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# Some remarks on O($a$) improved twisted mass QCD*

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Twisted mass QCD (tmQCD) has been introduced as a solution to the problem of unphysical fermion zero modes in lattice QCD with quarks of the Wilson type. We here argue that O($a$) improvement of the tmQCD action and simple quark bilinear operators can be more economical than in the standard framework. In particular, an improved and renormalized estimator of the pion decay constant in two-flavour QCD is available, given only the Sheikholeslami-Wohlert coefficient $c_{\rm sw}$ and an estimate of the critical mass $m_{\rm c}$.

## 1. INTRODUCTION

Twisted mass QCD (tmQCD) [1,2] is a theoretically sound method to eliminate unphysical fermionic zero modes, which are at the origin of both conceptual and technical problems in lattice QCD with quarks of the Wilson type. In the continuum limit, tmQCD is equivalent to QCD with a standard mass term, provided the parameters and correlation functions are correctly matched [2]. This physical equivalence implies that the angle $\alpha$, defined by the ratio of twisted to standard mass parameter,

$$\tan(\alpha) = \mu_{\rm R}/m_{\rm R}, \tag{1}$$

is unphysical. Furthermore, the matching of theories defined at different values of $\alpha$ induces a mapping between composite fields, which can be used to circumvent certain lattice renormalization problems. Examples for this are the definition of the order parameter of chiral symmetry, the pion decay constant, and some matrix elements of the effective weak hamiltonian [2,3].

Here we want to review some features of O($a$) improved tmQCD [4,5]. We restrict attention to the action and quark bilinear operators which appear in the PCAC and PCVC relations, i.e. the simplest Ward identities associated with chiral and flavour symmetries.

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## 2. SET-UP

The lattice Dirac operator of two-flavour QCD with a chirally twisted mass term is given by

$$D_{\rm twist} \stackrel{\rm def}{=} D_{\rm W} + m_0 + i\mu_{\rm q}\gamma_5\tau^3. \tag{2}$$

where $D_{\rm W}$ denotes the massless Wilson-Dirac operator and the Pauli matrix $\tau^3$ acts in flavour space. Assuming that O($a$) improvement has been implemented in the massless theory [6], the O($a$) improved bare parameters of the action are given by

$$\tilde{g}_0^2 = g_0^2(1 + b_{\rm g} a m_{\rm q}), \tag{3}$$
$$\widetilde{m}_{\rm q} = m_{\rm q} + b_{\rm m} a m_{\rm q}^2 + \tilde{b}_{\rm m} a \mu_{\rm q}^2, \tag{4}$$
$$\widetilde{\mu}_{\rm q} = \mu_{\rm q}(1 + b_\mu a m_{\rm q}), \tag{5}$$

where $m_{\rm q} = m_0 - m_{\rm c}$ is the subtracted bare mass. In a mass-independent scheme the renormalized parameters then take the form

$$g_{\rm R}^2 = \tilde{g}_0^2 Z_{\rm g}(\tilde{g}_0^2, a\mu), \tag{6}$$
$$m_{\rm R} = \widetilde{m}_{\rm q} Z_{\rm m}(\tilde{g}_0^2, a\mu), \tag{7}$$
$$\mu_{\rm R} = \widetilde{\mu}_{\rm q} Z_\mu(\tilde{g}_0^2, a\mu). \tag{8}$$

O($a$) improvement of the action thus introduces the improvement coefficients $b_\mu$ and $\tilde{b}_{\rm m}$, in addition to the standard coefficients $b_{\rm m}$ and $b_{\rm g}$. At this point we note that none of the coefficients proportional to $m_{\rm q}$ is needed if $m_{\rm q} = {\rm O}(a)$, as their contribution to physical quantities is then of O($a^2$) and thus negligible in the spirit of O($a$) improvement. Furthermore, eq. (7) then implies $m_{\rm R} = {\rm O}(a)$ so that the physical quark mass

$(m_\mathrm{R}^2+\mu_\mathrm{R}^2)^{1/2}$ is essentially defined by $\mu_\mathrm{R}$, and the angle $\alpha$ is close to $\pi/2$.

We now consider the $\mathrm{O}(a)$ improved bare composite fields [6,4]

$$(A_\mathrm{I})^a_\mu = A^a_\mu + c_\mathrm{A} a\tilde{\partial}_\mu P^a + a\mu_\mathrm{q}\tilde{b}_\mathrm{A}\varepsilon^{3ab}V^b_\mu, \quad (9)$$
$$(V_\mathrm{I})^a_\mu = V^a_\mu + c_\mathrm{V} a\tilde{\partial}_\nu T^a_{\mu\nu} + a\mu_\mathrm{q}\tilde{b}_\mathrm{V}\varepsilon^{3ab}A^b_\mu, \quad (10)$$
$$(P_\mathrm{I})^a = P^a, \quad (11)$$

with isospin index $a = 1,2$. Renormalized improved operators are then multiplicatively related to the improved bare ones,

$$(A_\mathrm{R})^a_\mu = Z_\mathrm{A}(\tilde{g}_0^2)(1+b_\mathrm{A} am_\mathrm{q})(A_\mathrm{I})^a_\mu, \quad (12)$$
$$(V_\mathrm{R})^a_\mu = Z_\mathrm{V}(\tilde{g}_0^2)(1+b_\mathrm{V} am_\mathrm{q})(V_\mathrm{I})^a_\mu, \quad (13)$$
$$(P_\mathrm{R})^a = Z_\mathrm{P}(\tilde{g}_0^2, a\mu)(1+b_\mathrm{P} am_\mathrm{q})(P_\mathrm{I})^a. \quad (14)$$

While the improvement coefficients may be considered functions of the bare coupling $g_0$, consistent $\mathrm{O}(a)$ improvement implies that the $Z$-factors are functions of the improved bare coupling $\tilde{g}_0^2$ [6]. As non-perturbative determinations of $b_\mathrm{g}$ are not available (cf., however, [7]), this renders chiral extrapolations of $\mathrm{O}(a)$ improved matrix elements difficult[2]. Compared to the standard framework at $\mu_\mathrm{q} = 0$ we note that in $\mathrm{O}(a)$ improved tmQCD there are only two new coefficients ($\tilde{b}_\mathrm{V}$ and $\tilde{b}_\mathrm{A}$) required to improve the above operators. Considering again $\alpha = \pi/2$ this means that the massive theory is improved with a single new coefficient $\tilde{b}_\mathrm{m}$ in the action (rather than two, $b_\mathrm{g}$ and $b_\mathrm{m}$), while the operators are improved with $\tilde{b}_\mathrm{A}$ and $\tilde{b}_\mathrm{V}$ as compared to $b_\mathrm{V}, b_\mathrm{A}, b_\mathrm{P}$. This comparison becomes even more favourable if one takes into account a generic

## 3. REDUNDANCY OF IMPROVEMENT COEFFICIENTS

To illustrate this point we consider the renormalized 2-point functions

$$G_\mathrm{A}(x-y) = \left\langle (A_\mathrm{R})^1_0(x)(P_\mathrm{R})^1(y)\right\rangle, \quad (15)$$
$$G_\mathrm{V}(x-y) = \left\langle (V_\mathrm{R})^2_0(x)(P_\mathrm{R})^1(y)\right\rangle, \quad (16)$$

which, for the proper choice of the improvement coefficients, are expected to converge to their continuum limit with $\mathrm{O}(a^2)$ corrections. If all improvement coefficients were necessary one would expect uncancelled $\mathrm{O}(a)$ effects to arise if any of the improvement coefficients is modified by terms of $\mathrm{O}(1)$. Denoting such shifts to $\tilde{b}_\mathrm{m}, b_\mu, \tilde{b}_\mathrm{A}, \tilde{b}_\mathrm{V}$ by $\Delta\tilde{b}_\mathrm{m}$ etc., we find that the induced $\mathrm{O}(a)$ effect in $G_\mathrm{A}$ is of the form

$$\Delta G_\mathrm{A}(x) \propto a\mu_\mathrm{R}\Big[\Delta\tilde{b}_\mathrm{m}\mu_\mathrm{R}\frac{\partial}{\partial m_\mathrm{R}}G_\mathrm{A}(x) + \Delta b_\mu c_1 m_\mathrm{R}\frac{\partial}{\partial \mu_\mathrm{R}}G_\mathrm{A}(x) + \Delta\tilde{b}_\mathrm{A} c_2 G_\mathrm{V}(x)\Big], \quad (17)$$

with some constants $c_1, c_2$ which can be easily worked out [4]. Now, due to the identity in the continuum limit,

$$\left(m_\mathrm{R}\frac{\partial}{\partial\mu_\mathrm{R}} - \mu_\mathrm{R}\frac{\partial}{\partial m_\mathrm{R}}\right)G_\mathrm{A}(x) = -G_\mathrm{V}(x), \quad (18)$$

one concludes that $\Delta\tilde{b}_\mathrm{m}$, $\Delta b_\mu$ and $\Delta\tilde{b}_\mathrm{A}$ need not vanish separately for $G_\mathrm{A}(x)$ to remain $\mathrm{O}(a)$ improved. We find this to be a generic feature of tmQCD, which can be traced back to the equivalence of correlation functions of tmQCD and QCD in the continuum limit. In fact, the l.h.s. of eq.(18) is nothing else but the derivative with respect to the unphysical parameter $\alpha$.

$\mathrm{O}(a)$ improved tmQCD as introduced above may hence be regarded as a one-parameter family of $\mathrm{O}(a)$ improved theories. Choosing $\tilde{b}_\mathrm{m}$ as the free parameter, we set

$$\tilde{b}_\mathrm{m} = -\tfrac{1}{2}, \quad (19)$$

*exactly.* The other coefficients are then fixed, and in perturbation theory given by [$C_\mathrm{F} = (N^2-1)/2N$]

$$b_\mu = -0.103(3)\,C_\mathrm{F} g_0^2 + \mathrm{O}(g_0^4), \quad (20)$$
$$\tilde{b}_\mathrm{A} = 0.086(4)\,C_\mathrm{F} g_0^2 + \mathrm{O}(g_0^4), \quad (21)$$
$$\tilde{b}_\mathrm{V} = 0.074(3)\,C_\mathrm{F} g_0^2 + \mathrm{O}(g_0^4). \quad (22)$$

Indeed, the choice (19) is partially motivated by the fact that the tree level values of these coefficients then vanish. Beyond perturbation theory, $\tilde{b}_\mathrm{A}$ can be determined through the PCAC relation [8], $\tilde{b}_\mathrm{V}$ can be obtained by imposing the physical parity symmetry in tmQCD at finite $a$, whereas the PCVC relation involves the combination $b_\mathrm{P} + b_\mu$. We also note that tmQCD offers new ways to determine some of the standard

[2] This problem does not occur in the quenched approximation where $b_\mathrm{g} = 0$.

coefficients such as $b_{\rm A}$, by matching appropriate correlation functions of tmQCD and standard QCD. This is not too surprising if one recalls that tmQCD and QCD are, in the continuum, related by a chiral symmetry transformation, and given the fact that most of the standard coefficients are determined by chiral Ward identities [9].

## 4. O($a$) IMPROVED $F_\pi$

In tmQCD the flavour symmetry is only softly broken by the twisted mass term. As a consequence, there exists a vector current $\tilde{V}_\mu^a$, which satisfies the PCVC relation exactly,

$$\partial_\mu^* \tilde{V}_\mu^a = -2\mu_{\rm q}\varepsilon^{3ab}P^b. \qquad (23)$$

This vector current is protected against renormalization, which implies $Z_\mu = Z_{\rm P}^{-1}$ in any scheme which respects the Ward identities. Recalling that the vector current in tmQCD at $m_{\rm R}=0$ is interpreted as the physical axial current, the pion decay constant $F_\pi$ can be extracted from the asymptotic behaviour of the correlation function,

$$\mu_{\rm q}\tilde{G}_{\rm P}(x_0) = a^3\sum_{\mathbf{x}}\mu_{\rm q}\left\langle P^1(x)(P_{\rm R})^1(0)\right\rangle, \qquad (24)$$

for large times $x_0$, after division by $m_\pi^2$ and by the wave function renormalization of the interpolating field $P^1$. Given the renormalization properties of both $P^1$ and $\mu_{\rm q}$ we conclude that $F_\pi$ is obtained with O($a^2$) errors only. First studies of $F_\pi$ using this method in the quenched approximation have been presented in [10].

Generalizing this procedure to non-vanishing $m_{\rm R}$, one considers the r.h.s. of the physical PCAC relation,

$$\sqrt{m_{\rm R}^2+\mu_{\rm R}^2}\left\langle P_{\rm R}^1(x)P_{\rm R}^1(0)\right\rangle. \qquad (25)$$

From this more general expression one infers that $F_\pi$ remains O($a$) improved even if $m_{\rm R} = {\rm O}(a)$ rather than O($a^2$) [11].

## 5. CONCLUSIONS

We have argued that O($a$) improvement of tmQCD at $m_{\rm R}=0 \Leftrightarrow \alpha=\pi/2$ is more economical than in the standard theory. No additional free parameters arise in the action, so that $m_{\rm R}=0$ can be satisfied up to O($a^2$), provided the coefficients $c_{\rm sw}, c_{\rm A}$ are known[3]. Moreover, an O($a$) improved estimate of $F_\pi$ can even be obtained with the knowledge of $c_{\rm sw}$ alone. This is to be confronted with the standard situation where also $Z_{\rm A}, b_{\rm g}, c_{\rm A}, b_{\rm A}$ are required. An interesting aspect of O($a$) improved tmQCD is the absence of a quark mass dependent rescaling of $g_0$, which allows for chiral extrapolations to be done at fixed $g_0$, whilst maintaining O($a$) improvement.

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[3] The knowledge of $c_{\rm A}$ is required if one needs an O($a$) improved estimate of the critical mass from the PCAC relation.