i\theta/N_f]\mathcal{M}$. We normally do not consider QCD with a non-zero vacuum angle, and here it is also just introduced at a preliminary step, as a source of topological charge ν, and as a means for computing the partition function in sectors of fixed ν [4]:

$$Z_\nu = \int_{U(N_f)} dU \;(\det U)^\nu \exp\left[V\Sigma \mathrm{ReTr}(\mathcal{M}U^\dagger)\right] \quad (2)$$

The projection onto fixed topological charge has been used as the additional U(1)-factor that extends the integration over the zero-momentum modes to U(N_f). Note that Z_ν does not depend on m_i, V and Σ independently, but only on the combination

$$\mu_i \;\equiv\; m_i V\Sigma \;. \quad (3)$$

We will see that this finite-volume scaling is reflected in the Dirac operator spectrum.

There are two distinct advantages for considering the theory at fixed topological charge ν. The first is that the zero-dimensional group integral can be done analytically for all N_f and ν [6][3],

$$Z_\nu(\{\mu_i\}) = \det[\mu_i^{j-1} I_{\nu+j-1}(\mu_i)] / \prod_{i>j}^{N_f} (\mu_i^2 - \mu_j^2) \quad (4)$$

where $I_n(x)$ is a modified Bessel function, and the matrix in the numerator is of size $N_f \times N_f$. The second is the big surprise: at fixed ν there is a connection to Random Matrix Theory.

2. The relation to Random Matrix Theory

As observed by Shuryak and Verbaarschot [7], the group integral (2) can be rewritten as a Random Matrix Theory partition function which has an uncanny resemblance to the original QCD path integral. Consider

$$\tilde{Z}_\nu \equiv \int dW \prod_{f=1}^{N_f} \det(iM + m_f) \exp\left[-\frac{N}{2} \mathrm{Tr}\, V(M^2)\right] \quad (5)$$

where

$$M = \begin{pmatrix} 0 & W^\dagger \\ W & 0 \end{pmatrix} . \quad (6)$$

Here the integral is over complex matrices W of rectangular size $N \times (N+\nu)$. The $\tilde{m}_i$'s are dimensionless numbers, and the potential $V(M^2)$ is unspecificied at this point. The matrix M anticommutes with diag$\{1_N, -1_{N+\nu}\}$, and as a consequence the eigenvalues of M occur in pairs $\pm\tilde{\lambda}$ whenever $\tilde{\lambda} \neq 0$. Because of the rectangular nature of W, the matrix M also has precisely ν zero modes. The intuitive idea is the analogy with the determinant of the Dirac operator for complex representation fermions: the matrix M has ν zero modes, it is chiral, and in (5) one integrates over complex matrices.

But one is *not* simply trying to replace the path integral over gauge potentials $A_\mu(x)$ by zero-dimensional matrices. Instead, the precise relationship is as follows. Take a "microscopic limit" of eq. (5) in which $\tilde{\mu}_i \equiv m_i(2N)\tilde{\rho}(0)$ is kept fixed as $N \to \infty$. In that limit, and up to an irrelevant (μ_i-independent) normalization,

$$Z_\nu[\{\mu_i\}] = \tilde{Z}_\nu[\{\tilde{\mu}_i\}]\Big|_{\tilde{\mu}_i=\mu_i} . \quad (7)$$

[3]An intermediate calculation by Berezin and Karpelevich, from which this integral can be derived, dates back to 1958. I thank T. Wettig for informing me of this.

This was first demonstrated in ref. [7] for the case of a Gaussian potential in the $\beta = 2$ chiral ensemble. It follows from a series of universality theorems that the identity holds for any generic choice of $V(M)$ [8], and that a fine tuning is required to reach other universality classes that are of no obvious relevance here. Basically, the domain of universality is determined by the condition that $\tilde{\rho}(0) \neq 0$, as could have been expected from the manner in which the microscopic limit is taken. Identities similar to (7) exist for the two other classes of chiral symmetry breaking [10]. As outlined above, the chiral Random Matrix Theory ensembles are different, but there are analogous universality proofs for those cases [11].

Having two partition functions coincide as in eq. (7) may not seem to give much information. But one should rather view $Z_\nu[\{\mu_i\}]$ as a generating function for the chiral condensate. Then the statement (7) is highly non-trivial since it means that in the infinite-volume limit where $\mu_i = m_i\Sigma V$ is kept fixed we can just as well compute the chiral condensate from Random Matrix Theory. This observation is also the basis for finally understanding why it is even possible to compute the microscopic Dirac operator spectrum from Random Matrix Theory. More about this below.

3. The microscopic Dirac operator spectrum

Although the two partition functions (2) and (5) coincide in the microscopic limit, it is far from obvious that the eigenvalues $\tilde{\lambda}$ of the matrix M should be related to the eigenvalues of the Dirac operator λ. The eigenvalue spectrum of the Random Matrix Theory is *very* different from that of the Dirac operator: for a Gaussian potential the eigenvalue density of (5) is the famous Wigner semi-circle, while the spectrum of the Dirac operator is expected to approach $\rho(\lambda) \sim \lambda^3$ near the UV cut-off. One may search for *universality* that makes such differences irrelevant. Here it means going to a scale near the origin where both spectra, on that scale, are trivially identical (namely constant, near $\lambda \sim 0$). This is possible, because by the Banks-Casher relation $\Sigma = \pi\rho(0)$

and the assumption of spontaneous chiral symmetry breaking we know that $\rho(0) \neq 0$. Similarly, we must require of the potential in (5) that it leads to $\tilde{\rho}(0) \neq 0$. Then one can define microscopically rescaled variables, which for the Dirac operator eigenvalues are $\zeta \equiv \lambda V \Sigma$ (cf. eq. (3)), and for Random Matrix Theory $\tilde{\zeta} \equiv \tilde{\lambda} 2N\pi\tilde{\rho}(0)$. A microscopic spectral density (and similarly for higher spectral correlators) of the Dirac operator is analogously $\rho_s(\zeta) \equiv \rho(\zeta/(V\Sigma))/V$, which has a finite well-defined limit as $V \to \infty$ [7]. The microscopic density in the Random Matrix Theory context is defined analogously, and it can be computed by various technqiues. The first analytical expression, for N_f massless flavors in the $\beta = 2$ universality class, was obtained in ref. [13]:

$$\rho_s(\zeta) = \frac{\zeta}{2}\left[J_{N_f+\nu}(\zeta)^2 - J_{N_f+\nu-1}(\zeta)J_{N_f+\nu+1}(\zeta)\right] \quad (8)$$

which, as a first check, was found to reproduce the Leytwyler-Smilga spectral sum rule for *e.g.* $\langle \sum_n 1/\zeta_n^2 \rangle$ [13,14]. Corresponding expressions exist for $\beta = 1, 4$ [15].

Quark masses can be included without difficulty by performing a "double-microscopic limit" in which both m_i and eigenvalues λ are blown up as in (3). The precise analytical form of the spectral correlators can then be worked out explicitly [16,17].

There is a very compact formulation of all these results. A central object is the so-called kernel $K(\zeta_a, \zeta_b)$ from which all spectral correlation functions can be computed:

$$\rho_s(\zeta_1, \ldots, \zeta_k) = \det_{ab} K(\zeta_a, \zeta_b) \quad (9)$$

A "Master Formula" for this kernel is [12],

$$K(\zeta, \zeta') = C\sqrt{\zeta\zeta'}\prod_f^{N_f}\sqrt{(\zeta^2+\mu_f^2)(\zeta'^2+\mu_f^2)} \times Z_\nu^{(N_f+2)}(\{\mu\}, i\zeta, i\zeta')/Z_\nu^{(N_f)}(\{\mu\}) \,. \quad (10)$$

This formula encompasses all cases for $\beta = 2$, any number N_f of fermions with masses μ_f, and any ν. Moreover, although it is derived within Random Matrix Theory, we have used the identity (7) to express the r.h.s. entirely in terms of field theory partition functions. There are closely related formulas for the $\beta = 4$ case [12].

One can also compute an infinite series of individual eigenvalue distributions, beginning with the smallest. They too can be expressed in terms of the effective partition functions [18]. The analytical formula is known in all generality: it gives the k'th eigenvalue distribution for all three universality classes, for any number of fermions N_f, and in an arbitrary sector of topological charge ν.[4] To give a simple example, the quenched distribution of the first eigenvalue in a sector of arbitrary ν reads $(i, j = 1, \ldots, \nu)$ [18]

$$P_{\rm min}(\zeta) = \frac{\zeta}{2}e^{-\zeta^2/4}\det[I_{2+i-j}(\zeta)] \quad (11)$$

for the $\beta = 2$ universality class.

One can perform the sum over topological charge explicitly for all spectral correlation functions, including the microscopic spectral density itself [19]. The result can for $N_f > 0$ be written in terms of the effective field theory partition functions at vacuum angle $\theta = \pi$ when k is odd:

$$\rho_s(\zeta_1, \ldots, \zeta_k) = (-1)^{k[N_f/2]}\prod_{j<l}^{k}(\zeta_j^2 - \zeta_l^2)^2 \times \prod_{i=1}^{k}(|\zeta_i|\prod_f^{N_f}(\zeta_i^2+\mu_i^2))\frac{Z^{(N_f+2k)}(\{\mu\},\{i\zeta_i\})_{\theta=k\pi}}{Z^{(N_f)}(\{\mu\})_{\theta=0}} \quad (12)$$

4. Derivation from chiral Lagrangians

All microscopic spectral correlation functions, and all probability distributions of individual eigenvalues, can thus be expressed in terms of effective field theory partition functions. This puts universality of these quantities on a very simple footing. It also gives a strong hint that in fact Random Matrix Theory is not needed at all. These results can indeed be derived entirely within the chiral Lagrangian framework. It hinges on the crucial identity (7), and an extension of it which we will discuss next.

One way to compute the spectral density of the Random Matrix Theory (5) is through the resolvent

$$R_\nu(m_v) \equiv \left\langle {\rm Tr}\frac{1}{M - m_v}\right\rangle \,, \quad (13)$$

[4] The only exception is the $\beta = 1$ class where, for technical reasons, ν must be odd in the general formula.

where m_v is an external parameter. In the microscopic limit this is, in view of eq. (7), nothing but the partially quenched chiral condensate. To be precise, consider the supersymmetric method for computing this quantity from the chiral Lagrangian [20]: One adds a quark of mass $m_v + j$, and a bosonic quark of mass m_v. Then the partially quenched chiral condensate is $\partial_j \ln Z_\nu|_{j=0}$. In a similar way, one can modify the Random Matrix Theory (5) and multiply the integrand by $\det[M + \tilde{m}_v + \tilde{j}]/\det[M + \tilde{m}_v]$. Taking the microscopic limit, this leads to a supersymmetric generalization of the identity (7), where the chiral Lagrangian is the supersymmetric one associated with partial quenching. Naively this Lagrangian would be based on the Goldstone supermanifold $U(N_f + 1|1)$, but in fact the proper choice of integration domain is a more subtle issue, as discussed at length in [21,23]. This way of proceeding is only in order to establish why the Random Matrix Theory approach really does give exact answers for the microscopic Dirac operator spectrum. If one is not interested in this, one can proceed completely within the chiral Lagrangian framework, and simply compute the partially quenched chiral condensate directly. As a simple $\beta = 2$ example, consider the fully quenched case where, with $\mu_v = \mu$, one finds [21]:

$$\frac{\Sigma_\nu(\mu)}{\Sigma} = \mu\left(I_\nu(\mu)K_\nu(\mu) + I_{\nu+1}(\mu)K_{\nu-1}(\mu)\right) + \frac{\nu}{\mu} \quad (14)$$

The microscopic spectral density of the Dirac operator is given by [21]

$$\rho_s(\zeta) = \frac{1}{2\pi}\text{Disc } \Sigma_\nu(\mu)|_{\mu=i\zeta} \quad (15)$$

which precisely agrees with (8)[5].

Similar expressions exist for higher spectral k-point functions. See ref. [22] for the $\beta = 2$ case. There is again exact agreement with the Random Matrix Theory results.

5. Replicas

It is interesting that one can get quite far by an alternative formulation of partial quenching based on the replica method. The fermion determinant is then removed by analytic continuation in the number of flavors, so that eventually one can take the limit $N_v \to 0$. Here N_v denotes a number of additional unphysical flavors added to the theory with N_f physical quarks.

The first observation is that now the identity (7) as it stands is all that is needed in order to show equivalence to the Random Matrix Theory approach. The integration in eq. (2) is over $\mathrm{U}(N_f + N_v)$, and the limit $N_v \to 0$ is taken after differentiating w.r.t. μ_v. Small-mass and large-mass expansions can be performed in this way [24,25]. As an example, for the quenched chiral condensate the replica method yields precisely small-mass and large-mass series expansions of the expression (14), whose inversion in turns yields the same microscopic spectral density as was originally derived from Random Matrix Theory. It is a highly non-trivial check on the consistency of the whole formalism that these results all coincide.

6. Lattice results

This review has not proceeded in historical order. The derivations of the microscopic Dirac operator spectrum based directly on the partially quenched chiral Lagrangian came after the first lattice study [26] already had demonstrated the viability of the Random Matrix Theory approach. Since then there have been numerous lattice comparisons with the exact analytical expressions, for different gauge groups, different representation quarks in both quenched and full theories [27,28,3]. It is impossible to do justice to all that work here, but a few points can be highlighted.

A quite systematic study of the smallest eigenvalue distributions in all three universality classes and for zero and non-zero topological charge showed perfect agreement with the analytical predictions [29]. The distributututions of a whole sequence of smallest eigenvalues have been compared with the analytical formulas in ref. [30]. One sees clearly how the microscopic spectral density is built up of the individual distributions.

Also higher-point spectral correlators have been checked [26,31], again with spectacularly

[5] The microscopic spectral density by definition does not include the zero modes.

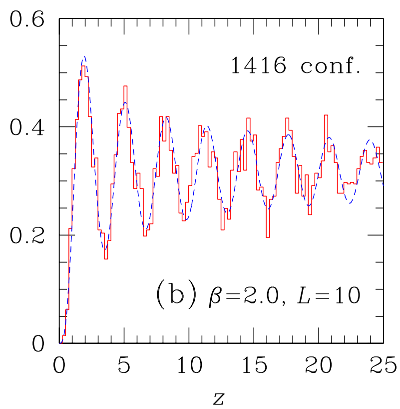

Figure 1. The first Monte Carlo measurement of the microscopic spectral density of the Dirac operator [26], for gauge group SU(2) and quenched staggered fermions. The dashed curve is the analytical prediction.

good agreement with the analytical predictions. Extensions to the regime of chiral perturbation theory have also been made [32].

Lattice calculations with staggered fermions compare only with the analytical expressions for the sector with $\nu = 0$. This makes sense, since it has been known for long [33] that staggered fermions do not have exact zero modes at usual gauge couplings. Even if one selects distinct topologically non-trivial gauge field sectors, the microscopic Dirac spectrum seems completely unaffected, and always agrees with the $\nu = 0$ predictions [34]. There is a much more direct way of understanding this. The symmetry breaking patterns of staggered fermions away from the continuum always have an additional U(1) factor[6] which for the zero momentum modes is completely equivalent to a projection on the $\nu = 0$ sector [35,3]. So we *should* only compare with what can be viewed as $\nu = 0$ predictions of the theory without the extra U(1) factor. If one sees sensitivity to topology [36], then continuum flavor symmetries are beginning to be recovered.

Instead of looking directly at the smallest Dirac operator eigenvalues, one can also consider derived quantities such as the (mass-dependent) chiral condensate [37–41]

$$\frac{\Sigma_\nu(\mu)}{\Sigma} = 2\mu \int_0^\infty d\zeta \frac{\rho_s^{(\nu)}(\zeta,\mu)}{\zeta^2+\mu^2} + \frac{\nu}{\mu} \ , \qquad (16)$$

or higher chiral susceptibilities, such as [38]

$$\frac{\omega_\nu(\mu)}{V\Sigma^2} = 4\mu^2 \int_0^\infty d\zeta \frac{\rho_s^{(\nu)}(\zeta,\mu)}{(\zeta^2+\mu^2)^2} + \frac{2\nu}{\mu^2} \qquad (17)$$

In the quenched and partially quenched cases there is a one-to-one correspondence between $\Sigma(\mu)_\nu$ and the microscopic spectral density $\rho_s^{(\nu)}(\zeta)$. But: $\Sigma_\nu(\mu)$ has a short-distance singularity that is not taken into account in the above description. At fixed UV cut-off Λ this is not a problem when one takes the infinite-volume limit, as one should in order to compare with (14). Corrections are of the form $Am\Lambda^2 + Bm^3 \ln\Lambda$, where A and B are constants. This looks scary, but we are taking the limit in which $\mu = m\Sigma V$ is kept fixed as $V \to \infty$. So these UV corrections actually vanish; they are suppressed by inverse factors of the volume, and really read $(A/\Sigma)(\mu/V)\Lambda^2 + (B/\Sigma^3)(\mu^3/V^3)\ln\Lambda$. If the volume V is not large enough these terms can be annoying, and then one should subtract them.

There are cases where very, very few Dirac operator eigenvalues play any rôle in building up the chiral condensate (14) in the interesting mass range. As an example, let us compare the exact result (14) with what one gets by keeping only *one single Dirac operator eigenvalue* in the integral of eq. (16) for $\nu = 0$. We then simply replace $\rho_s^{(0)}(\zeta)$ in the integrand of (16) by the probability distribution of just one single Dirac operator eigenvalue, the smallest, $P^{(0)}_{\rm min}(\zeta) = (\zeta/2)e^{-\zeta^2/2}$. This may seem like an absurd truncation, and of course an unnecessary one, since we know the full analytical form of $\rho_s^{(0)}(\zeta)$. But the comparison between this approximation, and the exact answer is shown in figure 2.

The two curves are basically indistinguishable (up to $\mu \sim 0.1$ the difference is less than 5%, decreasing fast as μ is lowered). If one sees good

[6] They are $U(N)\times U(N) \to U(N)$ [complex], $U(2N) \to Sp(2N)$ [real], and $U(2N) \to SO(2N)$ [pseudo-real]

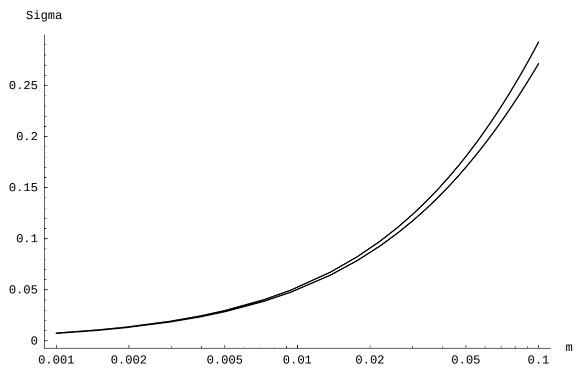

Figure 2. The exact quenched chiral condensate (here m stands for μ) for $\nu = 0$, compared with keeping only one single eigenvalue in eq. (16).

scaling of the $\nu = 0$ quenched chiral condensate in such a mass range, then one is probing, and having good statistics of, only one single Dirac operator eigenvalue. Of course, we could just as well have included all the other eigenvalues, and then the relation would be exact. For extracting Σ in this way it seems more logical to measure the distributions of the Dirac operator eigenvalues directly. When $\nu \neq 0$ the eigenvalues are shifted further away from the origin, the integral (16) is not as infrared sensitive, and the brutal approximation of keeping only the first eigenvalue is not nearly as good. The error is easily assessed if we rewrite, for $\nu \neq 0$ [38],

$$\frac{\Sigma_\nu(\mu)}{\Sigma} = \frac{\nu}{\mu} + 2\mu \left\langle \sum_{n>0} \frac{1}{\zeta_n^2} \right\rangle + \dots \ , \qquad (18)$$

which shows that to leading order one is simply trying to measure the first Leutwyler-Smilga sum rule [4], $\langle \sum_n 1/\zeta_n^2 \rangle = 1/(4\nu)$. It seems advantageous to instead measure the first few eigenvalue distributions, or, for lower statistics data, at least individual averages. That way one avoids contaminating the result with the larger eigenvalues which at the limited finite volume will have distributions that are incompatible with the analytical formulae.

7. Beyond Random Matrix Theory

We have emphasized that the results reviewed here are exact in the sense that they can be made as accurate as we wish by tuning the quark masses m_i and the volume V. It immediately suggests that it is possible to compute corrections to this scaling, which would improve convergence and bring in more terms from the chiral Lagrangian. This is indeed the case. A computational framework was laid out by Gasser and Leutwyler, who called it the ϵ-expansion [5].[7]

The idea is simple. Instead of keeping only the zero-momentum mode in the chiral Lagrangian, one includes the modes of non-zero momentum in a perturbative manner. Thus, one starts with the full chiral Lagrangian and then separates out the zero-momentum modes of $U(x)$:

$$U(x) \ = \ u e^{i\sqrt{2}\xi(x)/F} u \qquad (19)$$

where $u \in \mathrm{SU}(N_f)$ is a space-time independent collective field, and $\xi(x)$ contains only modes of non-zero momentum. To leading order, where one simply ignores $\xi(x)$, this gives the effective partition function (1) after identifying $U \equiv u^2$. To next order the kinetic energy term contributes, and one resorts to the usual loop expansion in the fluctuation field $\xi(x)$. The modes of zero momentum are still treated *exactly.* So this expansion is a combination of the exact, non-perturbative, leading-order result for $V \to \infty$, and a perturbative expansion where each term should be suppressed by powers of $1/L \equiv 1/V^{1/4}$. In 4 dimensions the expansion is even better behaved, as the first correction goes as $1/L^2$, rather than just $1/L$ [5,42]. Leading terms are what we alternatively could obtain from Random Matrix Theory. The corrections take us beyond.

The one-loop correction to the chiral condensate computed in this way can be re-absorbed into a volume-dependent $\Sigma_{eff}(V)$ [5],

$$\frac{\Sigma_{eff}(V)}{\Sigma} = 1 + \frac{N_f^2 - 1}{N_f} \frac{1}{F^2} \frac{\beta_1(L_i/L)}{L^2} \qquad (20)$$

where β_1 depends on the geometry [42,45]. This

[7] The quantity ϵ counts orders in $1/L$, where L is the linear extent of the volume. There is no relation to the expansion around critical dimensions.

carries directly over to the Dirac operator eigenvalues, on account of the relation (15). From the small $1/L^2$-correction one can thus, with sufficient statistics, also measure the pion decay constant F from the eigenvalue distributions.

The quenched ϵ-expansion meets the usual difficulties of the quenched theory. We outline the problem here using the replica version of quenched chiral perturbation theory [43] because it is simpler to explain. Then (19) still defines the split between modes of zero and non-zero momentum, but now $U(x) \in \mathrm{U}(N_v)$. Let $\Xi(x) \equiv \mathrm{Tr}\xi(x)$ denote the flavor singlet fluctuations. The lowest-order chiral Lagrangian in a sector of fixed topology reads [35]

$$\mathrm{Tr}\left[\frac{1}{2}\partial_\mu\xi(x)\partial^\mu\xi(x) + \frac{m_0^2}{6}\Xi(x)^2 + \frac{\alpha}{6}\partial_\mu\Xi(x)\partial^\mu\Xi(x) - \frac{\Sigma}{2}\,\mathcal{M}(U+U^\dagger)\left(V - \frac{1}{F^2}\int dx\;\xi(x)^2\right)\right] \quad . \qquad (21)$$

and the propagator of $\Xi(x)$ is thus modified due to the m_0-term (the precise form is given in [43]), and one again takes the limit $N_v \to 0$ after having differentiated with respect to the sources. A new scale is introduced by the m_0-term, and in order to proceed perturbatively one must assume that $m_0^2/(4\pi F^2)$ is small. At one-loop level the quenched analog of the effective volume-dependent $\Sigma_{eff}(V)$ is [35]

$$\frac{\Sigma_{eff}(V)}{\Sigma} = 1 - \frac{1}{3F^2}\left[\frac{\alpha\beta_1}{L^2} - \frac{m_0^2}{8\pi^2}\ln(L)\right] \qquad (22)$$

In the full theory logarithms are typically multiplied by m_π^2, and hence in this regime suppressed by at least $1/L^2$. Here it survives at fixed m_0, indicating difficulties with the whole expansion. It is not a quenched chiral logarithm, but, in this regime, a quenched finite-volume logarithm that signals the limitations of the expansion.

The ϵ-expansion has also been worked out for correlation functions of the full theory [42,44], and very recently these results have been generalized to the quenched case [46]. By measuring correlation functions in this regime one can extract the low-energy constants of QCD.

8. Conclusions

The microscopic tail of the Dirac operator spectrum is computable by a precise comparison with the associated effective field theory. The result is a series of exact expressions for observables in finite-volume gauge theories. This is a highly unusual situation for such strongly coupled theories, and one can make good use of these results. In particular:

• Reliability of fermion algorithms can be checked by comparing with the exact expressions.

• Gauge field topology gives distinct predictions, and is probed at the "quantum level".

• Physical observables derivable from the microscopic spectral correlation functions are known in exact analytical forms. Simple models, ansätze, or unknown extrapolations for the small-mass finite-volume behavior can be replaced by the correct analytical expressions.

• By going to *unphysical regimes* we can extract *physical observables.* This includes unphysical volumes, and, if need be, fixed gauge field topology. The most extreme example was given: From the distribution of just one single Dirac eigenvalue we learn **i**) whether spontaneous chiral symmetry occurs in the infinite-volume theory, **ii**) which symmetry breaking class it belongs to, **iii**) the value of the infinite-volume chiral condensate Σ, and finally, by looking even closer, **iv**) the pion decay constant F. If one is willing to push it, in principle the infinite series of parameters of the chiral Lagrangian can be probed by the Dirac operator eigenvalues.

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