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Finite-volume meson propagators in quenched chiral perturbation theory

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We compute meson propagators in finite-volume quenched chiral perturbation theory.

Recent progress in the formulation of chiral fermions on the lattice makes it possible to approach the regime of the physical light quark masses. However, lattice volumes $V = L^4$ needed to ensure negligible finite size effects are prohibitively large for such light quarks. Only when L is very large compared to the Compton wavelength of the lightest particles of the theory (i.e. the pions) are the volume effects exponentially suppressed. Fortunately chiral perturbation theory (χPT) can predict the volume effects that occur in the regime $m_\pi L \leq 1$. In this way the finite volume becomes a distinct advantage, and a finite-size scaling analysis is actually a very useful tool to extract the physical low energy constants. This has been demonstrated in simpler models [1].

The χPT predictions in full QCD for the finite size scaling of quantities such as the chiral condensate or the propagators of scalar and vector densities have been computed in [2,3]. Unfortunately, lattice simulations in the regime of very light quark masses are presently only possible in the quenched approximation, and the predictions of chiral perturbation theory are quite drastically modified in this approximation [4]. We present here some results for the propagators of

the scalar and pseudoscalar densities at finite volume in quenched χPT (QχPT). Full details will be presented in a forthcoming publication.

The low-energy limit of QCD with light quarks is a chiral Lagrangian of the Goldstone bosons resulting from spontaneous chiral symmetry breaking. In an infinite volume, the effective Lagrangian is approximated by an expansion in powers of the pion momentum p and mass m_π. This is the standard *p-expansion*. In a finite volume the p-expansion is still good if the Compton wavelength of the particle is much smaller than the size of the box ($L \gg \frac{1}{m_\pi}$). In the opposite limit, $L \ll \frac{1}{m_\pi}$, the p-expansion breaks down due to propagation of pions with zero momenta [2]. A convenient expansion for this regime is the so-called *ϵ-expansion* in which $m_\pi \sim p^2 \sim \epsilon^2$ [2]. By the Gell-Mann–Oakes–Renner relation this corresponds to keeping $\mu_i \equiv m_i \Sigma V$ of order unity while V is taken large. The zero mode of the pion can be isolated by factorizing $U(x) = U_0 \exp i\sqrt{2}\xi(x)/F_\pi$ into the constant collective field U_0 and the pion fluctuations $\xi(x)$. The difficulty then comes from the fact that the integral over U_0 needs to be done exactly, while ordinary χPT applies to the non-zero mode integration. At leading order one obtains the suitably

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normalized partition function

$$\mathcal{Z} = \int_{SU(N_f)} dU_0 \exp\left[\frac{\Sigma}{2} V \text{Tr} M \left(U_0 + U_0^+\right)\right].$$

where M is the quark mass matrix. It is interesting to consider averages in sectors of fixed topology ν as well [5]. To the same order

$$\mathcal{Z}_\nu = \int_{U(N_f)} dU_0 \left(\det U_0\right)^\nu \exp\left[\frac{\Sigma}{2} V \text{Tr} M \left(U_0 + U_0^+\right)\right].$$

The ϵ-expansion can be worked out also in the case of QχPT. We have considered two methods to take the quenched limit. The first is the supersymmetric method [4], suitably refined to be valid at the non-perturbative level [6]. Here the fermionic determinant of N_v valence quarks is cancelled by adding to the theory N_v flavors of ghost bosonic quarks. Assuming a supergroup generalization of the chiral symmetry breaking pattern of QCD, the field $U(x)$ becomes an element of a graded group with both Goldstone bosons and Goldstone fermions. The integration over these fields is taken on what is called the maximal Riemannian submanifold of the supergroup $\text{Gl}(N_v|N_v)$ [6]. This is a combination of compact (as for the usual Goldstone bosons) and non-compact integration domain (for the Goldstone bosons of the ghost-ghost block). The factorization of $U(x)$ is still as above, but $\xi(x)$ is no longer traceless since the singlet field does not decouple in the quenched approximation. To leading order in the ϵ-expansion the partition function in a fixed topological sector is

$$\mathcal{Z}_\nu = \int dU_0 d\xi \, (\text{Sdet} U_0)^\nu \exp\left\{\frac{\Sigma}{2} V \text{Str} M(U_0 + U_0^{-1})\right.$$
$$\left. + \int d^4x \left[-\frac{1}{2}\text{Str}\left(\partial_\mu \xi \partial_\mu \xi\right) + \frac{m_0^2}{6}\Xi^2 + \frac{\alpha}{6}\partial_\mu \Xi \partial_\mu \Xi\right]\right\},$$

with $\Xi(x) = \text{Str}\xi(x)$ and Str denotes the supertrace.

An alternative to the supersymmetric method is the replica method [7], which amounts to taking the $N_f \to 0$ limit of the full QCD result. Chiral perturbation theory done this way is equivalent to that of the supersymmetric method, but the Feynman rules are somewhat simpler. Its shortcoming in the present context is that the integral over the zero momentum modes in general can only be performed through series expansions [8]. As a first check, we have confirmed the calculation of the leading correction to the chiral condensate in the ϵ-expansion [9], now using the supersymmetric method. Likewise, as a check on the results presented below we have performed all calculations both ways, and thus compared the resulting series expansions from the replica method with the closed expressions obtained by the supersymmetric method. In all cases we find complete agreement.

Here we present the first quenched results for correlation functions in the ϵ-expansion. We begin with the scalar and pseudoscalar correlation functions $s^a(x) \equiv \langle S^a(x)S^a(0)\rangle$ and $p^a(x) \equiv \langle P^a(x)P^a(0)\rangle$ to $\mathcal{O}(\epsilon^2)$, which is the leading order contribution to the space-time dependent terms. The $\mathcal{O}(1)$ contributions are constant. In a sector with fixed topology we find for the space integrals of flavor singlets

$$s^0(t) = \text{const} - \frac{\Sigma^2}{2F_\pi^2}\left[G(t)a_- - \Delta(t)\frac{a_+ + a_- - 4}{2}\right]$$
$$p^0(t) = \text{const} + \frac{\Sigma^2}{2F_\pi^2}\left[G(t)a_+ - \Delta(t)\frac{a_+ + a_- + 4}{2}\right]$$

where $V = L^3 T$ and

$$a_+ = 4\frac{\Sigma'_\nu(\mu)}{\Sigma} + 4 + 4\frac{\nu^2}{\mu^2},$$
$$a_- = -\frac{4}{\mu}\frac{\Sigma_\nu(\mu)}{\Sigma} + 4\frac{\nu^2}{\mu^2},$$

with the chiral condensate [6],

$$\frac{\Sigma_\nu(\mu)}{\Sigma} = \mu(I_\nu(\mu)K_\nu(\mu) + I_{\nu+1}(\mu)K_{\nu-1}(\mu)) + \frac{\nu}{\mu}.$$

The functions $G(t)$ and $\Delta(t)$ are, with $\tau = \frac{t}{T}$,

$$
\begin{aligned}
\Delta(t) &= Th_1(\tau) \\
G(t) &= -\frac{m_0^2}{3}T^3 h_2(\tau) + \frac{\alpha}{3}Th_1(\tau) \\
h_1(\tau) &= \frac{1}{2}\left[(\tau - \frac{1}{2})^2 - \frac{1}{12}\right] \\
h_2(\tau) &= \frac{1}{24}\left[\tau^2(1-\tau)^2 - \frac{1}{30}\right].
\end{aligned}
$$

The origin of the functions $h_{1,2}(\tau)$ has been discussed in detail in the literature [10]. As a check on the above result, in the limit $\mu \to \infty$ we find

$$
p^0(t) \to +\frac{2\Sigma^2}{F_\pi^2}\left[G(t) - \Delta(t)\right],
$$

which coincides with the leading order result from the p-expansion. This is simply tree level propagation of the flavor singlet.

We have also computed the flavored correlation functions. The calculation is more involved because in the supersymmetric formulation it is necessary to work with at least $N_v = 2$ [11] since the sources contain two flavors. All details will be given in a forthcoming publication. Here we just show a plot of the pseudoscalar correlation function $p^a(t)$, $a = 1, 2, 3$ in the ϵ and p-expansions, for different values of the quark mass close to the range of validity of both regimes[4]. As in the unquenched case, there is an overlap region at $M_\pi L \sim 1$ where both expansions are good if $m\Sigma V \gg 1$.

The volume dependence of the above formulae provides an extra handle to extract the low-energy constants Σ, F_π, m_0^2 and α in this regime. This technique has already been successfully applied to extract the constant Σ from the spectral density of the Dirac operator and from the quark condensate at finite volume [12].

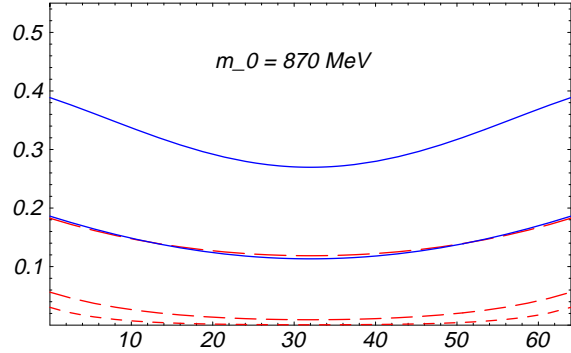[4]The constant terms have here been included only to leading order in the ϵ expansion.

Figure 1. Flavor-non-singlet $p^a(t)$ in lattice units with $a^{-1} \sim 1.5\text{GeV}, V = 32^3 \times 64$. Shown is the p-expansion (dashed lines, $m = 10, 3, 0.48\text{MeV}$) and the ϵ-expansion (solid lines, for $m = 0.48, 0.2\text{MeV}$). As m is decreased the curves go upward, as expected. In the intermediate mass region ($m \sim 0.5\text{Mev}$) there is nice agreement between the two expansions.

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